Systems of coupled Poisson equations with critical growth in unbounded domains^{*}

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Abstract

We establish the existence of a nontrivial solution to systems of coupled Poisson equations with critical growth in unbounded domains. The proofs rely on a generalized linking theorem due to Krysewski and Szulkin [9], and on a concentration-compactness argument since the Palais-Smale condition fails at all critical levels.

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1 Introduction

In this paper, we consider the following systems of coupled Poisson equations with critical growth in unbounded domains

$$\begin{aligned} -\Delta u &= |v|^{2^*-2}v \\ -\Delta v &= |u|^{2^*-2}u \\ u, v &\in D_0^{1,2}(\mathbb{R}^N \setminus \mathbb{Z}^N); \end{aligned}$$

and

$$\begin{aligned} -\Delta u &= \gamma v + |v|^{2^*-2}v \\ -\Delta v &= \lambda u + |u|^{2^*-2}u \\ u, v &\in H_0^1(\Omega), \end{aligned}$$

where $\Omega := \mathbb{R}^l \times \omega$ is a cylindrical domain with ω a bounded domain of $\mathbb{R}^{N-l}$ for $1 \leq l \leq N-1$, and $0 < \gamma, \lambda < \lambda_1(\Omega)$, with $\lambda_1(\Omega)$ the best constant in the Poincaré inequality. Those problems have a variational formulation with respectively the functionals:

$$\begin{aligned} \varphi(u, v) &:= \int_{\Omega} \nabla u \cdot \nabla v - \frac{|u|^{2^*}}{2^*} - \frac{|v|^{2^*}}{2^*} dx \\ \varphi_1(u, v) &:= \int_{\Omega} \nabla u \cdot \nabla v - \frac{\gamma u^2}{2} - \frac{\lambda v^2}{2} - \frac{|u|^{2^*}}{2^*} - \frac{|v|^{2^*}}{2^*} dx \end{aligned}$$

Observe that the first part is a strongly indefinite operator. In particular, those functionals have the form

$$\frac{\|Q(u, v)\|^2}{2} - \frac{\|P(u, v)\|^2}{2} - \psi(u, v),$$

where the Hilbert space $X = Y \oplus Z$ with Y and Z infinite dimensional subspaces, and P and Q are respectively the orthogonal projections on Y and Z . Therefore, the classical min-max results can not be applied. In particular, since φ and φ_1 are invariant under $\mathbb{Z}^N$ -translations and $\mathbb{R}^l$ -translations respectively, the Palais-Smale condition fails at all critical levels.

This type of problems on bounded domains were studied in the subcritical growth case by Husholf and van der Vorst [7] in using the Indefinite Functional Theorem due to Benci and Rabinowitz [1]; and by Felmer and Wang [6] who obtained multiplicity results in using Galerkin type methods. The critical growth case was studied by Husholf, Mitidieri and van der Vorst [8] where they used a dual formulation due to Clarke and Ekeland [2].

Here, we will use concentration-compactness techniques and we will applied a result obtained by Kryszewski and Szulkin [9] in 1996, which is a generalized linking theorem for a suitable functional defined on a Hilbert space $X = Y \oplus Z$ with Y a separable subspace of X which could be infinite dimensional and $Z := Y^\perp$. In their proof, they considered another topology on Y and they introduced a generalized degree theory. We state a corollary of their result that will be used in what follows (see also [13]).

Let $P : X \rightarrow Y, Q : X \rightarrow Z$ be the orthogonal projections. Now, let $\rho > r > 0$ and let $z \in Z$ be such that $\|z\| = 1$. Define

$$M := \{u = y + \lambda z : \|u\| \leq \rho, \lambda \geq 0, y \in Y\}, \quad (1.1)$$

$$M_0 := \{u = y + \lambda z : y \in Y, \|u\| = \rho \text{ and } \lambda \geq 0 \text{ or } \|u\| \leq \rho \text{ and } \lambda = 0\}, \quad (1.2)$$

$$N := \{u \in Z : \|u\| = \rho\}. \quad (1.3)$$

Theorem 1.1. (Kryszewski-Szulkin, 1996). *Let $\psi \in C^1(X, \mathbb{R})$ be weakly sequentially lower semicontinuous, bounded below and such that ψ' is weakly sequentially continuous. If*

$$\varphi(u) := \frac{\|Qu\|^2}{2} - \frac{\|Pu\|^2}{2} - \psi(u)$$

satisfies

$$b := \inf_N \varphi > 0 = \sup_{M_0} \varphi, \quad d := \sup_M \varphi < \infty, \quad (1.4)$$

then there exists $c \in [b, d]$ and a sequence $(u_n) \subset X$ such that

$$\varphi(u_n) \rightarrow c, \quad \varphi'(u_n) \rightarrow 0.$$

2 Preliminaries

In the sequel $N \geq 3$ and $2^* := 2N/(N-2)$. We consider the Hilbert space

$$D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\},$$

endowed with the inner product

$$\int_{\mathbb{R}^N} \nabla u(x) \cdot \nabla v(x) dx,$$

and the associated norm noted $\|u\|$. The Sobolev imbedding Theorem asserts that the imbedding

$$D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$$

is continuous.

For a domain $\Omega \subset \mathbb{R}^N$, we denote by $D_0^{1,2}(\Omega)$ the closure of $D(\Omega)$ in $D^{1,2}(\mathbb{R}^N)$. Obviously $D^{1,2}(\mathbb{R}^N) = D_0^{1,2}(\mathbb{R}^N)$, and when the Poincaré inequality is satisfied, $D_0^{1,2}(\Omega) = H_0^1(\Omega)$.

Let us recall that the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is not compact, but we have the following well known result, see Wang and Willem [12] for a generalization.

Lemma 2.1. *If $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$, then $u_n \rightarrow u$ in $L_{loc}^2(\mathbb{R}^N)$.*

The following lemma due to Ramos, Wang and Willem [11] gives sufficient conditions to ensure the convergence to 0 in $L^{2^*}(\mathbb{R}^N)$ of a sequence in $H^1(\mathbb{R}^N)$. This type of results was firstly established by P. L. Lions [10] for an exponent $p < 2^*$. See also Colin [3] or [4] for a similar result in weighted spaces on a cylindrical domain.

Lemma 2.2. *Let $r > 0$. If (u_n) is bounded in $H^1(\mathbb{R}^N)$ and if*

$$\sup_{x \in \mathbb{R}^N} \int_{B(x,r)} |u_n|^{2^*} dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

then $u_n \rightarrow 0$ in $L^{2^}(\mathbb{R}^N)$.*

Here, we establish an analogous result in $D_0^{1,2}(\mathbb{R}^N \setminus \mathbb{Z}^N)$. The choice of the domain $\mathbb{R}^N \setminus \mathbb{Z}^N$ could seem surprising but it is, in some sense, the biggest domain in which we are able to find a weak solution using our method. If a more regular domain is needed, notice that all the following results are also true for domains of the form:

$$\mathbb{R}^N \setminus \bigcup_{a \in \mathbb{Z}^N} \overline{\tau_a(\omega)},$$

where $\tau_a(x) = x + a$ is a translation applied to a domain ω such that $0 \in \omega \subset B(0, R)$, $R < 1/2$ and such that $\partial\omega$ is of class C^1 . Indeed, these domains are invariant under $\mathbb{Z}^N$ -translations and have a C^1 -boundary.

Lemma 2.3. *Let $(u_n) \subset D_0^{1,2}(\mathbb{R}^N \setminus \mathbb{Z}^N)$ be a bounded sequence. If*

$$\sup_{a \in \mathbb{Z}^N} \int_{B(a, \sqrt{N})} |u_n|^{2^*} \rightarrow 0, \quad \text{when } n \rightarrow \infty,$$

then $u_n \rightarrow 0$ in $L^{2^}(\mathbb{R}^N)$.*

In order to prove this result we will need the following notations and results. Let $\Omega := \mathbb{R}^l \times \omega$, where ω is a domain of $\mathbb{R}^{N-l}$ with $l \in \{1, \dots, N-1\}$. Let us denote

$$\Omega_+ :=]0, \infty[^l \times \omega,$$

and for each function $u \in C^\infty(\Omega_+)$, we define its symmetric extension u^* on Ω , and for each i such that $1 \leq i \leq l$, an another extension $u^{\square i}$ by:

$$u^*(x) = u^*(y_1, \dots, y_l, z_{l+1}, \dots, z_N) = \begin{cases} u(x) & \text{if } x \in \Omega_+, \\ u(|y_1|, \dots, |y_l|, z_{l+1}, \dots, z_N) & \text{otherwise,} \end{cases}$$

$$u^{\square i}(x) = u^{\square i}(y_1, \dots, y_l, z_{l+1}, \dots, z_N) = \begin{cases} u(|y_1|, \dots, |y_l|, z_{l+1}, \dots, z_N) & \text{if } y_i > 0, \\ -u(|y_1|, \dots, |y_l|, z_{l+1}, \dots, z_N) & \text{if } y_i < 0, \end{cases}$$

where $x = (y, z)$, with $y \in \mathbb{R}^l$ and $z \in \omega$.

Proposition 2.4. (Symmetric extension) Let Ω be as above and $1 \leq p < \infty$. If $u \in D(\Omega)$ and in choosing to denote $u_+ := u|_{\Omega_+}$, then $u_+^* \in W^{1,p}(\Omega)$ and more precisely we have the following equalities:

$$\begin{aligned} \frac{\partial u_+^*}{\partial y_i} &= \left(\frac{\partial u}{\partial y_i} \right)^{\square_i} \quad \text{where } 1 \leq i \leq l, \\ \frac{\partial u_+^*}{\partial z_j} &= \left(\frac{\partial u}{\partial z_j} \right)^* \quad \text{where } l+1 \leq j \leq N. \end{aligned}$$

Proposition 2.5. (Chain rule) Let U and U' be two open sets of $\mathbb{R}^N$, $p \geq 1$ and $H : U' \rightarrow U$ a bijective application such that

$$H \in C^1(U'), \quad H^{-1} \in C^1(U), \quad \text{Jac}H \in L^\infty(U'), \quad \text{Jac}H^{-1} \in L^\infty(U),$$

where $\text{Jac}H$ denote the Jacobian matrix. Let $u \in W^{1,p}(U)$, then $u \circ H \in W^{1,p}(U')$ and, in addition,

$$(u \circ H)'(y) = u'(H(y)) \circ H'(y).$$

Proof of Lemma 2.3. Since the problem is invariant by $\mathbb{Z}^N$ translations, we may assume for now that $a = 0$. Let us consider $u \in D(\mathbb{R}^N \setminus \mathbb{Z}^N)$ and let $\Psi \in D(\mathbb{R}^N)$ be such that $0 \leq \Psi \leq 1$ and

$$\Psi(x) := \begin{cases} 1 & \text{if } x \in B(0, \sqrt{N}) \\ 0 & \text{if } |x| > \sqrt{N} + 1. \end{cases}$$

The Sobolev imbedding Theorem implies that

$$\left(\int_{B(0, \sqrt{N})} |u|^{2^*} dx \right)^{2/2^*} \leq C \int_{B(0, \sqrt{N}+1)} |\nabla(\Psi u)|^2 dx. \quad (2.1)$$

In order to bound $\int |\nabla(\Psi u)|^2$ by $\int |\nabla u|^2$, we have to consider the following decomposition

$$\begin{aligned} \int_{B(0, \sqrt{N}+1)} |\nabla(\Psi u)|^2 dx &= \int_{B(0, \sqrt{N}+1)} \Psi^2 |\nabla u|^2 dx + \int_{B(0, \sqrt{N}+1)} u^2 |\nabla \Psi|^2 dx \\ &\quad + 2 \int_{B(0, \sqrt{N}+1)} u \Psi (\nabla u \cdot \nabla \Psi). \end{aligned} \quad (2.2)$$

Working on the third term, we obtain:

$$\begin{aligned} \int_{B(0, \sqrt{N}+1)} u \Psi (\nabla u \cdot \nabla \Psi) dx &= \frac{1}{2} \int_{B(0, \sqrt{N}+1)} (\nabla u^2 \cdot \Psi \nabla \Psi) dx \\ &= -\frac{1}{2} \int_{B(0, \sqrt{N}+1)} u^2 \text{div}(\Psi \nabla \Psi) dx \\ &\leq C \int_{B(0, \sqrt{N}+1)} u^2 dx. \end{aligned} \quad (2.3)$$

Please note that to avoid the use of too many different letters, we will denote by C all real positive constants.

In a next step, we have to bound

$$\int_{B(0, \sqrt{N}+1)} u^2 dx \quad \text{by} \quad \int_{B(0, \sqrt{N}+1)} |\nabla u|^2 dx.$$

For $r > 0$, denote by P_r the polar coordinates transformation

$$P_r :]0, r(\sqrt{N} + 1)[\times]0, 2\pi[\times]-\frac{\pi}{2}, \frac{\pi}{2}[\stackrel{N-2}{\mapsto} B(0, r(\sqrt{N} + 1)) \setminus \mathfrak{N}_r,$$

where $\mathfrak{N}_r$ is a null set. It is also clear that $\text{Jac}P_r$ is bounded. Lemmas 2.4 and 2.5 imply that we can extend $u \circ P_1$ with respect to the first variable in order to obtain the symmetric extension $(u \circ P_1)^*$ over the new domain:

$$]0, 2\sqrt{N} + 2[\times]0, 2\pi[\times]-\frac{\pi}{2}, \frac{\pi}{2}[\stackrel{N-2}{\mapsto}.$$

It follows that $u^* := (u \circ P_1)^* \circ P_2^{-1}$ is defined on the ball $B(0, 2\sqrt{N}+2)$ but a null set, and $u^* \in D_0^{1,2}(B(0, 2\sqrt{N}+2))$ which is an extension of u having, in addition, the following property:

$$\int_{B(0, 2\sqrt{N}+2)} |\nabla u^*|^2 dx \leq C \int_{B(0, \sqrt{N}+1)} |\nabla u|^2 dx.$$

This, the fact that $u = u^*$ on $B(0, \sqrt{N}+1)$, and the Poincaré inequality applied to the function u^* lead to

$$\int_{B(0, \sqrt{N}+1)} u^2 dx \leq C \int_{B(0, \sqrt{N}+1)} |\nabla u|^2 dx. \quad (2.4)$$

In combining (2.1)–(2.4) and in using the $\mathbb{Z}^N$ invariance, we obtain for every $a \in \mathbb{Z}^N$ and for every $u \in D_0^{1,2}(\mathbb{R}^N \setminus \mathbb{Z}^N)$, by a density argument

$$\int_{B(a, \sqrt{N}+1)} |u|^{2^*} dx \leq C \left(\int_{B(a, \sqrt{N}+1)} |\nabla u|^2 dx \right)^{2^*/2},$$

and hence, for any $\lambda > 0$,

$$\int_{B(a, \sqrt{N})} |u|^{2^*} dx \leq C^\lambda \left(\int_{B(a, \sqrt{N}+1)} |\nabla u|^2 dx \right)^{(2^*/2)\lambda} \left(\int_{B(a, \sqrt{N})} |u|^{2^*} dx \right)^{1-\lambda}.$$

In choosing $\lambda = 2/2^*$ we may write, since every $x \in \mathbb{R}$ is contained in at most $(\sqrt{N})^N$ balls,

$$\int_{\mathbb{R}^N} |u|^{2^*} dx \leq C \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \sup_{a \in \mathbb{Z}^N} \left(\int_{B(a, \sqrt{N})} |u|^{2^*} dx \right)^{(2^*-2)/2^*}.$$

Under the assumptions of the lemma, we conclude that $u_n \rightarrow 0$ in $L^{2^*}(\mathbb{R}^N)$. $\square$

3 Applications

In this section, we start in presenting an application to the following system of two coupled Poisson equations:

$$\begin{cases} -\Delta u &= |v|^{2^*-2}v \\ -\Delta v &= |u|^{2^*-2}u \\ u, v &\in D_0^{1,2}(\mathbb{R}^N \setminus \mathbb{Z}^N). \end{cases} \quad (3.1)$$

First of all, we establish some general results. Let Ω be a domain in $\mathbb{R}^N$. Denote $X := D_0^{1,2}(\Omega) \times D_0^{1,2}(\Omega)$ the Hilbert space endowed with the scalar product:

$$\langle (u, v), (u_1, v_1) \rangle := \int_{\Omega} \nabla u \cdot \nabla u_1 + \nabla v \cdot \nabla v_1 dx, \quad (3.2)$$

If we set

$$Y := \{(-v, v) \in X\} \quad \text{and} \quad Z := \{(u, u) \in X\}, \quad (3.3)$$

it is easy to check that $X = Y \oplus Z$ since

$$(u, v) = \frac{1}{2}(u + v, u + v) + \frac{1}{2}(-v + u, v - u).$$

Let us denote by P (resp. Q) the projection of X onto Y (resp. Z).

Define the functional $\varphi : X \rightarrow \mathbb{R}$ by

$$\begin{aligned} \varphi(u, v) &:= \int_{\Omega} \nabla u \cdot \nabla v - \frac{|u|^{2^*}}{2^*} - \frac{|v|^{2^*}}{2^*} dx \\ &= \frac{\|Q(u, v)\|^2}{2} - \frac{\|P(u, v)\|^2}{2} - \psi(u, v), \end{aligned}$$

where

$$\psi(u, v) := \int_{\Omega} \frac{|u|^{2^*}}{2^*} + \frac{|v|^{2^*}}{2^*} dx.$$

Observe that the problem (3.1) allows a variational formulation since its solutions are critical points of φ for $\Omega = \mathbb{R}^N \setminus \mathbb{Z}^N$.

Lemma 3.1. *The function ψ is C^1 and weakly sequentially lower semi-continuous. Moreover, for every $(u, v), (w, z) \in X$,*

$$\langle \varphi'(u, v), (w, z) \rangle = \int_{\Omega} \nabla u \cdot \nabla z + \nabla v \cdot \nabla w - |u|^{2^*-2}uw - |v|^{2^*-2}vz \, dx.$$

Proof. It is clear that ψ is weakly sequentially lower semi-continuous since the embedding $X \hookrightarrow L^{2^*}(\Omega) \times L^{2^*}(\Omega)$ is linear continuous and the norm on $L^{2^*}(\Omega)$ is weakly sequentially continuous.

Let $(u, v), (w, z) \in X$. For $x \in \Omega$ and $|t| \in]0, 1[$, there exists $\lambda \in]0, 1[$ such that

$$\begin{aligned} \frac{||u(x) + tw(x)|^{2^*} - |u(x)|^{2^*}|}{2^*|t|} &= ||u(x) + \lambda tw(x)|^{2^*-2}(u(x) + \lambda tw(x))w(x)| \\ &= |u(x) + \lambda tw(x)|^{2^*-1}|w(x)| \\ &\leq (|u(x)| + |w(x)|)^{2^*-1}|w(x)|. \end{aligned}$$

The term on the right hand side is in L^1 by the Hölder inequality. The Lebesgue dominated convergence Theorem implies that

$$\lim_{t \rightarrow 0} \frac{1}{2^*|t|} \int_{\Omega} |u(x) + tw(x)|^{2^*} - |u(x)|^{2^*} \, dx = \int_{\Omega} |u(x)|^{2^*-2}u(x)w(x) \, dx.$$

Similarly,

$$\lim_{t \rightarrow 0} \frac{1}{2^*|t|} \int_{\Omega} |v(x) + tz(x)|^{2^*} - |v(x)|^{2^*} \, dx = \int_{\Omega} |v(x)|^{2^*-2}v(x)z(x) \, dx.$$

Now, assume that $(u_n, v_n) \rightarrow (u, v)$ in X . From the continuous embedding $D_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$, we deduce that

$$(|u_n|^{2^*-2}u_n, |v_n|^{2^*-2}v_n) \rightarrow (|u|^{2^*-2}u, |v|^{2^*-2}v) \quad \text{in } L^{2^*/(2^*-1)} \times L^{2^*/(2^*-1)},$$

and,

$$\begin{aligned} &\left| \int_{\Omega} |u_n(x)|^{2^*-2}u_n(x)w(x) - |u(x)|^{2^*-2}u(x)w(x) \right. \\ &\quad \left. + |v_n(x)|^{2^*-2}v_n(x)z(x) - |v(x)|^{2^*-2}v(x)z(x) \, dx \right| \\ &\leq \|w\|_{L^{2^*}} \left(\int_{\Omega} \left| |u_n(x)|^{2^*-2}u_n(x) - |u(x)|^{2^*-2}u(x) \right|^{2^*/2^*-1} \, dx \right)^{(2^*-1)/2^*} \\ &\quad + \|z\|_{L^{2^*}} \left(\int_{\Omega} \left| |v_n(x)|^{2^*-2}v_n(x) - |v(x)|^{2^*-2}v(x) \right|^{2^*/2^*-1} \, dx \right)^{(2^*-1)/2^*} \\ &\leq k \left(\left(\int_{\Omega} \left| |u_n(x)|^{2^*-2}u_n(x) - |u(x)|^{2^*-2}u(x) \right|^{2^*/2^*-1} \, dx \right)^{(2^*-1)/2^*} \right. \\ &\quad \left. + \left(\int_{\Omega} \left| |v_n(x)|^{2^*-2}v_n(x) - |v(x)|^{2^*-2}v(x) \right|^{2^*/2^*-1} \, dx \right)^{(2^*-1)/2^*} \right), \end{aligned}$$

for every $(w, z) \in X$ such that $\|(w, z)\| \leq 1$. This shows that the Gâteaux derivative of ψ is continuous and hence ψ is C^1 .

On the other hand,

$$\lim_{t \rightarrow 0} \frac{1}{|t|} \int_{\Omega} \nabla(u + tw) \cdot \nabla(v + tz) - \nabla u \cdot \nabla v \, dx = \langle (u, v), (z, w) \rangle.$$

This Gâteaux derivative is obviously continuous; so φ is C^1 and

$$\langle \varphi'(u, v), (w, z) \rangle = \int_{\Omega} \nabla u \cdot \nabla z + \nabla v \cdot \nabla w - |u|^{2^*-2}uw - |v|^{2^*-2}vz \, dx.$$

□

Lemma 3.2. *The map ψ' is weakly sequentially continuous.*

Proof. Suppose that $(u_n, v_n) \rightharpoonup (u, v)$ in X . So, $\{u_n\}$ and $\{v_n\}$ are bounded in $L^{2^*}(\Omega)$ and hence $\{|u_n|^{2^*-2}u_n\}$ and $\{|v_n|^{2^*-2}v_n\}$ are bounded in $(L^{2^*}(\Omega))'$. Lemma 2.1 implies that $u_n \rightarrow u$ and $v_n \rightarrow v$ in $L^2_{loc}(\Omega)$. Therefore, for every $w, z \in D(\Omega)$,

$$\int_{\Omega} |u_n|^{2^*-2}u_n w + |v_n|^{2^*-2}v_n z \, dx \rightarrow \int_{\Omega} |u|^{2^*-2}u w + |v|^{2^*-2}v z \, dx;$$

i.e. $\langle \psi'(u_n, v_n), (w, z) \rangle \rightarrow \langle \psi'(u, v), (w, z) \rangle$. Moreover, $\{\psi'(u_n, v_n)\}$ is bounded in X , so $\psi'(u_n, v_n) \rightharpoonup \psi'(u, v)$. $\square$

Now, we want to show that φ satisfies (1.4). Fix $(z, z) \in Z$ such that $\|(z, z)\| = 1$.

Lemma 3.3. *There exists $r > 0$ such that*

$$b := \inf_{\substack{(u,u) \in Z \\ \|(u,u)\|=r}} \varphi(u, u) > 0 = \min_{\substack{(u,u) \in Z \\ \|(u,u)\| \leq r}} \varphi(u, u). \quad (3.4)$$

Moreover, there exists $\rho > r$ such that

$$\max_{M_0} \varphi = 0 \quad \text{and} \quad d := \sup_M \varphi < \infty, \quad (3.5)$$

where M and M_0 are given respectively by (1.1) and (1.2).

Proof. The Sobolev embedding Theorem of $D_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ implies directly (3.4) since for $(u, u) \in Z$,

$$\varphi(u, u) \geq \frac{\|(u, u)\|^2}{2} - C\|(u, u)\|^{2^*}.$$

Observe that on Y , we have

$$\varphi(-v, v) = \frac{-\|(-v, v)\|^2}{2} - \frac{2}{2^*} \int_{\Omega} |v|^{2^*} \, dx \leq 0.$$

Denote by W the closure of $Y \oplus \mathbb{R}(z, z)$ in $L^{2^*}(\Omega) \times L^{2^*}(\Omega)$. Since there exists a continuous projection of W onto $\mathbb{R}(z, z)$ and all the norms are equivalent on the latter space, we have

$$\begin{aligned} \varphi((-v, v) + \lambda(z, z)) &= -\frac{1}{2}\|(-v, v)\|^2 + \frac{\lambda^2}{2}\|(z, z)\|^2 \\ &\quad - \frac{1}{2^*} \left(\|-v + \lambda z\|_{2^*}^{2^*} + \|v + \lambda z\|_{2^*}^{2^*} \right) \\ &\leq -\frac{1}{2}\|(-v, v)\|^2 + \frac{\lambda^2}{2} - C\lambda^{2^*}. \end{aligned}$$

It follows that for $w \in Y \oplus \mathbb{R}(z, z)$

$$\varphi(w) \rightarrow -\infty \quad \text{whenever} \quad \|w\| \rightarrow \infty$$

and so, for some $\rho > r$, $\max_{M_0} \varphi = 0$.

Finally, the Cauchy-Schwarz inequality and the Sobolev inequality imply that φ maps bounded sets into bounded sets, hence $\sup_M \varphi < \infty$. $\square$

Lemma 3.4. *There exists $c \in [b, d]$ and a bounded sequence $\{(u_n, v_n)\}$ in X such that*

$$\varphi(u_n, v_n) \rightarrow c > 0, \quad \varphi'(u_n, v_n) \rightarrow 0. \quad (3.6)$$

Proof. It follows from Theorem 1.1 and Lemmas 3.1–3.3 that there exist $c \in [b, d]$ and a sequence $\{(u_n, v_n)\}$ in X satisfying (3.6).

Observe that

$$\begin{aligned} \varphi(u_n, v_n) - \frac{1}{2} \langle \varphi'(u_n, v_n), (u_n, v_n) \rangle &= \int_{\Omega} \nabla u_n \cdot \nabla v_n - \frac{|u_n|^{2^*}}{2^*} - \frac{|v_n|^{2^*}}{2^*} \\ &\quad - \frac{1}{2} (2 \nabla u_n \cdot \nabla v_n - |u_n|^{2^*} - |v_n|^{2^*}) \\ &= \mu \left(\|u_n\|_{L^{2^*}}^{2^*} + \|v_n\|_{L^{2^*}}^{2^*} \right), \end{aligned}$$

with $\mu = 1/2 - 1/2^*$. So, for $\varepsilon > 0$, and $n \in \mathbb{N}$ large enough,

$$c + \varepsilon + \varepsilon \|(u_n, v_n)\| \geq \mu \left(\|u_n\|_{L^{2^*}}^{2^*} + \|v_n\|_{L^{2^*}}^{2^*} \right) \geq c - \varepsilon - \varepsilon \|(u_n, v_n)\|. \quad (3.7)$$

On the other hand,

$$\begin{aligned} & \|Q(u_n, v_n)\|^{2-\varepsilon} \|Q(u_n, v_n)\| \\ & \leq \left| \|Q(u_n, v_n)\|^2 - \frac{1}{2} \langle \varphi'(u_n, v_n), (u_n + v_n, u_n + v_n) \rangle \right| \\ & = \int_{\Omega} \left(|u_n|^{2^*-2} u_n + |v_n|^{2^*-2} v_n \right) \left(\frac{u_n + v_n}{2} \right) \\ & \leq \|Q(u_n, v_n)\|_{L^{2^*} \times L^{2^*}} \left(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1} \right) \\ & \leq k \|Q(u_n, v_n)\| \left(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1} \right). \end{aligned} \quad (3.8)$$

Similarly,

$$\begin{aligned} & \|P(u_n, v_n)\|^{2-\varepsilon} \|P(u_n, v_n)\| \\ & \leq \left| -\|P(u_n, v_n)\|^2 - \frac{1}{2} \langle \varphi'(u_n, v_n), (u_n - v_n, v_n - u_n) \rangle \right| \\ & \leq k \|P(u_n, v_n)\| \left(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1} \right). \end{aligned} \quad (3.9)$$

Therefore,

$$\|Q(u_n, v_n)\| - \varepsilon \leq k \left(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1} \right), \quad (3.10)$$

$$\|P(u_n, v_n)\| - \varepsilon \leq k \left(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1} \right). \quad (3.11)$$

Adding (3.10) and (3.11), and combining the result with (3.7) leads to

$$\|(u_n, v_n)\| - 2\varepsilon \leq c_1 + c_2 \|(u_n, v_n)\|^{(2^*-1)/2^*};$$

so, $\{(u_n, v_n)\}$ is bounded in X . □

We are now ready to establish the existence of a solution to the problem (3.1).

Theorem 3.5. *Problem (3.1) has a non trivial solution.*

Proof. Take $\Omega = \mathbb{R}^N \setminus \mathbb{Z}^N$. By lemma 3.4, there exists a bounded sequence $(u_n, v_n) \subset X$ satisfying (3.6)

Now, let us assume that:

$$\begin{aligned} \delta_1 &:= \overline{\lim}_{n \rightarrow \infty} \sup_{a \in \mathbb{Z}^N} \int_{B(a, \sqrt{N})} |u_n|^{2^*} = 0 \\ \delta_2 &:= \overline{\lim}_{n \rightarrow \infty} \sup_{a \in \mathbb{Z}^N} \int_{B(a, \sqrt{N})} |v_n|^{2^*} = 0 \end{aligned}$$

then Lemma 2.3 implies that $u_n, v_n \rightarrow 0$ in $L^{2^*}(\mathbb{R}^N)$. This fact and (3.7) lead to a contradiction since $c > 0$.

Therefore, we must have $\delta := \max\{\delta_1, \delta_2\} > 0$. Taking a subsequence if necessary, we may assume the existence of $a_n \in \mathbb{Z}^N$ such that

$$\int_{B(a_n, \sqrt{N})} |u_n|^{2^*} + |v_n|^{2^*} dx > \frac{\delta}{2}.$$

The sequence $(\hat{u}_n, \hat{v}_n)$ defined by $\hat{u}_n(x) := u_n(x + a_n)$ and $\hat{v}_n(x) := v_n(x + a_n)$ is such that

$$\int_{B(0, \sqrt{N})} |\hat{u}_n|^{2^*} + |\hat{v}_n|^{2^*} dx > \frac{\delta}{2}$$

and satisfies (3.6) by $\mathbb{Z}^N$ invariance. Taking again a subsequence, if needed, we may assume that

$$(\hat{u}_n, \hat{v}_n) \rightharpoonup (u, v) \quad \text{in } X.$$

Since $\hat{u}_n \rightarrow u$, $\hat{v}_n \rightarrow v$ in $L^2_{loc}(\mathbb{R}^N)$, then $(u, v) \neq 0$. Finally, the weakly sequentially continuity of φ' gives

$$\|\varphi'(u, v)\| \leq \liminf_{n \rightarrow \infty} \|\varphi'(\hat{u}_n, \hat{v}_n)\| = 0.$$

Consequently (u, v) is a non trivial solution of (3.1). $\square$

Now, we will study the following problem in $\Omega := \mathbb{R}^l \times \omega$ a cylindrical domain with ω a bounded domain of $\mathbb{R}^{N-l}$ with for $1 \leq l \leq N-1$:

$$\begin{cases} -\Delta u = \gamma v + |v|^{2^*-2}v \\ -\Delta v = \lambda u + |u|^{2^*-2}u \\ u, v \in H_0^1(\Omega), \end{cases} \quad (3.12)$$

where $0 < \gamma, \lambda < \lambda_1(\Omega)$, with $\lambda_1(\Omega)$ the best constant in the Poincaré inequality,

$$\lambda_1(\Omega) := \inf \left\{ \int_{\Omega} |\nabla u|^2 dx : u \in H_0^1(\Omega), \int_{\Omega} u^2 dx = 1 \right\}.$$

It is shown in [5] that $\lambda_1(\Omega) > 0$.

The Poincaré inequality implies that the usual norm on $X := H_0^1(\Omega) \times H_0^1(\Omega)$ is equivalent to the norm induced by the scalar product defined in (3.2). If we set Y and Z as in (3.3), and we denote by P (resp. Q) the projection of X onto Y (resp. Z), the problem (3.12) allows a variational formulation since its solutions are critical points of the functional $\varphi_1 : X \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \varphi_1(u, v) &:= \int_{\Omega} \nabla u \cdot \nabla v - \frac{\gamma u^2}{2} - \frac{\lambda v^2}{2} - \frac{|u|^{2^*}}{2^*} - \frac{|v|^{2^*}}{2^*} dx \\ &= \frac{\|Q(u, v)\|^2}{2} - \frac{\|P(u, v)\|^2}{2} - \psi(u, v) - \hat{\psi}(u, v) \\ &= \varphi(u, v) - \hat{\psi}(u, v), \end{aligned}$$

where

$$\hat{\psi}(u, v) := \int_{\Omega} \gamma \frac{u^2}{2} + \lambda \frac{v^2}{2} dx.$$

and where φ and ψ are defined at the beginning of this section. It is easy to see that $\hat{\psi}$ is C^1 , weakly sequentially lower semi-continuous, and $\hat{\psi}'$ is weakly sequentially continuous. Also,

$$\langle \hat{\psi}'(u, v), (w, z) \rangle = \int_{\Omega} \gamma u w + \lambda v z dx.$$

The following result establishes the existence of a bounded Palais-Smale sequence of φ_1 . However, in writing $x := (y, z)$, where $y \in \mathbb{R}^l$ and $z \in \omega$, we observe that for each $(u, v) \in X$ and each $a \in \mathbb{R}^l$

$$\varphi_1(u, v) = \varphi_1(\hat{u}, \hat{v})$$

where $\hat{u} := u(y + a, z)$, $\hat{v} := v(y + a, z)$. So φ_1 is invariant under $\mathbb{R}^l$ -translations and consequently, the Palais-Smale condition fails at all critical levels.

Lemma 3.6. *There exists $c > 0$ and a bounded sequence $\{(u_n, v_n)\}$ in X such that*

$$\varphi_1(u_n, v_n) \rightarrow c > 0, \quad \varphi'_1(u_n, v_n) \rightarrow 0. \quad (3.13)$$

Proof. Set $\beta := (\gamma + \lambda)/2$. In using the Poincaré inequality and the Sobolev embedding Theorem, it follows the existence of a constant $C \geq 0$ such that for every $(u, u) \in Z$

$$\varphi_1(u, u) \geq \left(\frac{\lambda_1(\Omega) - \beta}{2\lambda_1(\Omega)} \right) \|(u, u)\|^2 - C\|(u, u)\|^{2^*}.$$

Therefore, there exists $r > 0$ such that

$$\inf_{\substack{(u, u) \in Z \\ \|(u, u)\| = r}} \varphi_1(u, u) > 0 = \min_{\substack{(u, u) \in Z \\ \|(u, u)\| \leq r}} \varphi_1(u, u). \quad (3.14)$$

The fact that for $(u, v) \in X$

$$\varphi_1(u, v) \leq \varphi(u, v)$$

allows us to use the arguments contained in the proof of Lemma 3.3 in order to show that there exists $\rho > r$ such that

$$\max_{M_0} \varphi_1 = 0 \quad \text{and} \quad \sup_M \varphi_1 < \infty, \quad (3.15)$$

where M and M_0 are given respectively by (1.1) and (1.2).

Theorem 1.1 ensures the existence of a sequence $\{(u_n, v_n)\}$ in X satisfying (3.13) for some $c > 0$. It remains to show that the sequence is bounded. Let us observe that

$$\varphi_1(u_n, v_n) - \frac{1}{2} \langle \varphi'_1(u_n, v_n), (u_n, v_n) \rangle = \varphi(u_n, v_n) - \frac{1}{2} \langle \varphi'(u_n, v_n), (u_n, v_n) \rangle,$$

which implies (3.7).

Without loss of generality, we may assume that $\lambda \geq \gamma$. A straightforward computation leads to

$$\begin{aligned} \frac{1}{2} \langle \hat{\psi}'(u_n, v_n), (u_n + v_n, u_n + v_n) \rangle &\leq \frac{\lambda}{2} \|u_n + v_n\|_{L^2}^2 \\ &\leq \frac{\lambda}{2\lambda_1(\Omega)} \|\nabla(u_n + v_n)\|_{L^2}^2 \\ &= \frac{\lambda}{4\lambda_1(\Omega)} \|(u_n + v_n, u_n + v_n)\|^2 \\ &= \frac{\lambda}{\lambda_1(\Omega)} \|Q(u_n, v_n)\|^2. \end{aligned}$$

Similarly, we have

$$\frac{1}{2} \langle \hat{\psi}'(u_n, v_n), (u_n - v_n, v_n - u_n) \rangle \leq \frac{\lambda}{\lambda_1(\Omega)} \|P(u_n, v_n)\|^2.$$

In combining respectively the last two equations with (3.8) and (3.9), the inequalities (3.10) and (3.11) become respectively:

$$\begin{aligned} \left(\frac{\lambda_1(\Omega) - \lambda}{\lambda_1(\Omega)} \right) \|Q(u_n, v_n)\| - \varepsilon &\leq k \left(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1} \right), \\ \left(\frac{\lambda_1(\Omega) - \lambda}{\lambda_1(\Omega)} \right) \|P(u_n, v_n)\| - \varepsilon &\leq k \left(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1} \right), \end{aligned}$$

and the conclusion follows as in the proof of Lemma 3.4. □

We are now ready to establish the existence of a solution to the problem (3.12).

Theorem 3.7. *If $\gamma, \lambda \in]0, \lambda_1(\Omega)[$, then Problem (3.12) has a non trivial solution.*

Proof. Let $(u_n, v_n) \subset X$ be a bounded sequence given in Lemma 3.6, and define

$$\begin{aligned} \delta_1 &:= \overline{\lim}_{n \rightarrow \infty} \sup_{x \in \Omega} \int_{B(x,1)} |u_n|^{2^*}, \\ \delta_2 &:= \overline{\lim}_{n \rightarrow \infty} \sup_{x \in \Omega} \int_{B(x,1)} |v_n|^{2^*}. \end{aligned}$$

In arguing as in the proof of Theorem 3.5 with Lemma 2.2, we show that $\delta := \max\{\delta_1, \delta_2\} > 0$, and hence we deduce the existence of a sequence $x_n = (y_n, z_n) \in \Omega$ such that

$$\int_{B(x_n,1)} |u_n|^{2^*} + |v_n|^{2^*} dx > \frac{\delta}{2}.$$

The sequence $(\hat{u}_n, \hat{v}_n)$ defined by $\hat{u}_n(x) := u_n(y + y_n, z)$ and $\hat{v}_n(x) := v_n(y + y_n, z)$ is such that

$$\int_{B((0,z_n),1)} |\hat{u}_n|^{2^*} + |\hat{v}_n|^{2^*} dx > \frac{\delta}{2},$$

and satisfies (3.13) by $\mathbb{R}^l$ invariance. By letting $R := \max_{z \in \omega} |z|$, we obtain

$$\int_{B(0, R+1)} |\hat{u}_n|^{2^*} + |\hat{v}_n|^{2^*} dx > \frac{\delta}{2}.$$

Finally, taking a subsequence, if needed, we may assume that

$$(\hat{u}_n, \hat{v}_n) \rightharpoonup (u, v) \quad \text{in } X.$$

As in the proof of Theorem 3.5, we conclude that (u, v) is a non trivial solution of (3.12). $\square$

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