On trigonometric analogue of Atiyah-Hitchin bracket

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Abstract

The space of rational functions has natural Poisson structure discovered by Atiyah and Hitchin. We consider its derivation by means of quadratic r-matrix structure which naturally arises for the corresponding scattering problem and provide a trigonometric analogue.

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1 Poisson brackets on space of rational maps

In the work by Atiayh and Hitchin [1] it was introduced a symplectic structure on the space R_N of rational functions of the form

$$S(\lambda) = \sum_{i=0}^{N-1} \frac{\rho_i}{\lambda - \beta_i} \tag{1.1}$$

To describe this symplectic structure we represent function $S(\lambda)$ as a ratio of two polynomials

$$S(\lambda) = \frac{p(\lambda)}{q(\lambda)} \tag{1.2}$$

where $q(\lambda) = (\lambda - \beta_1) \ldots (\lambda - \beta_N)$. Then the Atiyah-Hitchin symplectic form looks like follows:

$$\omega = \sum_{i=1}^{N} \frac{dp(\beta_i) \wedge d\beta_i}{p(\beta_i)} \tag{1.3}$$

The symplectic structure implies the following Poisson brackets

$$\{p(\beta_m), p(\beta_n)\} = 0, \qquad \{\beta_m, \beta_n\} = 0 \tag{1.4}$$

$$\{p(\beta_m), \beta_n\} = p(\beta_m) {\delta_m}^n \tag{1.5}$$

These Poisson brackets in turn imply brackets between polynomials $p(\lambda)$ and $q(\lambda)$:

$$\{q(\lambda), q(\mu)\} = 0, \qquad \{p(\lambda), p(\mu)\} = 0 \tag{1.6}$$

$$\{p(\lambda), q(\mu)\} = \frac{p(\lambda)q(\mu) - q(\lambda)p(\mu)}{\lambda - \mu} \tag{1.7}$$

It was shown by Gekhtman and Faybusovich in [3] that the relations (1.6) and (1.7) are equivalent to the following brackets on the space of rational functions R_N (1.3):

$$\{S(\lambda), S(\mu)\} = \frac{(S(\lambda) - S(\mu))^2}{\lambda - \mu} \tag{1.8}$$

In fact, the bracket (1.8) can be obviously extended to the space of all rational functions being the ratio of two polynomials of arbitrary degree. However the requirement of distinct roots remains essential. As it was mentioned by K.Takasaki [4] , the Gekhtman-Faybusovich bracket (1.8) is naturally related to the rational quadratic Sklyanin bracket

$$\{\overset{1}{T}(\lambda), \overset{2}{T}(\mu)\} = [\overset{12}{r}(\lambda - \mu), \overset{1}{T}(\lambda)\overset{2}{T}(\mu)] \tag{1.9}$$

where

$$\overset{12}{r}(\lambda) = \frac{\Pi}{\lambda} = \frac{I \otimes I + \sigma_\alpha \otimes \sigma_\alpha}{2\lambda} \tag{1.10}$$

is classical rational r-matrix and $\Pi(u \otimes v) = v \otimes u$ is a permutation matrix.
If we parametrize matrix $T(\lambda)$ to be:

$$T(\lambda) = \begin{pmatrix} p(\lambda) & \tilde{p}(\lambda) \\ q(\lambda) & \tilde{q}(\lambda) \end{pmatrix}$$

then the Sklyanin bracket (1.9) implies the following Poisson brackets between the matrix elements of $T(\lambda)$:

$$\{q(\lambda), q(\mu)\} = 0, \qquad \{p(\lambda), p(\mu)\} = 0 \tag{1.11}$$

$$\{p(\lambda), q(\mu)\} = \frac{p(\lambda)q(\mu) - q(\lambda)p(\mu)}{\lambda - \mu} \tag{1.12}$$

$$\{\tilde{q}(\lambda), \tilde{q}(\mu)\} = 0, \qquad \{\tilde{p}(\lambda), \tilde{p}(\mu)\} = 0 \tag{1.13}$$

$$\{\tilde{p}(\lambda), \tilde{q}(\mu)\} = \frac{\tilde{p}(\lambda)\tilde{q}(\mu) - \tilde{q}(\lambda)\tilde{p}(\mu)}{\lambda - \mu} \tag{1.14}$$

$$\{\tilde{q}(\lambda), q(\mu)\} = \frac{\tilde{q}(\lambda)q(\mu) - q(\lambda)\tilde{q}(\mu)}{\lambda - \mu}, \qquad \{\tilde{p}(\lambda), p(\mu)\} = \frac{\tilde{p}(\lambda)p(\mu) - p(\lambda)\tilde{p}(\mu)}{\lambda - \mu} \tag{1.15}$$

$$\{p(\lambda), \tilde{q}(\mu)\} = \frac{q(\lambda)\tilde{p}(\mu) - \tilde{p}(\lambda)q(\mu)}{\lambda - \mu} \tag{1.16}$$

$$\{\tilde{p}(\lambda), q(\mu)\} = \frac{\tilde{q}(\lambda)p(\mu) - p(\lambda)\tilde{q}(\mu)}{\lambda - \mu} \tag{1.17}$$

Notice, that the brackets (1.11) and (1.12) coincide with the Gekhtman-Faybusovich brackets (1.6, 1.7). Therefore, the Atiyah-Hitchin Poisson structure coincides with the Sklyanin bracket between elements T_{11} and T_{21} for polynomial dependence of matrix T on λ.

This coincidence leads to several natural questions. The Sklyanin bracket (1.9) arises in inverse scattering method as Poisson structure on scattering matrix $T(\lambda)$ implied by fundamental Poisson structure between physical field in models of non-linear Schrödinger type. On the other hand, the Atiyah-Hitchin bracket is also defined as a bracket on the space of scattering matrices related to solutions of the Bogomolny equations [1].

However, in Atiyah-Hitchin framework it remains unclear how this Poisson structure is related to fundamental Poisson bracket between physical fields A and ϕ; close relationship between Atiyah-Hitchin and Sklyanin brackets suggests that there should exist some kind of derivation of the Atiyah-Hitchin structure from the brackets on A and ϕ; however, so far we were unable to find it. Instead in these note we shall extend this observation to trigonometric case and show how to derive natural Poisson structure on the space of trigonometric rational functions starting from Sklyanin bracket with trigonometric r-matrix.

2 Trigonometric generalizations of Atiyah-Hitchin and Gekhtman-Faybusovich brackets

Now, consider a space of meromorphic functions $p(\lambda)/q(\lambda)$ being the ratio of trigonometric polynomials

$$p(\lambda) = \prod_{i=1}^{N-1} sin(\lambda - \alpha_i) \tag{2.1}$$

$$q(\lambda) = \prod_{k=1}^{N} sin(\lambda - \lambda_k) \tag{2.2}$$

The symplectic structure here is of the same form as before

$$\Omega = \sum_{i=1}^{N} \frac{dp(\lambda_i) \wedge d\lambda_i}{p(\lambda_i)} \tag{2.3}$$

Consider the following 2×2 matrix

$$T(\lambda) = \begin{pmatrix} p(\lambda) & \tilde{p}(\lambda) \\ q(\lambda) & \tilde{q}(\lambda) \end{pmatrix}$$

with polynomials $p(\lambda)$ and $q(\lambda)$ are of the form as above . Then with respect to trigonometric r-matrix defined as

$$r(\lambda) = \frac{1}{2sin(\lambda)}[\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + cos(\lambda)\sigma_3 \otimes \sigma_3] \tag{2.4}$$

the quadratic Sklyanin Poisson bracket on matrices $T(\lambda)$ and $T(\mu)$ provides the following relations on the space of trigonometric polynomials

$$\{p(\lambda), p(\mu)\} = 0 \tag{2.5}$$

$$\{q(\lambda), q(\mu)\} = 0 \tag{2.6}$$

$$\{q(\mu), p(\lambda)\} = \frac{q(\lambda)p(\mu) - cos(\lambda-\mu)q(\lambda)p(\mu)}{sin(\lambda-\mu)} \tag{2.7}$$

which lead in their turn to the following generalization of the Gekhtman-Faybusovich bracket on the space of rational trigonometric functions:

$$\{S(\lambda), S(\mu)\} = \frac{(S(\lambda)-S(\mu))^2}{sin(\lambda-\mu)} + 2S(\lambda)S(\mu)(ctg(\lambda-\mu)-1) \tag{2.8}$$

It is easy to see that in rational limit this bracket coincide with (1.8)

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