Correlated MIMO Channel Capacity*

T. Ratnarajah[†] [‡] R. Vaillancourt[†][§]

CRM-2937

November 2003

*Partially supported by the Natural Sciences and Engineering Research Council of Canada and the Centre de recherches mathématiques of the Université de Montréal.

†Department of Mathematicis and Statistics, University of Ottawa, 585 King Edward Ave., Ottawa, ON K1N 6N5 Canada

‡t.ratnarajah@ieee.org

§remi@uottawa.ca

Abstract

In this paper, the correlated multiple-input, multiple-output (MIMO) channel capacity is studied by using the eigenvalue densities of complex central Wishart matrices. These densities are represented by complex hypergeometric functions of matrix arguments, which can be expressed in terms of complex zonal polynomials. We derive a close form ergodic capacity formula for correlated MIMO channels. It is shown how the channel correlation degrades the capacity of the communication system.

To appear in Proceeding of the Thirty-Seven Asilomar Conference on Signals, Systems and Computers, Pacific Grove, California, Nov. 9–12, 2003.

Résumé

On étudie la capacité d'un canal multi-entrée multi-sortie (MIMO) corrélé au moyen des densités des valeurs propres de matrices de Wishart centrales complexes. On représente ces densités au moyen de fonctions hypergéométriques d'arguments matriciels, qu'on peut aussi exprimer au moyen de polynômes zonaux centraux. On dérive un formule pour la capacité ergodique, sous forme close, pour les canaux MIMO. On montre que la corrélation du canal réduit la capacité du système de communication.

1 Introduction

Let an $n \times m$ complex Gaussian random matrix A be distributed as $\mathrm{A} \sim \mathcal{C}N(0, I_n \otimes \Sigma)$ with mean $\mathcal{E}\{\mathrm{A}\} = 0$ and covariance $\mathrm{cov}\{\mathrm{A}\} = I_n \otimes \Sigma$. Then the matrix $\mathrm{W} = \mathrm{A}^H\mathrm{A}$ is called a complex central Wishart matrix and its distribution is denoted by $\mathcal{C}W_m(n, \Sigma)$.

In this work, we investigate the eigenvalue densities of complex central Wishart matrices and their applications. In contrast to the literature in [6], we consider that the elements of random matrices are complex Gaussian distributed with zero mean and arbitrary covariance matrices. This will enable us to consider the beautiful but difficult theory of complex zonal polynomials, which are symmetric polynomials in the eigenvalues of a complex matrix [4]. Complex zonal polynomials enable us to represent the eigenvalue densities of these complex central Wishart matrices as infinite series.

The theory of these complex central Wishart matrices is used to evaluate the capacity of MIMO wireless communication systems. Note that the capacity of a communication channel expresses the maximum rate at which information can be reliably conveyed by the channel [1]. Let us denote the number of inputs (or transmitters) and number of outputs (or receivers) of the MIMO wireless communications system by n_t and n_r, respectively, and assume the channel coefficients are distributed as complex Gaussian and correlated at the transmitter end. This means the channel coefficients from two different transmitter antennas to a single receiver antenna can be correlated. This channel correlation, which degrades capacity [2], depends on the physical parameters of a MIMO system and the scatterer characteristics. The physical parameters include the antenna arrangement and spacing, the angle spread, the angle of arrival, etc. One of the objectives of this paper is to evaluate this capacity degradation for the correlated channel matrix. This will be done by deriving closed-form ergodic capacity formulas for correlated channels and their numerical evaluation.

This paper is organized as follows. The correlated MIMO channel capacity is studied in Section 2 and the computational method is given in Section 3.

2 The Correlated Channel Capacity

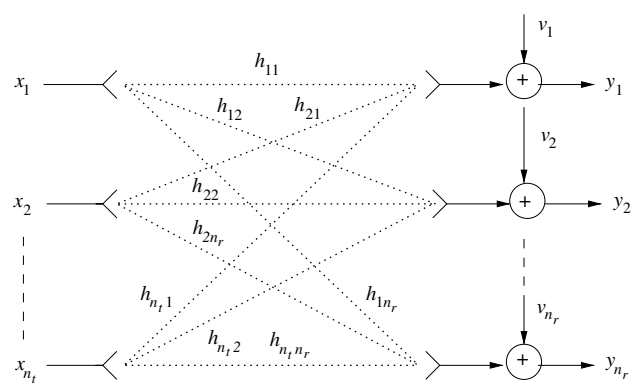

Figure 1: A MIMO communication system.

The complex signal received at the jth output can be written as

$$y_j = \sum_{i=1}^{n_t} h_{ij} x_i + v_j, \tag{1}$$

where h_{ij} is the complex channel coefficient between input i and output j, x_i is the complex signal at the ith input and v_j is complex Gaussian noise, as shown in Figure 1. The signal vector received at the output can be written as

$$\begin{bmatrix} y_1 \\ \vdots \\ y_{n_r} \end{bmatrix} = \begin{bmatrix} h_{11} & \cdots & h_{n_t1} \\ \vdots & \vdots & \vdots \\ h_{1n_r} & \cdots & h_{n_tn_r} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n_t} \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_{n_r} \end{bmatrix},$$

i.e., in vector notation,

$$y = Hx + v, \tag{2}$$

where $y, v \in \mathbb{C}^{n_r}$, $H \in \mathbb{C}^{n_r \times n_t}$, and $x \in \mathbb{C}^{n_t}$. The total power of the input is constrained to ρ,

$$\mathcal{E}\{x^H x\} \leq \rho \quad \text{or} \quad \operatorname{tr} \mathcal{E}\{x x^H\} \leq \rho.$$

Let $n_r \geq n_t$ and the channel matrix be distributed as complex Gaussian with zero mean and covariance matrix $I_{n_r} \otimes \Sigma$, i.e., $\mathrm{H} \sim \mathcal{C}N(0, I_{n_r} \otimes \Sigma)$, where Σ represents the channel correlation at the transmitter end. This means that the covariance matrix of the rows of H is given by an $n_t \times n_t$ Hermitian matrix Σ. Note that the off-diagonal elements of Σ are nonzero for correlated channels. If $\Sigma = I_{n_t}$ (or $\sigma^2 I_{n_t}$) then the channel is called *uncorrelated Rayleigh distributed channel*. We assume that the realization of H is known only to the receiver but not to the transmitter and power is distributed equally over all transmitting antennas. Moreover, if we assume a block-fading model and coding over many independent fading intervals, then the Shannon or ergodic capacity of the random MIMO channel is given in [6] by

$$\begin{aligned} C &= \mathcal{E}_{\mathrm{H}} \left\{ \log\det\left(I_{n_t} + (\rho/n_t)\mathrm{H}^H\mathrm{H}\right)\right\} \\ &= \mathcal{E}_{\mathrm{W}} \left\{ \log\det\left(I_{n_t} + (\rho/n_t)\mathrm{W}\right)\right\}, \end{aligned} \tag{3}$$

where expectation is evaluated using a complex central Wishart distribution, denoted by $\mathrm{W} = \mathrm{H}^H\mathrm{H} \sim \mathcal{C}W_{n_t}(n_r, \Sigma)$. The density is given in [3] by

$$f(W) = \frac{1}{\mathcal{C}\Gamma_{n_t}(n_r)(\det\Sigma)^{n_r}} (\det W)^{n_r - n_t} \operatorname{etr}\left(-\Sigma^{-1}W\right),$$

where $\operatorname{etr}(\cdot) \equiv e^{\operatorname{tr}(\cdot)} \equiv \exp\operatorname{tr}(\cdot)$. Let $\lambda_1 > \cdots > \lambda_{n_t}$ be the eigenvalues of W and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n_t})$. Then the capacity can also be computed using the joint eigenvalue density $f(\Lambda)$ or the single unordered eigenvalue density $f(\lambda)$, i.e.,

$$\begin{aligned} C &= \mathcal{E}_{\Lambda}\left\{\log\left(\prod_{k=1}^{n_t}\left[1 + (\rho/n_t)\lambda_k\right]\right)\right\} \\ &= \sum_{k=1}^{n_t} \mathcal{E}_{\lambda_k}\left\{\log(1 + (\rho/n_t)\lambda_k)\right\} \\ &= n_t \mathcal{E}_{\lambda}\left\{\log(1 + (\rho/n_t)\lambda)\right\}. \end{aligned} \tag{4}$$

The joint eigenvalues density of a complex central Wishart matrix is given by

$$\begin{aligned} f(\Lambda) = &\frac{\pi^{n_t(n_t-1)}(\det\Sigma)^{-n_r}}{\mathcal{C}\Gamma_{n_t}(n_t)\mathcal{C}\Gamma_{n_t}(n_r)} \prod_{k=1}^{n_t} \lambda_k^{n_r - n_t} \prod_{k<l}^{n_t} (\lambda_k - \lambda_l)^2 \\ &\times {}_0F_0^{(n_t)}\left(-\Sigma^{-1}, \Lambda\right). \end{aligned} \tag{5}$$

Next we define the hypergeometric function ${}_0F_0^{(n_t)}(\cdot,\cdot)$. Let $\kappa = (k_1, \ldots, k_{n_t})$ be a partition of the integer k with $k_1 \geq \cdots \geq k_{n_t} \geq 0$ and $k = k_1 + \cdots + k_{n_t}$. The function ${}_0F_0^{(n_t)}(\cdot,\cdot)$ of two complex matrix arguments is defined by

$${}_0F_0^{(n_t)}(X, Y) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(X) C_\kappa(Y)}{k!\; C_\kappa(I_{n_t})}, \tag{6}$$

where $X, Y \in \mathbb{C}^{n_t \times n_t}$ and $\sum_\kappa$ denotes summation over all partitions κ of k. The complex zonal polynomial $C_\kappa(X)$ of a complex matrix X is defined by

$$C_\kappa(X) = \chi_{[\kappa]}(1)\chi_{[\kappa]}(X), \tag{7}$$

where $\chi_{[\kappa]}(1)$ is the dimension of the representation $[\kappa]$ of the symmetric group given by

$$\chi_{[\kappa]}(1) = k!\, \frac{\prod_{i<j}^{n_t}(k_i - k_j - i + j)}{\prod_{i=1}^{n_t}(k_i + n_t - i)!}, \tag{8}$$

and $\chi_{[\kappa]}(X)$ is the character of the representation $[\kappa]$ of the linear group given as a symmetric function of the eigenvalues, $\mu_1, \ldots, \mu_{n_t}$, of X by

$$\chi_{[\kappa]}(X) = \frac{\det\left[\left(\mu_i^{k_j + n_t - j}\right)\right]}{\det\left[\left(\mu_i^{n_t - j}\right)\right]}. \tag{9}$$

Moreover, $C_\kappa(I_{n_t})$ is given by

$$C_\kappa(I_{n_t}) = 2^{2k} k! \left[\frac{1}{2}\, n_t\right]_\kappa \frac{\prod_{i<j}^{r}(2k_i - 2k_j - i + j)}{\prod_{i=1}^{r}(2k_i + r - i)!},$$

where

$$\left[\frac{1}{2} n_t\right]_\kappa = \prod_{i=1}^{r} \left(\frac{1}{2}(n_t - i + 1)\right)_{k_i}$$

and $(a)_k = a(a+1)\cdots(a+k-1)$. Note that the partition κ of k has r nonzero parts. From the joint eigenvalue density (5) we can obtain a single unordered eigenvalue density $f(\lambda)$ by dividing $f(\Lambda)$ by $n_t!$ and integrating with respect to $\lambda_2, \ldots, \lambda_{n_t}$, see [5].

3 Correlated $n_r \times 2$ channel matrix

In this section, a numerical evaluation of a correlated Rayleigh $n_r \times 2$ channel capacity is given. Thus, we assume that we have a two-input ($n_t = 2$), n_r-output wireless communication system operating over a correlated Rayleigh fading environment. The joint eigenvalue density of a central Wishart matrix depends on the population covariance matrix Σ only through its eigenvalues $\upsilon_1, \ldots, \upsilon_{n_t}$, i.e.,

$$ {}_0F_0^{(n_t)}\left(-\Sigma^{-1}, \Lambda\right) = {}_0F_0^{(n_t)}\left(-\Upsilon^{-1}, \Lambda\right), \tag{10}$$

where $\Upsilon = \mathrm{diag}(\upsilon_1, \ldots, \upsilon_{n_t})$. Let $n_t = 2$ and $\Upsilon^{-1} = \mathrm{diag}(a_1, a_2)$. Then we have

$$\begin{aligned} {}_0F_0^{(2)}(-\Upsilon^{-1}, \Lambda) = & \frac{1}{(a_2 - a_1)(\lambda_1 - \lambda_2)} \\ & \times \left[\exp\left\{-(a_1\lambda_1 + a_2\lambda_2)\right\} - \exp\left\{-(a_1\lambda_2 + a_2\lambda_1)\right\}\right]. \end{aligned} \tag{11}$$

The following theorem gives the correlated Rayleigh channel capacity for an $n_r \times 2$ matrix.

Theorem 1 *Consider a two-input correlated Rayleigh channel,* $\mathrm{H} \sim \mathcal{C}N(0, I_{n_r} \otimes \Sigma)$*, with* $n_r \geq 2$*. If the input power is constrained by* ρ*, then the capacity* C *is given by*

$$\begin{aligned} & \frac{a_1^{n_r} a_2}{(a_2 - a_1)\Gamma(n_r)} \int_0^\infty \log[1 + (\rho/2)\lambda]\lambda^{n_r - 1} e^{-a_1\lambda} d\lambda - \\ & \frac{a_1 a_2^{n_r}}{(a_2 - a_1)\Gamma(n_r)} \int_0^\infty \log[1 + (\rho/2)\lambda]\lambda^{n_r - 1} e^{-a_2\lambda} d\lambda - \\ & \frac{a_1^{n_r}}{(a_2 - a_1)\Gamma(n_r - 1)} \int_0^\infty \log[1 + (\rho/2)\lambda]\lambda^{n_r - 2} e^{-a_1\lambda} d\lambda + \\ & \frac{a_2^{n_r}}{(a_2 - a_1)\Gamma(n_r - 1)} \int_0^\infty \log[1 + (\rho/2)\lambda]\lambda^{n_r - 2} e^{-a_2\lambda} d\lambda, \end{aligned}$$

where λ *is an unordered eigenvalue of* $W = H^H H$ *and* (a_1, a_2) *are eigenvalues of* Σ^{-1}*.*

Proof. Using Equation (11), the unordered eigenvalue density of W is given by

$$\begin{aligned} f(\lambda_1, \lambda_2) = & \frac{(a_1 a_2)^{n_r} (\lambda_1 \lambda_2)^{n_r - 2} (\lambda_1 - \lambda_2)}{2(a_2 - a_1)\Gamma(n_r)\Gamma(n_r - 1)} \\ & \times \left[e^{-a_1\lambda_1 - a_2\lambda_2} - e^{-a_1\lambda_2 - a_2\lambda_1}\right]. \end{aligned}$$

Now, integrating with respect to λ_2 and noting that

$$\int_0^\infty x^{a-1} e^{-x/b} dx = \Gamma(a) b^a,$$

we obtain the density of λ_1 (say λ). Thus we have

$$\begin{aligned} f(\lambda) \quad = \quad & \frac{1}{2(a_2 - a_1)} \left\{ \frac{a_1^{n_r} a_2 \lambda^{n_r - 1} e^{-a_1\lambda}}{\Gamma(n_r)} \right. \\ & - \frac{a_1 a_2^{n_r} \lambda^{n_r - 1} e^{-a_2\lambda}}{\Gamma(n_r)} - \frac{a_1^{n_r} \lambda^{n_r - 2} e^{-a_1\lambda}}{\Gamma(n_r - 1)} \\ & \left. + \frac{a_2^{n_r} \lambda^{n_r - 2} e^{-a_2\lambda}}{\Gamma(n_r - 1)} \right\}. \end{aligned}$$

It is easy to see that $\int_0^\infty f(\lambda)d\lambda = 1$. Finally, evaluating equation (4) with $f(\lambda)$ gives Theorem 1. □

Table 1 shows the capacity in nats[5] for an $n_r \times 2$ correlated Rayleigh fading channel matrix with correlation coefficient 0.9. Note that each column represents different levels of input power or signal-to-noise ratio (SNR) in dB. Note that we choose the channel covariance matrix as

$$\Sigma = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$$

and its eigenvalues are 1.9 and 0.1. Hence $\Upsilon = \text{diag}(1.9, 0.1)$ and $a_1 = 1/1.9$, $a_2 = 1/0.1$. It should be noted that the off-diagonal element of Σ gives the correlation between the channel coefficient from different transmitter antennas to a single receiver antenna, i.e.,

$$\mathcal{E}\{h_{ij}h_{kl}^*\} = \begin{cases} 0.9 & i \neq k = 1, 2, \ j = l = 1, \ldots, n_r \\ 0 & \text{otherwise.} \end{cases}$$

We called this off-diagonal element as a channel correlation coefficient or correlation coefficient. Figure 2 shows

	ρ in dB			
n_r	0 dB	10 dB	20 dB	30 dB
2	1.0326	3.1157	6.2023	9.7221
4	1.6408	4.4118	8.4439	12.9577
6	2.0685	5.1852	9.4266	13.9863
8	2.4033	5.7454	10.0862	14.6604
10	2.6804	6.1838	10.5817	15.1635

Table 1: The capacity in nats for a two-input, n_r-output communication system operating over a correlated Rayleigh fading channel, where ρ is SNR in dB and the correlation coefficient is equal to 0.9.

the capacity in nats vs the correlation coefficient. Figure 3 shows the capacity in nats vs SNR for the correlation coefficient 0.2 and Figure 4 shows the capacity vs n_r. From these tables and figures we note the following: (i) the capacity is decreasing with increasing channel correlation, (ii) the capacity is increasing with increasing n_r and SNR. Due to space limitation we only give the capacity evaluation of a correlated Rayleigh $n_r \times 2$ channel matrix, see [5] for details.

Next, the capacity formula of an uncorrelated Rayleigh $n_r \times 2$ channel matrix is given. In other words, we assumed we have a two-input ($n_t = 2$), n_r-output wireless communication system operating over an uncorrelated Rayleigh fading environment, which is a typical fixed wireless environment. The following theorem gives the capacity expression.

Theorem 2 *Consider a two-input uncorrelated Rayleigh channel, i.e.,* $\mathrm{H} \sim \mathcal{C}N(0, I_{n_r} \otimes \sigma^2 I_2)$, *with* $n_r \geq 2$. *If the input power is constrained by* ρ, *then the capacity* C *is given by*

$$\frac{(\sigma^2)^{-n_r-1}}{\Gamma(n_r)} \int_0^\infty \log[1 + (\rho/2)\lambda]\lambda^{n_r} e^{-\lambda/\sigma^2} d\lambda -$$
$$\frac{2(\sigma^2)^{-n_r}}{\Gamma(n_r - 1)} \int_0^\infty \log[1 + (\rho/2)\lambda]\lambda^{n_r-1} e^{-\lambda/\sigma^2} d\lambda +$$
$$\frac{(\sigma^2)^{-n_r+1}\Gamma(n_r + 1)}{\Gamma(n_r)\Gamma(n_r - 1)} \int_0^\infty \log[1 + (\rho/2)\lambda]\lambda^{n_r-2} e^{-\lambda/\sigma^2} d\lambda,$$

where λ *is an unordered eigenvalue of* $W = H^H H$.

Proof. In this case the unordered eigenvalue density of W is given in [5] by

$$f(\lambda_1, \lambda_2) = \frac{(\sigma^2)^{-2n_r}(\lambda_1\lambda_2)^{n_r-2}(\lambda_1 - \lambda_2)^2}{2\Gamma(n_r)\Gamma(n_r - 1)} e^{-(\lambda_1+\lambda_2)/\sigma^2}.$$

[5] In Theorem 1, if we use $\log_e$ then the capacity is measured in nats. If we use $\log_2$ then the capacity is measured in bits. Thus, one nat is equal to e bits/sec/Hz ($e = 2.718\ldots$).

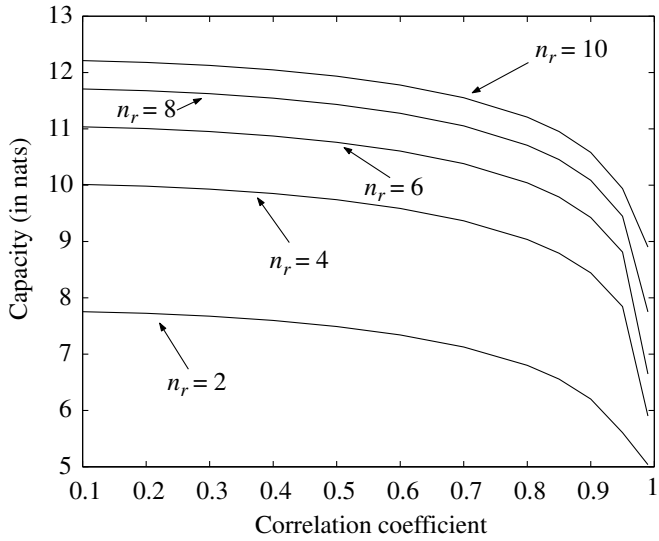

Figure 2: Capacity vs channel correlation coefficient for SNR= 20dB, $n_t = 2$, and $n_r = 2, 4, 6, 8, 10$, i.e., H is an $n_r \times 2$ correlated Rayleigh fading channel matrix.

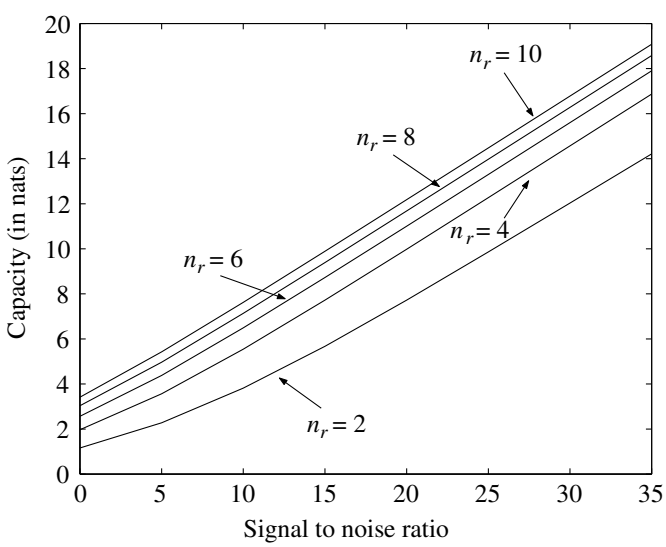

Figure 3: Capacity vs SNR for correlation coefficient 0.2, $n_t = 2$, and $n_r = 2, 4, 6, 8, 10$, i.e., H is an $n_r \times 2$ correlated Rayleigh fading channel matrix.

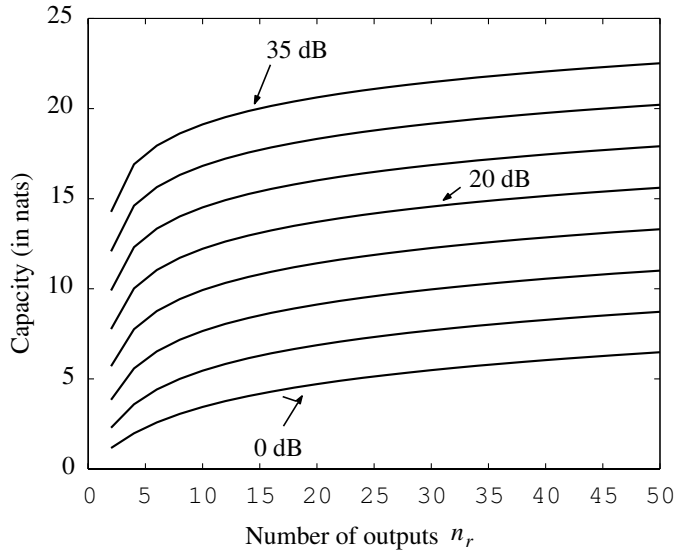

Figure 4: Capacity vs number of outputs for SNR= $0, 5, 10, 15, 20, 25, 30, 35$dB. Note that H is an $n_r \times 2$ correlated Rayleigh fading channel matrix with correlation coefficient equal to 0.9.

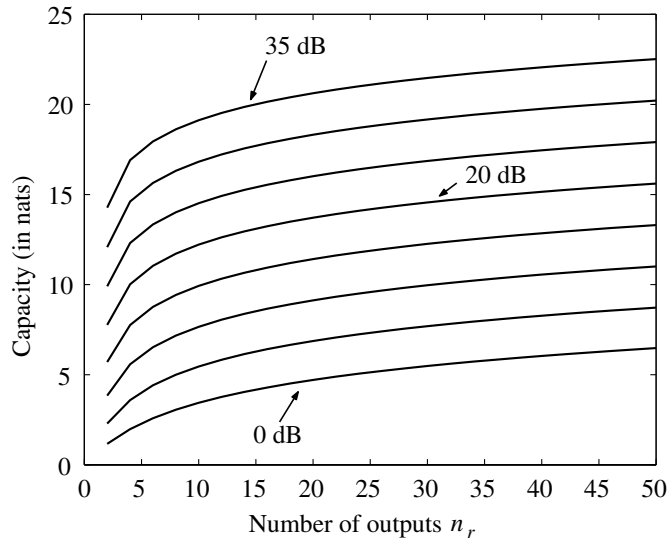

Figure 5: Capacity vs number of outputs for SNR= $0, 5, 10, 15, 20, 25, 30, 35$ dB. Note that H is an $n_r \times 2$ uncorrelated Rayleigh fading channel matrix.

Integrating with respect to λ_2 we obtain the density of λ_1 (say λ),

$$f(\lambda) = \frac{(\sigma^2)^{-n_r-1}}{2\Gamma(n_r)}\lambda^{n_r}\, e^{-\lambda/\sigma^2} - \frac{(\sigma^2)^{-n_r}}{\Gamma(n_r-1)}\lambda^{n_r-1}\, e^{-\lambda/\sigma^2} + \frac{(\sigma^2)^{-n_r+1}\Gamma(n_r+1)}{2\Gamma(n_r)\Gamma(n_r-1)}\lambda^{n_r-2}\, e^{-\lambda/\sigma^2}.$$

It is easy to see that $\int_0^\infty f(\lambda)\, d\lambda = 1$. Finally, evaluating (4) with $f(\lambda)$ gives Theorem 2. □

	ρ in dB			
n_r	0 dB	10 dB	20 dB	30 dB
2	1.1671	3.8382	7.7633	12.0815
4	1.9831	5.5788	10.0227	14.6102
6	2.5857	6.5274	11.0462	15.6425
8	3.0573	7.1725	11.7191	16.3183
10	3.4425	7.6605	12.2214	16.8221

Table 2: The capacity in nats for a two-input, n_r-output communication system operating over an uncorrelated Rayleigh fading channel, where ρ is SNR in dB.

Table 2 shows the capacity in nats for an $n_r \times 2$ uncorrelated Rayleigh fading channel matrix with different levels of input power. Figure 5 shows the capacity in nats vs n_r for different SNR. It is clearly seen from the table and figure that the capacity is increasing with increasing n_r and SNR. Moreover, comparing Table 1 and Table 2 we can evaluate the capacity degradation due channel correlations. Note that, if $\sigma^2 = 1$, then a similar uncorrelated Rayleigh channel capacity result can be obtained from [6].

4 Conclusion

In this paper, joint and single unordered eigenvalue densities of complex central Wishart matrices are derived. These densities are used to derive formulas for the capacity of correlated and uncorrelated MIMO Rayleigh channels. The capacity of $n_r \times 2$ MIMO Rayleigh channel matrices are computed for both correlated and uncorrelated channels. This study shows how the channel correlation degrades the capacity of the communication system.

References

[1] R. B. Ash, *Information Theory*, Dover, New York, 1965.

[2] C. N. Chuah, D. Tse, J. M. Kahn and R. A. Valenzuela, "Capacity scaling in MIMO wireless systems under correlated fading", *IEEE Trans. on Information Theory*, Vol. 48, pp. 637–650, Mar. 2002.

[3] A. T. James, "Distributions of matrix variate and latent roots derived from normal samples", *Ann. Math. Statist.*, Vol. 35, pp. 475–501, 1964.

[4] R. J. Muirhead, *Aspects of Multivariate Statistical Theory*, Wiley, New York, 1982.

[5] T. Ratnarajah, R. Vaillancourt and M. Alvo, "Complex random matrices and Rayleigh Channel Capacity", *Communications in Information & Systems*, Vol. 3, No. 2, pp. 119-138, Oct. 2003 (http://www.ims.cuhk.edu.hk/~cis/).

[6] I. E. Telatar, "Capacity of multi-antenna Gaussian channels", *Eur. Trans. Telecom*, Vol. 10, pp. 585–595, 1999.