Bound-Preserving Operators and Bernstein-Type Inequalities

Dimiter Dryanov^{*} Richard Fournier[†]

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^{*}Department of Mathematics, University of Sofia, James Boucher 5, 1164 Sofia, Bulgaria
CURRENT ADDRESS: Département de mathématiques et de statistique, Université de Montréal, Montréal H3C 3J7,
Canada; dryanovd@dms.umontreal.ca

[†]Département de Mathématiques et de Statistique, Université de Montréal, Montréal H3C 3J7, Canada;
fournier@dms.umontreal.ca

Abstract

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$ and

$$\|p\| := \max_{z \in \partial\mathbb{D}} |p(z)|,$$

where $p(z) := \sum_{k=0}^n a_k(p) z^k$ is a polynomial of degree at most n and $a_k(p) \in \mathbb{C}$. The following sharpening of Bernstein's inequality

$$\|p'\| + \frac{2n}{n+2} |a_0(p)| \leq n \|p\|$$

has been published by Frappier, Rahman, and Ruscheweyh. Our main contribution concerns the equality case which has remained unsolved since 1982. We prove another inequality of Bernstein type that leads to an improvement of the upper bound for $\|p'\|$ under some additional condition.

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1 Introduction

Let $\mathbb{D}$ denote the unit disc $\{z : |z| < 1\}$ of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the set of functions analytic on $\mathbb{D}$. Let also $\mathcal{P}_n$ be the vector space of polynomials $p(z) := \sum_{k=0}^n a_k(p) z^k$ of degree at most n with coefficients $a_k(p) \in \mathbb{C}$. We define for $f \in H(\mathbb{D})$,

$$\|f\| := \sup_{z \in \mathbb{D}} |f(z)|.$$

According to the celebrated inequality of Bernstein (for references see [7] and the new book by Rahman and Schmeisser [8])

$$\|p'\| \leq n \|p\|, \quad p \in \mathcal{P}_n \quad (n \geq 1) \tag{1}$$

and equality holds in (1) if and only if $p(z)$ is a constant multiple of z^n .

In this paper, we prove the following sharpenings of (1).

Theorem 1 *Let*

$$d_n := \sup \{ d \geq 0 : \|p'\| + d |a_0(p)| \leq n \|p\|, \quad p \in \mathcal{P}_n \}.$$

Then $d_1 = 1$ and $d_n = 2n/(n+2)$ if $n \geq 2$. The equality

$$\|p'\| + d_n |a_0(p)| = n \|p\|$$

holds for all $p \in \mathcal{P}_1$ ($n = 1$) and only for $p(z) = a_n(p) z^n$ if $n \geq 2$.

Theorem 2 *Let*

$$\delta_n := \sup \{ \delta \geq 0 : \delta \|p'\| + |a_0(p)| \leq \|p\|, \quad p \in \mathcal{P}_n \}.$$

Then $\delta_1 = 1$, $\delta_2 = (1 + \sqrt{3})^{-1}$, $\delta_3 = (2 + \sqrt{10})^{-1}$ and in general

$$\frac{2}{n(n+1)} \leq \delta_n \leq \frac{2\sqrt{3}}{(n+1)\sqrt{n(n+2)}}.$$

The equality

$$\delta_n \|p'\| + |a_0(p)| = \|p\|$$

holds for all $p \in \mathcal{P}_1$ if $n = 1$ and only for constant polynomials if $n \geq 2$.

An obvious consequence of Theorem 2 is the next corollary.

Corollary 1 *Let $p \in \mathcal{P}_n$. Then*

$$\frac{2}{n(n+1)} \|p'\| + |a_0(p)| \leq \|p\|$$

and equality holds for all $p \in \mathcal{P}_1$ if $n = 1$ and only for constant polynomials if $n \geq 2$.

The inequality

$$\|p'\| + d_n |a_0(p)| \leq n \|p\|, \quad p \in \mathcal{P}_n \quad (2)$$

is not new and has been published in 1985 by Frappier, Rahman, and Ruscheweyh [4]. Our contribution concerns the case of equality which has remained unsolved since 1982 (see [9, pp. 125-126]).

The inequality

$$\delta_n \|p'\| + |a_0(p)| \leq \|p\|, \quad p \in \mathcal{P}_n \quad (3)$$

is a Bernstein type inequality but it can be also compared with

$$\frac{1}{2} \sec \left(\frac{\pi}{n+2} \right) |a_1(p)| + |a_0(p)| \leq \|p\|, \quad p \in \mathcal{P}_n$$

which has been obtained [2] recently as an extension of an inequality due to Visser [11]. Moreover, it has been proved in [2] that

$$\frac{1}{2} \sec \left(\frac{\pi}{n+2} \right) = \max \{ \gamma \in \mathbb{R} : \gamma |a_1(p)| + |a_0(p)| \leq \|p\|, \quad p \in \mathcal{P}_n \}.$$

Remark that (3) leads to an improvement of the upper bound for $\|p'\|$ obtained via (2) as soon as

$$\|p'\| \leq \frac{1}{\delta_n} (\|p\| - |a_0(p)|) < n \|p\| - \frac{2n}{n+2} |a_0(p)|,$$

i.e., as soon as

$$\frac{1 - n \delta_n}{1 - \frac{2n}{n+2} \delta_n} < \frac{|a_0(p)|}{\|p\|} \quad (n > 1).$$

2 Some Useful Lemmas

As in [4] (see also [9], Chapter 4), our proofs depend on a careful study of a class of bound-preserving operators on $\mathcal{P}_n$. We recall that for

$$f(z) := \sum_{n=0}^{\infty} a_n(f) z^n \in H(\mathbb{D}) \quad \text{and} \quad g(z) := \sum_{n=0}^{\infty} a_n(g) z^n \in H(\mathbb{D})$$

the convolution (or Hadamard product) $f * g$, defined by

$$f * g(z) := \sum_{n=0}^{\infty} a_n(f) a_n(g) z^n,$$

belongs also to $H(\mathbb{D})$. Let for $n \geq 1$

$$\mathcal{B}_n := \{ F \in H(\mathbb{D}) : F(0) = 1 \text{ and } \|F * p\| \leq \|p\| \text{ for all } p \in \mathcal{P}_n \}$$

and

$$\begin{aligned} \mathcal{B}_{\infty} &:= \bigcap_{n \geq 1} \mathcal{B}_n \\ &= \{ F \in H(\mathbb{D}) : F(0) = 1 \text{ and } \|F * f\| \leq \|f\| \text{ for all } f \in H(\mathbb{D}) \} \end{aligned}$$

The following results, due to Sheil - Small (see [10]), Carathéodory and Toeplitz (see [11, pp. 153-159] and [5, pp. 148-154]), and Szász (see [8, Chapter 16] and [9, Chapter 4]), give a complete description of the classes $\mathcal{B}_n$.

Lemma 1 (Sheil - Small) *We have*

$$G \in \mathcal{B}_\infty \Leftrightarrow G(z) = \int_{\partial\mathbb{D}} \frac{1}{1 - \zeta z} d\mu(\zeta)$$

for some probability measure $d\mu$ on $\partial\mathbb{D}$ and

$$F \in \mathcal{B}_n \Leftrightarrow F(z) = G(z) + o(z^n) \ (z \rightarrow 0), \text{ for some } G \in \mathcal{B}_\infty.$$

Lemma 2 (Carathéodory and Toeplitz) *Let $F(z) := 1 + \sum_{k=1}^\infty A_k z^k \in H(\mathbb{D})$. Then $F \in \mathcal{B}_\infty$ if either*

$$\mathfrak{Det}_n(A_1, \dots, A_n) := \det \begin{vmatrix} 1 & A_1 & \cdots & A_n \\ \bar{A}_1 & 1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_n & \bar{A}_{n-1} & \cdots & 1 \end{vmatrix} > 0 \quad (n = 1, 2, \dots)$$

or there exists a smallest integer $N \geq 1$ such that $\mathfrak{Det}_N(A_1, \dots, A_N) = 0$. In the latter case

$$\mathfrak{Det}_n(A_1, \dots, A_n) = 0 \quad \text{for all } n \geq N$$

and

$$F(z) = \sum_{j=1}^N \frac{\lambda_j}{1 - \xi_j z},$$

where $\lambda_j > 0$ and the N complex numbers $\xi_j \in \partial\mathbb{D}$ are distinct.

Lemma 3 (Szász) *Let $F(z) := 1 + \sum_{k=1}^n A_k z^k$. Then $F \in \mathcal{B}_n$ if and only if the Toeplitz-Hermite form*

$$(z_0, z_1, \dots, z_n) \rightarrow \sum_{\mu, \nu=0}^n A_{\mu-\nu} z_\mu \bar{z}_\nu \quad (A_0 := 1, A_{-k} := \bar{A}_k)$$

is non-negative definite.

The next statement shows how Lemma 3 can be applied.

Proposition 1 *A necessary and sufficient condition for a Toeplitz - Hermite form to be positive is that the principal minors of the corresponding hermitian matrix are all positive.*

Finally, in the discussion of equality cases of our results we use Jack's lemma [6] that reads as follows:

Lemma 4 *Let $f \in H(\mathbb{D})$ be non - constant and*

$$|f(r)| = \max_{|z|=r} |f(z)| \quad (r < 1).$$

Then

$$\frac{r f'(r)}{f(r)} \geq \frac{|f(r)| - |f(0)|}{|f(r)| + |f(0)|} > 0.$$

3 Proof of Theorem 1

The result is obvious when $n = 1$ and we shall fix once for all $n \geq 2$. For the sake of completeness, we include a variant of the elegant proof first published in [9]. Let $p \in \mathcal{P}_n$ and $d \geq 0$. Then we have

$$\begin{aligned}
\|p'\| + d|a_0(p)| &\leq n\|p\| \Leftrightarrow \frac{1}{n}\|p'\| + \frac{d}{n}|a_0(p)| \leq \|p\| \\
&\Leftrightarrow \left| \sum_{j=0}^{n-1} \frac{n-j}{n} a_{n-j} z^j + \frac{d}{n} a_0(p) e^{i\theta} z^n \right| \leq \|p\| \quad (z \in \mathbb{D}, \theta \in \mathbb{R}) \\
&\Leftrightarrow \left| \sum_{j=0}^n a_{n-j} z^j * \left(\sum_{j=0}^{n-1} \frac{n-j}{n} z^j + \frac{d}{n} e^{i\theta} z^n \right) \right| \leq \|p\| \\
&\Leftrightarrow \left| \tilde{p}(z) * \left(\sum_{j=0}^{n-1} \frac{n-j}{n} z^j + \frac{d}{n} e^{i\theta} z^n \right) \right| \leq \|\tilde{p}\|.
\end{aligned} \tag{4}$$

Here $\tilde{p}(z) := z^n p(1/z)$ and we have used the fact that $\|p\| = \|\tilde{p}\|$. The mapping $p \rightarrow \tilde{p}$ is an isomorphism of $\mathcal{P}_n$ and we obtain

$$d_n = \sup \left\{ d \geq 0 : \sum_{j=0}^{n-1} \frac{n-j}{n} z^j + \frac{d}{n} e^{i\theta} z^n \in \mathcal{B}_n, \text{ for all real } \theta \right\}.$$

According to a result of Fejér [3], we have $P_n \in \mathcal{B}_\infty$, where

$$P_n(z) := \sum_{j=0}^{n-1} \frac{n-j}{n} z^j$$

and by Lemma 2 (note that P_n is a polynomial!)

$$\mathfrak{Det}_k(P_n) := \mathfrak{Det}_k\left(\frac{n-1}{n}, \dots, \frac{n-k}{n}\right) > 0 \quad (k = 1, 2, \dots, n-1).$$

By Lemma 2, given

$$F_{d,\theta}(z) := P_n(z) + \frac{d e^{i\theta}}{n} z^n \quad (\theta \in \mathbb{R}),$$

the following holds:

$$F_{d,\theta} \in \mathcal{B}_n \Rightarrow \mathfrak{Det}_n(d, \theta) := \mathfrak{Det}_n\left(\frac{n-1}{n}, \dots, \frac{1}{n}, \frac{d e^{i\theta}}{n}\right) \geq 0.$$

On the other hand, by Lemma 3 and Proposition 1 we have

$$\mathfrak{Det}_n(d, \theta) > 0 \Rightarrow F_{d,\theta} \in \mathcal{B}_n.$$

Simple calculations show that

$$\begin{aligned}
\mathfrak{Det}_n(d, \theta) &= \frac{2^{n-1}}{n^n} \left[1 + \frac{\cos \theta}{n} d - \frac{n+2}{4n} d^2 \right] \\
&\geq \frac{2^{n-1}}{n^n} \left[1 - \frac{d}{n} - \frac{n+2}{4n} d^2 \right].
\end{aligned}$$

It follows that $d_n = \frac{2n}{n+2}$. In particular,

$$\mathfrak{Det}_n(d_n, \theta) = 0 \Leftrightarrow \theta = \pi \pmod{2\pi}. \quad (5)$$

Let us assume that for some polynomial $p \in \mathcal{P}_n$

$$\|p'\| + d_n |a_0(p)| = n \|p\|. \quad (6)$$

According to (4), we have

$$|\tilde{p} * F_{d_n, \theta}(z)| \leq \|\tilde{p}\| \quad (\theta \in \mathbb{R}),$$

and there are *an extremal direction* (a real number) θ_n and *an extremal point* $z_n \in \partial\mathbb{D}$ such that

$$|\tilde{p} * F_{d_n, \theta_n}(z_n)| = \left| \tilde{p}(z) * \left(\sum_{j=0}^{n-1} \frac{n-j}{n} z^j + \frac{d_n}{n} e^{i\theta_n} z^n \right) \right|_{z=z_n} = \|\tilde{p}\|.$$

In view of Lemma 3 and Proposition 1

$$\mathfrak{Det}_n(d_n, \theta_n) \geq 0.$$

Case 1. Let us suppose that

$$\mathfrak{Det}_n(d_n, \theta_n) > 0.$$

For a fixed θ_n , $\mathfrak{Det}_n(d, \theta_n)$ is a continuous function of d . Then, for a sufficiently small $\eta > 0$ we must also have $\mathfrak{Det}_n(d_n + \eta, \theta_n) > 0$ and according to Lemma 3 and Proposition 1

$$\begin{aligned} \frac{1}{n} \|p'\| + \frac{d_n + \eta}{n} |a_0(p)| &= |\tilde{p} * F_{d_n + \eta, \theta_n}(z_n)| \\ &= \left| \tilde{p}(z) * \left(\sum_{j=0}^{n-1} \frac{n-j}{n} z^j + \frac{d_n + \eta}{n} e^{i\theta_n} z^n \right) \right|_{z=z_n} \leq \|\tilde{p}\|. \end{aligned}$$

In view of (6), this is possible only when $a_0(p) = 0$. However, it is well known that the only polynomials from $\{p : p \in \mathcal{P}_n, a_0(p) = 0\}$ extremal for (2) have the form $p(z) = a_n(p) z^n$.

Case 2. Here we assume

$$\mathfrak{Det}_n(d_n, \theta_n) = 0.$$

Then, in view of the explicit expression for $\mathfrak{Det}_n(d_n, \theta_n)$ we have

$$\theta_n = \pi \pmod{2\pi}$$

and according to Lemma 2

$$F_{d_n, \theta_n}(z) = \sum_{j=1}^n \frac{\lambda_j}{1 - \zeta_j z} + o(z^n) \quad (z \rightarrow 0),$$

where $\lambda_j > 0$ and the n complex numbers $\zeta_j \in \partial\mathbb{D}$ are distinct.

This leads us to the following system of equations

$$\begin{cases} \sum_{j=1}^n \zeta_j^k \lambda_j &= \frac{n-k}{n}, \quad (k = 0, \dots, n-1) \\ \sum_{j=1}^n \zeta_j^n \lambda_j &= -\frac{d_n}{n} \end{cases} \quad (7)$$

together with the constraints $\lambda_j > 0$ ($j = 1, \dots, n$).

Case 2.1. Suppose that the set $\{\zeta_j\}_{j=1}^n$ is not closed under conjugation. The following observation is crucial: The system (7) is equivalent to the system

$$\begin{cases} \sum_{j=1}^n \bar{\zeta}_j^k \lambda_j = \frac{n-k}{n} & (k = 0, \dots, n-1) \\ \sum_{j=1}^n \bar{\zeta}_j^n \lambda_j = -\frac{d_n}{n}. \end{cases}$$

In view of this, the polynomial $\tilde{p}$ would attain its maximum modulus $\|\tilde{p}\|$ on $\partial\mathbb{D}$ at each point $z_n \zeta_j$ ($j = 1, \dots, n$) and also at another point $z_n \zeta_*$, $\zeta_* \in \{\bar{\zeta}_j\}_{j=1}^n$, and $\zeta_* \neq \zeta_j$ ($j = 1, \dots, n$). Under this condition, it is easily seen that $\tilde{p}(z) = a_{n-k}(p) z^k$ for some $k \in \{0, 1, \dots, n\}$. So, if the set $\{\zeta_j\}_{j=1}^n$ is not closed under conjugation, then the only extremal polynomials for the inequality (2) must have the form $p(z) = a_k(p) z^k$ for $0 \leq k \leq n$. Assuming that $p(z) = a_k(p) z^k$, $a_k(p) \neq 0$ ($0 \leq k \leq n-1$) we are led to

$$\|p'\| + d_n |a_0(p)| = k |a_k(p)| + d_n |a_0(p)| < n \|p\| \quad (n \geq 2)$$

which is not possible. Hence, the only extremal polynomials for (2) have the form $p(z) = a_n(p) z^n$.

Case 2.2. Suppose now that the set $\{\zeta_j\}_{j=1}^n$ is closed under conjugation.

Let us write the last n equations of the system (7) as

$$\mathfrak{B} \vec{\Lambda} = \vec{C},$$

where

$$\vec{\Lambda} := \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}, \quad \vec{C} := \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

$$c_k := \frac{n-k}{n} \quad (k = 1, \dots, n-1), \quad c_n = -\frac{d_n}{n},$$

and $\mathfrak{B}$ is the invertible matrix

$$\mathfrak{B} := \begin{pmatrix} \zeta_1 & \zeta_2 & \dots & \zeta_n \\ \zeta_1^2 & \zeta_2^2 & \dots & \zeta_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_1^n & \zeta_2^n & \dots & \zeta_n^n \end{pmatrix}.$$

By known properties of Vandermonde matrices [1, p. 64], if

$$\left(L_0^{(j)}, L_1^{(j)}, \dots, L_{n-1}^{(j)} \right)^T$$

stands for column $\#j$ of $(\mathfrak{B}^T)^{-1} = (\mathfrak{B}^{-1})^T$, then

$$\sum_{k=0}^{n-1} L_k^{(j)} z^k \equiv \frac{W(z)}{(z - \zeta_j) \zeta_j W'(\zeta_j)},$$

where $W(z) := \prod_{s=1}^n (z - \zeta_s)$. Therefore, the row $\#j$ of $\mathfrak{B}^{-1}$ equals

$$(L_0^{(j)}, L_1^{(j)}, \dots, L_{n-1}^{(j)})$$

and the unique solution of (7) is

$$\begin{aligned}
\lambda_j &= \sum_{k=0}^{n-1} L_k^{(j)} c_{k+1} = \frac{W(z)}{(z - \zeta_j) \zeta_j W'(\zeta_j)} * \sum_{k=0}^{n-1} c_{k+1} z^k \Big|_{z=1} \\
&= \frac{W(z)}{(z - \zeta_j) \zeta_j W'(\zeta_j)} * \left(\sum_{k=1}^{n-1} \frac{n-k}{n} z^{k-1} - \frac{d_n}{n} z^{n-1} \right) \Big|_{z=1} \\
&= \frac{1}{n} \left(n \sum_{k=0}^{n-2} L_k^{(j)} - \sum_{k=0}^{n-2} (k+1) L_k^{(j)} - d_n L_{n-1}^{(j)} \right).
\end{aligned}$$

After some short computations we get

$$\begin{aligned}
\lambda_j &= \frac{W(1)}{n} \frac{1}{\zeta_j W'(\zeta_j)} \frac{\zeta_j}{(1 - \zeta_j)^2} \times \left\{ \frac{2d_n}{W(1)} - (n-1) \right. \\
&\quad \left. + \frac{W'(1)}{W(1)} + \zeta_j \left(-\frac{d_n}{W(1)} \right) + \bar{\zeta}_j \left(n - \frac{W'(1)}{W(1)} - \frac{d_n}{W(1)} \right) \right\}.
\end{aligned} \tag{8}$$

We have

$$\|\tilde{p}\| = |\tilde{p} * F_{d_n, \theta_n}(z_n)| = \left| \sum_{j=1}^n \lambda_j \tilde{p}(\zeta_j z_n) \right| = \sum_{j=1}^n \lambda_j |\tilde{p}(\zeta_j z_n)| = \|\tilde{p}\|.$$

This means that at each point $\zeta_j z_n$, the value $\tilde{p}(\zeta_j z_n)$ has constant argument and $|\tilde{p}(\zeta_j z_n)| = \|\tilde{p}\|$. Without any loss of generality, we may suppose that $z_n = 1$ and $\tilde{p}(\zeta_j) = \|\tilde{p}\|$ ($j = 1, \dots, n$), i.e., we have the representation

$$\tilde{p}(z) = cW(z) + \|\tilde{p}\| \quad (c \in \mathbb{C}).$$

Assuming $c \neq 0$ ($\tilde{p}$ is non - constant!) we now use Lemma 4 to obtain

$$0 < \zeta_j \frac{\tilde{p}'(\zeta_j)}{\tilde{p}(\zeta_j)} = \frac{c \zeta_j W'(\zeta_j)}{\|\tilde{p}\|}. \tag{9}$$

The combination of (8) and (9), together with the fact that $\lambda_j > 0$, yields that there exists $\beta_* \in \mathbb{R}$ which does not depend of $j \in \{1, 2, \dots, n\}$, such that

$$\arg \beta(\zeta_j) \equiv \beta_*,$$

where

$$\beta(u) := \frac{2d_n}{W(1)} - (n-1) + \frac{W'(1)}{W(1)} + u \left(-\frac{d_n}{W(1)} \right) + \bar{u} \left(n - \frac{W'(1)}{W(1)} - \frac{d_n}{W(1)} \right).$$

Case 2.2.1. We shall first assume that

$$\left| \frac{d_n}{W(1)} \right| \neq \left| n - \frac{W'(1)}{W(1)} - \frac{d_n}{W(1)} \right|.$$

Then the image of the unit circle $\partial\mathbb{D}$ by the function $\beta(u)$ is a once-covered non-degenerated ellipse and such a curve meets any line at most twice. This means that for $n > 2$ it is impossible $\arg \beta(\zeta_j)$ to have a constant value.

Case 2.2.2. We now need to discuss the case where

$$\left| \frac{d_n}{W(1)} \right| = \left| n - \frac{W'(1)}{W(1)} - \frac{d_n}{W(1)} \right|.$$

By assumption, the point set $\{\zeta_j\}_{j=1}^n$ is closed under conjugation. Hence, $W(1)$ and $W'(1)$ are real. Then we have $nW(1) = W'(1)$ or else $nW(1) = W'(1) + 2d_n$. It follows that either

$$n = \frac{W'(1)}{W(1)} = \sum_{j=1}^n \Re \left(\frac{1}{1 - \zeta_j} \right) = \frac{n}{2}$$

which is impossible or else

$$\beta(\zeta_j) = 1 - \zeta_j \frac{d_n}{W(1)} + \frac{1}{\zeta_j} \frac{d_n}{W(1)} = 1 - 2i \Im(\zeta_j) \frac{d_n}{W(1)}.$$

This implies that $\Im(\beta(\zeta_j))$ is a constant and clearly this can occur only when $n = 2$.

We complete the proof of Theorem 1 by considering the case $n = 2$, $p \in \mathcal{P}_2$. From the system (7), taking into account that $\bar{\zeta}_2 = \zeta_1$, we obtain $\zeta_1 = 1/2 + i\sqrt{3}/2$ and $\beta(\zeta_1) = 3/2 - i\sqrt{3}/2 = \bar{\beta}(\zeta_2)$. So,

$$\arg(\beta(\zeta_1)) \neq \arg(\beta(\zeta_2)).$$

Hence, $\tilde{p}$ is a constant, i.e., the only extremal polynomials $p \in \mathcal{P}_2$ ($n = 2$) for (2) are $p(z) = a_2 z^2$.

4 Proof of Theorem 2

We consider the case $n \geq 2$. Clearly,

$$\begin{aligned} \delta \|p'\| + |a_0(p)| &\leq \|p\| \Leftrightarrow \left| a_0(p) + \sum_{k=1}^n \delta e^{i\theta} k a_k(p) z^k \right| \leq \|p\| \\ &\Leftrightarrow \left| p(z) * \left(1 + \delta e^{i\theta} \sum_{k=1}^n k z^k \right) \right| \leq \|p\| \quad (z \in \mathbb{D}, \theta \in \mathbb{R}). \end{aligned}$$

and by Lemma 1

$$\delta_n = \sup \left\{ \delta \geq 0 : 1 + \delta e^{i\theta} \sum_{k=1}^n k z^k \in \mathcal{B}_n \text{ for all real } \theta \right\}.$$

According to Lemma 2, Lemma 3, and Proposition 1 we need to study the determinants

$$\mathfrak{Det}_m(\delta e^{i\theta}) := \det \begin{vmatrix} 1 & \delta e^{i\theta} & \cdots & m\delta e^{i\theta} \\ \delta e^{-i\theta} & 1 & \cdots & (m-1)\delta e^{i\theta} \\ \vdots & \vdots & \ddots & \vdots \\ m\delta e^{-i\theta} & (m-1)\delta e^{-i\theta} & \cdots & 1 \end{vmatrix}$$

for $m = 1, 2, \dots, n$.

We shall prove that there exists a positive number $\tilde{\delta}_n$ such that

1. $\mathfrak{Det}_m(\delta e^{i\theta}) > 0$ for $m = 1, 2, \dots, n-1$, $\delta \in [0, \tilde{\delta}_n]$, and $\theta \in \mathbb{R}$;

2. $\mathfrak{Det}_n(\delta e^{i\theta}) > 0$ for $\delta \in [0, \tilde{\delta}_n)$ and $\theta \in \mathbb{R}$;
3. $\mathfrak{Det}_n(-\tilde{\delta}_n) = 0$ and $\mathfrak{Det}_n(\tilde{\delta}_n e^{i\theta}) > 0$ for $\theta \neq \pi \pmod{2\pi}$;
4. $\mathfrak{Det}_n(-\tilde{\delta}_n - \varepsilon) < 0$ for $\varepsilon \in (0, \varepsilon_o)$, $\varepsilon_o > 0$ and ε_o sufficiently small.

Then by Lemma 2, Lemma 3, and Proposition 1 we shall conclude that

$$\delta_n = \tilde{\delta}_n.$$

We define, for each $m \geq 1$ and $\theta \in \mathbb{R}$, an $m \times m$ matrix $\mathcal{M}_m := \mathcal{M}_m(\delta \cos \theta)$

$$(\mathcal{M}_m)_{sj} := \begin{cases} -(2s+2)j\delta \cos \theta + (s+1)(j+1) & \text{if } 1 \leq j < s \leq m \\ -(2s+2)j\delta \cos \theta + (s+1)(j+1) + 1 & \text{if } 1 \leq j = s \leq m. \end{cases}$$

The following recursion formula will be useful in our study :

$$\mathfrak{Det}_m(\delta e^{i\theta}) = \mathfrak{Det}_{m-1}(\delta e^{i\theta}) - \delta^2 \det(\mathcal{M}_{m-1}(\delta \cos \theta)). \quad (10)$$

The above identity can be obtain by performing a sequence of elementary operations on the Toeplitz matrix

$$\mathfrak{Mat}_m(\delta e^{i\theta}) := \begin{pmatrix} 1 & \delta e^{i\theta} & \cdots & m\delta e^{i\theta} \\ \delta e^{-i\theta} & 1 & \cdots & (m-1)\delta e^{i\theta} \\ \vdots & \vdots & \ddots & \vdots \\ m\delta e^{-i\theta} & (m-1)\delta e^{-i\theta} & \cdots & 1 \end{pmatrix}.$$

1° Add proper multiples of row 1 of $\mathfrak{Mat}_m$ to obtain a $(m+1) \times (m+1)$ matrix $\mathfrak{Mat}'_m$ with $(\mathfrak{Mat}'_m)_{s1} = 0$ for $1 < s \leq m+1$. Delete row 1 and column 1 of $\mathfrak{Mat}'_m$ to obtain an $m \times m$ matrix $\mathfrak{Mat}''_m$ with $\mathfrak{Det}_m(\delta e^{i\theta}) = \det(\mathfrak{Mat}''_m)$.

2° For each $2 \leq j \leq m$, add $-j$ times column number one of $\mathfrak{Mat}''_m$ to column number j of $\mathfrak{Mat}''_m$. Call $\mathfrak{Mat}'''_m$ the matrix so obtained.

3° We have

$$\mathfrak{Det}_m(\delta e^{i\theta}) = \det(\mathfrak{Mat}'''_m) = \mathfrak{Det}_{m-1}(\delta e^{i\theta}) + \det(\mathcal{M}'_m)$$

where $\mathcal{M}'_m$ is an $m \times m$ matrix. Use now row one of $\mathcal{M}'_m$ to cancel all entries in column one of $\mathcal{M}'_m$ except entry $(\mathcal{M}'_m)_{11}$ which is equal to $-\delta^2$. Finally, we obtain (10).

Clearly, $\det(\mathcal{M}_m(\delta \cos \theta))$ is a polynomial of degree m in the variable $\delta \cos \theta$; for example $\det(\mathcal{M}_1) = -4\delta \cos \theta + 5$, $\det(\mathcal{M}_2) = 12\delta^2 \cos^2 \theta - 28\delta \cos \theta + 14$ and the following recursion formula holds for $m \geq 3$:

$$\begin{aligned} \det(\mathcal{M}_m) &= [-2\delta \cos \theta (1 + 1/m) + 1 + (1 + 1/m)^2] \det(\mathcal{M}_{m-1}) \\ &\quad - (1 + 1/m)^2 \det(\mathcal{M}_{m-2}). \end{aligned}$$

Let us define

$$\det(\mathcal{M}_m(\delta \cos \theta)) := \sum_{k=0}^m a_{k,m} \delta^k \cos^k \theta.$$

We easily compute $a_{m,m} = (-1)^m 2^m (m+1)$ and we shall prove that

$$\text{sign}(a_{k,m}) = (-1)^k \quad (0 \leq k \leq m). \quad (11)$$

We have

$$a_{0,m} = a_{0,m-1} + (1 + 1/m)^2 (a_{0,m-1} - a_{0,m-2}) \quad (m \geq 2).$$

It follows by induction on m that ($a_{0,j} := 0$ if $j \leq 0$)

$$a_{0,m} > 0 \quad \text{and} \quad a_{0,m} > a_{0,m-1} \quad (m \geq 0).$$

We also have

$$a_{1,m} = -2(1 + 1/m) a_{0,m-1} + a_{1,m-1} + (1 + 1/m)^2 (a_{1,m-1} - a_{1,m-2}),$$

where $m \geq 1$ and $a_{1,j} := 0$ if $j < 1$. It follows by induction on m that

$$a_{1,m} < 0 \quad \text{and} \quad a_{1,m} < a_{1,m-1} \quad (m \geq 1).$$

As a last example

$$a_{2,m} = -2(1 + 1/m) a_{1,m-1} + a_{2,m-1} + (1 + 1/m)^2 (a_{2,m-1} - a_{2,m-2})$$

for $m \geq 2$. It should be now clear that a double induction on m and k shall yield (11). It now follows from (10) and (11) that

$$\mathfrak{Det}_m(\delta e^{i\theta}) \geq \mathfrak{Det}_m(-\delta) = 1 - \delta^2 - \delta^2 \sum_{j=1}^{m-1} \det(\mathcal{M}_j(-\delta)). \quad (12)$$

where $\delta > 0$ and $\theta \in \mathbb{R}$. However, the sequence $\{\delta_m\}_{m \geq 1}$ is strictly decreasing. We conclude that

$$\tilde{\delta}_n = \delta_n$$

is the only positive root of the equation

$$\mathfrak{Det}_n(-\delta) = 0,$$

and further

$$\mathfrak{Det}_n(\delta_n e^{i\theta}) = 0 \Leftrightarrow \theta = \pi \pmod{2\pi}, \quad (13)$$

and $\mathfrak{Det}_n(\delta_n e^{i\theta}) > 0$ for $\theta \neq \pi \pmod{2\pi}$. Clearly, $\mathfrak{Det}_1(-\delta) = 1 - \delta^2$ and $\delta_1 = 1$. Some easy computations lead to $\mathfrak{Det}_2(-\delta) = 1 - 6\delta^2 - 4\delta^3$ and $\mathfrak{Det}_3(-\delta) = 1 - 20\delta^2 - 32\delta^3 - 12\delta^4$ with $\delta_3 = (2 + \sqrt{10})^{-1}$. However, it does not seem easy to obtain an explicit formula for δ_n expressed as a function of n . We shall content ourselves with some growth estimates.

It is seen from (12) and (13) that $\mathfrak{Det}(-\delta)$ is a polynomial in δ whose coefficients, except for the constant one, are negative. It is also easy to compute

$$\begin{aligned} \mathfrak{Det}_n(-\delta) &= 1 - \left(\frac{1}{6} \sum_{k=1}^n k(k+1)(2k+1) \right) + o(\delta^2), \quad \delta \rightarrow 0 \\ &= 1 - \frac{n(n+1)^2(n+2)}{12} \delta^2 + o(\delta^2), \quad \delta \rightarrow 0 \end{aligned}$$

and therefore

$$1 - \frac{n(n+1)^2(n+2)}{12} \delta_n^2 > \mathfrak{Det}_n(-\delta_n) = 0$$

i.e.,

$$\delta_n \leq \frac{2\sqrt{3}}{(n+1)\sqrt{n(n+2)}}.$$

We also have for $\delta \neq 0$,

$$\mathfrak{Det}_n(\delta) = \delta^n \begin{pmatrix} 1 - (1 - 1/\delta) & 1 & \cdots & n \\ 1 & 1 - (1 - 1/\delta) & \cdots & n-1 \\ 2 & 1 & \cdots & n-2 \\ \vdots & \vdots & \ddots & \vdots \\ n & n-1 & \cdots & 1 - (1 - 1/\delta) \end{pmatrix}.$$

and any root δ of $\mathfrak{Det}_n(\delta) = 0$ is such that $1 - 1/\delta$ is an eigenvalue of the symmetric matrix

$$\begin{pmatrix} 1 & 1 & \cdots & n \\ 1 & 1 & \cdots & n-1 \\ 2 & 1 & \cdots & n-2 \\ \vdots & \vdots & \ddots & \vdots \\ n & n-1 & \cdots & 1 \end{pmatrix}.$$

In particular, all the roots of $\mathfrak{Det}_n(-\delta) = 0$, except $\delta_n > 0$, are negative. Applying Gerschgorin's estimates to the first row of the above matrix we obtain

$$\frac{1}{\delta_n} = |1 - (1 - 1/\delta_n)| \leq \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

and we conclude that

$$\frac{2}{n(n+1)} \leq \delta_n < \frac{2\sqrt{3}}{n(n+1)}.$$

We shall now determine all polynomials $p \in \mathcal{P}_n$, $n > 1$, such that

$$\delta_n \|p'\| + |a_0(p)| = \|p\|. \quad (14)$$

We define

$$F_{\delta, \theta} : +1 + \delta e^{i\theta} \sum_{k=1}^n k z^k.$$

It follows that if (14) holds for $p \in \mathcal{P}_n$, $n \geq 1$, then

$$\begin{aligned} |p * F_{\delta_n, \theta_n}(z_n)| &= |a_0(p) + \delta_n e^{i\theta_n} z_n p'(z_n)| \\ &= \delta_n |p'(z_n)| + |a_0(p)| \\ &= \delta_n \|p'\| + |a_0(p)| = \|p\| \end{aligned} \quad (15)$$

for some real θ_n and $z_n \in \partial\mathbb{D}$.

We have that the sequence $\{\delta_n\}_{n \geq 1}$ is strictly decreasing, δ_n is the only positive root of the equation $\mathfrak{Det}_n(-\delta) = 0$, δ_n is a simple zero of $\mathfrak{Det}_n(-\delta)$, and

$$\delta_n = \min \{ |z| : \mathfrak{Det}_n(z) = 0, z \in \mathbb{C} \}.$$

So, analogously to the proof of Theorem 1, we can claim that *the direction* θ_n must be such that $\mathfrak{Det}_n(\delta_n e^{i\theta_n}) = 0$, i.e., $\theta_n = \pi \pmod{2\pi}$. Then, by Lemma 2 we have

$$F_{\delta_n, \theta_n}(z) = 1 - \delta_n \sum_{k=1}^n k z^k = \sum_{j=1}^n \frac{\lambda_j}{1 - z\zeta_j} + o(z^n) \quad (z \rightarrow 0)$$

with $\lambda_j > 0$ and distinct $\zeta_j \in \partial\mathbb{D}$. By (15) we obtain $|p(z_n\zeta_j)| = \|p\|$ ($j = 1, 2, \dots, n$) and we assume without loss of generality that $z_n = 1$ and

$$p(z) = cW(z) + \|p\| \quad \left(c \neq 0, c \in \mathbb{C}, W(z) := \prod_{j=1}^n (z - \zeta_j) \right) \quad (16)$$

We now consider the system of equations

$$\sum_{j=1}^n \lambda_j \zeta_j^k = -\delta_n k \quad (k = 0, 1, 2, \dots, n). \quad (17)$$

As in the proof of theorem 1 we have

$$\begin{aligned} 0 < \lambda_j &= -\delta_n \sum_{k=0}^{n-1} L_k^{(j)}(k+1) \\ &= \frac{-\delta_n}{\zeta_j W'(\zeta_j)} \left(\frac{-\zeta_j}{(1 - \zeta_j)^2} W(1) + \frac{1}{1 - \zeta_j} W'(1) \right) \\ &= \frac{\delta_n}{\zeta_j W'(\zeta_j)} \frac{\zeta_j}{(1 - \zeta_j)^2} (W(1) + (\bar{\zeta}_j - 1) W'(1)). \end{aligned} \quad (18)$$

By (17) and Jack's lemma, $\arg(\zeta_j W'(\zeta_j))$ does not depend on j and we deduct from (18) that

$$\arg(W(1) + (\bar{\zeta}_j - 1) W'(1))$$

is a constant for $j = 1, \dots, n$. This can possibly occur only when $n = 2$.

Let now p be a polynomial in $\mathcal{P}_2$ such that

$$\delta_2 \|p'\| + |a_0(p)| = (1 + \sqrt{3})^{-1} \|p'\| + |a_0(p)| = \|p\|.$$

Without loss of generality we may assume that

$$p(z) = c(z - e^{i\theta})(z - e^{-i\theta}) + \|p\|.$$

We then use (17) to obtain $\cos \theta = -\delta_2$ and $p(z) \equiv cz^2 + 2\delta_2 cz + c + \|p\|$. Therefore, $\|p'\| = 2|c| + 2\delta_2 |c|$ and

$$\delta_2 \|p'\| + |a_0(p)| = \|p\| \Leftrightarrow |c + \|p\|| = \|p\| - |c|.$$

This means that

$$|a_0(p)| + |a_2(p)| = |c + \|p\|| + |c| = \|p\|.$$

We have equality in Visser's inequality (see [2] and [11]) for $n = 2$ and therefore $c = 0$, which amounts to p being a constant.

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