Normalizability, Synchronicity and Relative Exactness for Vector Fields in $\mathbb{C}^2$

C. Christopher, P. Mardešić and C. Rousseau

CRM-2910

February 2003

This work was supported by NSERC and FCAR in Canada.

School of Mathematics and Statistics, University of Plymouth, Plymouth, Devon PL4 8AA, U.K.

Laboratoire de Topologie, Unité mixte de recherche 5584 du C.N.R.S., Université de Bourgogne, B.P. 47 870, 21078 Dijon Cedex, France

Département de mathématiques et de statistique and CRM, Université de Montréal, C.P. 6128, succursale Centre-Ville, Montréal, Québec, H3C 3J7, Canada

Abstract

In this paper we study the necessary and sufficient condition under which an orbitally normalizable vector field of saddle or saddle-node type in $\mathbb{C}^2$ is analytically conjugate to its formal normal form (i.e., normalizable).

We first express this condition in terms of the relative exactness of a certain 1-form derived from comparing the *time form* of the vector field with the time-form of the normal form. We then show that this condition is equivalent to a *synchronicity condition*: the vanishing of the integral of this 1-form along certain asymptotic cycles defined by the vector field. This can be seen as a generalization of the classical theorem of Poincaré saying that a center is isochronous (i.e. synchronous to the linear center) if and only if it is linearizable.

The results, in fact, allow us in many cases to compare any two vector fields which differ by a multiplicative factor. In these cases we obtain that the two vector fields which are multiples of each other are analytically conjugate if and only if their time forms are synchronous.

Mathematical Subject Classification. subjclass 37C15, 34C20

Résumé

Dans cet article nous étudions les conditions nécessaires et suffisantes pour qu'un champ de vecteurs orbitalement normalisable au voisinage d'un col ou col-nœud de $\mathbb{C}^2$ soit analytiquement conjugué à sa forme normale (c.-à-d., normalisable). Nous exprimons d'abord cette condition en termes de l'exactitude relative d'une 1-forme obtenue en comparant la *1-forme temps* avec la 1-forme temps de la forme normale. Nous montrons ensuite que cette condition est équivalente à une condition de synchronicité: l'intégrale de cette 1-forme s'annule le long de cycles asymptotiques du champ de vecteurs. Ceci peut être vu comme une généralisation du théorème de Poincaré montrant qu'un centre est isochrone (c.-à-d., synchrone au champ linéaire) si et seulement il est linéarisable.

1. Introduction

In this paper we consider the analytic system

$$\begin{aligned} \dot{x} &= x + f(x,y) = x + o(|(x,y)|) \\ \dot{y} &= -\lambda y + g(x,y) = -\lambda y + o(|(x,y)|), \quad \lambda \geq 0 \end{aligned} \tag{1.1}$$

in $\mathbb{C}^2$ with a saddle point or saddle-node at the origin. If the origin of (1.1) is orbitally normalizable or integrable (*i.e.* orbitally linearizable), we wish to understand what conditions guarantee that the origin is in fact normalizable or linearizable. That is, we are concerned with the "time" element of the dynamics as well as the "orbital" element. However, we will ignore the effects of a purely uniform dilation of time, so that the form (1.1) does not represent a restriction on the possible eigenvalues of the vector field.

If we choose a 1-form ω which is dual to the vector field above then it is known that, when $d\omega$ does not vanish at the origin, a multiplicative factor can be absorbed into the 1-form by a change of variables. In the case of vector fields the situation is more complex. In fact, to recover a vector field by duality we need, not only ω, but a two form Ω such that $i_V\Omega = \omega$. Changes of variable which simplify ω will also change the dualizing form Ω, and it is not hard to show that there are obstructions to normalizing the vector field which do not appear in the case of 1-forms.

In the simplest case, where the saddle comes from a real center with eigenvalues $\pm i$, then the classical theorem of Poincaré states that the center is in fact linearizable if and only if the period function is locally a constant. Our aim in this paper is to generalize this result to all orbitally normalizable systems of the form (1.1). That is, we assume the system can be conjugated with a system in normal form up to some non-constant multiple, and seek conditions which guarantee that this multiple can be chosen to be unity.

In fact, the results we give here are not restricted to orbitally normalizable vector fields, but apply to the comparison of any two vector fields which differ by multiplication by a nonzero function, except that in the case where λ is irrational, we require that the critical point be integrable. However, since the applications to normalizability are probably the more interesting, we emphasize these in some detail.

In Section 2 we give the definitions of normalizability etc. which we shall use in this paper, and then in Section 3 we give formal normal forms for vector fields with respect to formal conjugacy. These are expressed as the classical orbital normal forms for vector fields together with a multiplicative factor which represents the resonant terms which cannot be absorbed by a change of variables. In the case when λ is a positive rational, the classification of vector fields up to conjugacy has be given by Voronin. Our aim here is much more specific, however.

In Section 4, we return to the problem of characterizing the analytic conjugacy of two orbitally equivalent vector fields. We first show that two vector fields V_1 and $V_2 = hV_1$ are analytically conjugate (we say that they can be "synchronized") if and only if

$$V_1(g) = \frac{1}{h} - 1 \tag{1.2}$$

for some analytic function g. This result looks classical, but we have only found references to it in the unpublished thesis of Natali Pazii [P], and in the recent preprint [T1] in the formal case. Let ω be a 1-form dual to V_1 (and hence V_2), and dt_1 and dt_2 be the *time-forms* for V_1 and V_2. By definition, the dt_i are meromorphic 1-forms (defined up to a multiple of ω) such that $i_{V_i}dt_i = 1$. The condition (1.2) can then be expressed in terms of the relative exactness of the difference of the two time forms $\eta = dt_2 - dt_1$ with respect to ω. That is, we want

$$\eta = k\omega + dh, \tag{1.3}$$

with k and h analytic. In fact, η may be initially meromorphic, but we can always arrange η to be holomorphic by a suitable change of coordinates applied to the V_i (see Corollary 4.2).

In Section 5 we show that the relative exactness of a differential form η with respect to ω is characterized in terms of the vanishing of the integrals of η along all asymptotic cycles contained in the leaves of ω. This is a result of general interest. When λ is rational, the results are particular cases of the work by Berthier and Loray [BL], in the case of a resonant saddle, and by Teyssier [T1] in the saddle-node case. The case for integrable critical points with irrational λ has been covered in a more general context by Berthier and Cerveau [BC], but their results require some diophantine condition on λ. We give the result here for general λ in the planar case. We call this property *synchronicity* in analogy to the term *isochronicity*, used when comparing a center of a vector field with the linear center, and say that dt_1 is synchronous to dt_2.

We thus prove the following theorem.

Main Theorem.

(1) *Let V_1 and $V_2 = hV_1$ be two analytic vector fields of the form* (1.1) *which differ by multiplication by a nonzero function h, and let η be the difference of their time forms. If we furthermore assume that if λ is irrational then V_1 (or V_2) has a local nontrivial first integral (i.e. is orbitally linearizable, the following are equivalent:*
  (a) *V_1 and V_2 are analytically conjugate;*
  (b) *η is relatively exact with respect to some form ω, dual to the V_i;*
  (c) *the integral of the form η vanishes along every asymptotic cycle of the vector fields V_i (that is, the two vector fields are synchronous).*

(2) *In particular any integrable (resp. orbitally normalizable) system* (1.1) *is linearizable (resp. normalizable) if and only its time form is synchronous to the time form of its formal normal form.*

When $\lambda = 1$, this is just the classical theorem of Poincaré mentioned above.

It is the authors hope that the study of time dependence for integrable and normalizable systems will enrich the understanding of the analytic behavior of these critical points just as the study of isochronous centers has enriched the study of centers for planar vector fields.

2. Preliminaries

We recall some standard definitions on conjugacy and orbital equivalence for germs of vector fields.

Definition 2.1.

i) *Two germs of vector fields are formally (resp. analytically) conjugate if one can be transformed to the other by a formal (resp. analytic) change of coordinates.*

ii) *Two germs of vector fields are formally (resp. analytically) orbitally equivalent if one is formally (resp. analytically) conjugate to a formal (resp. analytic) multiple of the other.*

Definition 2.2.

i) *A normal form of a germ of a vector field is a vector field containing no non-resonant monomials.*

ii) *A germ of a vector field is normalizable if it is analytically conjugate to a normal form.*

iii) *A vector field is orbitally normalizable if it is analytically orbitally equivalent to a normal form.*

The case of a germ of vector field formally conjugate (resp. formally orbitally equivalent) to the linear normal form for rational values of λ is special, as the notions of formal conjugacy (resp. formal orbital equivalence) and analytic conjugacy (resp. orbital equivalence) are equivalent [B].

In more detail, we have the following definitions.

Definitions 2.3.

(i) *The system* (1.1) *is integrable at the origin if and only if it is orbitally linearizable, i.e. there exists an analytic change of coordinates*

$$(X, Y) = (x + \phi(x, y), y + \psi(x, y)) = (x + o(x, y), y + o(x, y)) \tag{2.1}$$

bringing the system (1.1) *to the system*

$$\begin{aligned} \dot{X} &= Xh(X, Y) \\ \dot{Y} &= -\lambda Y h(X, Y), \end{aligned} \tag{2.2}$$

with $h(X, Y) = 1 + O(X, Y)$. If $h(X, Y) = 1$, then the system is linearizable at the origin.

(ii) *For $\lambda = \frac{p}{q} \in \mathbb{Q}^+$ the system is orbitally normalizable at the origin if there exists an analytic change of coordinates of the form* (2.1) *transforming* (1.1) *to the semi-normal form*

$$\begin{aligned} \dot{X} &= Xk_1(U)h(X, Y) \\ \dot{Y} &= -\lambda Y k_2(U)h(X, Y), \end{aligned} \tag{2.3}$$

where k_1, k_2 and h are analytic functions such that $h(0, 0) = 1$, $U = X^pY^q$ and $k_1(0) = k_2(0) = 1$. If $h(X, Y) = 1$, then the system is normalizable at the origin.

(iii) *For $\lambda = 0$ the system is orbitally normalizable at the origin if there exists an analytic change of coordinates of the form* (2.1) *transforming* (1.1) *to the semi-normal form*

$$\begin{aligned}\dot{X} &= Xk_1(Y)h(X,Y)\\ \dot{Y} &= k_2(Y)h(X,Y),\end{aligned} \tag{2.4}$$

where k_1, k_2 and h are analytic functions such that $k_1(0) = 1$ and $h(0,0) = 1$. If $h(X,Y) = 1$, then the system is normalizable at the origin.

Remarks 2.4.

(1) *The case (iii) above is nothing more than case (ii) taking $p = 0$ and $q = 1$. We separate it for ease of comparison. We continue this policy throughout the paper.*
(2) *For $\lambda \neq 0$ the system* (1.1) *is integrable if and only if the holonomy of any separatrix is linearizable. This follows from the theorems of Mattei-Moussu* [MM].
(3) *For $\lambda \neq 0$ (resp $\lambda = 0$) the system* (1.1) *is orbitally normalizable if and only if the holonomy of any separatrix (resp. of the strong separatrix) is embedable i.e. given by the time-one flow of a vector field in a neighborhood of the origin in $\mathbb{C}$ composed with a rotation (see for instance* [MR, CMR]*).*

3. Formal normal forms for orbital equivalence and conjugacy of vector fields

In this section we give reduced formal normal forms for a critical point of saddle or saddle-node type. The following normal forms for formal (orbital) equivalence are well-known.

Proposition 3.1.

Let (1.1) *be a critical point of saddle or saddle-node type.*

(i) *If $\lambda \neq 0$ is irrational, then* (1.1) *is formally conjugate to the linear vector field*

$$\begin{aligned}\dot{X} &= X\\ \dot{Y} &= -\lambda Y.\end{aligned} \tag{3.1}$$

(ii) *If $\lambda = p/q \neq 0$ is rational, $p, q \in \mathbb{Z}$ relatively prime, then* (1.1) *is either formally orbitally equivalent to the linear form* (3.1) *(in fact, it is analytically orbitally equivalent to* (3.1) *in this case) or formally orbitally equivalent to a vector field of the form*

$$\begin{aligned}\dot{X} &= X(1 + aU^k)\\ \dot{Y} &= -\frac{p}{q}Y(1 + (a-1)U^k).\end{aligned} \tag{3.2}$$

(iii) *If $\lambda = 0$ and the origin is an isolated singular point, then* (1.1) *is formally orbitally equivalent to a vector field of the form*

$$\begin{aligned}\dot{X} &= X(1 + aY^k)\\ \dot{Y} &= Y^{k+1}.\end{aligned} \tag{3.3}$$

A formally integrable system is formally linearizable. However, this is no longer true in the analytic category. In fact, in [CMR] we give a sharp condition on irrational λ to ensure that any integrable system of the form (1.1) is linearizable. The condition is that λ is not a Cremer number. That is, the denominators q_n in its continuous fraction expansion satisfy

$$\limsup_{n\to\infty} \frac{1}{q_n} \log q_{n+1} < +\infty.$$

The condition of being non Cremer is weaker than the Brjuno condition, which guarantees linearizability of all (1.1) for an irrational Brjuno number λ.

We now study the normal forms under *formal conjugacy* i.e. the equivalence relation induced by formal changes of coordinates without multiplication by a function (i.e. a generalized "time scaling").

Proposition 3.2. *Let* (1.1) *be a saddle or saddle-node type vector field.*

(i) *If* $\lambda \neq 0$ *is irrational, then* (1.1) *is formally conjugate to the linear vector field*

$$\begin{aligned} \dot{X} &= X \\ \dot{Y} &= -\lambda Y. \end{aligned} \tag{3.4}$$

(ii) *If* $\lambda = p/q \neq 0$ *is rational,* $p, q \in \mathbb{Z}$ *relatively prime, and*

$$U = X^p Y^q, \tag{3.5}$$

then one of the following cases is satisfied for (1.1)*:*

(iia) If (1.1) *is integrable then it is analytically conjugate to a vector field of the form*

$$\begin{aligned} \dot{X} &= X(1 + \sum_{i=1}^{\infty} a_i U^i) \\ \dot{Y} &= -\frac{p}{q} Y(1 + \sum_{i=1}^{\infty} a_i U^i), \end{aligned} \tag{3.6}$$

for some $a_i \in \mathbb{C}$.

(iib) If (1.1) *is not integrable then it is formally conjugate to a vector field of the form*

$$\begin{aligned} \dot{X} &= X(1 + aU^k)(1 + a_1 U + \cdots a_k U^k) \\ \dot{Y} &= -\frac{p}{q} Y(1 + (a-1)U^k)(1 + a_1 U + \cdots a_k U^k), \end{aligned} \tag{3.7}$$

for some $a_1, \ldots, a_k \in \mathbb{C}$. *If* (1.1) *is not integrable but is normalizable, then it is analytically conjugate to* (3.7)

(iii) *If* $\lambda = 0$ *and the origin is an isolated singular point of* (1.1)*, then* (1.1) *is formally conjugate to a vector field of the form*

$$\begin{aligned} \dot{X} &= X(1 + aY^k)(1 + a_1 Y + \cdots a_k Y^k), \\ \dot{Y} &= Y^{k+1}(1 + a_1 Y + \cdots a_k Y^k), \end{aligned} \tag{3.8}$$

for some $a_1, \ldots, a_k \in \mathbb{C}$. *If* $\lambda = 0$ *and* (1.1) *is normalizable, then it is analytically conjugate to* (3.8)

Only forms (iib) and (iii) need a proof, the others being well-known. To prove these we need to study the changes of coordinates which preserve the orbital normal forms. For completeness, we give formulas for (i) and (iia) also.

Let $T_{r,s}$ denote the transformation $(X, Y) = (rx, sy)$ and let $h(x, y)$ be a nonzero analytic function.

Proposition 3.3.

(i)/(iia) *Consider the system*

$$\begin{aligned} \dot{x} &= xh(x, y), \\ \dot{y} &= -\lambda y h(x, y). \end{aligned} \tag{3.9}$$

Changes of coordinates which preserve the orbital form of this system are compositions of $T_{r(u),s(u)}$, *with* r *and* s *analytic functions of* u, *and transformations of the form*

$$\begin{aligned} X &= xe^{g(x,y)}, \\ Y &= ye^{-\lambda g(x,y)}, \end{aligned} \tag{3.10}$$

with $g(x, y)$ *analytic, where* $u = x^p y^q$ *when* $\lambda = p/q$ *and* 0 *otherwise.*

(iib) *Consider the system*

$$\begin{aligned} \dot{x} &= x(1 + au^k)h(x, y), \\ \dot{y} &= -\frac{p}{q} y(1 + (a-1)u^k)h(x, y), \end{aligned} \tag{3.11}$$

where $u = x^p y^q$. Changes of coordinate, preserving the orbital normal form of (3.11) *are compositions of $T_{r,s}$, with $r^{pk}s^{qk} = 1$, and transformations of the form*

$$X = x(1 - pku^k g(x,y))^{-a/(pk)} e^g = xm(x,y),$$
$$Y = y(1 - pku^k g(x,y))^{(a-1)/(qk)} e^{-gp/q} = yn(x,y), \tag{3.12}$$

where $g(x,y)$ is an analytic function.

(iii) *Consider the system*

$$\dot{x} = x(1 + ay^k)h(x,y),$$
$$\dot{y} = y^{k+1}h(x,y). \tag{3.13}$$

Changes of coordinate, preserving the orbital normal form of (3.13) *are compositions of $T_{r,s}$, with $s^k = 1$, and transformations of the form*

$$X = x(1 - ky^k g(x,y))^{-a/k} e^g = xm(x,y),$$
$$Y = y(1 - ky^k g(x,y))^{-1/k} = yn(x,y), \tag{3.14}$$

where $g(x,y)$ is an analytic function.

(iv) *Denoting V the vector fields given by* (3.9), (3.11) *and* (3.13) *respectively and $V_0 = V/h$ the vector field of the orbital normal form, the transformations* (3.10), (3.12) *and* (3.14) *are given by the $g(x,y)$-time flow of the orbital normal form vector field V_0. They have the effect of changing the vector field hV_0 to the vector field $r(X,Y)V'$, where V' is just the vector field V_0 under the direct substitution $x = X$ and $y = Y$ and $r(X,Y)$ is obtained from $h(x,y)(1 + V_0(g))$ under the substitutions* (3.10), (3.12) *and* (3.14) *above.*

Proof. The statement (iv) of the proposition can be proved by direct, but tedious calculation. A conceptual proof is given in Proposition 4.1 below, but note that the transformation given there is the inverse of the one given here.

For the rest of the proposition, we prove only (iib), as the proof of (i)/(iia) and (iii) are very similar. A change of coordinates which preserves the orbital normal form must preserve the invariant coordinate axes, and so must be of the form

$$X = xm(x,y) = x(r + O(x,y))$$
$$Y = yn(x,y) = y(s + O(x,y)). \tag{3.15}$$

The new system has the form

$$\dot{X} = X(1 + aX^k)H(X,Y),$$
$$\dot{Y} = -\frac{p}{q}Y(1 + (a-1)U^k)H(X,Y), \tag{3.16}$$

where $U = X^pY^q$. The system (3.11) has the first integral

$$F(x,y) = x^{pk(a-1)}y^{qka}e^{-1/x^{pk}y^{qk}} = u^{ka}e^{-1/u^k}x^{-pk} = u^{k(a-1)}e^{-1/u^k}y^{kq}. \tag{3.17}$$

We claim that the first integral must be preserved, that is

$$F(X(x,y), Y(x,y)) = CF(x,y) \tag{3.18}$$

for some constant C. Indeed, $F(X(x,y), Y(x,y))$ is also a first integral of (3.11), so the quotient $F(X(x,y), Y(x,y))/F(x,y)$ is also a first integral of (3.11). It has the form

$$\frac{F(X(x,y), Y(x,y))}{F(x,y)} = K(x,y)e^{1/u^k - 1/U^k}, \tag{3.19}$$

with

$$K = m^{pk(a-1)}n^{qka} = r^{pk(a-1)}s^{qka} + O(x,y). \tag{3.20}$$

Therefore taking the logarithm of (3.19), we obtain a meromorphic first integral of (3.11). However, (3.11) being non integrable (because of the presence of resonant term), any meromorphic first integral is trivial (i.e. constant).

Let

$$pk\, g(x,y) = \frac{1}{u^k} - \frac{1}{U^k} = \frac{1}{u^k}\frac{m^{pk}n^{qk} - 1}{m^{pk}n^{qk}}, \tag{3.21}$$

then from above, $\log(K) + pk\, g$ is a constant function, and hence $g(x,y)$ is analytic. The value of the quotient in (3.19) is then $C = r^{pk(a-1)}s^{qka}e^{pk\, g_0}$, with $g_0 = g(0,0)$. Rearranging (3.21) gives

$$m^{pk}n^{qk} = (1 - pku^k g(x,y))^{-1}. \tag{3.22}$$

Hence $n = m^{-\frac{p}{q}}(1 - pku^k g(x,y))^{-\frac{1}{qk}}$. Putting this relation in (3.19) we obtain

$$m^{-pk}(1 - pku^k g)^{-a}e^{pk\, g} = r^{pk(a-1)}s^{qka}e^{pk\, g_0}. \tag{3.23}$$

Evaluating this equation at the origin, we see that $r^{pka}s^{qka} = 1$. This gives $m/r = (1 - pku^k g)^{-\frac{a}{pk}}e^{g-g_0}$, and $n/s = (1 - pku^k g)^{\frac{a-1}{qk}}e^{-\frac{p}{q}(g-g_0)}$. After composing with $T_{e^{g_0}, e^{-pg_0/q}}$, we obtain our result. □

Proof of Proposition 3.2.

Once again, we present the proofs for only (iib), as the cases (i), (iia) and (iii) follow along similar lines.

>From Proposition 3.3 (iv) the system (3.11) is transformed to

$$\begin{aligned} \dot{X} &= X(1 + aU^k)r(x,y), \\ \dot{Y} &= -\frac{p}{q}Y(1 + (a-1)U^k)r(x,y), \end{aligned} \tag{3.24}$$

where $r(x,y)$ is given by

$$r = h(x,y)(1 + V_0(g)), \tag{3.25}$$

with

$$V_0 = x(1 + au^k)\frac{\partial}{\partial x} - \frac{p}{q}y(1 + (a-1)u^k)\frac{\partial}{\partial y}. \tag{3.26}$$

We therefore need to solve

$$V_0(g) = \frac{1}{h(x,y)}P(U) - 1, \tag{3.27}$$

for some choice of $P(U) = \sum_{i=0}^k a_i U^i$ with $a_0 = 1$. Formally, it is clear that if the right hand side of (3.27) contains no term in u^i for $i = 0, \ldots, k$, then there is a unique solution. However, we know that $U = u(1 - pku^k g(u))^{-1/k}$, and so

$$\sum_{i=0}^k a_i U^k = \sum_{i=0}^k a_i u^k + O(u^{k+1}). \tag{3.28}$$

We write

$$1/h(x,y) = \sum_{i,j \geq 0} b_{i,j}x^i y^j. \tag{3.29}$$

Then, in order to have a formal solution the coefficients a_i of $P(U)$ must be chosen as $a_0 = 1$ and

$$a_i = -\sum_{j=0}^{i-1} a_j b_{i-j,i-j}. \tag{3.30}$$

To prove the final statement of (iib) we can assume that $h = h(u)$, so that (3.27) becomes

$$V_0(g(u)) = \frac{1}{h(u)} \sum_{i=0}^{k} a_i U^k - 1, \tag{3.31}$$

with $U = u(1 - pku^k g(u))^{-1/k}$. To show that (3.31) has an analytic solution reduces to finding an analytic solution to the differential equation

$$pu^{k+1} g'(u) = \frac{1}{h(u)} P(u(1 - pku^k g(u))^{-1/k}) - 1 = u^{k+1} R(u, g(u)), \tag{3.32}$$

for some analytic function R. But this follows directly from the standard existence and uniqueness results for differential equations.

In the case (iia) we have $U = u$ and $V_0 = x\frac{\partial}{\partial x} - \frac{p}{q} y \frac{\partial}{\partial y}$. The corresponding equation to (3.27), which brings (3.9) with $\lambda = p/q$ to the form (3.6) can always be solved analytically. This demonstrates the remark made after Definition 2.2. If λ is not rational (case(i)), then there can arise obstructions due to small divisors.

Results in the same spirit can be found in [VG], although their reduced normal form is not exactly the same as ours.

Corollary 3.4. *There are exactly k obstructions to the existence of a formal change of variables transforming the system* (3.11) *to the form*

$$\begin{aligned} \dot{x} &= x(1 + au^k) \\ \dot{y} &= -\lambda y(1 + (a-1)u^k). \end{aligned} \tag{3.33}$$

They are given by the nonvanishing of the coefficients $a_1, \ldots, a_k$ in (3.7)*. A similar statement holds for system* (3.13)*.*

4. Linearizability, normalizability and relative exactness.

In this section we find conditions for an orbitally normalizable saddle or saddle-node to be normalizable, i.e. conjugate to one of the formal normal forms of Proposition 3.2, in terms of the relative exactness of a time-form which we will give below. However, since it is no extra work, we shall deal with the general case of deciding when two vector fields with the same orbits, V_1 and $V_2 = hV_1$ are analytically conjugate.

Proposition 4.1. *Let V be an analytic vector field in the neighbourhood of the origin, and $\psi_t(x, y)$ represent its flow, with $\psi_0(x, y) = (x, y)$, then the local diffeomorphism $\alpha : (x, y) \mapsto \psi_{g(x,y)}(x, y)$ pulls back the vector field V to the vector field $(1 + V(g))^{-1} V$.*

Proof. The Proposition is proved in the formal case in [T1], and is also in the 1999 Ph.D. thesis of Natali Pazii. Let α_t represent the map

$$\alpha_t : (x, y) \mapsto \psi_{tg(x,y)}(x, y). \tag{4.1}$$

If we let α_t^* represent the pull-back induced by α_t then we have

$$\frac{d}{dt} \alpha_t^*(\phi(x, y)) = g(x, y) \alpha_t^*(V\phi). \tag{4.2}$$

Now, since α_t is a diffeomorphism, the pull back of the vector field, $\alpha_t^*(V)$, makes sense, and since α_t also preserves orbits, then

$$\alpha_t^*(V) = m(x, y, t)V, \tag{4.3}$$

for some function m with $m(x, y, 0) = 1$. Thus,

$$\alpha_t^*(V\phi) = mV\alpha_t^*(\phi). \tag{4.4}$$

Differentiating this expression with respect to t, we get

$$g\alpha_t^*(V^2\phi) = \frac{dm}{dt}V\alpha_t^*(\phi) + mV(g\alpha_t^*(V\phi)). \tag{4.5}$$

This gives, using (4.4),

$$gmV\alpha_t^*(V\phi) = \frac{dm}{dt}\frac{1}{m}\alpha_t^*(V(\phi)) + mV(g)\alpha_t^*(V\phi) + mgV\alpha_t^*(V\phi), \tag{4.6}$$

from which we can deduce that

$$\frac{d}{dt}\left(\frac{1}{m}\right) = V(g), \tag{4.7}$$

and hence $m = (1 + tV(g))^{-1}$. Putting $t = 1$ completes the theorem. □

Corollary 4.2. *If $V = hV_0$ is a vector field of the form (1.1), with $h = 1 + O(x, y)$ then, when $\lambda \neq 0$ there exists an analytic change of coordinates which brings the vector field to the form*

$$\tilde{V} = (1 + xl(x, y))V \tag{4.8}$$

where l is analytic. In particular, an integrable system can always be brought to the form

$$\begin{aligned} \dot{x} &= xh(x, y) \\ \dot{y} &= -\lambda y h(x, y) \end{aligned} \tag{4.9}$$

with

$$h = 1 + xl(x, y) \tag{4.10}$$

for some analytic function l. If $\lambda = 0$ then there exists a change of coordinates which replaces h by a function with no terms in y^s, $s > k$.

Proof. If $\lambda \neq 0$, we can write $1/h = 1 + h_1(y) + xm(x, y)$, for some analytic function h_1 and m. From Proposition 4.1, we only need to show that there is some function g with

$$V_0(g) = h_1(y) + x\tilde{m} \tag{4.11}$$

and then pull back via α defined in Proposition 4.1. However, it is easy to see that we can find such an g of the form $g_1(y)$. The case when $\lambda = 0$ is similar. □

Definitions 4.3.

Let V be an analytic vector field in the neighbourhood of a point p, and ω a reduced 1-form dual to V (ie $i_V\omega = 0$).

(1) *The time form dt associated to V is any meromorphic 1-form such that $i_V dt = 1$. Thus, time forms are only defined up to addition of meromorphic multiples of ω.*
(2) *In the case when V has an orbitally normalized saddle, we let dt_{norm} be the time form of the formal normal vector field associated to V. More precisely,*

$$dt_{norm} = \begin{cases} \frac{dx}{x} & \textit{for} \quad (3.9) \\ \frac{dx}{x(1+au^k)P(u)} & \textit{for} \quad (3.11) \\ \frac{dx}{x(1+ay^k)P(y)} & \textit{for} \quad (3.13). \end{cases} \tag{4.12}$$

where $P(u)$ and $P(y)$ are defined as in the proof of Proposition 3.2.

(3) *We say that a 1-form η is relatively exact with respect to ω if there exist analytic functions g and m such that*

$$\eta = dg + m\omega. \tag{4.13}$$

Note that, if we start with a meromorphic form η and apply the transformation of Corollary 4.2 to the vector fields, we can always assume that η is holomorphic.

Theorem 4.4.

(1) *Let* V_1 *and* $V_2 = \tilde{h}V_1$ *be two vector fields, and take* η *to be the difference of their time forms*

$$\eta = dt_2 - dt_1, \tag{4.14}$$

then V_1 *and* V_2 *are analytically conjugate if and only if* η *is relatively exact with respect to some 1-form* ω *dual to* V_1 *(or* V_2*).*

(2) *Let* V *be a vector field of the form (1.1) which is orbitally normalizable, and take* η *to be the difference between its time form and the time form of its normalized system,*

$$\eta = dt_{norm} - dt, \tag{4.15}$$

then V *is analytically conjugate to its normal form if and only if* η *is relatively exact with respect to some 1-form* ω *dual to* V*.*

Proof.

(1) From Proposition 4.1 it is sufficient to show that there exists some g such that $V_1(g) = 1/\tilde{h} - 1$. If V_1 is of the form $Q_1\partial_x + Q_2\partial_y$, then taking $\omega = Q_1\, dy - Q_2\, dx$ we see that this is equivalent to

$$\eta \wedge \omega = dg \wedge \omega, \tag{4.16}$$

where η is defined as above. The claim now follows from Lemma 4.5 below, whose proof is easy.

(2) To apply (1) we take $V_1 = hV_0$ and $V_2 = P(u)V_0$, so that $\tilde{h}(x, y) = P(u)/h(x, y)$. Thus (1) shows that (4.16) is equivalent to the existence of some g satisfying $V_1(g) = h/P - 1$, which is exactly the condition required for analytic conjugacy of V_1 with its normal form V_2.

Lemma 4.5. *Let* ω *be a one-form in* $\mathbb{C}^2$ *having isolated singularities. A holomorphic one-form* η *is relatively exact with respect to the form* ω *if and only if there exists a holomorphic function* g *satisfying* (4.16).

5. Relative exactness and asymptotic cycles

In this final section we shall show that the notion of relative exactness of a 1-form η with respect to a differential form ω is equivalent to the vanishing of the integral of η along certain asymptotic cycles defined on the leaves of the foliation $\omega = 0$.

Definition 5.1. *Let* $[\alpha_n, \beta_n] \subset \mathbb{R}$ *be a sequence of intervals and let* $\gamma_n : [\alpha_n, \beta_n] \to \mathbb{C}^2$ *be a sequence of curves, all coinciding on the intersection of their domains.*

(i) *We say that the sequence* $\gamma = (\gamma_n)$*,* $n \in \mathbb{N}$*, is an asymptotic cycle if*

$$\lim \gamma(\alpha_n) = \lim \gamma(\beta_n). \tag{5.1}$$

(ii) *We define*

$$\int_\gamma \eta = \lim_{n\to\infty} \int_{\gamma_n} \eta.$$

Theorem 5.2. *Let* ω *be a 1-form dual to a vector field of the form* (1.1)*. If* λ *is irrational, we assume that the vector field is integrable. Then any analytic 1-form* η *is relatively exact with respect to* ω *if and only if*

$$\int_\gamma \eta = 0, \tag{5.2}$$

for any asymptotic cycle γ *belonging to a leaf of the foliation defined by* $\omega = 0$*.*

The claim of the theorem has been proven in the saddle resonant case by Berthier and Cerveau [BC] (integrable case) and Berthier and Loray [BL] (nonintegrable case) and in the saddle-node case by Teyssier [T1]. The only remaining case is the case of an integrable non resonant saddle. We shall give this proof below.

Definition 5.3. *Given two vector fields V_1 and $V_2 = hV_1$ with time forms dt_1 and dt_2, we say that the two time forms are synchronous if the integral of their difference is zero along every asymptotic cycle γ belonging to a leaf of the foliation defined by $\omega = 0$, where ω is some 1-form dual to V_1 (or V_2). We say that a time form is isochronous if it is synchronous to the time form of a linear system.*

Theorem 5.4.

(1) *Let V_1 and $V_2 = hV_1$ be two analytic vector fields in the form (1.1), and assume that if λ is irrational then the V_i are integrable. Then the two vector fields are analytically conjugate if and only if their time forms are synchronous.*

(2) *Let V be an orbitally normalizable vector field in the form* (3.9), (3.11) *or* (3.13). *Then V is normalizable if and only if its time form is synchronous with dt_{norm} given by* (4.12). *In particular, an integrable system of the form* (1.1) *is isochronous if and only if it is synchronous to the linear form dX/X, for some variable X which vanishes on the separatrix tangent to the x-axis of* (1.1).

Proof. These results follow directly from Theorem 4.4 and Theorem 5.2 once we show that we can replace the η's in Theorem 4.4 with analytic functions. However, this is clear from Corollary 4.2 and the fact that the η's are the difference of two such time forms. The last statement follows from the fact that, transforming to a normalized system (4.8) we have $X = xW(x, y)$, for some non vanishing function W, and $dX/X = dx/x+dW/W$, so the relative exactness with respect to X is equivalent to the relative exactness with respect to x. Unfortunately, there does not seem to be an equivalent "coordinate free" description of normalizability. □

Proof of Theorem 5.2. As stated above, we only need to prove this for an integrable non-resonant saddle. The direct implication is obvious. We prove only the converse. In this case one can get a formal solution of (4.16) as in the rational case (without using the isochronicity condition). However, as shown in [BC] and [CMR], Theorem A, the formal solution does not converge in general. This is why we adopt an approach different from the approach in the rational case.

We have to define a holomorphic function g satisfying (4.16) in a neighborhood of the origin. By linear scaling we can assume that the polydisc $\Omega = \{(x, y) \in \mathbb{C}^2 : |x| \leq 1, |y| \leq 1\}$ belongs to the domain of convergence of the form η. Put $g(1, 1) = 0$. Let γ_c be the curve lying in the leaf of the foliation $\omega_\lambda = 0$, passing through the point $(1, c)$, given by

$$\gamma_c(\theta) = (e^{i\theta}, ce^{-i\lambda\theta}), \quad \theta \in \mathbb{R}. \tag{5.3}$$

Put

$$g(\gamma_1(\theta)) = \int_0^\theta \gamma_1^*\eta. \tag{5.4}$$

This defines the function g on a dense set of the torus $T = \{(x, y) \in \mathbb{C}^2 : |x| = 1, |y| = 1\}$. Next, by the hypothesis (5.2), the function g can be extended without ambiguity by continuity to the torus T.

Denote by $D = \{y \in \mathbb{C} : |y| < 1\}$, the unitary disc, $\bar{D}$ its closure and S its boundary. Note that g satisfies the condition

$$g(1, ce^{2\pi i\lambda}) = g(1, c) + \int_0^{2\pi} \gamma_c^*\eta, \tag{5.5}$$

for c on the circle S. We next want to extend g to a continuous function on the disc $\{1\} \times \bar{D}$,

$$g(1, c) = u(c), \tag{5.6}$$

with u holomorphic in D. Moreover, we want condition (5.5) to hold for all points $c \in \bar{D}$.

Initially, the function u is continuous complex valued defined on the circle S. Applying separately the existence theorem for solutions of Dirichlet's problem for the real and imaginary part of u, we extend u to a continuous function on $\bar{D}$, *harmonic* in D. Let g be given by (5.6), $c \in \bar{D}$. We claim that (5.5) holds for this extended function g. Indeed, consider the function

$$\psi(c) = u(ce^{2\pi i\lambda}) - u(c) - \int_0^{2\pi} \gamma_c^* \eta. \tag{5.7}$$

The function ψ is a harmonic function in D, as the last term in (5.7) is a holomorphic function in c. Moreover, $\psi(c)$ vanishes for $c \in S$. Now by the uniqueness of solutions of Dirichlet's problem it follows that ψ is identically zero on $\bar{D}$. This proves that(5.5) holds on $\bar{D}$. We claim next that u is in fact holomorphic in D. In order to prove it, introduce the differential operators

$$\partial = \frac{1}{2}(\frac{\partial}{\partial c'} + i\frac{\partial}{\partial c''}), \quad \bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial c'} - i\frac{\partial}{\partial c''}), \tag{5.8}$$

where c' and c'' are the real and imaginary part of c. As the last term in the definition of ψ is holomorphic, it follows, from the vanishing of ψ, that

$$\bar{\partial}(u(ce^{2\pi i\lambda})) = \bar{\partial}(u(c)). \tag{5.9}$$

Now, since λ is irrational, the numbers $ce^{2\pi ik\lambda}$, $k \in \mathbb{N}$, are dense on the circle of radius $|c|$, so (5.9) shows that the function $\bar{\partial}(u(c))$ depends only on $|c|$. Say $\bar{\partial}(u(c)) = \phi(|c|)$. However, since u is harmonic, then $\partial(\phi(|c|)) = 0$. This can only happen, if ϕ is a constant. We have so far proven that $u(c) = v(c) + k\bar{c}$, where v is a holomorphic function and $k \in \mathbb{C}$. To show that $k = 0$, we integrate relation(5.7) along $|c| = 1$. As ψ vanishes in D and moreover v and the last term in (5.7) are holomorphic, we get $2\pi ik(e^{-2\pi i\lambda} - 1) = 0$, so $k = 0$ and we have shown that u is holomorphic on D.

We now extend g to a complement of the y-axis. Given any (x, y), denote $\mathcal{L}_{(x,y)}$ the leaf of the linear foliation $\omega = 0$ passing through (x, y). For (x, y), $x \neq 0$, belonging to a neighborhood of the origin, the leaf $\mathcal{L}_{(x,y)}$ cuts the disc $\{1\} \times D$ infinitely many times. Let $\gamma_{(x,y)}$ be a curve in the leaf $\mathcal{L}_{(x,y)}$ starting at a point $(1, c)$, $c = x^\lambda y \in D$, and connecting it to (x, y). The leaf also cuts the disc $\{1\} \times D$ at points $(1, ce^{2\pi i\lambda k})$ with $k \in \mathbb{Z}$. We put

$$g(x, y) = g(1, c) + \int_{\gamma_{(x,y)}} \eta. \tag{5.10}$$

We claim that g is well defined and does not depend on the choice of $\gamma_{(x,y)}$. Indeed, let $\tilde{\gamma}_{(x,y)}$ be another choice of $\gamma_{(x,y)}$ starting at $(1, \tilde{c}) \in D$ with $\tilde{c} = c\exp(2\pi ik\lambda)$. Note that the path obtained by taking $\gamma_{(x,y)}$ followed by $(\tilde{\gamma}_{(x,y)})^{-1}$ is homotopic to the path $\gamma_c : [0, 2k\pi] \to \mathcal{L}_{(x,y)}$, given by $\gamma_c(\theta) = (e^{i\theta}, ce^{-i\lambda\theta})$, for some $k \in \mathbb{Z}$. This follows from the simple connectedness of the leaf $\mathcal{L}_{(x,y)}$.

By induction (5.5) gives

$$g(1, ce^{2\pi ik\lambda}) = g(1, c) + \int_0^{2k\pi} \gamma_c^* \eta, \quad k \in \mathbb{Z}. \tag{5.11}$$

This shows that

$$g(1, c) + \int_{\gamma_{(x,y)}} \eta = g(1, \tilde{c}) + \int_{\tilde{\gamma}_{(x,y)}} \eta \tag{5.12}$$

and hence the function g is well defined on a neighborhood of the origin from which the y-axis has been deleted. Moreover, g is holomorphic, as the initial value $g(1, c)$ depends holomorphically on $c = x^\lambda y$ and g is extended by integration of the holomorphic form η.

In order to extend holomorphically g to the y-axis, note that a point (x, y) close to the y-axis can be linked to a point in the disc $\{1\} \times D$ by a path γ belonging to the leaf $\mathcal{L}_{(x,y)}$ whose length is uniformly bounded. This can be seen by taking the path obtained by following first

$$\gamma_1(\theta) = (xe^{-i\theta}, ye^{i\lambda\theta}), \tag{5.13}$$

for θ varying from 0 to $arg(x) < 2\pi$ and then following

$$\gamma_2(r) = (r, \frac{yx^\lambda}{r^\lambda}), \tag{5.14}$$

for r varying from $|x|$ to 1. The form η is bounded in the fixed neighborhood of the origin which we have chosen. As g is also holomorphic (hence bounded) on $\{1\} \times D$, it now follows from the definition (5.10) of g that it is bounded on a fixed neighborhood of the origin from which the y-axis has been deleted. By the removable singularity theorem, we can now extend g holomorphically to a full neighborhood of the origin.

Relation (4.16) follows from the definition of g. Indeed, it suffices to verify this relation locally in the complement of the origin, where ω is different from zero. By a local change of coordinates $(z, w) = (z(x, y), w(x, y))$, it can be assumed that $\omega = dw$. Taking a section $z = k$ transverse to the leaves of the foliation and noting that g is obtained by integrating η along the leaves $w = const$, it follows that the dz coordinates of dg and η coincide. Hence $\eta - dg$ is collinear to ω. Now (4.16) is proved and the proof of the Theorem is completed. □

Open Question. Can the proof of Theorem 5.2 be extended to cover 1-forms ω dual to non-integrable vector fields of the form (1.1) when λ is irrational?

Acknowledgements

We would like to thank Frank Loray, Loïc Teyssier and Sergei Yakovenko for useful discussions or suggestions.

The first and second author thank the Centre de Recherches Mathématiques where part of this work was done during their visit.

References

BC. M. Berthier, D. Cerveau, *Quelques calculs de cohomologie relative*, Ann. scient. Éc. Norm. Sup. **26** (1993), 403-424.

BL. M. Berthier, F. Loray, *Cohomologie relative des formes résonnantes non dégénérées*, Asymptotic Analysis **15** (1997), 41-54.

B. A.D. Brjuno, *Analytic form of differential equations*, Trans. Moscow Math. Soc. **25** (1971), 131-288.

Ca. C. Camacho, *On the local structure of conformal mappings and holomorphic fields in* $\mathbb{C}^2$, Astérisque **59-60** (1978), 83-94.

CMR. C. Christopher, P. Mardešić and C. Rousseau, *Normalizable, integrable and linearizable saddle points in complex quadratic systems in* $\mathbb{C}^2$, preprint CRM, to appear in Jour. Dynam and Control Syst. (2002).

Du. H. Dulac, *Sur les cycles limites*, Bull. Soc. Math. France **51** (1923), 45-188.

K. V. Kostov, *Versal deformations of differential forms of degree* α *on the line*, Functional Anal. Appl. **18** (1984), 335–337.

MM. J.-F. Mattei and R. Moussu, *Holonomie et intégrales premières*, Ann. Scient. Éc. Norm. Sup., 4^e série **13** (1980), 469–523.

MR. J. Martinet and J.-P. Ramis, *Classification analytique des équations non linéaires résonnantes du premier ordre*, Ann. Sc. E.N.S. 4^e série, **16** (1983), 571–621.

P. N. Pazii, *Local analytic classification of equations of Sobolev's type*, Thesis, Cheliabynsk, Russia (1999).

PMY. R. Pérez-Marco and J.-C. Yoccoz, *Germes de feuilletages holomorphes à holonomie prescrite*, Astérisque **222** (1994), 345–371.

VM. S.M. Voronin, Yu.I. Meshcheryakova, *Analytic classification of typical degenerate elementary singular points of germs of holomorphic vector fields in the complex plane*, Izvestia Vuzov, **1** (2002,).

VG. S.M. Voronin, A.A. Grinchy, *Analytic classification of saddle resonant singular points of holomorphic vector fields on the complex plane*, preprint, 1–28.

T1. L. Teyssier, *Équation homologique et cycles asymptotiques d'une singularité nœud-col.*, preprint, 1–18.

T2. L. Teyssier, *Analytical classification of singular saddle-node vector fields.*, preprint, 1–25.

Y. J.-C. Yoccoz, *Théorème de Siegel, nombres de Brjuno et polynômes quadratiques*, Astérisque **231** (1995), 3–88.