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The Hausdorff measure functions: A new way to characterize fractal sets

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Abstract

We introduce a new method based on Hausdorff measure spectrum function (HMSF) which provides a more precise way for tracing the geometrical organization of a fractal set. The HMSF does carry a huge amount of information about the set to likely be explored in a chosen way. Depending on the nature of the set, we propose two ways to extract this information. We apply these methods to typical fractals as well as to synthetic models of porous media. This results in a complete distinction between same fractal dimension sets.

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Various physical processes and structures share the same fractal dimension in spite of their different appearance. The most used and popularized concept in fractal geometry was the fractal dimension. Only in the last decade, additional tools to get rid of the degeneracy character of fractal dimension have been developed. Among them, few have been devoted to fractal features of 'texture', a broad concept called lacunarity by Mandelbrot (1983). Lacunarity has been introduced qualitatively, as a notion of texture based on the tendency of a set to get gaps and intended to distinguish between two same dimensional fractal sets. This texture notion, in its general, even loose quantitative meaning, is strongly related to the shapes of those gaps as well as to their distribution. Lacunarity has been related to a range of physical phenomena where the mass distribution of physical structures is intrinsically involved. Examples are as various as cosmology in the distribution of galaxies, radar images, geology and material sciences (Mandelbrot, 1983; Plotnick et al., 1996; Pietronero, 1992; Hildgen et al., 1997). It is largely documented that there is a need to develop methods that account for fine structure (Arneodo et al., 2000; Gefen et al., 1984) since most of the properties depend, in addition to the fractal dimensionality, on other geometrical factors related to texture. In porous media, it is well known that microstructure fluctuations can have important consequences on bulk mechanical and rheological

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properties. Fat fractals—whose main peculiarity, compared to thin fractals, consists in a finite (non-zero) Lebesgue measure of their support—are considered to be good representations of the micro-architecture of porous solids. Indeed, the empty holes of fat fractal have size-dependant power distribution similar to porous materials (Bulgakov, 1992; Umberger and Farmer, 1985). To mathematically quantify this loose notion of lacunarity, several methods have been proposed. The gliding-box algorithm (GBA) is one of them (Allain and Cloitre, 1991). GBA has been derived from the box-counting method by gliding a box over the set, one unit at a time in a discrete manner. Despite the popularity of this algorithm, it still proved to be degenerate in rather simple cases (Fabio et al., 1994), where two deterministic regular shapes of the same fractal dimension have not been resolved. In this fundamental context of characterization of the fine structure of a fractal, one would like to know if there is a way to build a more powerful quantitative and characteristic tool to unveil the information carried by the structure. Having this purpose in mind, we have been led, through the notion of lacunarity (Mandelbrot, 1983), to study the Hausdorff measure of the intersection of sets of same dimension with their translates (Nekka and Li, 2002). In fact, Mandelbrot briefly mentioned the potential relation between lacunarity and the intersection of a set with its translate (Mandelbrot, 1983, p. 317). From this vague yet suggestive idea, we introduced the translation method as a new way to study the fine geometrical details in a structure. The method we propose here is based on the Hausdorff (or Lebesgue when these two measures coincide) measure of the translation of the set through itself in a continuous manner. Since the translation is made continuously on each point (local) and that Hausdorff measure (global) is estimated, the measure function obtained is able to extract the whole information within the structure. In a concern of generality and since Lebesgue measure can degenerate, we first apply the more general notion of Hausdorff measure to show the potential of the method. At this point, it should be mentioned that the indicator function of the intersection of a set with its translates can be viewed as a two-point joint moment (autocovariance) within the set's indicator function. This explains in a way why the measure function introduced here naturally completes the information obtained from pointwise descriptors. To explain the underlying mechanism and test our method, we will use Cantor-like sets of same fractal dimension as a tractable model, which belong to thin sets and are the most studied fractals. As a physical application, we use the so called fat fractals which are known to have fractal dimension equal to one in 1-D and which, as mentioned above, are good models to simulate porous media.

In (Nekka and Li, 2002), we have set up the fundamental properties of Hausdorff measure of the intersection produced by these translations and proposed to use, in a subsequent study, this spectrum of measures to distinguish between sets having the same fractal dimension. As a first step, we will use here what we call the Hausdorff measure spectrum function (HMSF) of the intersection of a fractal set with its translate to characterize the geometry of the set. This function has the advantage of being easy to manipulate and allows for a deep study of the set by revealing thus the whole picture of its geometrical structure. In fact, this Hausdorff measure function does carry a huge amount of information about the set to likely be exploited in a chosen way. In the case of thin sets, we propose to distinguish them by taking advantage of the property of translation invariance (Nekka and Li, 2002) verified by the HMSF of these sets. We suggest two successive steps to process the characterization of the set. The first one is based on what we call here the translation invariance based method (TIBM). The second one consists in comparing the dispersion of translation shifts of this function for different sets at a fixed measure level, which we call the fixed level based method (FLBM). The latter one is necessary only in case of insufficiency of the first step. FLBM associates an index to sets which allows for their differentiation as well as for evaluation of their degree of homogeneity. For fat fractals, we quantify the fluctuations in their HMSF using the regularization dimension (RD) proposed by Levy-vehel which proved to be more stable than the box-counting dimension (Roueff and Levy-vehel, 1998).

In (Nekka and Li, 2002), we found that the Hausdorff measure of the intersection of triadic Cantor set with its translates forms a discrete spectrum of the form $\{0, (1/2^n)_{n\in\mathbb{Z}^+}\}$ when t varies between −1 and 1 at its well-known Hausdorff dimension $s = \log 2/\log 3$. This discreteness of Hausdorff measure can also be theoretically justified for other Cantor-type fractal sets and is practically true for a large amount of self-similar sets.

To exploit the HMSF within a classification purpose, we illustrate our method using known fractal sets, especially triadic Cantor sets. However, since a direct computation of HMSF is time consuming and laborious, we propose three approximation algorithms based solely on the similarity properties of fractals, whose generator can be decomposed in identical parts, to compute the HMSF. Thus, the algorithms can be directly applied to the five examples of Cantor-type sets in Fig. 5. As the obtained values by the algorithms at a generation n are the exact values of the final set, it is more convenient to use Hausdorff metric to evaluate the approximation as done in (Peitgen et al., 1992). In fractal geometry, the Hausdorff distance is used to measure the distance between two sets. As an example, in the case of triadic Cantor set, the Hausdorff distance between the approximation set of HMSF of generation n and the graph of HMSF is less than $1/3^n + 1/2^{n+1}$ using our proposed algorithm. For the case of fractal sets whose generator parts are not identical, a more complex formalism can be developed based on the similarity properties of these sets. This will be the context in a more general work.

To get the HMSF of a general fractal set, one can first estimate its fractal dimension and then use it to substitute for the Hausdorff measure involved in the HMSF expression. However, we are presently working on the potential of HMSF (as a function of s) to extract the fractal dimension and then use this dimension value within the HMSF for the characterization of the set.

1. The first one is the similarity algorithm. It is built upon the similarity properties of a fractal set which are inherited by the HMSF itself. Let C be the triadic Cantor set generated from the unit interval [0, 1], and consider the intersection of C with its translation $C + t$, that we denote by $I(t) = C \cap (C + t)$, where t varies between −1 and 1. Let $C = C_l \cup C_r$, where C_l and C_r are respectively the identical left and right parts of C. We have $C \cap (C + t) = (C_l \cup C_r) \cap ((C_l + t) \cup (C_r + t))$, which results in the union of four parts giving rise to the general equation:

$$\mathscr{M}(t) = (\mathscr{M}(3t-2) + 2\mathscr{M}(3t) + \mathscr{M}(3t+2))/2 \qquad (1)$$

where $\mathscr{M}(t) = \mathscr{H}^s(I(t))$, whose spectrum gives the corresponding HMSF. It is easy to see that the graph of $\mathscr{M}(t)$ can be generated from the initial set $\{(-1,0),(0,1),(1,0)\}$ by means of three affine maps s_i, $i = 0,1,2$, where $s_i(X) = A_i(X) + B_i$; $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A_0 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 = A_2 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $B_0 = 0$, $B_2 = -B_1 = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$. If we denote by G the graph of $\mathscr{M}(t)$, then G is the invariant set (attractor) by the union of three affine maps: s_0, s_1, s_2, i.e.

$$G = s_0(G) \cup s_1(G) \cup s_2(G)$$

The process is shown in Fig. 1.

2. The second method is based on interpolation. First, recall that HMSF is symmetric and discontinuous everywhere, but that it can be approximated by a sequence of continuous functions (Nekka and Li, 2002). We know that the triadic Cantor set is given by $C = \bigcap_{n=0}^{\infty} C_n$. Let us construct the sequence of the approximating functions of $\mathscr{M}(t)$. The initial unit weight will be conserved through subsequent generations (in an iterative process, generations refer to the successive steps involved in the process). Denote by m_n the measure of uniformly distributing weights on each subinterval of C_n. Then, we construct a sequence of measure functions of $C_n \cap (C_n + t)$, which is given by:

$$\mathscr{M}_n(t) = m_n(C_n \cap (C_n + t))$$

We have $\mathscr{M}_n(t) \to \mathscr{M}(t)$. The approximation is shown in Fig. 2.

Moreover, this approximation by continuous functions allows for an interpolation based method to construct the HMSF. A checking for

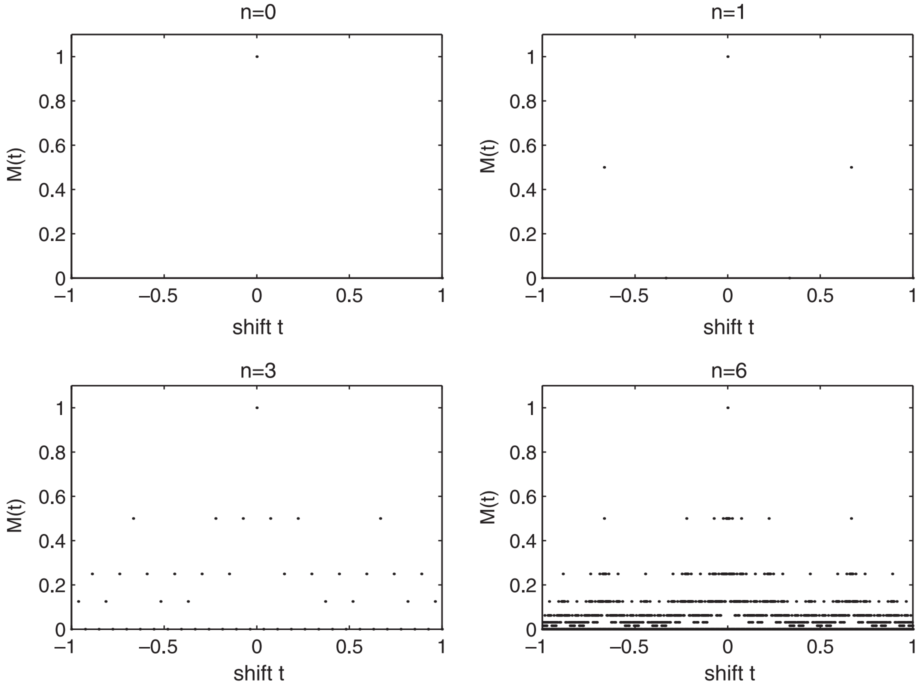

Fig. 1. The construction of the HMSF using the similarity algorithm for generations: 0, 1, 3, 6.

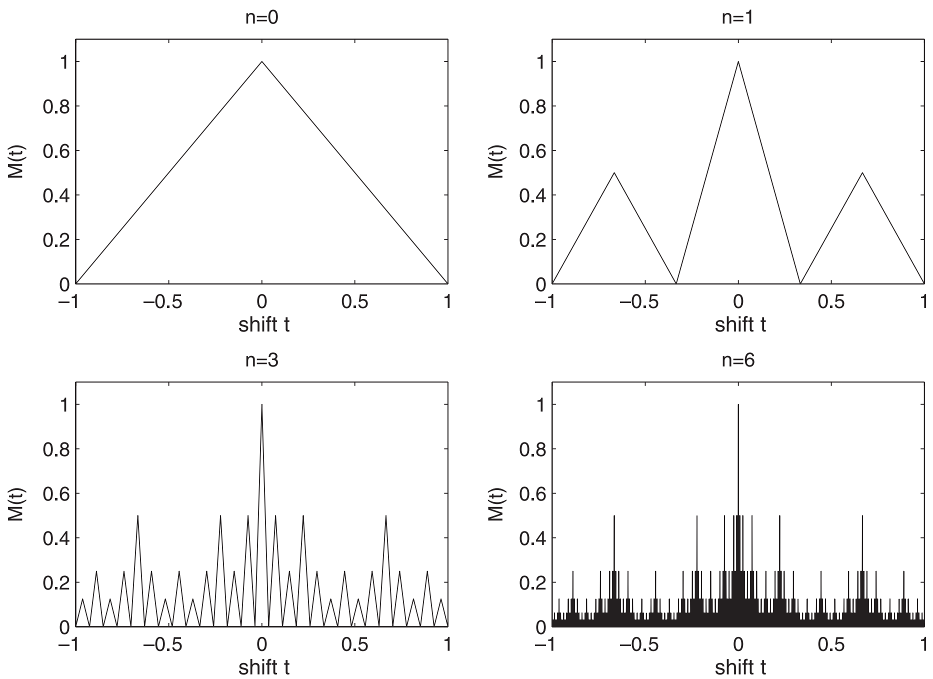

Fig. 2. The approximation of the HMSF by a sequence of continuous functions; generations: 0, 1, 3, 6.

several steps is enough to determine the interpolation based on the scaling properties satisfied by HMSF. It is easy to see that the interpolation will converge to the real function $\mathscr{M}(t)$. We omit the detail here and just mention that Fig. 2 can also be used to illustrate the interpolation procedure if we consider each broken point as the point for interpolation.

3. Finally, if we only need to determine the HMSF at a given point, we can exploit the similarity property of the HMSF to write $\mathscr{M}(t)$ in the following recursive form:

$$\mathscr{M}(t) = \begin{cases} \mathscr{M}(|t| - 2/3)/2 & \text{if } 1/3 \leqslant |t| \leqslant 1; \\ \mathscr{M}(|t| - 2/3^2)/2 & \text{if } 1/3^2 \leqslant |t| \leqslant 1/3; \\ \cdots & \\ \mathscr{M}(|t| - 2/3^{n+1})/2 & \text{if } 1/3^{n+1} \leqslant |t| \leqslant 1/3^n, \\ \cdots & \end{cases}$$

knowing that $\mathscr{M}(0) = 1$ and $-1 \leqslant t \leqslant 1$.

Then one can quickly determine the numerical values of $\mathscr{M}(t)$ for any given t without having knowledge of the geometry of HMSF.

Now that we know how to approximate the HMSF by different methods, let us show how to use it to differentiate between sets having the same fractal dimension (in this example $\log 2/\log 3$). For thin sets, the Cantor sets are constructed from the initiator $I = [0, 1]$; we illustrate the procedure of their construction in Fig. 5.

Fig. 1 shows the several iteration of the HMSF of the fractal sets (a). Fig. 3 shows the sixth iteration of the HMSF of the remaining fractal sets (b)–(e). We propose here two different ways to exploit the HMSF in order to distinguish between these sets. With TIBM, we take the translation invariant values of HMSF corresponding to values preserved by translation. Each value corresponds to a level, which is a set of points representing a fixed HMSF value for different shifts (see Fig. 1). The graph of these levels, in terms of the shift number, are illustrated in Fig. 4.

We see that TIBM succeeds in distinguishing between (a), (c) and (b) (or (d)). However, TIBM levels are the same for (a) and (e) as well as for (b) and (d). This last fact does not allow one to conclude that (a) and (e) or (b) and (d) are the same. We have yet to go a step further in our exploration and use the FLBM. This method compares, for a given level, the HMSF values of the concerned sets. In fact, the first level, which contains the

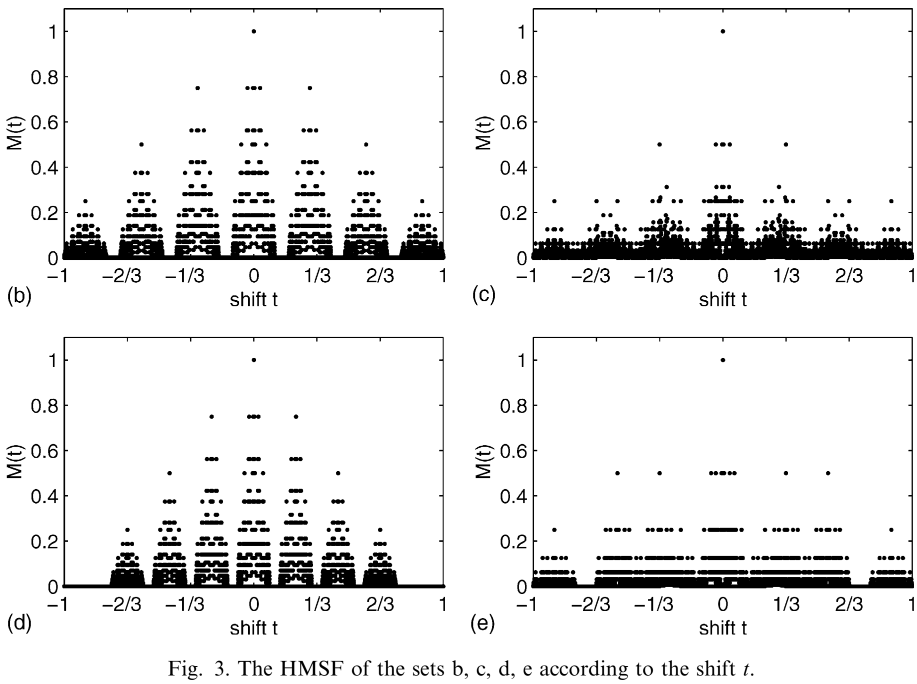

Fig. 3. The HMSF of the sets b, c, d, e according to the shift t.

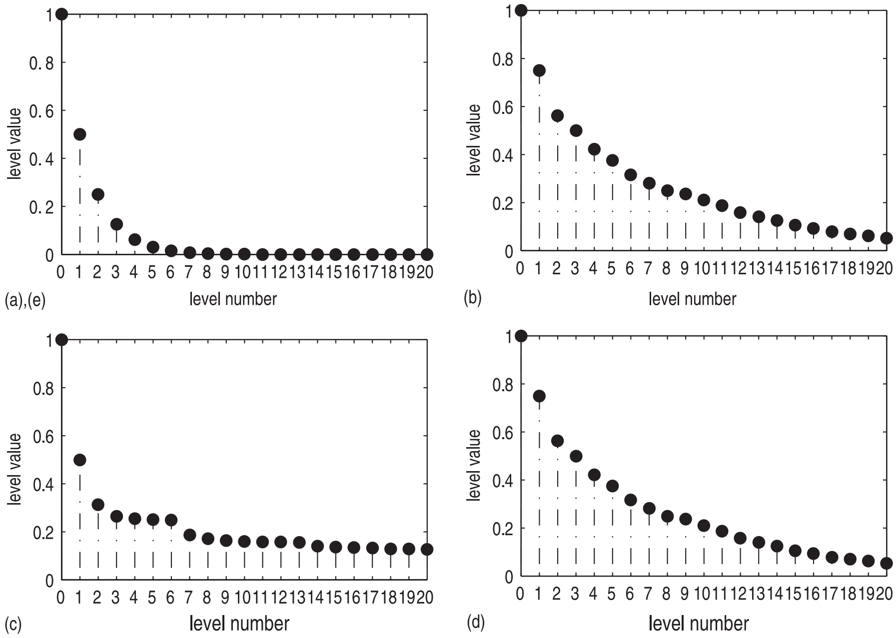

Fig. 4. The Hausdorff measure levels where the translational invariance of the Hausdorff measure is preserved.

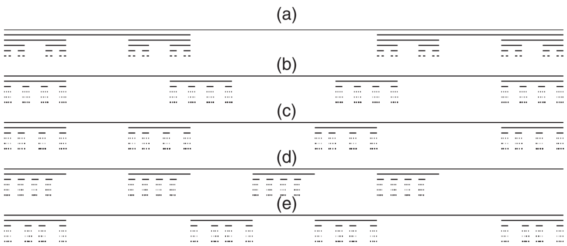

Fig. 5. Construction of the Cantor-like sets of dimension $\log 2/\log 3$.

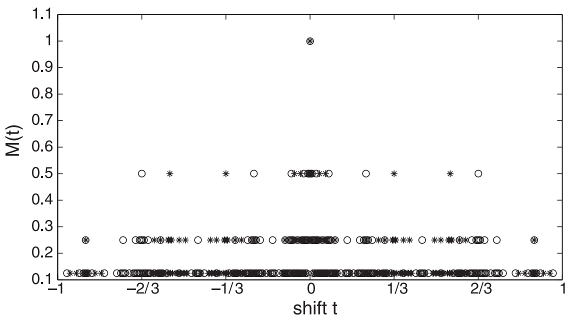

Fig. 6. The first four levels of Hausdorff measure function, represented by (○) and (∗) for the Cantor set (a) and (e) respectively.

whole information of HMSF, is very enough. In Fig. 6, we plotted the first four fixed levels (from 0 to 3) of the HMSF of (a) and (e). Graphically, the difference is already obvious on level one. This difference can be quantified by averaging weighted distances between shift values and their limit point at the first level, giving thus level indexes associated to each set. For example, using a dyadic sequence weights $\{1/2^i\}_i$, we get the value 0.5962 for (a) and 0.6248 for (e). This index, from one part, is able to differentiate between sets and, from the other part, indicates the degree of homogeneity of the set: the higher the index, the more homogeneous is the set. Finally, for sets (b) and (d), one observes that they have the same HMSF once the support corresponding to (d) is reported, by a rescaling centered on 0, on the same support of (b) (Fig. 3). This equivalence is naturally also reflected in the FLMB. If their supports of HMSF are rescaled to the same size, (b) and (d) can not be distinguished by FLMB. In fact, it has no sense to differentiate (b) and (d) since these two sets are similar in their geometrical structure. This also suggests that FLMB should be based on the same rescaled support size of HMSF. The five examples taken here are by no means restrictive since the methods used to distinguish between them are only based on their similarity properties. We chose them since they are good candidates to build a rigorous mathematical formalism behind the method. The difference between the construction of thin Cantor sets and their fat versions can be seen in a simple way. In the nth generation of a thin Cantor set, the length of the central deleted part of each interval is a proportion of a^n to the total length. However, for fat fractals, this central deleted part is a fraction a^n of the length of each of the latter interval. We have considered four fat fractals with a being equal to: 1/3, 1/4, 1/5, and 1/15. Given that the Hausdorff as well as the box-counting dimensions of these fat fractals are 1, these dimensions cannot be used to distinguish between them. Hence, we have used the regulation dimension since, by construction, it is more sensitive to variations (Roueff and Levy-vehel, 1998). We quantified RD of these fat fractals and their HMSF, respectively. We found that, when RD is applied directly to the sets, the differentiation by this dimension is less effective compared to when it is applied to their HMSF (Figs. 7 and 8). Indeed, in the latter case, the difference in RD values in terms of the hole size is amplified. Moreover, RD of HMSF has a more linear and monotonic behavior than RD when directly estimated on the sets.

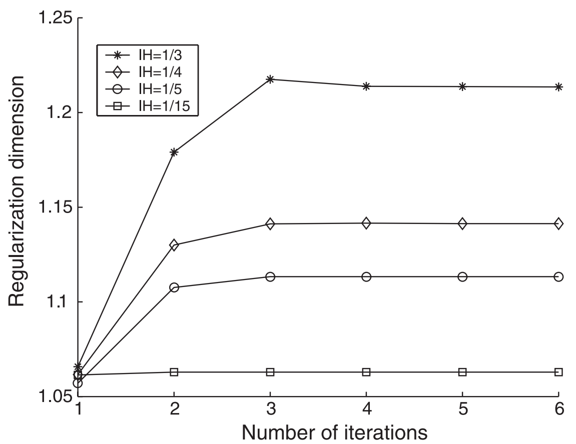

Fig. 7. Differentiating four fat fractals of dimension one by the regularization dimension of their HMSF.

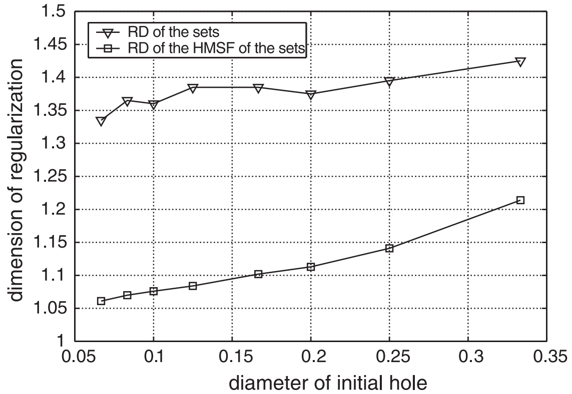

Fig. 8. Comparison of RD applied directly on the sets and on their HMSF.

In conclusion, we presented here a new method of classification of fractal sets, based on Hausdorff measure. This method offers a more precise description of the fine structure of complex sets generally undistinguishable by existing methods. Work on additional and more general examples is in progress. We are also pursuing this work on the potential of our method to characterize the Hausdorff dimension.

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