

Modulus of analytic classification  
for unfoldings of generic  
parabolic diffeomorphisms

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### **Abstract**

In this paper we give a complete modulus of analytic classification under weak equivalence for generic analytic 1-parameter unfoldings of diffeomorphisms with a generic parabolic point. The modulus is composed of a canonical parameter associated to the family, together with an unfolding of the Ecalle-Voronin modulus. We then study the fixed points bifurcating from a parabolic point with nontrivial Ecalle-Voronin modulus and show that some of the non-hyperbolic resonant ones are non linearizable.

*Subject Mathematical Classification.* 34C,58F

### **Résumé**

Dans cet article on donne un module complet de classification analytique pour l'équivalence faible des familles génériques à un paramètre déployant un difféomorphisme avec un point fixe parabolique générique. Le module est composé d'un paramètre canonique associé à la famille et d'un déploiement du module d'Ecalle-Voronin. On étudie ensuite les points fixes bifurcant d'un point fixe parabolique avec module d'Ecalle-Voronin non trivial et on montre que certains points fixes non hyperboliques sont non linéarisables pour des résonances particulières.



## 1. INTRODUCTION

In this paper we consider generic parabolic points of germs of diffeomorphisms of  $\mathbb{C}$ . A fixed point of a diffeomorphism  $f : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$  is *parabolic* if  $f'(0) = 1$ , i.e. the point is a multiple fixed point. It is *generic* if  $f''(0) \neq 0$ , i.e. the fixed point is of multiplicity 2. Hence, up to rescaling of  $z$  the diffeomorphism can be written as

$$f(z) = z + z^2 + o(z^2). \quad (1.1)$$

All generic parabolic germs are topologically equivalent with the dynamics described by Figure 1.

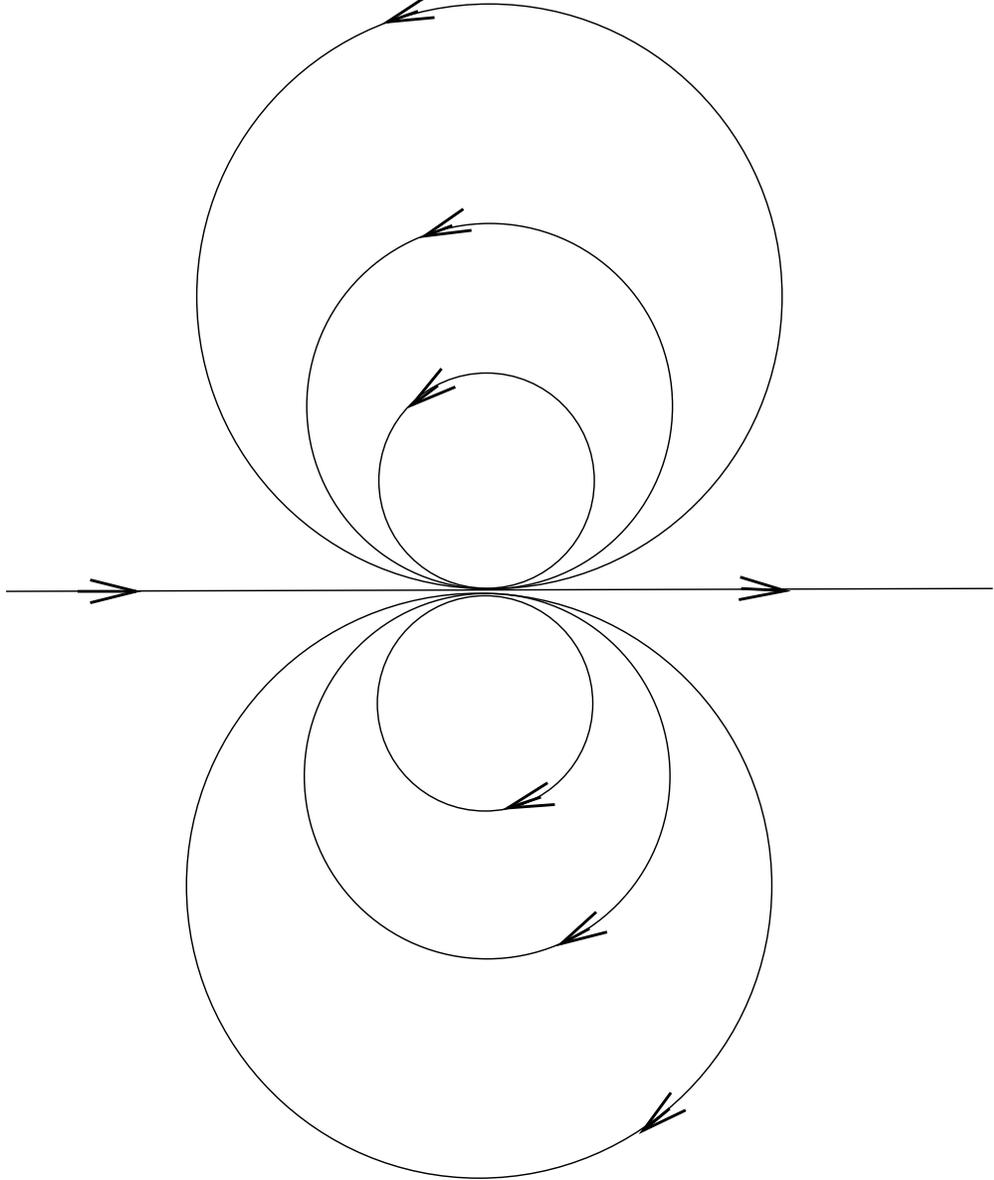


FIGURE 1. Dynamics of a generic parabolic germ

They are *topologically* equivalent to the function  $g(z) = \frac{z}{1-z}$ , which is the time-one map of the vector field  $Y = z^2 \frac{\partial}{\partial z}$  in a full neighborhood of the origin.

If we are now interested in the analytic classification of generic diffeomorphisms with a parabolic point the natural “model” for such a diffeomorphism is the time-one map of the flow of a vector field

$$\frac{z^2}{1+az} \frac{\partial}{\partial z}. \quad (1.2)$$

Indeed any parabolic germ is *formally* equivalent to the time-one flow of (1.2).

Generically a series normalizing a germ with a parabolic fixed point (i.e. embedding it in a flow) diverges. Why? The dynamics of the parabolic point is very complicated with the modulus space being a functional space. The model is too poor and has no room to encode the complexity. The only way for the system to express its complexity is through the divergence of the normalizing series. The modulus space has been described by Ecalle [E] and Voronin [V].

A parabolic point of a diffeomorphism being a double fixed point it is natural to unfold it and to consider its dynamics as the limit situation of the dynamics of the diffeomorphism in a neighborhood of the two fixed points of the unfolding. This point of view has been first conjectured by Arnold and then developed by Martinet [M], Glutsyuk [G], Ramis [R] and Duval [D] considered particular cases. These authors have restricted themselves to cones in the parameter space where the bifurcating fixed points are hyperbolic.

A parallel study has been performed independently by Lavaurs [L], Shishikura [S] and Oudkerk [O] in conic regions of the parameter space which complement the regions studied by Martinet and Glutsyuk. However, these authors have not exploited their study to deduce the dynamics of the bifurcation fixed points.

In this paper we consider the full problem of analytic classification of generic families unfolding a generic parabolic point of a germ of analytic diffeomorphism:

$$f_\epsilon(z) = z + (z^2 - \epsilon)(1 + \dots). \quad (1.3)$$

We give a complete modulus of analytic classification of families under weak equivalence defined as follows:

**Definition 1.1.** *Two germs  $f_{1,\epsilon_1}(z)$  and  $f_{2,\epsilon_2}$  of analytic families of diffeomorphisms are weakly equivalent if there exists a germ of bijective map  $K = (h, H)$ ,  $(\epsilon_1, z) \mapsto (h(\epsilon_1), H(\epsilon_1, z))$  fibered over the parameter space where*

- i)  $h : \epsilon_1 \mapsto \epsilon_2 = h(\epsilon_1)$  is a germ of holomorphic diffeomorphism preserving the origin.
- ii) For each  $\epsilon_1$  in a sufficiently small neighborhood of the origin there exists a representative  $H_{\epsilon_1}(z) = H(\epsilon_1, z)$  of the germ depending analytically on  $z$  such that  $H_{\epsilon_1}$  conjugates  $f_{1,\epsilon_1}$  and  $f_{2,h(\epsilon_1)}$  over a ball of small radius  $r > 0$ :

$$f_{2,h(\epsilon_1)}(H_{\epsilon_1}(z)) = H_{\epsilon_1}(f_{1,\epsilon_1}(z)). \quad (1.3)$$

*Remark.* In practice we will construct families of conjugacy maps  $H_{\epsilon_1}$  which will be multivalued in  $\epsilon_1$ . They will depend analytically on  $\hat{\epsilon}_1 \neq 0$ , where  $\hat{\epsilon}_1$  will be varying in a sector of opening greater than  $2\pi$  and small positive radius of the universal covering of  $\epsilon_1$ -space. Moreover, the conjugation map will be continuous in  $\hat{\epsilon}_1$  near  $\epsilon_1 = 0$

In fact, our modulus is defined on sectors of opening  $4(\pi - \delta)$  for arbitrarily small  $\delta$  in the universal covering of the parameter space. It depends analytically on  $\epsilon \neq 0$  and continuously on  $\epsilon = 0$  and converges to the Ecalle-Voronin modulus. The obstruction to a sector of wider opening in  $\epsilon$ -space is so drastic that we conjecture our modulus to be 1-summable in  $\sqrt{\epsilon}$ .

The method we use to obtain the analytic classification is the following. We study the analytic structure of the space of orbits, by comparing it to the space of orbits of the “model”, i.e. the time-one map of the flow of

$$\frac{z^2 - \epsilon}{1 + a(\epsilon)z} \frac{\partial}{\partial z}. \quad (1.4)$$

We start by adequately “preparing” the family of diffeomorphisms via an analytic change in the variable and in the parameter  $(z, \epsilon) \mapsto (\bar{z}, \bar{\epsilon})$ . As in [IY] the change of parameter is chosen so that the multipliers of the fixed points of the family of diffeomorphisms be equal to the multipliers of the time-one flow of the model. The new parameter  $\bar{\epsilon}$  is “canonical” and is an analytic invariant of the prepared family. This parameter is part of our modulus. It allows to reduce the full problem of classification to classification

problems for each fixed value of the canonical parameter. For each value of the canonical parameter the space of orbits is described by two spheres which are glued together in the neighborhood of 0 (resp.  $\infty$ ) by means of two germs of conformal diffeomorphisms  $\psi_\epsilon^0$  (resp.  $\psi_\epsilon^\infty$ ). This germs can be taken analytic in  $\epsilon \neq 0$  and continuous in  $\epsilon = 0$ . Moreover,  $(\psi_\epsilon^0, \psi_\epsilon^\infty)$  tends in the limit to the Ecalle-Voronin modulus  $(\psi_0^0, \psi_0^\infty)$ . As for the Ecalle-Voronin modulus, in order to describe the analytic type of the space of orbits we must take the equivalence class of pairs  $(\psi_\epsilon^0, \psi_\epsilon^\infty)$  under changes of coordinates on the spheres fixing 0 and  $\infty$ . In the usual presentations of the analytic invariant of a parabolic germ one adds the formal invariant  $a$  in (1.2). However  $a$  and  $a(\epsilon)$  in (1.4) can be recovered from the pair  $(\psi_\epsilon^0, \psi_\epsilon^\infty)$  and represent a shift in the eigenvalues at the two singular points  $\pm\sqrt{\epsilon}$ .

In general, if one considers a bifurcation from a situation satisfying a generic condition then this yields strong restrictions on the dynamics in the bifurcating systems while one can say nothing on the dynamics close to a very degenerate system.

In the context of our study a parabolic point satisfies a genericity condition when its modulus is not trivial, i.e. the diffeomorphism cannot be embedded in a flow. The unfolded system may have two hyperbolic points, i.e. trivial dynamics at the level of singular points. The interesting direction is the *Siegel direction*, i.e. the direction in parameter space where at least one of the multipliers is on the unit circle. We show that in the unfolding of a parabolic point with non trivial modulus the diffeomorphism necessarily has non linearizable fixed points at least for sequences of parameter values converging to the critical value of the parameter. For these parameter values one of the fixed points is resonant. This comes from the fact that we can construct a return map in the neighborhood of the fixed points. The renormalized return map as considered by Yoccoz [Y] near one singular point is constructed using  $\psi_\epsilon^0$  or  $\psi_\epsilon^\infty$  alone. This explains why there is an obstruction to make a full turn in the parameter space allowing to pass twice through the Siegel direction: the fixed points are exchanged, while their dynamics is strongly controlled respectively by  $\psi_0^0$  (resp.  $\psi_0^\infty$ ).

This result shows a link between the notions of linearizability and normalizability (a point is normalizable if it has trivial Ecalle-Voronin modulus). In [CMR] the authors has initiated a program to study the global organization of strata of integrable and linearizable saddle points inside families of polynomial vector fields

$$\begin{aligned} \dot{x} &= x + \sum_{i+j=2}^n a_{ij} x^i y^j \\ \dot{y} &= -\lambda y + \sum_{i+j=2}^n b_{ij} x^i y^j. \end{aligned} \tag{1.5}$$

with parameter space  $(\lambda, a_{ij}, b_{ij})$ , where  $\lambda \in \mathbb{R}^+$  and  $a_{ij}, b_{ij} \in \mathbb{C}$ . It follows from our results that the normalizable points are “organizing centers” for the strata of integrable and linearizable points and explain phenomena which looked pathological: integrability strata ending spontaneously at a parameter value, drop in the dimension of the strata of integrable systems on sequences of parameter values.

We will make a complete study of a saddle-node in a forthcoming publication. Here we discuss only one question whose answer is a direct consequence of our results. It was shown by Martinet and Ramis that the analytic classification of a saddle-node is equivalent to the analytic classification of the strong separatrix. However Martinet and Ramis raised the question why the modulus space is not the full modulus space for parabolic diffeomorphisms. An explanation of this comes from the unfolding in the Siegel direction. Considering the unfolded modulus we have that  $\psi_\epsilon^\infty$  (resp.  $\psi_\epsilon^0$ ) controls the dynamics of the node (resp. saddle). It is known that the function  $\psi_0^\infty(w) = aw + b$  is a germ of affine map at  $\infty$ . Poincaré has proved that a node is always linearizable except when it is resonant in which case it is necessarily at least normalizable. Hence its holonomy map is either linearizable when the node is non resonant. When the node is resonant its holonomy map is either linearizable or embeddable. We show that among nonlinear germs, an affine map  $\psi_0^\infty(w) = aw + b$  is the only germ of diffeomorphism at infinity that can reflect the dynamics of a node in the unfolded vector field. Indeed, if we take for  $\psi_0^\infty$  any germ at infinity different from an affine map then necessarily the bifurcating node would have a non linearizable holonomy map when non resonant.

The paper is organized as follows. In Section 2 we “prepare” the family of diffeomorphisms. The proof of the classification theorem goes through constructing changes of coordinate bringing the diffeomorphism to the model on adequate domains. This study is spread through end of Section 2 (preliminary considerations) and Section 3: construction of Fatou coordinates. Our construction is a refinement of the construction of Shishikura [S]: this refinement allows us to cover a full neighborhood of the parameter space. It unifies the approaches of Lavaurs [L] and Glutsyuk [G]. In Section 4 we prove the theorem of analytic classification of analytic families unfolding a parabolic point and we give equivalent forms of the modulus in terms of return maps near the two fixed points. The point of view of Section 4 is what we call the Lavaurs point of view as it was first introduced in Lavaurs’ thesis [L]: it consists in embedding the system in the model in a region located between the fixed points and in reading the obstruction to a full embedding as a ramification around the singular points. In Section 5 we describe the modulus of analytic classification in the Glutsyuk point of view: there we embed the family in the model in a neighborhood of each of the singular points and read the obstruction to a full embedding in the intersection of the two neighborhoods. The Glutsyuk point of view is however only valid when the fixed points are hyperbolic. In Section 6 we compare the Lavaurs and Glutsyuk points of view. Section 7 shows how the Lavaurs and Glutsyuk points of view glue together naturally on segments through the origin in parameter space. In Section 8 we show the non linearizability of some of the bifurcating fixed points from a parabolic point with non trivial modulus. In Section 9 we discuss the question of Martinet and Ramis on the saddle-node. In Section 10 we show on an example how the Ecalle-Voronin modulus plays a role of organizing center for the strata of integrable systems in a family of the form (1.5). Finally, in Section 11 we list a number of open problems and conjectures. An appendix shows how the coefficients of the modulus in the Glutsyuk point of view tend in the limit to the coefficients of the Ecalle-Voronin modulus.

## 2. GENERALITIES

### 2.1. The prepared family and the model.

We want to compare the family of diffeomorphisms with the time-one map of a family of vector fields of the form  $\frac{z^2 - \epsilon}{1 + a(\epsilon)z} \frac{\partial}{\partial z}$  which is our “model” . For this purpose we need to prepare the family.

**Proposition 2.1.** *We consider a generic 1-parameter analytic family of local diffeomorphisms depending on a small parameter  $\epsilon$  with a double fixed point at the origin for  $\epsilon = 0$ :*

$$f_\epsilon(z) = z - \epsilon + c_1(\epsilon)z^2 + o(z^2) \tag{2.1}$$

with  $c_1(0) \neq 0$ . There exists a change of coordinate  $(z, \epsilon) \mapsto (\bar{z}, \bar{\epsilon})$  transforming the family to the prepared form

$$\bar{f}_{\bar{\epsilon}}(\bar{z}) = \bar{z} + (\bar{z}^2 - \bar{\epsilon})[1 + \bar{\beta}(\bar{\epsilon}) + \bar{A}(\bar{\epsilon})\bar{z} + (\bar{z}^2 - \bar{\epsilon})\bar{Q}(\bar{z}, \bar{\epsilon})], \tag{2.2}$$

where

(i) the multipliers at the fixed points  $P_0: \bar{z} = -\sqrt{\bar{\epsilon}}$  and  $P_\infty: \bar{z} = \sqrt{\bar{\epsilon}}$  are given by

$$\begin{aligned} \lambda_0 &= \bar{f}'_{\bar{\epsilon}}(-\sqrt{\bar{\epsilon}}) = 1 - 2\sqrt{\bar{\epsilon}}(1 + \bar{\beta}(\bar{\epsilon}) - \bar{A}(\bar{\epsilon})\sqrt{\bar{\epsilon}}) \\ \lambda_\infty &= \bar{f}'_{\bar{\epsilon}}(+\sqrt{\bar{\epsilon}}) = 1 + 2\sqrt{\bar{\epsilon}}(1 + \bar{\beta}(\bar{\epsilon}) + \bar{A}(\bar{\epsilon})\sqrt{\bar{\epsilon}}). \end{aligned} \tag{2.3}$$

(ii) Let  $\mu_{0,\infty} = \ln \lambda_{0,\infty}$ : they are analytic functions of  $\sqrt{\bar{\epsilon}}$ . The functions  $\bar{A}(\bar{\epsilon})$  and  $\bar{\beta}(\bar{\epsilon})$  are such that  $\mu_{0,\infty}$  satisfy

$$\frac{1}{\sqrt{\bar{\epsilon}}} = \frac{1}{\mu_\infty} - \frac{1}{\mu_0}. \tag{2.4}$$

(iii) Let  $\bar{a}(\bar{\epsilon})$  be defined by

$$\bar{a}(\bar{\epsilon}) = \frac{1}{\mu_\infty} + \frac{1}{\mu_0}. \tag{2.5}$$

Then  $\mu_0$  (resp.  $\mu_\infty$ ) is the eigenvalue at  $P_0$  (resp.  $P_\infty$ ) of the vector field

$$\frac{\bar{z}^2 - \bar{\epsilon}}{1 + \bar{a}(\bar{\epsilon})\bar{z}} \frac{\partial}{\partial \bar{z}}. \tag{2.6}$$

*Proof.* Ideas of the proof are borrowed from [IY]. Using a translation in  $z$  and dilatation in  $z$  and  $\epsilon$  we can suppose that the initial family has the two fixed points located at  $z = \pm\sqrt{\epsilon}$  and that  $c_1(0) = 1$ , i.e. that we start with a family:

$$f_\epsilon(z) = z + (z^2 - \epsilon)h(z, \epsilon) \quad (2.7)$$

where

$$h(z, \epsilon) = 1 + \alpha_0 z + o(z) + O(\epsilon) = 1 + \alpha_0 z + k(z, \epsilon). \quad (2.8)$$

By the Weierstrass division theorem we have  $k(z, \epsilon) = Q(z, \epsilon)(z^2 - \epsilon) + B(\epsilon)z + C(\epsilon)$ , where  $B(0) = C(0) = 0$ . The multipliers at  $P_0$  (resp.  $P_\infty$ ) are given by  $\lambda_0 = 1 - 2\sqrt{\epsilon}[1 + C(\epsilon) - (\alpha_0 + B(\epsilon))\sqrt{\epsilon}]$  (resp.  $\lambda_\infty = 1 + 2\sqrt{\epsilon}[1 + C(\epsilon) + (\alpha_0 + B(\epsilon))\sqrt{\epsilon}]$ ).

The eigenvalues of the singular points  $\sqrt{\bar{\epsilon}}$  (resp.  $-\sqrt{\bar{\epsilon}}$ ) of (2.6) are given by

$$\mu_\infty = \frac{2\sqrt{\bar{\epsilon}}}{1 + \bar{a}(\bar{\epsilon})\sqrt{\bar{\epsilon}}}, \quad \mu_0 = -\frac{2\sqrt{\bar{\epsilon}}}{1 - \bar{a}(\bar{\epsilon})\sqrt{\bar{\epsilon}}}, \quad (2.9)$$

and satisfy (2.4) and (2.5). We want to have  $\lambda_{0,\infty} = \exp(\mu_{0,\infty})$ . Let us first try to solve this with  $\bar{z} = z$  and  $\bar{\epsilon} = \epsilon$ . The equation (2.5) tells how to choose  $\bar{a}(\bar{\epsilon})$  in (2.6). In general it is not possible to realize  $\lambda_{0,\infty} = \exp(\mu_{0,\infty})$  without additional scaling in  $z$  and  $\epsilon$ . A scaling in  $z$  of the form  $\bar{z} = z(1 + b(\epsilon))$ , with  $b(\epsilon) = O(\epsilon)$  chosen later, will change the family (2.7) to the form

$$\bar{f}_\epsilon(\bar{z}) = \bar{z} + (\bar{z}^2 - \epsilon(1 + b(\epsilon))^2) \left( \frac{1 + C(\epsilon)}{1 + b(\epsilon)} + \frac{a_0 + B(\epsilon)}{(1 + b(\epsilon))^2} \bar{z} + (\bar{z}^2 - \epsilon(1 + b(\epsilon))^2) \bar{K}(\bar{z}, \epsilon) \right). \quad (2.10)$$

We ask that the new multipliers at  $\pm(1 + b(\epsilon))\sqrt{\bar{\epsilon}}$  still be given by  $\ln \mu_{\infty,0}(\epsilon)$ . The new parameter should obviously be  $\bar{\epsilon} = (1 + b(\epsilon))^2 \epsilon$ . The equation (2.4) is now solvable and yields  $b(\epsilon) = O(\epsilon)$ . Replacing in the first equation yields

$$\bar{a}(\epsilon) = -\alpha_0 + O(\epsilon) = -\bar{A}(\epsilon) + O(\epsilon), \quad (2.11)$$

where  $\alpha_0$  is in (2.8).  $\square$

**Definition 2.2.** *We call a family*

$$f_\epsilon(z) = z + (z^2 - \epsilon)[1 + \bar{\beta}(\epsilon) + \bar{A}(\epsilon)z + (z^2 - \epsilon)Q(z, \epsilon)]. \quad (2.12)$$

*a prepared family if it is a one-parameter family of the form (2.1) satisfying (i)-(iii) of Proposition 2.1. Changing the parameter to  $\nu = \sqrt{\epsilon}$  we also consider prepared families  $g_\nu = f_{\nu^2}$  i.e.*

$$g_\nu(z) = z + (z^2 - \nu^2)[1 + \underline{a}(\nu) + A(\nu)z + (z^2 - \nu^2)Q(z, \nu)]. \quad (2.13)$$

We compare our family  $g_\nu$  to the *model*, which is a family of diffeomorphisms that are fully iterable and given by the time-one maps of the family of vector fields

$$\frac{z^2 - \nu^2}{1 + a(\nu)z} \frac{\partial}{\partial z}, \quad (2.14)$$

with  $a(\nu) = \bar{a}(\nu^2) = -A(\nu) + O(\nu)$ .

As was done in [IY] we get

**Corollary 2.3.**

- (1) For a prepared family (2.13) the quantities  $\nu$  and  $a(\nu)$  are analytic invariants.
- (2) Any equivalence between two prepared families must preserve the parameter  $\nu$ . Hence two prepared families  $g_{i,\nu}$ ,  $i = 1, 2$ , are weakly equivalent if there exists a family  $H_\nu$  of germs of diffeomorphisms conjugating  $g_{1,\nu}$  and  $g_{2,\nu}$  for each  $\nu$  in a sector of opening greater than  $\pi$ .

*Proof.* The multipliers  $\mu_0$  and  $\mu_\infty$  are analytic invariants. The conclusion follows as they characterize  $\nu$  and  $a(\nu)$  in (2.4) and (2.5). Moreover if we cover a sector in  $\nu$  of opening greater than  $\pi$  we cover a sector of opening greater than  $2\pi$  in the  $\epsilon$ -space.  $\square$

**2.2. The charts in parameter space.**

We want to describe the dynamics of a prepared family  $g_\nu$  in a neighborhood  $U = U_r = B(0, r)$  of the origin for values of the parameter in a neighborhood  $V = V(\rho) = B(0, \rho)$  of the origin. To obtain results for the germ of a family, we will consider values  $r, \rho \rightarrow 0$  and restrict ourselves to values  $0 < r \leq r_0, 0 < \rho \leq \rho_0$  where  $r_0, \rho_0 > 0$ . These bounds  $r_0, \rho_0$  will be chosen later sufficiently small, but will be maintained fixed.

In order to describe the results continuously in the parameter we have to restrict  $\nu$  to sectorial regions (Figure 2). For  $0 < \delta \ll \pi/2$ , we consider two types of sectorial regions on which we use different points of view to describe the dynamics. The two *Lavaurs sectorial regions* are regions of opening  $2\pi - 2\delta$  centered on half-lines directed by  $\exp(i(\pi/2 + k\pi))$ ,  $k = 0, 1$ ,

$$V_{\delta,k}^L(\rho) = \{\nu; |\nu| < \rho, \arg \nu \in (-\pi/2 + \delta + k\pi, 3\pi/2 - \delta + k\pi)\}. \quad (2.15)$$

The two *Glutsyuk sectorial regions* are regions of opening  $\pi - 2\delta$  centered on half-lines directed by  $\exp(ik\pi)$ ,  $k = 0, 1$ ,

$$V_{\delta,k}^G(\rho) = \{\nu; |\nu| < \rho, \arg \nu \in (-\pi/2 + \delta + k\pi, \pi/2 - \delta + k\pi)\}. \quad (2.16)$$

*Remark.*

- i) The intersection of the two Glutsyuk sectorial regions is the origin only.
- ii) The two Lavaurs sectorial regions intersect on the union of the two Glutsyuk sectorial regions.
- iii) In practice we will limit ourselves to one sectorial Glutsyuk region and one sectorial Lavaurs region (corresponding to  $k = 0$ ).

**Notation.** Unless necessary we will only discuss the Lavaurs sectorial region  $V_{\delta,0}^L(\rho)$  and to simplify the notation we will write  $V_{\delta,0}^L(\rho) = V_\delta^L = V^L$ . Similarly we will only discuss the Glutsyuk sectorial region  $V_{\delta,0}^G(\rho)$  and to simplify the notation we will write  $V_{\delta,0}^G(\rho) = V_\delta^G = V^G$ .

In the  $\epsilon$ -plane,  $V_\delta^L$  (resp.  $V_\delta^G$ ) corresponds to a sectorial neighborhood of the origin in  $\epsilon$ -space with  $\arg(\epsilon) \in (-\pi + 2\delta, 3\pi - 2\delta)$  (resp.  $\arg(\epsilon) \in (-\pi + 2\delta, \pi - 2\delta)$ ). The size of the neighborhood  $U$  (resp.  $V$ ) of the origin in  $z$  (resp.  $\nu$ )-space, i.e. the values of  $r$  and  $\rho$  will be chosen in function of  $\delta$ : we want to take  $\delta$  arbitrarily small. The smaller  $\delta$ , the smaller we need to take  $r$  and  $\rho$  so that several inequality conditions (conditions (2.29),(3.4),(3.16) below) be satisfied.

**2.3. The domain of the lifted diffeomorphism.**

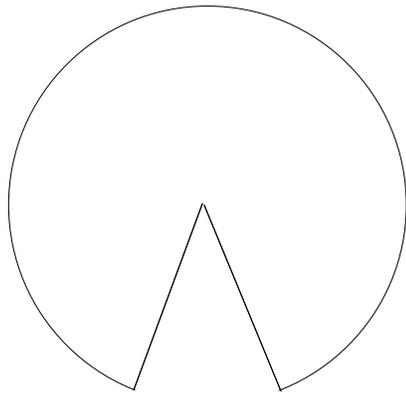
The singular points of the vector field (2.14) are given by  $\pm\nu$ . We call them :  $P_0 = -\nu$  and  $P_\infty = +\nu$  and their multipliers will be denoted by

$$\mu_0(\nu) = -\frac{2\nu}{1 - a(\nu)\nu}, \quad \mu_\infty(\nu) = \frac{2\nu}{1 + a(\nu)\nu}. \quad (2.17)$$

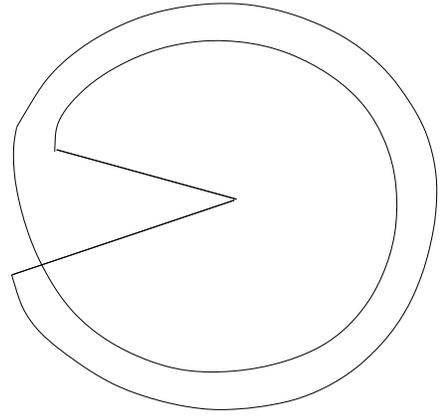
Let  $T_\alpha$  be the translation by  $\alpha$ , i.e.

$$T_\alpha(Z) = Z + \alpha. \quad (2.18)$$

Let

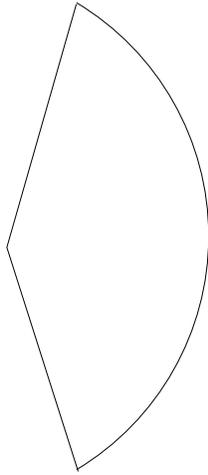


in  $\nu$ -space

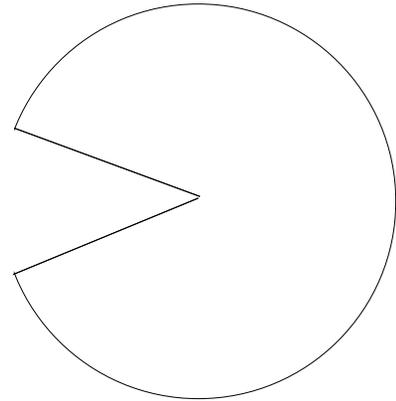


in  $\epsilon$ -space

$$V_{\delta,0}^L(\rho)$$



in  $\epsilon$ -space



in  $\epsilon$ -space

$$V_{\delta,0}^G(\rho)$$

FIGURE 2. The sectorial regions in  $\nu$ -space and  $\epsilon$ -space

$$\alpha(\nu) = \frac{\pi i}{\nu}. \tag{2.19}$$

**Proposition 2.4.**

(1) The mapping  $p_\nu : \mathbb{C} \rightarrow \mathbb{CP}^1 \setminus \{-\nu, \nu\}$ ,  $\nu \in V$ , given by

$$p_\nu(Z) = \begin{cases} \nu \frac{1+e^{2\nu Z}}{1-e^{2\nu Z}}, & p_\nu(\frac{k\pi i}{\nu}) = \infty, \quad k \in \mathbb{Z} \\ -\frac{1}{Z} & \nu = 0, \end{cases} \tag{2.20}$$

yields the universal covering of  $\mathbb{CP}^1 \setminus \{-\nu, \nu\}$  with covering map  $T_{\alpha(\nu)}$ .

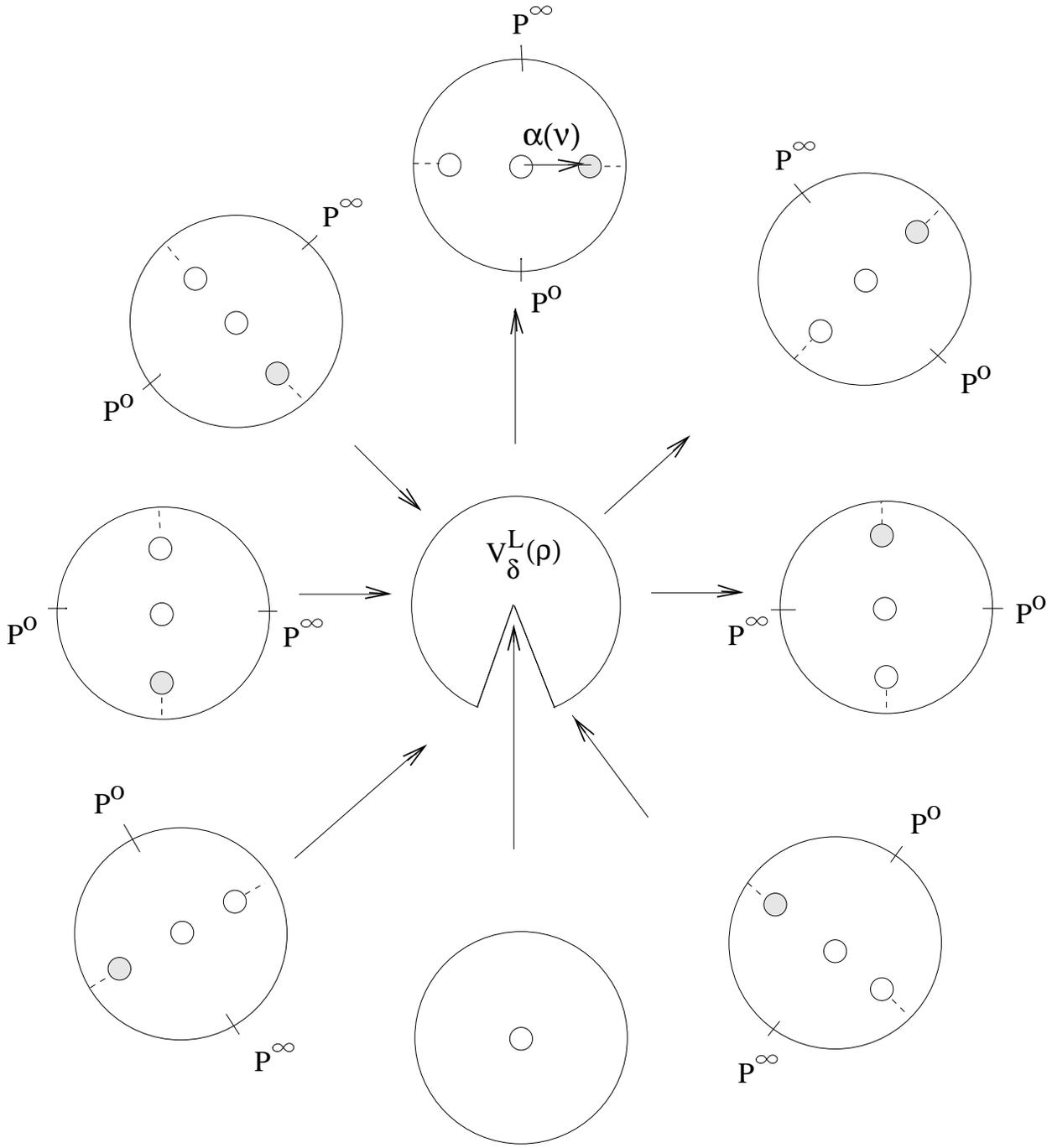


FIGURE 3. The domain  $\hat{U}_\nu$  for  $\nu \in V_\delta^L(\rho)$

(2) Its inverse is the multivalued function

$$q_\nu(z) = \begin{cases} \frac{1}{2\nu} \log \frac{z-\nu}{z+\nu} & \nu \neq 0 \\ -\frac{1}{z} & \nu = 0. \end{cases} \quad (2.21)$$

Let  $B_\nu$  be the connected component of  $\mathbb{C} \setminus p_\nu^{-1}(U)$  which contains 0 and is sent bijectively to the complement of  $U$  by  $p_\nu$ . Then

$$p_\nu^{-1}(U) = \hat{U}_\nu = \mathbb{C} \setminus \cup_{k \in \mathbb{Z}} T_{\alpha(\nu)}^k(B_\nu), \quad (2.22)$$

is the universal covering of  $U$  with covering map  $T_{\alpha(\nu)}$ .

(3) The function  $p_\nu(Z)$  tends to  $-\frac{1}{Z}$ , uniformly on any compact of  $\mathbb{C}$ .

We lift the function  $g_\nu$  of  $z$  to a function  $G_\nu$  of  $Z$  verifying

$$T_{\alpha(\nu)} \circ G_\nu = G_\nu \circ T_{\alpha(\nu)}. \quad (2.23)$$

The function  $G_\nu$  is defined on  $\hat{U}_\nu$ , i.e. on the complex plane minus an infinite number of aligned equidistant holes  $T_{\alpha(\nu)}^k(B_\nu)$ .

We call the hole  $B_\nu$  the *principal hole* and put  $B_\nu^k = T_{\alpha(\nu)}^k(B_\nu)$ . We will work essentially with the holes  $B_\nu^k$ ,  $k = -1, 0, 1$ , which we will call respectively  $B_\nu^-, B_\nu, B_\nu^+$ . The holes are aligned on the straight line directed by  $i/\nu$  and the distance of centers of two adjacent holes is  $\pi/|\nu|$ . They have the same size (the size of  $B_\nu$ ) which depends continuously on  $\nu$ . Moreover  $B_0$  is a ball of radius equivalent to  $\frac{1}{r}$  when  $r \rightarrow 0$ .

We denote  $P^0$  and  $P^\infty$  points located at infinity in the direction orthogonal to the line of holes (Figure 3) and corresponding by  $p_\nu$  to the singular points  $P_0$  and  $P_\infty$ .

**Proposition 2.5.** *The map  $G_\nu$  is a small perturbation of the translation  $Z \rightarrow Z + 1$  in  $C^1$  topology. More precisely, there exists  $K > 0$ , such that for  $z$  and  $\nu$  sufficiently small (1) and (2) hold*

$$(1) \quad |G_\nu(Z) - Z - 1| < K \max(r, \rho). \quad (2.24)$$

$$(2) \quad |G'_\nu(Z) - 1| < K \max(r, \rho)^2. \quad (2.25)$$

*Proof.*

(1) The function  $G_\nu(Z)$  can be written in the form

$$\begin{aligned} G_\nu(Z) &= \frac{1}{2\nu} \log \frac{(z - \nu)(1 + (z + \nu)(1 + h_\nu(z)))}{(z + \nu)(1 + (z - \nu)(1 + h_\nu(z)))} \\ &= Z + \frac{1}{2\nu} \log \left( 1 + \frac{2\nu(1 + h_\nu(z))}{1 + (z - \nu)(1 + h_\nu(z))} \right) \\ &= Z + 1 + O(z, \nu), \end{aligned} \quad (2.26)$$

since  $h_\nu(z) = O(|z| + |\nu|)$

(2) Let  $R_\nu(Z) = \frac{1 + h_\nu(z)}{1 + (z - \nu)(1 + h_\nu(z))}$ . Then

$$\frac{dG}{dZ} = 1 + \frac{1}{1 + 2\nu R_\nu} \frac{dR_\nu}{dz}(z) \frac{dz}{dZ}. \quad (2.27)$$

The quantity  $\frac{dR_\nu}{dz}$  is bounded for small  $(z, \nu)$ . Also

$$\frac{dz}{dZ} = \frac{1}{\frac{dZ}{dz}} = z^2 - \nu^2 = o(|z| + |\nu|). \quad (2.28)$$

□

In the rest of the paper we consider only neighborhoods  $U$  and  $V$  of the origin in the  $z$ -plane and the  $\nu$ -plane of respective radii  $r$  and  $\rho$  such that the function  $M(r, \rho) = K \max(r, \rho)$  verifies

$$M(r, \rho) \leq \delta/4. \quad (2.29)$$

This means in practice  $r, \rho \leq \frac{\delta}{4K}$ .

### 3. MAXIMAL DOMAINS OF INTEGRABILITY AND FATOU COORDINATES.

In this section, we show that there exist coordinates called *Fatou coordinates* valid in certain domains in  $\hat{U}$  (*translation domains*) which transform  $G_\nu$  to a translation by one. The Fatou coordinates are not globally defined as univalued functions on  $\hat{U}_\nu$ . For given  $G_\nu$  there are several *equally good* choices of maximal translation domains. These domains are obtained by extending certain *admissible strips* determined by the relative position of some lines (*admissible lines*) with respect to the holes.

#### 3.1. Admissible lines, admissible strips and translation domains.

Let  $g_\nu$  be an unfolding of a parabolic germ, let  $G_\nu$  be the corresponding lifted map. The condition (2.29) implies that, for  $\nu \in V$ ,  $G_\nu$  verifies on  $\hat{U}_\nu$

$$\begin{aligned} |G_\nu(Z) - (Z + 1)| &< \delta/4 \\ |G'_\nu(Z) - 1| &< \delta/4. \end{aligned} \tag{3.1}$$

We generalize here an idea of Shishikura[S] and consider slanted lines  $l$  such that the image  $G_\nu(l)$  of  $l$  is on the right of  $l$ . Then  $l \cup G_\nu(l)$  is the boundary of a strip which is a fundamental domain for the dynamics. Shishikura [S] restricted himself to vertical lines. He used the strip to conjugate holomorphically  $G_\nu$  with a translation but he limited himself to  $\arg \nu$  in some interval  $(\delta, \pi - \delta)$  and  $\delta > 0$  small. We need to generalize somewhat this idea to cover larger intervals in  $\arg \nu$ , of size almost  $2\pi$ .

**Lemma 3.1.** *Let us suppose that  $3M(r_0, \rho_0) < 1$  and let  $r \leq r_0$ ,  $\rho \leq \rho_0$ . Let us consider the function  $\theta_0(r, \rho)$  defined by  $\sin \theta_0(r, \rho) = 3M(r, \rho)$  where  $0 < \theta_0(r, \rho) < \pi/2$ , ( $\theta_0(r, \rho)$  is equivalent to  $3M(r, \rho)$  when  $(r, \rho) \rightarrow (0, 0)$ ). Let  $\nu \in V(\rho)$ .*

*One considers a line  $l$  such that  $l$  and  $G_\nu(l)$  are both contained in  $\hat{U}_\nu$ . If the angle  $\theta(l)$  of the line with the horizontal axis verifies*

$$\theta_0(r, \rho) \leq \theta(l) \leq \pi - \theta_0(r, \rho), \tag{3.2}$$

*then  $G_\nu(l)$  is located to the right of  $l$ . If  $r_0$  is sufficiently small the strip  $\hat{C}_\nu(l)$  bounded by  $l$  and  $G_\nu(l)$  is contained in  $\hat{U}_\nu$ .*

**Definition 3.2.** *A line satisfying the hypotheses of Lemma 3.1 is called an *admissible line* for  $G_\nu$  (and for the given values of  $r, \rho$ ). The strip bounded by  $l$  and  $G_\nu(l)$  is called an *admissible strip* (Figure 4). We denote it by  $\hat{C}_\nu(l)$ .*

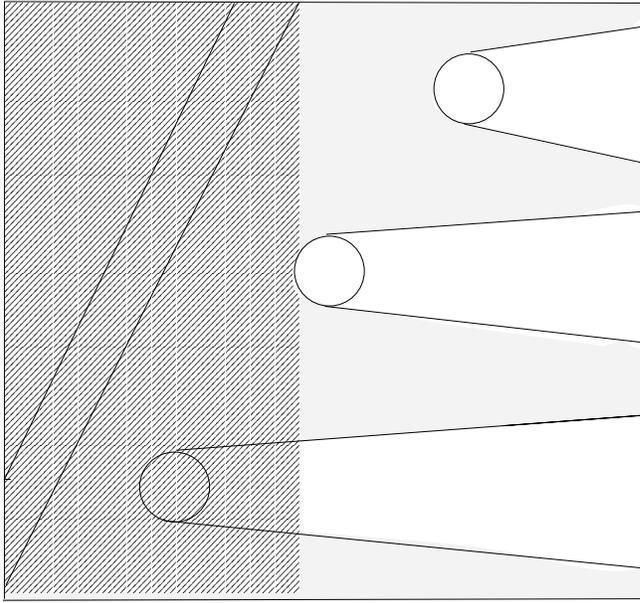
Let us now consider domains in  $\hat{U}_\nu$  obtained by saturation of admissible strips under iterations of  $G_\nu$ :

**Definition 3.3.** *Let  $l$  be an admissible line for  $G_\nu$ . The translation domain associated with  $l$  is the set*

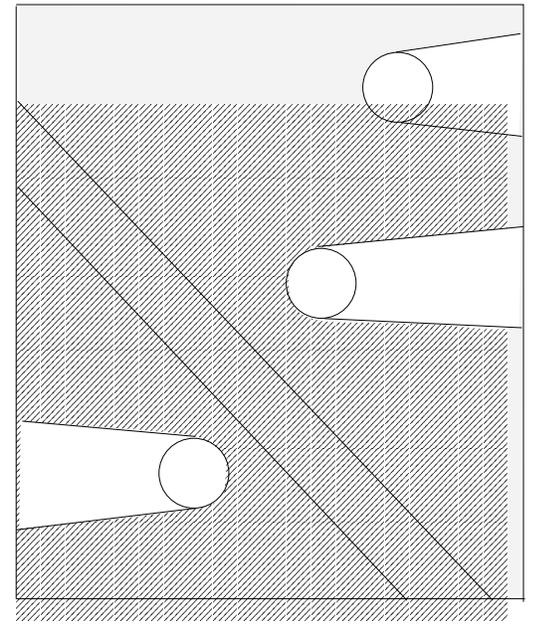
$$Q_\nu(l) = \{Z \in \hat{U}_\nu \mid \exists n \in \mathbb{Z}, G_\nu^n(Z) \in \hat{C}_\nu(l), \forall i \in [0, n] \subset \mathbb{Z}, G_\nu^i(Z) \in \hat{U}_\nu\}. \tag{3.3}$$

#### Proposition 3.4.

- (1) *The domain  $Q_\nu(l)$  is a simply connected open subset of  $\hat{U}_\nu$ .*
- (2)  *$\hat{C}_\nu(l) \setminus l$  is a fundamental domain for  $G_\nu$  restricted to  $Q_\nu(l)$  : each  $G_\nu$ -orbit in  $Q_\nu(l)$  has one and only one point in this subset.*
- (3) *If  $l'$  is another admissible line, then  $l' \subset Q_\nu(l)$  if and only if  $l \subset Q_\nu(l')$ . This defines an equivalence relation among the admissible lines for  $G_\nu$ , each equivalence class corresponding to a different translation domain.*



Glutsyuk admissible  
strip and translation  
domain



Lavaurs admissible  
strip and translation  
domain

FIGURE 4. Two types of admissible lines and strips

**Definition 3.5.** *Let  $\nu \neq 0$ . Then*

- (1) *A Glutsyuk translation domain (Figure 4(i)) is a domain associated with an admissible line parallel to the direction  $\alpha(\nu)$  of the covering map  $T_{\alpha(\nu)}$  located on one side of the alignment of holes (notation  $Q_\nu^G$ ). A Glutsyuk translation domain is attractive (resp. repelling) if it contains a half-plane to the right (resp. to the left) of the alignment of holes. We denote  $Q_\nu^{G,0}$  (resp.  $Q_\nu^{G,\infty}$ ) these two domains (See Figure 5).*
- (2) *A Lavaurs translation domain (Figure 4(ii)) is a domain associated with an admissible line passing between the fundamental hole and one of its two adjacent holes (notation  $Q_\nu^L$ ). Let be  $S_\nu$  the sector of angle  $< \pi$ , with sides oriented by the complex numbers  $-1, \alpha(\nu)$ . We will consider more precisely the domains associated with an admissible line passing between the fundamental hole and the hole  $B_\nu^+$  (resp.  $B_\nu^-$ ) and whose direction is passing outside the sector  $S_\nu$ . We call them the positive (resp. negative) Lavaurs translation domain and denote them  $Q_\nu^{L,+}$  (resp.  $Q_\nu^{L,-}$ ) (See Figure 6).*

*For  $\nu = 0$  the Glutsyuk and Lavaurs translation domains coincide. They are given by translation domains associated with any admissible line.*

**Remark 3.6.**

- (1) The choice of the admissible line in the definition of the Lavaurs translation domains  $Q_\nu^{L,\pm}$  is made such that the line can be chosen in a continuous way in function of  $\nu$ . This will be explained in the Section 3.4.
- (2) As they are defined, the Lavaurs domains  $Q_\nu^{L,\pm}$  are unique if they exist. This follows easily from the fact that two admissible lines with direction passing outside the sector  $S_\nu$ , are homotopic through the set of admissible lines. Of course, it may exist different domains corresponding to admissible lines with direction passing inside the sector  $S_\nu$ . In fact the definition 3.5 is restricted to the sectorial regions  $V_{\delta,0}^G(\rho)$  and  $V_{\delta,0}^L(\rho)$ , as it was said in Paragraph 2.2. Completely similar definitions can be given for the sectorial regions  $V_{\delta,1}^G(\rho)$  and  $V_{\delta,1}^L(\rho)$ . The other possibilities of Lavaurs translation domains that we mentioned above, would be the domains  $Q_\nu^{L,\pm}$  associated to

these sectorial regions and for this reason we do not need to introduce them for  $\nu \in V_{\delta,0}^L(\rho)$ . (They are defined when  $\arg(\nu) = -\pi/2$  and not defined for  $\arg(\nu)$  is near  $\pi/2$ ).

**Proposition 3.7.** *Let us consider  $r_0, \rho_0$  sufficiently small, such that  $\sin 4M(r_0, \rho_0) > 3M(r_0, \rho_0)$ . Consider  $r \leq r_0, \rho \leq \rho_0$  and  $\nu \in V_{\delta}^G(\rho)$ . Then, there exist two unique Glutsyuk translation domains  $Q_{\nu}^{G,0}$  and  $Q_{\nu}^{G,\infty}$  (Figure 5). The domain  $Q_{\nu}^{G,0}$  (resp.  $Q_{\nu}^{G,\infty}$ ) is associated to lines on the side of  $P^0$ , (resp.  $P^{\infty}$ ). The two domains are invariant by the translation  $T_{\alpha(\nu)}$ .*

*Proof.* Let  $\ell$  be a line parallel to the direction  $\alpha(\nu)$ , i.e.  $\theta(\ell) = \arg(\alpha(\nu))$ . If  $\nu \in V_{\delta}^G(\rho)$ , then  $\arg(\alpha(\nu))$  belongs to  $(\delta, \pi - \delta)$ . As the function  $s \rightarrow \frac{\sin 4s}{3s}$  is monotone on  $(0, \pi/2)$ , we have that  $\sin 4M(r, \rho) > 3M(r, \rho)$ , for all  $r \leq r_0, \rho \leq \rho_0$ . It follows from the definition of  $\theta_0(r, \rho) : \sin \theta_0(r, \rho) = 3M(r, \rho)$  and (2.29) that  $\theta_0(r, \rho) < 4M(r, \rho) \leq \delta$ . Then the line  $\ell$  will be admissible as soon as  $G_{\nu}(\ell) \subset \hat{U}_{\nu}$ , i.e. for  $r_0$  small enough. Clearly, one obtains two different Glutsyuk translation domains  $Q_{\nu}^{G,0}$  (resp.  $Q_{\nu}^{G,\infty}$ ) associated to lines on the side of  $P^0$  (resp.  $P^{\infty}$ ). The invariance by the translation  $T_{\alpha(\nu)}$  follows from the invariance of the strip  $\hat{C}_{\nu}(\ell)$ .  $\square$

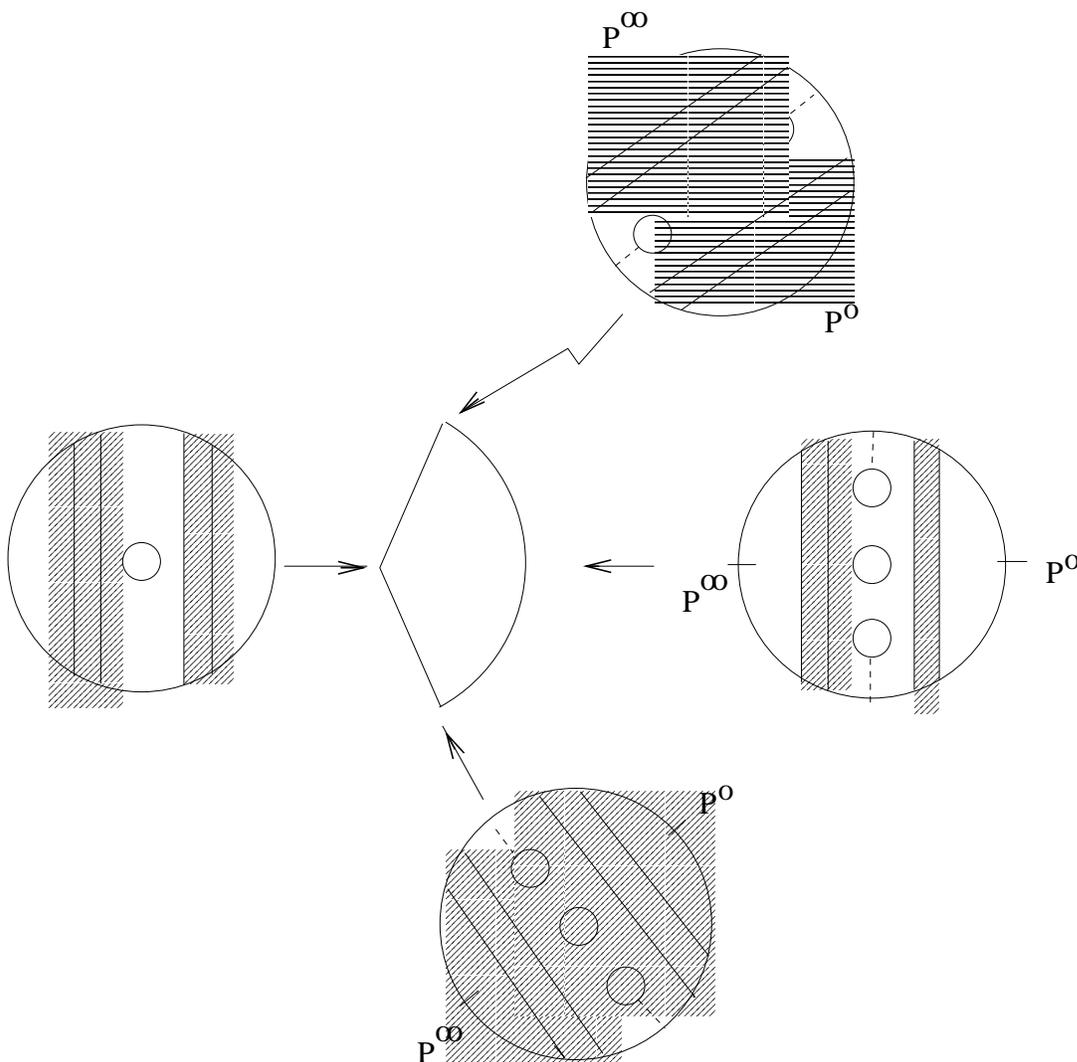


FIGURE 5. Pairs of Glutsyuk translation domains: we only draw associated strips

**Proposition 3.8.** *Let us suppose that  $r_0$  and  $\rho_0$  are small enough and that  $r < r_0, \rho < \rho_0, \delta$  satisfy the following conditions :*

$$\rho < Kr^2 \quad \text{and} \quad 2\theta_0(r, \rho) < \delta \tag{3.4}$$

where  $K$  is the constant introduced in Proposition 2.5. Then, for  $\nu \in V_\delta^L(\rho)$  there exist two unique families of Lavaurs translation domains  $Q_\nu^{L,\pm}$ , (Figure 6) associated with admissible lines passing between the fundamental hole and the hole  $B_\nu^\pm$  respectively and depending continuously on  $\nu$ .

*Proof.* The reason for the unicity of  $V_\delta^L(\rho)$  is explained in the remark 3.6. We want now some indication for the existence of these domains. Recall that we consider  $\nu \in (-\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta)$  which corresponds to an angle of the line of holes  $\alpha = \arg(\alpha(\nu)) \in (\delta, \pi - \delta)$ . Of course it is sufficient to look at the limit situation : passing an admissible line  $\ell$  when  $\arg(\nu)$  tends toward  $-\pi/2 + \delta$  or  $3\pi/2 - \delta$ . First, one has to choose  $r_0$  and  $\rho_0$  small enough to fulfill the condition (2.29). Next, taking a  $k > 1$ , one supposes that the line  $\ell$  has an angle  $\theta = k\theta_0$ , and that  $r_0, \rho_0$  are small enough to have the holes of radius less than  $\frac{k}{r}$ . To pass an admissible line between two consecutive holes, it then suffices that

$$\frac{\pi}{\rho} \sin(\delta - k\theta_0) - \frac{2k}{r} > \sin k\theta_0 + M. \quad (3.5)$$

We recall that  $M = M(r, \rho) = K \text{Sup}\{r, \rho\}$  and that  $\theta_0(r, \rho)$  is defined by the formula  $\sin \theta_0 = 3M$ . Using the hypothesis that  $2\theta_0(r, \rho) < \delta$ , it is easy to see that the condition (3.5) is implied by the condition  $\rho < M(r, \rho)r$  for a well-chosen constant  $k$  and  $r_0, \rho_0$  small enough. Finally, this last condition is equivalent to  $\rho < Kr^2$  if  $r_0, \rho_0$  are small enough.  $\square$

*Remarks.*

- (1) As Lavaurs sectorial domains intersect on Glutsyuk sectorial domains we have six translation domains on a Glutsyuk sectorial domain  $V_{\delta,k}^P$  while we may have only two outside the Glutsyuk sectorial domains.
- (2) When  $\arg \nu$  approaches  $-\pi/2 + \delta$  or  $3\pi/2 + \delta$  we have two pairs of Lavaurs domains generated by parallel lines (one on the left and one on the right of the fundamental hole). Their relative positions with respect to  $P^0$  and  $P^\infty$  are however opposite. This suggests that although our construction is continuous in  $\nu$  for  $\nu \in V_\delta$  with arbitrarily small  $\delta > 0$  a drastic discontinuous phenomenon may happen at the limit when  $\delta \rightarrow 0$ .

### 3.2. Existence of Fatou coordinates.

Let  $Q_\nu = Q_\nu(\ell)$  be a translation domain generated by an admissible line  $\ell$ . Because the strip  $\hat{C}_\nu(\ell) \setminus \ell$  is a fundamental domain for  $G_\nu$  on  $Q_\nu$ , the diffeomorphism  $G_\nu$  is in fact topologically conjugate on  $Q_\nu$  to the translation  $T_1(Z) = Z + 1$  (this explains the name ‘‘translation domain’’ chosen for  $Q_\nu$ ). One can construct such a conjugacy as follows. Choose any homeomorphism  $\Phi_0$  of  $\hat{C}_\nu(\ell)$  with the strip  $C_0 = \{Z = X + iY \mid 0 \leq X \leq 1\}$ , such that for  $\forall Z \in \ell$ ,  $\Phi_0(Z) + 1 = \Phi_0 \circ G_\nu(Z)$ . Then one can extend  $\Phi_0$  into an homeomorphism  $\Phi$  of  $Q_\nu$  onto a simply connected open subset of  $\mathbb{C}$  in the following way : if  $Z \in Q_\nu$  there a unique  $n \in \mathbb{Z}$  such that  $G_\nu^n(Z) \in \hat{C}_\nu(\ell) \setminus \ell$ ; we then define  $\Phi(Z) = \Phi_0 \circ G_\nu^n(Z) - n$ . By construction,  $\Phi$  is a homeomorphism which conjugates  $G_\nu|_{Q_\nu}$  with the translation  $T_1$  : for any  $Z \in Q_\nu \cap G_\nu^{-1}(Q_\nu)$  one has  $\Phi \circ G_\nu(Z) = \Phi(Z) + 1$ .

Using the same technique as Shishikura in [S] we prove now that it is possible to find a holomorphic diffeomorphism  $\Phi$  conjugating  $G_\nu$  with  $T_1$ . Let

$$\alpha_{0,\infty}(\nu) = \frac{2i\pi}{\mu_{0,\infty}(\nu)}. \quad (3.6)$$

Note that for  $a(\nu) = 0$  in (2.14), we have  $\alpha_0(\nu) = \alpha_\infty(\nu) = \alpha(\nu)$ .

**Theorem 3.9.** *Let  $Q_\nu$  be any translation domain and  $Z_0(\nu) \in Q_\nu$ .*

- (1) *There exists a holomorphic diffeomorphism  $\Phi_\nu : Q_\nu \rightarrow \mathbb{C}$ , such that*

$$\Phi_\nu(G_\nu(Z)) = \Phi_\nu(Z) + 1, \quad (3.7)$$

for  $Z \in Q_\nu \cap G_\nu^{-1}(Q_\nu)$ .

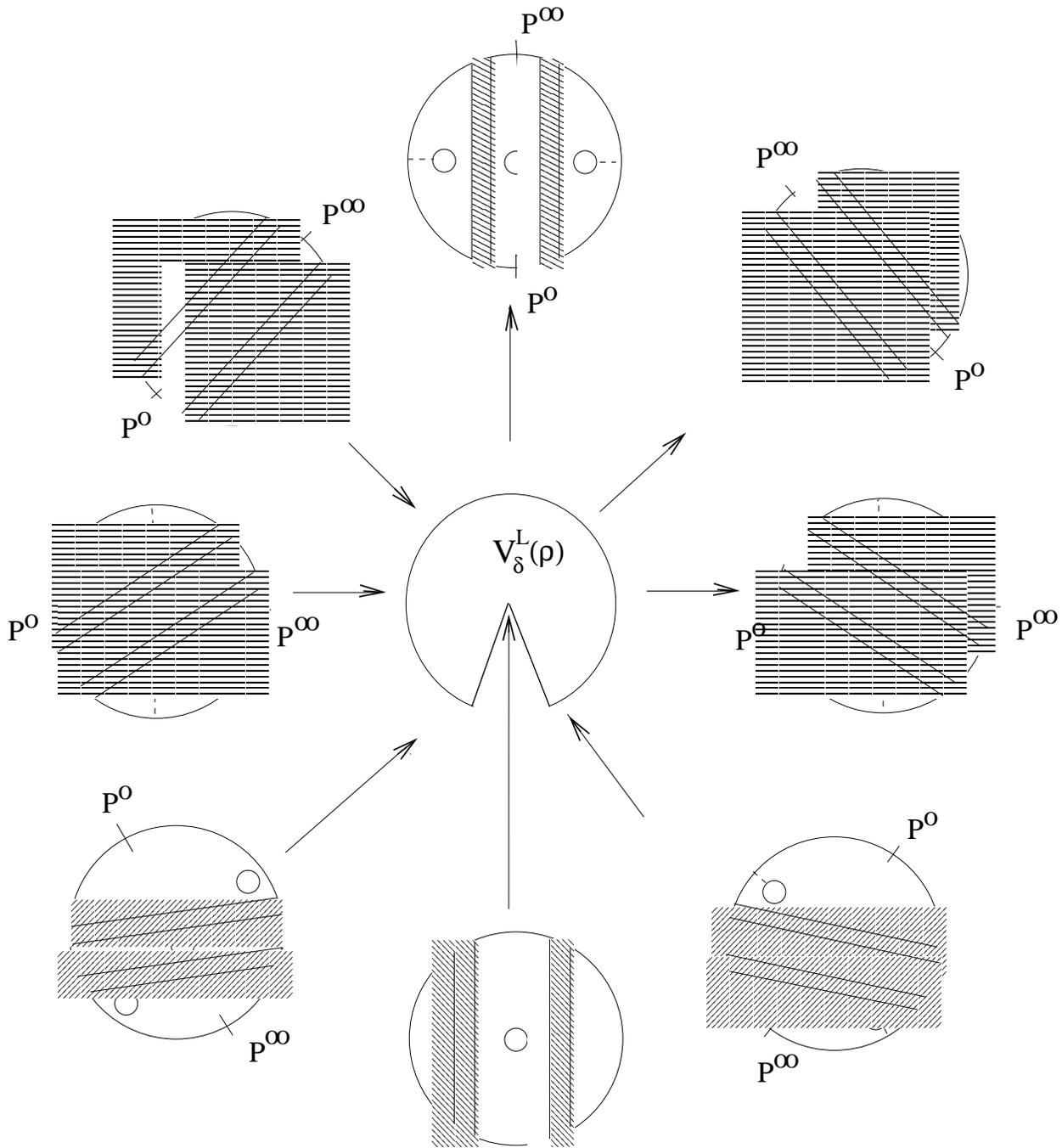


FIGURE 6. Pairs of Lavaurs translation domains: we only draw associated strips

- (2) If  $\Phi_{1,\nu}$  and  $\Phi_{2,\nu}$  are two solutions of (3.7), then there exists  $A \in \mathbb{C}$  such that  $\Phi_{2,\nu}(Z) = A + \Phi_{1,\nu}(Z)$ . In particular there is a unique holomorphic diffeomorphism  $\Phi_\nu$  satisfying (3.7) together with  $\Phi_\nu(Z_0(\nu)) = 0$ .
- (3) If  $Q_\nu^{G,0,\infty}$  is a Glutsyuk translation domain (then invariant by  $T_{\alpha(\nu)}$ ), and  $\Phi_\nu^{0,\infty}$  is the holomorphic map given in (3.7), we have :

$$\Phi_\nu^{0,\infty} \circ T_{\alpha(\nu)} = T_{\alpha_0,\infty(\nu)} \circ \Phi_\nu^{0,\infty}. \quad (3.8)$$

*Proof.*

- (1) The technique we use is identical to that of Shishikura [S]. It consists in constructing a quasi-conformal conjugacy of  $G_\nu$  to the translation by 1 and then use Ahlfors-Bers theorem to transform it into a conformal conjugacy.

All along the proof we do not mention the  $\nu$ -dependence. Let  $\ell$  be an admissible line in the translation domain  $Q$  and  $\hat{C}(\ell)$  the corresponding strip and let  $Z_1$  be any point of  $\ell$ . Points of  $\ell$  can be written as  $Z_1 + Ye^{i\theta}$ ,  $Y \in \mathbb{R}$ , where  $\theta = \theta(\ell)$  is the angle of  $\ell$  with  $\mathbb{R}$ . We recall that  $2M(r, \rho) \leq \theta(\ell) \leq \pi - 2M(r, \rho)$ . We define  $h_1 : C_0 = \{(X, Y) \in \mathbb{R}^2 \mid 0 \leq X \leq 1\} \rightarrow \hat{C}(\ell)$  by:

$$h_1(X, Y) = (1 - x)(Z_1 + Ye^{i\theta}) + XG_\nu(Z_1 + Ye^{i\theta}). \quad (3.9)$$

Then

$$\begin{aligned} \frac{\partial h_1}{\partial X} &= G_\nu(Z_1 + Ye^{i\theta}) - (Z_1 + Ye^{i\theta}) \\ \frac{\partial h_1}{\partial Y} &= Xe^{i\theta}G'_\nu(Z_1 + Ye^{i\theta}) + e^{i\theta}(1 - X). \end{aligned} \quad (3.10)$$

Using the inequalities (2.24) and (2.25), these formulas imply that  $\frac{\partial h_1}{\partial X} = 1 + u(X, Y)$ ,  $\frac{\partial h_1}{\partial Y} = e^{i\theta} + v(X, Y)$  with  $|u|, |v| \leq M(r, \rho)$ . Let  $\mu_{h_1} = \frac{\partial h_1}{\partial \bar{Z}} / \frac{\partial h_1}{\partial Z}$  be the dilatation coefficient field of  $h_1$ . One has

$$\frac{\partial h_1}{\partial \bar{Z}} = \frac{1}{2}[1 + u + i(e^{i\theta} + v)] \quad \text{and} \quad \frac{\partial h_1}{\partial Z} = \frac{1}{2}[1 + u - i(e^{i\theta} + v)] \quad (3.11)$$

When  $u, v \equiv 0$ , i.e. when  $G \equiv T_1$ ,  $\mu_{h_1}$  reduces to  $\mu^0 = \frac{1+ie^{i\theta}}{1-ie^{i\theta}} = \frac{i \cos \theta}{1 + \sin \theta}$  and

$$|\mu^0| \leq (1 + 3M(r, \rho))^{-1}.$$

as  $\sin \theta > \sin \theta_0 = 3M$ . From (3.11) one can write

$$\mu_{h_1} = \mu^0 \left(1 + \frac{u - iv}{1 - ie^{i\theta}}\right)^{-1} + \frac{u + iv}{1 - ie^{i\theta} + u - iv} \quad (3.12)$$

Let us remark that  $|1 - ie^{i\theta}| \geq \sqrt{2}$ . Then, from (3.12) one deduces ( $M = M(r, \rho)$ )

$$\begin{aligned} \|\mu_{h_1}\|_\infty &= \text{Sup}\{|\mu_{h_1}(z)| \mid z \in C_0\} \leq (1 + 3M)^{-1}(1 - \sqrt{2}M)^{-1} + \sqrt{2}M(1 - \sqrt{2}M)^{-1} \\ &= 1 - \frac{(3 - 2\sqrt{2})M - 6\sqrt{2}M^2}{(1 + 3M)(1 - 2\sqrt{2}M)} < 1, \end{aligned} \quad (3.13)$$

for  $M$  sufficiently small. So  $h_1$  is a quasi-conformal mapping on the strip  $C_0$  and satisfies  $h_1^{-1}(G_\nu(Z)) = h_1^{-1}(Z) + 1$  for  $Z \in \ell$  when  $M(r, \rho)$  is small enough. Moreover,  $\mu = \mu_{h_1}$  is a Beltrami field on  $C_0$ . (This just means that  $\mu$  is defined by a  $L^\infty$ -function with a norm strictly less than 1). One can also write that  $\mu = h_1^* \mu_0$ , where  $\mu_0$  is the standard Beltrami field on  $\mathbb{C}$  (defined by the function 0).

We extend  $\mu$  to all of  $\mathbb{C}$  by means of the translation  $T_1$ : the extended  $\mu$  is periodic of period 1, is in  $L^\infty(\mathbb{C})$  and has a  $L^\infty$ -norm :  $\|\mu\|_\infty = \|\mu_{h_1}\|_\infty < 1$  ( $\mu$  may have discontinuities along the lines  $\{\Re Z = c \mid c \in \mathbb{Z}\}$ ). Then this extended  $\mu$  is a Beltrami field on  $\mathbb{C}$ .

The universal covering

$$w = E(Z) = \exp(-2\pi i Z) \quad (3.14)$$

from  $\mathbb{C}$  to  $\mathbb{C}^*$  induces a holomorphic diffeomorphism from  $\mathbb{C}/T_1$  to  $\mathbb{C}^*$ . As  $\mu$  is invariant by  $T_1$  the map  $E$  induces a Beltrami field  $\tilde{\mu}$  on  $\mathbb{C}^*$  with the same norm :  $\mu = E^*(\tilde{\mu})$ . Considering the Riemann sphere  $S^2$  as  $\mathbb{C}^* \cup \{0, \infty\}$ , one can extend  $\tilde{\mu}$  on  $S^2$  by, for instance,  $\tilde{\mu}(0) = \tilde{\mu}(\infty) = 0$ . Then  $\tilde{\mu}$  defines a Beltrami field on the Riemann sphere.

By Ahlfors-Bers measurable mapping theorem there exists a unique quasi-conformal mapping  $\tilde{h}_2 : S^2 \rightarrow S^2$  such that  $\tilde{h}_2^* \mu_0 = \tilde{\mu}$ , and  $\tilde{h}_2(0) = 0, \tilde{h}_2(\infty) = \infty, \tilde{h}_2(1) = 1$ . As  $0, 1 \in E^{-1}(1)$  this map lifts to a quasi-conformal map  $h_2 : \mathbb{C} \rightarrow \mathbb{C}$  sending 0 to 0 and 1 to 1. Indeed, one can lift  $\tilde{h}_2$  into a map  $h_2$  such that  $h_2(0) = 0$ . The circle in  $S^2$  which turns one time around 0 or  $\infty$  lifts into the line segment  $[0, 1]$  in  $\mathbb{C}$ . This means that  $h_2(1) = 1$ . We have also that  $\text{Im}(h_2(X + iY)) \rightarrow \pm\infty$  when  $Y \rightarrow \pm\infty$ .

The most important property of  $h_2$  is that it commutes with  $T_1$ . To see this, consider the homeomorphism  $H_2 = h_2 \circ T_1 \circ h_2^{-1}$ . It induces the identity on  $S^2$  and must then be a power of the deck transformation  $T_1$  of the universal covering map  $E$ : i.e.  $H_2 = T_1^n$  for some  $n \in \mathbb{Z}$ . Now  $H_2(0) = h_2 \circ T_1(0) = h_2(1) = 1$ . This forces  $n = 1$  and then  $H_2 = T_1$ , i.e.  $h_2 \circ T_1 = T_1 \circ h_2$ .

We define  $\phi : \hat{C}(\ell) \rightarrow \mathbb{C}$  by  $\phi = h_2 \circ h_1^{-1}$ . This mapping  $\phi : \mathbb{C} \rightarrow \mathbb{C}$  is quasi-conformal and preserves the standard conformal structure. Hence it is conformal. For  $Z \in \ell$  one has  $T_1 \circ \phi(Z) = \phi \circ G_\nu(Z)$ . Then  $\phi$  extends in a map  $\Phi$  of  $Q$  into  $\mathbb{C}$  by  $\Phi(Z) = \phi \circ G_\nu^n(Z) - n$  where  $n \in \mathbb{Z}$  is such that  $G_\nu^n(Z) \in \hat{C}(\ell)$ . This map  $\Phi$  is a holomorphic diffeomorphism which verifies:  $\Phi \circ G_\nu = T_1 \circ \Phi$ .

(2) If  $\Phi_{i,\nu}$ ,  $i = 1, 2$ , satisfy (3.7), we have that  $\Phi_{2,\nu} \circ \Phi_{1,\nu}^{-1}(Z + 1) = \Phi_{2,\nu} \circ \Phi_{1,\nu}^{-1}(Z) + 1$ . From which it follows that  $\Phi_{2,\nu} \circ \Phi_{1,\nu}^{-1}$  is a translation.

(3) Suppose now that  $Q_\nu^G$  is a Glutsyuk translation domain. In this case  $Q_\nu^G$  is invariant by the translation  $T_{\alpha(\nu)}$  and admits admissible lines parallel to this direction (and only such lines!). Let  $\ell$  be one of them. Because  $G_\nu$  commutes with  $T_{\alpha(\nu)}$  it is the same for  $h_1$  in the above construction. It follows from this that the Beltrami field  $\mu$  is invariant by the two translations  $T_1$  and  $T_{\alpha(\nu)}$  which are of course non collinear. We repeat now an argument given above: the composition map  $h_2 \circ T_{\alpha(\nu)} \circ h_2^{-1}$  is a translation  $T_\beta$ . Then one has that  $\Phi_\nu \circ T_{\alpha(\nu)} = T_\beta \circ \Phi_\nu$ .

We want to prove that  $\beta = \alpha_{0,\infty}(\nu)$ . Consider the case of a domain  $Q_\nu^{G,0}$  for instance. Let us notice that the map  $\Phi_\nu$  conjugates the pair of commuting diffeomorphisms  $\{G_\nu, T_{\alpha(\nu)}\}$  with the pair of translations  $\{T_1, T_\beta\}$ . The map  $\Phi_\nu$  induces a holomorphic diffeomorphism between the quotient Riemann surfaces. On one side one has a complex torus of modulus equal to  $\beta$ . On the other side we can first take the quotient by the map  $T_{\alpha(\nu)}$  which is the deck transformation of the covering map  $p_\nu$ : we obtain the initial space  $U$  with coordinate  $z$ . Now, the second map  $G_\nu$  induces in this quotient the initial diffeomorphism  $g_\nu$  in a neighborhood  $U_\nu^0$  of the singular point  $P_0$  (because one takes the quotient by  $T_{\alpha(\nu)}$  of a domain  $Q_\nu^{G,0}$ ). In  $U_\nu^0$  the map  $g_\nu$  admits  $P_0$  as a global hyperbolic attracting point. The quotient of  $Q_\nu^{G,0}$  by the pair  $\{G_\nu, T_{\alpha(\nu)}\}$  is conformally equivalent to the quotient of  $U_\nu^0$  by  $g_\nu$ . But it is easy to prove that the quotient of the basin  $U_\nu^0$  of an hyperbolic attracting point  $P$  by the dynamics of  $g_\nu$  is a complex torus of modulus  $\frac{2\pi i}{\mu_0}$ , where  $g'_\nu(P_0) = \mu_0$ . A rough proof of this fact is as follows. One considers an annulus  $A$ , fundamental domain for  $g$  on  $U_\nu^0$ . Then  $U_\nu^0/g_\nu$  is conformally equivalent to  $A/g_\nu = A/\{(z \in \partial A) \sim (g_\nu(z) \in \partial A)\}$ . Of course one can replace  $A$  by any positive iterate  $g_\nu^n(A)$ . But when  $n \rightarrow +\infty$   $g_\nu^n(A)$  converges towards the linear map  $z \mapsto \mu_0 z$  and then the modulus of the quotient complex tori  $g_\nu^n(A)/g_\nu$  converges towards the modulus associated to the linear map, which is equal to  $\frac{2\pi i}{\mu_0}$ . But this quotient  $g_\nu^n(A)/g_\nu$  is conformally equivalent to the fixed Riemann surface  $U_\nu^0/g_\nu$ . Then this surface is a complex torus of modulus equal to  $\frac{2\pi i}{\mu_0}$ . It is exactly what we wanted to prove.  $\square$

**Definition 3.10.** A function  $\Phi_\nu$  constructed in Theorem 3.8 is called a Fatou coordinate associated with the translation domain  $Q_\nu$ . The base point of a Fatou coordinate is the point  $Z_0(\nu) = \Phi_\nu^{-1}(0)$ . In particular for  $\nu \neq 0$  we call the Fatou coordinates on a Lavaurs (resp. Glutsyuk translation domain) the Fatou Lavaurs coordinates (resp. Fatou Glutsyuk coordinates). For  $\nu = 0$ , there is no distinction between Fatou Lavaurs and Fatou Glutsyuk coordinates.

**Lemma 3.11.** If  $Q_\nu \subset \hat{U}_\nu$  is a translation domain of  $G_\nu$ , with Fatou coordinate  $\Phi_\nu$ , then for any  $k \in \mathbb{Z}$ , the domain  $T_{k\alpha(\nu)}Q_\nu$  is a translation domain of  $G_\nu$  with Fatou coordinate  $\Phi_\nu \circ T_{-k\alpha(\nu)}$ .

*Proof.* The claim follows by direct verification using the fact (2.23) that  $G_\nu$  commutes with  $T_{\alpha(\nu)}$ .

### 3.3. Embedding in a flow.

The existence of Fatou coordinates has an obvious, but very important consequence. It implies that the diffeomorphism  $g_\nu$  can be embedded in a flow on certain domains.

Indeed, consider the flow  $\frac{\partial}{\partial W}$  on  $\mathbb{C}$ .

**Proposition 3.12.** *Let  $G_\nu$  be the lift of an unfolding  $g_\nu$  of a parabolic germ. Let  $\Phi_\nu$  be a Fatou coordinate of  $G_\nu$  defined on a translation domain  $Q_\nu$ . Let  $\hat{\xi}_\nu = (\Phi_\nu)_*^{-1} \left( \frac{\partial}{\partial W} \right)$  be the vector field defined on  $Q_\nu$  obtained by transporting the vector field  $\frac{\partial}{\partial W}$  by  $(\Phi_\nu)^{-1}$  and let  $\hat{\xi}_\nu^t(Z)$  for  $t \in \mathbb{C}$  and  $Z \in Q_\nu$ , be its flow.*

- (i) *Then, the vector field  $\hat{\xi}_\nu$  is independent of the choice of the Fatou coordinate  $\Phi_\nu$  used to define it.  $G_\nu$  is the time-one map of the flow of  $\hat{\xi}_\nu$ . Let  $\Phi_\nu$  be the Fatou coordinate such that  $\Phi_\nu(Z_0(\nu)) = 0$  for any  $Z_0(\nu) \in Q_\nu$ . Then, for  $Z \in Q_\nu$ , one has  $W = \Phi_\nu(Z)$  if and only if  $Z = \hat{\xi}_\nu^W(Z_0)$ . In other words, the mapping  $W \rightarrow Z = \hat{\xi}_\nu^W(Z_0(\nu))$  is the inverse mapping of the Fatou coordinate mapping  $Z \rightarrow W = \Phi_\nu(Z)$  which verifies  $\Phi_\nu(Z_0(\nu)) = 0$ .*
- (ii) *Let us suppose now that  $Q_\nu$  is a Glutsyuk Translation domain  $Q_\nu^G$ . Then, a unique vector field  $\xi_\nu = (p_\nu)_*(\hat{\xi}_\nu)$  is well defined on  $p_\nu(Q_\nu^G)$  through transporting the vector field  $\hat{\xi}_\nu$  by  $p_\nu$ . The mapping  $g_\nu$  embeds as the time-one map of the flow of  $\xi_\nu$  on  $p_\nu(Q_\nu^G)$ .*

*Proof.*

- (i) As the vector field  $\frac{\partial}{\partial W}$  is invariant by the translations of  $\mathbb{C}$  and as two Fatou coordinates differ by a translation, the vector field  $\hat{\xi}_\nu$  is independent of the choice of the Fatou coordinate used to define it. As the diffeomorphism  $\Phi_\nu^{-1}$  conjugates the translation  $T_1$  with the mapping  $G_\nu$ , this mapping is the time-one map of the flow of  $\hat{\xi}_\nu$ . The last claim of (i) is just to write that the mapping  $\Phi_\nu$  conjugates the flow of  $\hat{\xi}_\nu$  by  $Z_0 \in Q_\nu$  with the flow of  $\frac{\partial}{\partial W}$  by  $0 \in \mathbb{C}$ .
- (ii) If  $Q_\nu$  is a Glutsyuk translation domain, we have shown in Theorem (3.9) that it conjugates  $T_{\alpha(\nu)}$  with some translation  $T_\beta$  of  $\mathbb{C}$ . As this translation leaves the vector field  $\frac{\partial}{\partial W}$  invariant by conjugacy, one has also that the mapping  $T_{\alpha(\nu)}$  leaves invariant the field  $\hat{\xi}_\nu$ . As  $T_{\alpha(\nu)}$  is the deck transformation of the covering map  $p_\nu$ , there exists a unique vector field  $\xi_\nu$  on  $p_\nu(Q_\nu)$  such that  $\xi_\nu = (p_\nu)_*(\hat{\xi}_\nu)$ . Then,  $p_\nu$  is a conjugacy between the flows of  $\hat{\xi}_\nu$  and  $\xi_\nu$ , which in particular, sends the time-one map  $G_\nu$  of  $\hat{\xi}_\nu$  on the time-one map of  $\xi_\nu$ . Then  $g_\nu$  is the time-one map of  $\xi_\nu$  on the domain  $p_\nu(Q_\nu)$ .

**Definition 3.13.**

- (1) *We call the vector field  $\hat{\xi}_\nu$  associated with a translation domain of  $G_\nu$  the Fatou vector field.*
- (2) *In the case of a Glutsyuk translation domain, we call Glutsyuk vector field the projection of the Fatou vector field by  $p_\nu$ . For a given value of  $\nu \in V_\delta^G(\rho)$  one have two such vector fields :  $\xi_{\nu,0}^G$  on the neighborhood  $p_\nu(Q_\nu^{G,0})$  of  $P_0$  and  $\xi_{\nu,\infty}^G$  on the neighborhood  $p_\nu(Q_\nu^{G,\infty})$  of  $P_\infty$ .*
- (3) *In the case of a Lavaurs translation domain, we call Lavaurs vector field the projection by  $p_\nu$  of the Fatou vector field  $\hat{\xi}_\nu^L$  restricted to a domain  $p_\nu^{-1}(D)$  where  $D$  is a simply connected domain of  $U \setminus \{P_0, P_\infty\}$ .*

**Proposition 3.14.** *All the different Lavaurs vector fields associated to a given Lavaurs translation domain glue together in a unique multivalued vector field  $\xi_\nu^L$ .*

*Remark.* The (ii) of the Proposition 3.11 gives a proof of the linearization theorem of Poincaré. Let us consider for instance the domain  $Q_\nu^{G,0}$ . The mapping  $\Phi_\nu$  induces on the quotient domain  $p_\nu(Q_\nu)$  a diffeomorphism  $\phi_\nu$  with a neighborhood of  $0 \in \mathbb{C}$  which conjugates  $g_\nu$  with the linear mapping  $w \mapsto \exp(\mu_0(\nu))w$ . This can be seen as follows.  $\Phi_\nu$  conjugates the pair of mappings  $(G_\nu, T_{\alpha(\nu)})$  with the pair  $(T_1, T_{\alpha_0(\nu)})$ , where  $\alpha_0(\nu) = \frac{2\pi i}{\mu_0(\nu)}$ . Then :  $E_1 \circ \Phi_\nu = \phi_\nu \circ p_\nu$ , where  $w = E_1(W) = \exp \frac{2\pi i}{\alpha_0(\nu)} W$ , and the translation  $T_1$  induces in the quotient of  $\mathbb{C}$  by  $T_{\alpha_0(\nu)}$  the linear diffeomorphism :  $w \mapsto (\exp \mu_0(\nu))w = \lambda_0 w$ .

### 3.4. Dependence on the parameters of Fatou coordinates and of Fatou vector fields.

**Theorem 3.15.** *Let  $g_\nu$  be a prepared unfolding (2.13) of a parabolic fixed point 0 and  $G_\nu$  the lifted unfolding. Let  $\delta > 0$  and  $r_0, \rho_0$  be given sufficiently small and let  $r < r_0, \rho < \rho_0$  chosen so that the conclusions of Propositions 3.6 and 3.7 hold. Let  $Q_\nu$  be a family of translation domains of the form  $Q_\nu^L$*

or  $Q_\nu^G$  for a given index  $\pm, 0$  or  $\infty$ . This family corresponds to the chosen value  $r$  and is parameterized by  $V_\delta$ . ( $V_\delta$  can be  $V_\delta^L(\rho)$  or  $V_\delta^G(\rho)$ ).

(1) The family  $(Q_\nu)_{\nu \in V_\delta}$  is continuous in the following sense. Let us consider

$$Q = \cup_{\nu \in V_\delta} (\{\nu\} \times Q_\nu) \subset \mathbb{C}^2 \quad (3.15)$$

Then  $Q$  is an open subset of  $V_\delta \times \mathbb{C}$ . Moreover  $\cap_{\nu \in V_\delta} Q_\nu \neq \emptyset$ .

(2) Let  $Z_0(\nu) \in Q_\nu$  depend holomorphically on  $\nu$  (including at  $\nu = 0$ ) and let  $\Phi_\nu$  be the Fatou coordinate defined on  $Q_\nu$  for  $\nu \in V_\delta$  and normalized by  $\Phi_\nu(Z_0(\nu)) = 0$ . An example is given by a constant function  $Z_0(\nu)$  corresponding to a point  $Z_0 \in \cap_{\nu \in V_\delta} Q_\nu$ .

Let  $\Phi : Q \rightarrow \mathbb{C}$  defined by  $\Phi(\nu, Z) = \Phi_\nu(Z)$ . The function  $\Phi$  is holomorphic in  $\text{Int}(Q)$  (i.e. for  $\nu \neq 0$ ), and continuous in  $Q$ .

is holomorphic in  $\text{Int}(Q)$  (i.e. for  $\nu \neq 0$ ), and continuous in  $Q$ .

(3) The Fatou vector field  $\hat{\xi}_\nu(Z)$  is holomorphic in  $(\nu, Z) \in \text{Int}(Q)$  (i.e. for  $\nu \neq 0$ ), and continuous in  $Q$ .

*Proof.*

(1) Let  $\nu_0 \in V_\delta \setminus \{0\}$ . If the family is of Lavaurs type and if  $\ell$  is an admissible line for  $\nu_0$ , this line remains admissible for any  $\nu$  near  $\nu_0$  in  $V_\delta \setminus \{0\}$ . If the family is of Glutsyuk type one can choose a family of admissible lines  $\ell_\nu$  depending continuously on  $\nu$  in some neighborhood of  $\nu_0$  in  $V_\delta \setminus \{0\}$ . In any case, one can suppose that we have a continuous family  $\ell_\nu$  defined for  $\nu$  near  $\nu_0$ . Let us consider now a  $Z_0 \in Q_{\nu_0}$  : there exist  $Z_1 \in \hat{C}_{\nu_0}(\ell)$  and  $n \in \mathbb{Z}$  such that  $G_{\nu_0}^n(Z_1) = Z_0$  and  $G_{\nu_0}^i(Z_0) \in Q_{\nu_0}$  for  $i \in [0, \dots, n]$ . These conditions remain valid, replacing  $\ell_{\nu_0}$  by  $\ell_\nu$ , if  $\nu$  is sufficiently near  $\nu_0$  and  $Z_0(\nu)$  sufficiently near  $Z_0$ . This proves that  $Q$  contains a neighborhood of  $(\nu_0, Z_0)$  in  $V_\delta \times \mathbb{C}$ .

Let us suppose now that  $\nu_0 = 0$ . Let  $Z_0 \in Q_0$  and  $\Gamma$  be a compact neighborhood of  $Z_0$  in  $Q_0$ . If  $|\nu| > 0$  is small enough, the holes different from the principal one lay outside a large open disk centered at 0 and containing  $\Gamma$ . It follows that if  $|\nu| > 0$  is small enough, one can choose a line  $\ell_\nu$  which is admissible for  $\nu_0 = 0$  as well as for  $\nu$ , and then  $\Gamma \subset Q_\nu$ . This proves that  $Q$  contains a neighborhood of  $(0, Z_0)$  in  $V_\delta \times \mathbb{C}$ .

For a Lavaurs family, one can choose a point  $Z_0(\nu) \equiv X \in \mathbb{R}$  ( $X > 0$  for a family  $Q_\nu^{L,+}$  and  $X < 0$  for a family  $Q_\nu^{L,-}$ ). For a Glutsyuk family one can choose a point  $p = iY$  with  $Y \in \mathbb{R}$ .

(2) In the Ahlfors-Bers theorem one can control the dependence of the quasi-conformal mapping in terms of the Beltrami field. First, the space of all Beltrami fields is identified the the open ball  $B_1$  of radius 1 in the complex Banach space  $L^\infty(S^2, \mathbb{C})$  ( $S^2$  is the Riemann sphere with its holomorphic structure). The set of all quasi-conformal mappings is also an open subset of some complex Banach space. Then one has the following [Le]:

*For any  $\mu \in B_1$  there exists a unique quasi-conformal mapping  $\phi_\mu$ , homeomorphism of  $S^2$  onto itself such that  $\mu$  is the field of dilatations of  $\phi_\mu$  and such that  $\phi_\mu$  verifies the normalization conditions :  $\phi_\mu(0) = 0$ ,  $\phi_\mu(\infty) = \infty$ ,  $\phi_\mu(1) = 1$ . Moreover the map  $\mu \rightarrow \phi_\mu$  is holomorphic.*

Let us return now to the proof of Theorem 3.15. Consider to begin with, a value  $\nu_0 \neq 0$ . As said above, one can work in the Lavaurs case with the same line  $\ell$  for all  $\nu \in \Gamma$ , where  $\Gamma$  is a small neighborhood of  $\nu_0$  in  $V_\delta \setminus \{0\}$ . In the Glutsyuk case, one can choose for  $\nu$  in a small neighborhood  $\Gamma$ , the holomorphic family of lines passing through some fixed point and parallel to the direction of the complex number  $\alpha(\nu)$ . In the two cases, the function  $h_1$  depends holomorphically on  $\nu$ . It is the same for the Beltrami field  $\tilde{\mu}$  that one constructs on the Riemann sphere. Then, the quasi-conformal mapping  $\tilde{h}_2$  depends also holomorphically on  $\nu$ , at it follows from the above form of the Ahlfors-Bers Theorem. If one looks at the way  $\Phi_\nu$  is deduced from  $h_1$  and  $\tilde{h}_2$  one gets easily that the map  $\nu \rightarrow \Phi_\nu$  is also holomorphic on  $\Gamma$ . This implies that the 2-variable function  $(Z, \nu) \rightarrow \Phi_\nu(Z)$  is holomorphic separately in each of its variables  $Z$  and  $\nu$  on the domain  $\text{Int}(\Omega)$ . Now, if a function is holomorphic separately in its two variables it is also holomorphic as a function of the two variables. A fine proof of this result was given by Tan Lei in the Appendix of [S].

Of course a consequence of the above form of the Ahlfors-Bers Theorem is that the if the Beltrami field  $\mu(u)$  depends continuously of some parameter  $u$ , then the quasi-conformal mapping

$\Phi_{\mu(u)}$  depends also continuously on  $u$ . As a consequence, if we lift the parameter  $\nu$  to the domain  $\hat{V}_\delta = \{(s, \eta) | se^{i\eta} \in V_\delta\}$ , by the same arguments as above, one obtains that the map  $(s, \eta) \rightarrow \Phi_{se^{i\eta}}$  is continuous in  $(s, \eta)$  in particular at the points of  $\{s = 0\}$ . But, as this map factorizes by  $\nu = se^{i\eta}$  (i.e., its value for  $(s = 0, \eta)$  is independent on  $\eta$ ), this implies that the map  $\nu \rightarrow \Phi_\nu$  is continuous at  $\nu = 0$ . Hence  $\Phi(Z, \nu)$  is continuous as a function of the two variables  $Z$  and  $\nu$  on the whole domain  $\Omega$ .

(3) The dependence of the Fatou field on  $(\nu, Z)$  is a direct consequence of (2) and Proposition 3.12.  $\square$

### 3.5. Admissible pairs of translation domains.

#### Definition 3.16.

- (1) For a given  $\nu \in V_\delta$  a pair of translation domains generated by two parallel admissible lines located on each side of the fundamental hole is called an admissible pair of translation domains. We can have admissible pairs of Lavaurs (resp. Glutsyuk) translation domains.
- (2) The Fatou coordinates associated to an admissible pair of translation domains once chosen a pair of base points, one in each domain, is called an admissible pair of Fatou coordinates.

**Definition 3.17.** The function  $\Phi : Q \rightarrow \mathbb{C} : \Phi(\nu, Z) = \Phi_\nu(Z)$  defined in Theorem 3.15 (2) for a holomorphic choice of base points  $Z_0(\nu)$ , is called a global Fatou coordinate.

We now specialize to Lavaurs translation domains and Glutsyuk translation domains.

#### Proposition 3.18.

- (1) We consider an admissible pair of Lavaurs translation domains  $Q_\nu^{L,\pm}$ . Then  $Q_\nu^{L,+} = T_{\alpha(\nu)}(Q_\nu^{L,-})$ . Moreover  $Q_\nu^{L,+} \cap Q_\nu^{L,-}$  contains open sets  $\hat{D}^0(\nu)$  and  $\hat{D}^\infty(\nu)$  whose quotient by  $G_\nu$  gives full neighborhoods of the two end points  $P^0$  and  $P^\infty$  in  $Q_\nu^{L,\pm}/G_\nu$ .
- (2) The two Fatou vector fields  $\hat{\xi}_\nu^\pm$  associated to the two translation domains are related by  $\hat{\xi}_\nu^+ = (T_{\alpha(\nu)})_* \hat{\xi}_\nu^-$ .

*Proof.* The two points are a direct consequence of the unicity of the domains  $Q_\nu^{L,\pm}$  and of the Fatou vector fields.

**Proposition 3.18.** Let  $\delta > 0$  and let an admissible pair of Glutsyuk translation domains  $Q_\nu^{G,0,\infty}$  be given, for  $\nu \in V_\delta^G$ . Then  $Q_\nu^{G,0} \cap Q_\nu^{G,\infty}$  contains an open set  $\hat{D}^a(\nu)$  whose quotient by  $G_\nu$  gives annular regions in  $Q_\nu^{G,0}/G_\nu$  and  $Q_\nu^{G,\infty}/G_\nu$ , if

$$\rho < \frac{\pi\delta}{3}r. \quad (3.16)$$

*Proof.* The proof is geometric (see Figure 7). Fixing the size  $r$  of  $U$  determines the size of the holes which are approximately balls of radius  $1/r$ . The distance between the centers of two consecutive holes is  $\frac{\pi i}{\nu}$ . If  $\nu = |\nu|e^{i\theta}$  then

$$\frac{\pi i}{\nu} = \frac{\pi \sin \theta}{|\nu|} + i \frac{\pi \cos \theta}{|\nu|}. \quad (3.17)$$

In order that the quotient of  $\hat{D}^a(\nu)$  by  $G_\nu$  be an annular region it is necessary that  $Q_\nu^{G,0} \cap Q_\nu^{G,\infty}$  contains a horizontal band. The vertical distance between the top of one hole and the bottom of the next higher hole is  $\frac{\pi \cos \theta}{|\nu|} - \frac{2}{r}$ . From Proposition 2.5 we know that the iterates of a point  $Z$  lie inside a cone  $C(Z)$  centered at the point and limited by lines of slopes  $\pm\delta/3$ . Hence we need to be able to pass a “thickened” line of slope  $\delta/3$  between the two holes. So we need

$$\frac{\delta}{3} < |\cot \theta| - \frac{2|\nu|}{\pi r}. \quad (3.18)$$

Slope  $\delta$

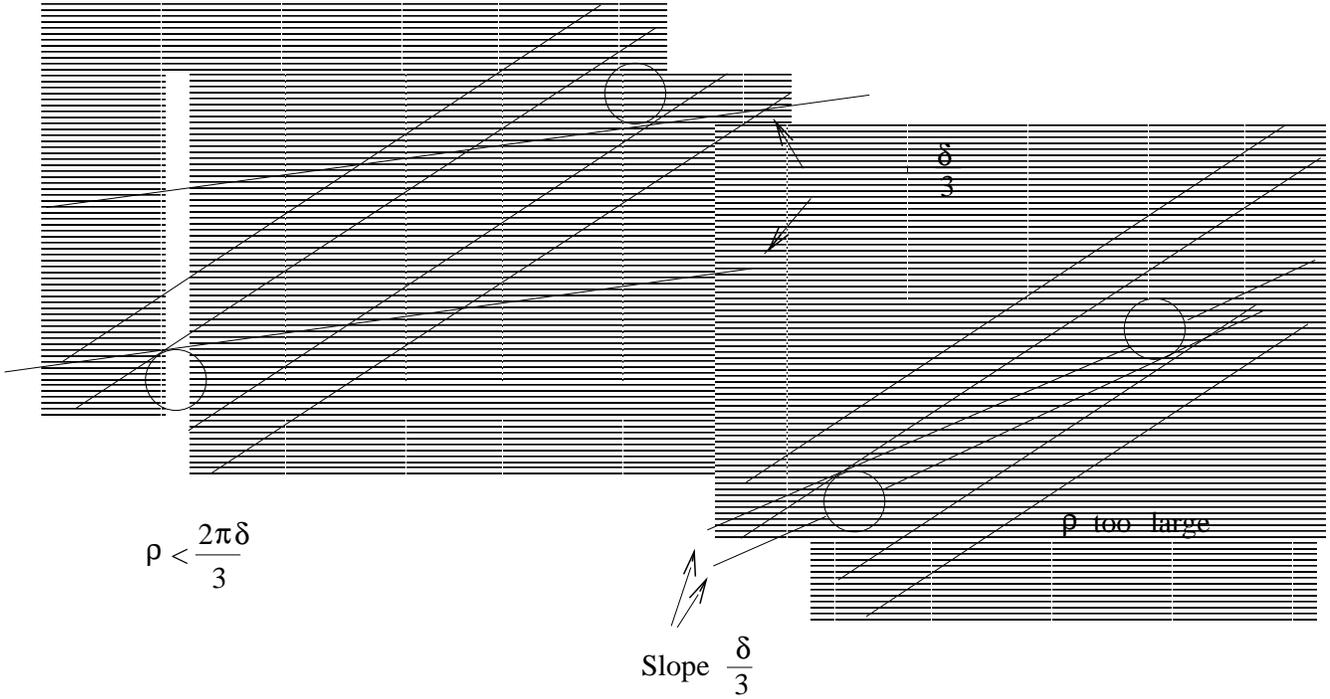


FIGURE 7. The necessity of the condition  $\rho < \frac{\pi\delta}{3}r$  in Proposition 3.18

A lower bound for  $|\cot \theta$  when  $\theta \in V_\delta^L(\rho)$  is  $\cot(\pi/2 - \delta) = \tan(\delta) > \delta$ . Hence we pass a “thickened” line of slope  $\delta/3$  between the two holes as soon as  $\frac{2|\nu|}{\pi r} < \delta - \delta/3 = 2\delta/3$ . This is the case as soon as  $\frac{|\nu|}{r} < \frac{\pi\delta}{3}$ .  $\square$

#### 4. MODULUS OF ANALYTIC CLASSIFICATION IN THE LAVAURS POINT OF VIEW

In this section we present a unified treatment of the dynamics of the family continuously in  $\nu$  for  $\nu \in V_\delta^L$ . We shall not mention everywhere the subscript  $L$ , writing for instance,  $Q_\nu^\pm$  for  $Q_\nu^{L,\pm}$ .

##### 4.1. Complete invariants of analytic classification.

We consider a pair of admissible translation domains  $Q_\nu^\pm$  each with a global Fatou coordinate  $\Phi^\pm$  with base points  $Z_0^\pm(\nu)$ . Let

$$\Psi = \Phi^+ \circ (\Phi^-)^{-1}, \quad (4.1)$$

which yields a family of functions

$$\Psi_\nu = \Phi_\nu^+ \circ (\Phi_\nu^-)^{-1} \quad (4.2)$$

##### Proposition 4.1.

- (1) The map  $\Psi_\nu$  in (4.2) commutes with the translation by one

$$\Psi_\nu \circ T_1 = T_1 \circ \Psi_\nu. \quad (4.3)$$

Hence, it induces a mapping  $\hat{\Psi}_\nu$  defined on an open set  $\Phi_\nu^+(Q_\nu^+ \cap Q_\nu^-)/\mathbb{Z}$  of the cylinder  $\mathbb{C}/\mathbb{Z}$  with values in  $\mathbb{C}/\mathbb{Z}$ .

- (2) Using the exponential function  $W \mapsto w = E(W) = \exp(-2i\pi W)$ , we can identify  $\mathbb{C}/\mathbb{Z}$  with the sphere minus two points :  $\mathbb{CP}^1 \setminus \{0, \infty\}$ . The upper end of the cylinder  $\mathbb{C}/\mathbb{Z}$ , corresponds to

$\infty \in \mathbb{CP}^1$  and the lower end to 0. Conjugating  $\hat{\Psi}_\nu$  with this map yields an analytic map  $\psi_\nu$  defined on the union of a neighborhood of 0 and a neighborhood of  $\infty$  on  $\mathbb{CP}^1$ :

$$\psi_\nu(w) = \exp\left(-2i\pi\hat{\Psi}_\nu\left(-\frac{1}{2i\pi}\log(w)\right)\right), \quad (4.4)$$

where

$$\psi_\nu(0) = 0, \quad \psi_\nu(\infty) = \infty. \quad (4.5)$$

- (3) We call  $\psi_\nu^0$  (resp.  $\psi_\nu^\infty$ ) the restriction of  $\psi_\nu$  to the neighborhood of 0 (resp.  $\infty$ ). The functions  $\psi_\nu^0$  and  $\psi_\nu^\infty$  depend analytically on  $\nu \neq 0$  and are continuous in  $\nu$  at  $\nu = 0$ .

**Definition 4.2.** Let  $\text{Diff}^0$  (resp.  $\text{Diff}^\infty$ ) be the set of germs of diffeomorphisms on  $\mathbb{CP}^1$  fixing 0 (resp.  $\infty$ ). We define an equivalence relation on families of pairs of diffeomorphisms  $(\psi^0, \psi^\infty)$  defined in the neighborhood of 0 and  $\infty$  on  $\mathbb{CP}^1$  by:

$$(\psi_1^0, \psi_1^\infty) \sim (\psi_2^0, \psi_2^\infty) \quad \text{iff} \quad \exists c_1, c_2 \in \mathbb{C}^* \begin{cases} \psi_2^0(w) = c_2\psi_1^0(c_1w) \\ \psi_2^\infty(w) = c_2\psi_1^\infty(c_1w). \end{cases} \quad (4.6)$$

Let

$$\mathcal{M} = \text{Diff}^0 \times \text{Diff}^\infty / \sim \quad (4.7)$$

be the quotient space.

**Corollary 4.3.** For a prepared germ of family  $\mathcal{G} = \{g_\nu\}_{\nu \in V_\delta^L}$  of the form (2.13), we have an application

$$m_{\mathcal{G}} : V_\delta^L \rightarrow \mathcal{M}, \quad \nu \mapsto m_{\mathcal{G}}(\nu), \quad (4.8)$$

where  $m_{\mathcal{G}}(\nu)$  is the equivalence class of  $(\psi_\nu^0, \psi_\nu^\infty)$ . This equivalence class depends just on  $\mathcal{G}$  and not on the choices of the base points  $Z_0^\pm(\nu)$ .

**Theorem 4.4.** We consider two prepared families  $f_{i,\epsilon}$ ,  $i = 1, 2$ , of the form (2.12) and the corresponding  $\mathcal{G}_i = \{g_{i,\nu}\}$ , where  $\nu = \sqrt{\epsilon}$ . We choose a common sector  $V_\delta^L$  with  $\delta$  small on which the previous analysis applies. Then the two families are weakly equivalent if and only if  $m_{\mathcal{G}_1} = m_{\mathcal{G}_2}$ .

**Definition 4.5.** For a prepared family  $\mathcal{G}$  we call the function  $m_{\mathcal{G}}$  in Corollary 4.5 the modulus of analytic classification.

**Theorem 4.6.** To any analytic family  $\mathcal{F}$  which is a generic unfolding of a parabolic point we associate a prepared analytic family  $\mathcal{G}$  and its modulus  $m_{\mathcal{G}}$ , which we call  $m_{\mathcal{F}}$ .

- (1)  $m_{\mathcal{F}}$  is well defined.
- (2) Two analytic families  $\mathcal{F}_i$ ,  $i = 1, 2$ , are weakly equivalent if and only if  $m_{\mathcal{F}_1} = m_{\mathcal{F}_2}$ , i.e.  $m_{\mathcal{F}}$  is a complete invariant of analytic classification.

**Theorem 4.7.** Two generic analytic families  $\mathcal{F}_i = \{f_{\epsilon_i}\}$ ,  $i = 1, 2$ , unfolding a generic parabolic point have the same modulus  $m$  if and only if there exists a weak equivalence  $(h, H_{\epsilon_1})$  between the two families, where

- i)  $h$  is analytic in  $\epsilon_i$ ;
- ii)  $H_{\epsilon_1}$  is holomorphic in  $z$ ;
- iii) let  $\epsilon$  be the parameter of an associated prepared family and  $\nu = \sqrt{\epsilon}$ . The function  $H_{\epsilon_1}$  depends holomorphically on  $\nu \neq 0$  for  $\arg(\nu) \in (-\pi/2 + \delta, 3\pi/2 - \delta)$  and continuously on  $\nu = 0$ .

*Remark 4.8.* As the moduli  $m_G$  or  $m_F$  depend analytically on the parameter  $\nu$  or  $\epsilon$  respectively, we can give stronger versions than in Theorems 4.4, 4.6, for the equivalence relation between moduli and for the weak equivalence of families. Let us consider Theorem 4.4 for instance. A weak equivalence of two families above any subsector  $V$  of the sector  $V_\delta$  of same radius as  $V_\delta$  and smaller opening will imply the equality of moduli above the whole sector  $V_\delta$ . Conversely, it suffices to have the equality of moduli above a subsector  $V$  of  $V_\delta$  to have a weak equivalence of the families over  $V_\delta$ . We can make a similar observation for Theorem 4.6.

The proof of Theorems 4.4 and 4.6 and 4.7 will be done in the rest of Section 4, together with equivalent forms of the modulus.

**Corollary 4.9.** *We consider a family  $g_\nu$  unfolding a singular point as in (2.13). If for  $\nu = 0$  the Ecalle-Voronin modulus  $(\psi^0, \psi^\infty)$  is not trivial i.e.  $\psi^0$  (resp.  $\psi^\infty$ ) is nonlinear, then the corresponding modulus is non trivial, i.e. the map  $\psi_\nu^0$  (resp.  $\psi_\nu^\infty$ ) is also nonlinear for small  $\nu$ .*

In general this property is not sufficient to prove that the dynamics around  $P_0$  or  $P_\infty$  is non trivial. It is however sufficient to prove the non linearizability of  $P_0$  or  $P_\infty$  for some particular resonant values of the multipliers of  $P_0$  or  $P_\infty$  on the unit circle. This will be studied in Section 8.

## 4.2. The Lavaurs phase.

The *Lavaurs map* comes from the comparison of the Fatou coordinates  $\Phi_\nu^+ \circ T_{\alpha(\nu)}$  and  $\Phi_\nu^-$ . These two maps are defined on the same translation domain  $Q_\nu^-$ . Hence, the map  $L_\nu = \Phi_\nu^- \circ (\Phi_\nu^+ \circ T_{\alpha(\nu)})^{-1}$  is an automorphism of  $\mathbb{C}/\mathbb{Z}$ , which by conjugating with  $W \mapsto E(W) = \exp(-2i\pi W)$  gives a holomorphic automorphism of  $\mathbb{C}\mathbb{P}^1$  preserving the origin 0 and the point at infinity. It is hence a linear mapping  $w \mapsto \tau(\nu)w$ ,  $w \in \mathbb{C}\mathbb{P}^1$ .

**Definition 4.10.** *The Lavaurs translation is the map*

$$L_\nu = \Phi_\nu^- \circ (\Phi_\nu^+ \circ T_{\alpha(\nu)})^{-1} = \Phi_\nu^- \circ T_{-\alpha(\nu)} \circ (\Phi_\nu^+)^{-1} = T_{\sigma(\nu)} \quad (4.9)$$

and  $\sigma(\nu)$  is called the *Lavaurs phase*. One has  $\tau(\nu) = E(\sigma(\nu)) \in \mathbb{C}^*$ .

## 4.3. Proof of Theorems 4.4, 4.6 and 4.7.

*Proof of Theorem 4.4.* Let two families be weakly equivalent then they have the same invariant. In Corollary 2.3 we have shown that the equivalence is over the identity, and then that it suffices to compare the two families for each value of  $\nu$ . For such a value of  $\nu$  the two  $g_{i,\nu}$  are equivalent. From an equivalence between the  $g_{i,\nu}$  we can construct an equivalence between the Fatou coordinates, etc, which will yield  $m_{G_1} = m_{G_2}$ . We do not write all details since we will give later a dynamic interpretation of the  $m_{G_i}(\nu)$ .

Conversely, we suppose  $m_{G_1} = m_{G_2}$  and we construct for each  $\nu$  an equivalence between  $g_{1,\nu}$  and  $g_{2,\nu}$ . In all the proof we drop the index  $\nu$  for the subsets  $Q$ , the Fatou coordinates and other functions. We consider two families of the form (2.13) with the same modulus of analytic classification. We can take  $r_0$  and  $\rho_0$  sufficiently small so that the previous analysis can be done simultaneously for the two families on the same neighborhoods. If  $(\Psi_i^0, \Psi_i^\infty)$ ,  $i = 1, 2$ , are representative of the modulus of the  $i$ -th family coming from Fatou coordinates  $\Phi_i^\pm$ , defined on the same translation domains  $Q^\pm$  for  $i = 1, 2$ , we can adjust the choice of coordinates so that  $\Psi_2^{0,\infty} = \Psi_1^{0,\infty}$  and  $L_1 = L_2 = L$ . Let  $Q^\pm$  be the domain of  $\Phi_2^\pm$ . Then the map  $H : Q^+ \cup Q^- \rightarrow \mathbb{C}$

$$Z \mapsto \begin{cases} (\Phi_1^+)^{-1} \circ \Phi_2^+(Z) & \text{if } Z \in Q^+, \\ (\Phi_1^-)^{-1} \circ \Phi_2^-(Z) & Z \in Q^-, \text{ the domain of } \Phi_2^- \end{cases} \quad (4.10)$$

is well defined since  $\Psi_2 = \Psi_1$ . We show that  $H$  commutes with  $T_\alpha$ . It suffices to show that

$$[(\Phi_1^\pm)^{-1} \circ \Phi_2^\pm] \circ T_\alpha = T_\alpha \circ [(\Phi_1^\pm)^{-1} \circ \Phi_2^\pm]. \quad (4.11)$$

From the definition of  $L$  and  $\Psi$  we have that:

$$\Psi_i \circ L_i = \Phi_i^+ \circ T_{-\alpha} \circ (\Phi_i^+)^{-1}, \quad i = 1, 2. \quad (4.12)$$

Then

$$\begin{aligned} [(\Phi_2^+)^{-1} \circ \Phi_1^+] \circ T_{-\alpha} &= (\Phi_2^+)^{-1} \circ \Psi_1 \circ L_1 \circ \Phi_1^+ \\ &= (\Phi_2^+)^{-1} \circ \Psi_2 \circ L_2 \circ \Phi_1^+ \\ &= (\Phi_2^+)^{-1} \circ [\Phi_2^+ \circ T_{-\alpha} \circ (\Phi_2^+)^{-1}] \circ \Phi_1^+ \\ &= T_{-\alpha} \circ [(\Phi_2^+)^{-1} \circ \Phi_1^+]. \end{aligned} \quad (4.13)$$

The other case

$$[(\Phi_2^-)^{-1} \circ \Phi_1^-] \circ T_{-\alpha} = T_{-\alpha} \circ [(\Phi_2^-)^{-1} \circ \Phi_1^-] \quad (4.14)$$

follows similarly by remarking that we also have

$$L_i \circ \Psi_i = \Phi_i^- \circ T_{-\alpha} \circ (\Phi_i^-)^{-1}. \quad (4.15)$$

Then (4.13) and (4.14) imply (4.11).

This allows to extend (4.10) to a map  $H$  defined on the union  $\tilde{Q} = \cup_{k \in \mathbb{Z}} T_\alpha^k(Q^-)$ , where  $\alpha = \alpha(\nu)$ . This domain is invariant by the translation  $T_\alpha$ , and  $H$  is a diffeomorphism sending the lifting of the first family to the lifting of the second family.

The diffeomorphism  $H$  induces an analytic equivalence  $h$  between the two families except at the singular points. Since the equivalence is bounded it can be extended at the singular points  $P_0, P_\infty$ . Finally it is easy to verify that the domain of definition of  $h$ , which is equal to  $p(Q^+) \cup \{P_0, P_\infty\}$ , contains a fixed ball of radius equivalent to  $r$ , centered at the origin of the  $z$ -plane and independent of  $\nu$ .  $\square$

*Proof of Theorem 4.6.*

- (1) Let  $\mathcal{F}$  be an analytic family and  $\mathcal{G}_i$ ,  $i = 1, 2$ , be two prepared families associated to it. Then the  $\mathcal{G}_i$  are equivalent over the identity and by Theorem 4.4 we have  $m_{\mathcal{G}_1} = m_{\mathcal{G}_2}$ .
- (2) Let  $\mathcal{F}_i$ ,  $i = 1, 2$ , be two families. We choose respective prepared families  $\mathcal{G}_i$ ,  $i = 1, 2$ , associated respectively to the  $\mathcal{F}_i$ . Then the  $\mathcal{F}_i$ ,  $i = 1, 2$ , are weakly equivalent if and only if the  $\mathcal{G}_i$ ,  $i = 1, 2$ , are weakly equivalent if and only if  $m_{\mathcal{F}_1} = m_{\mathcal{F}_2}$  by Theorem 4.4 and (1).  $\square$

*Remark.* Given an analytic family  $\mathcal{F}$ , the parameter  $\nu$  of an associated prepared family is canonical.

*Proof of Theorem 4.7.* The passage from a family to a prepared family is analytic in the parameter. So it is enough to work with prepared families. The equivalence between two prepared families is constructed as in the proof of Theorem 4.4 with global Fatou coordinates. Outside the singular points the holomorphic (resp. continuous) dependence in  $\nu$  for  $\nu \neq 0$  (resp.  $\nu = 0$ ) follows from Theorem 3.15 (2). Cauchy integral formula yields the same dependence at the singular points.  $\square$

#### 4.4. The first return maps.

We consider a global Fatou coordinate  $\Phi = (\Phi_\nu)_{\nu \in V_\delta}$ . We take a fundamental domain  $\hat{C}_\nu(\ell) = \hat{C}_\nu \subset Q_\nu$  limited by an admissible line (excluded) depending continuously on  $\nu$  and its image  $G_\nu(\ell)$  (included) and containing the base point  $Z_0(\nu)$ .

The union  $C_\nu = p_\nu(\hat{C}_\nu)$  with the endpoints in  $P_0$  and  $P_\infty$  is a crescent (Figure 8). Let  $z_0(\nu) = p_\nu(Z_0(\nu))$ . The quotient of  $C_\nu$  by  $g_\nu$  is a Riemann sphere  $S_\nu$ . The Fatou coordinate induces a holomorphic isomorphism  $\phi_\nu : S_\nu \rightarrow \mathbb{CP}^1$  sending  $P_0, P_\infty, z_0$  on  $0, \infty, 1$ .

The mapping  $p_\nu$  induces a holomorphic isomorphism between the quotient space  $Q_\nu/G_\nu$  (holomorphically isomorph to the cylinder  $\mathbb{C}^*$ ) with  $S_\nu \setminus \{0, \infty\}$ . It is natural to call  $P^0$  and  $P^\infty$  the corresponding ends of the cylinder  $Q_\nu/G_\nu$ . Then  $p_\nu$  extends into an holomorphic isomorphism (denoted again by  $p_\nu$ ), between the sphere  $\overline{Q_\nu/G_\nu} = (Q_\nu/G_\nu) \cup \{P^0, P^\infty\}$  and  $S_\nu$ , with  $p_\nu(P^0) = 0$  and  $p_\nu(P^\infty) = \infty$ . We shall identify these two surfaces through this isomorphism  $p_\nu$ . This identification defines an holomorphic parametrization on  $S_\nu$ , which is independent of the choice of the strip  $\hat{C}_\nu$ , made to define  $S_\nu$ .

**Proposition 4.11.** *For  $\nu \neq 0$  there exist first return maps  $\kappa_\nu^0$  and  $\kappa_\nu^\infty$  defined in neighborhoods of  $P_0$  and  $P_\infty$  on  $C_\nu$  under iteration of  $g_\nu$  which induce diffeomorphisms  $\hat{\kappa}_\nu^{0,\infty} : S_\nu \rightarrow S_\nu$  in neighborhoods of  $P_0$  and  $P_\infty$  fixing respectively  $P_0$  and  $P_\infty$ . Moreover they are induced by the map  $T_{-\alpha(\nu)}$ : as  $T_{-\alpha(\nu)}$  commutes with  $G_\nu$ , it induces holomorphic diffeomorphisms in neighborhoods of the endpoints  $P^0, P^\infty$  of  $Q_\nu/G_\nu$ , denoted again by  $T_{-\alpha(\nu)}$ , and  $p_\nu \circ T_{-\alpha(\nu)} = \hat{\kappa}_\nu^{0,\infty} \circ p_\nu$ .*

*Proof.* The proof is geometric. A crescent (depending holomorphically on  $\nu \neq 0$ ) as in Figure 8 is obtained as  $p_\nu(\hat{C}_\nu)$  where  $\hat{C}_\nu$  is a strip (depending holomorphically on  $\nu \neq 0$ ) as in Figure 6. Let us define  $K_\nu(Z) = G_\nu^N(z)$  where  $N > 1$  is the smallest integer such that  $G_\nu^N(Z) \in T_{\alpha(\nu)}(\hat{C}_\nu)$ , and  $G_\nu^i(Z) \in Q_\nu$  for  $-N \leq i \leq N$ . The domain of  $K_\nu$  is not connected yielding two maps  $K_\nu^{0,\infty}$  defined at the two ends of  $\hat{C}_\nu$ . The first return map near  $P_{0,\infty}$  are defined as  $q_\nu \circ K_\nu^{0,\infty} \circ p_\nu$ . The induced maps  $\hat{\kappa}_\nu^{0,\infty}$  on  $S_\nu$  are holomorphic and bounded in neighborhoods of 0 and  $\infty$  on  $S_\nu$ . Hence they are also holomorphic at 0 and  $\infty$  respectively.

It remains to prove that  $\hat{\kappa}_\nu^{0,\infty}$  are induced by the map  $T_{-\alpha(\nu)}$ . Let  $z$  be a point in  $C_\nu$  which belongs to the domain of definition of  $\kappa_\nu^{0,\infty}$ . It lifts to a point  $Z \in Q_\nu$  and also to the point  $T_{-\alpha(\nu)}(Z)$ , which belongs also to  $Q_\nu$  by definition. As  $T_{-\alpha(\nu)}(Z)$  commutes with  $G_\nu$ , the point  $K_\nu(T_{-\alpha(\nu)}(Z))$  is sent on the point  $\hat{\kappa}_\nu^{0,\infty}(z)$  by  $p_\nu$ . But the point  $K_\nu(T_{-\alpha(\nu)}(Z))$  is equivalent to the point  $T_{-\alpha(\nu)}(Z)$  in the quotient space  $Q_\nu/G_\nu$  and then  $T_{-\alpha(\nu)}$  induces  $\hat{\kappa}_\nu^{0,\infty}$  on  $S_\nu$ .  $\square$

*Remark.* The above proposition shows that if we lift the diffeomorphisms  $\hat{\kappa}_\nu^{0,\infty}$  on  $\overline{Q_\nu/G_\nu}$ , they do not depend on the choice of the strip  $\hat{C}_\nu$ .

We now consider two crescents  $C_\nu^\pm$  for an admissible pair of global Fatou coordinates  $\Phi^\pm$  and the corresponding spheres  $S_\nu^\pm$  with their parameterization  $\phi_\nu^\pm : S_\nu^\pm \rightarrow \mathbb{CP}^1$ , induced by  $\Phi_\nu^\pm$  through the formula  $\phi_\nu^\pm \circ p_\nu = E \circ \Phi_\nu^\pm$  (where  $E(Z) = \exp(-2\pi i Z)$ .)

**Proposition 4.12.**

- (1) *There exists a global transition  $\hat{l}_\nu : S_\nu^+ \rightarrow S_\nu^-$ , called the Lavaurs mapping. Then*

$$l_\nu = \phi_\nu^- \circ \hat{l}_\nu \circ (\phi_\nu^+)^{-1}, \quad l_\nu : w \mapsto \tau(\nu)w = \exp(-2\pi i \sigma(\nu))w, \quad (4.16)$$

where  $\sigma(\nu)$  is the Lavaurs phase. Its lifting by  $E$  is the Lavaurs translation  $L_\nu$  defined in (4.9).

- (2) *There exist transition functions defined from neighborhoods of  $P_0$  and  $P_\infty$  in  $S_\nu^-$  to neighborhoods of  $P_0$  and  $P_\infty$  in  $S_\nu^+$ . This induces local diffeomorphisms in the neighborhood of 0 (resp.  $\infty$ ) on  $\mathbb{CP}^1$ , which are equal to the modulus functions  $\psi_\nu^0$  (resp.  $\psi_\nu^\infty$ ). We recall that these modulus functions are the restriction of  $\phi_\nu^+ \circ (\phi_\nu^-)^{-1}$ .*
- (3) *The return maps  $\hat{\kappa}_\nu^{0,\infty}$  from  $S_\nu^-$  to  $S_\nu^-$  induce maps*

$$k_\nu^{0,\infty} = \phi_\nu^- \circ \hat{\kappa}_\nu^{0,\infty} \circ (\phi_\nu^-)^{-1} \quad (4.17)$$

in the neighborhoods of 0 and  $\infty$  on  $\mathbb{CP}^1$ . Then

$$k_\nu^{0,\infty} = l_\nu \circ \psi_\nu^{0,\infty}. \quad (4.18)$$

The applications  $k_\nu^{0,\infty}$  are the renormalized return maps. The functions  $l_\nu, \psi_\nu^{0,\infty}$  and  $k_\nu^{0,\infty}$  depend holomorphically on  $\nu \in V_\delta^L \setminus \{0\}$ .

*Proof.*

- (1) The crescents  $C_\nu^+$  and  $C_\nu^-$  lift into the two strips  $T_{-\alpha(\nu)}(\hat{C}_\nu^+)$  and  $\hat{C}_\nu^+$  respectively. These two strips are admissible in the same translation domain  $Q_\nu^-$ . Then a global transition is defined between them. This transition projects into a global transition  $\hat{l}_\nu : S_\nu^+ \rightarrow S_\nu^-$ . As in Proposition 4.11, it is easy to prove that the transition between  $T_{-\alpha(\nu)}(\hat{C}_\nu^+)$  and  $\hat{C}_\nu^-$  induces the map  $T_{-\alpha(\nu)}$  between  $Q_\nu^+/G_\nu$

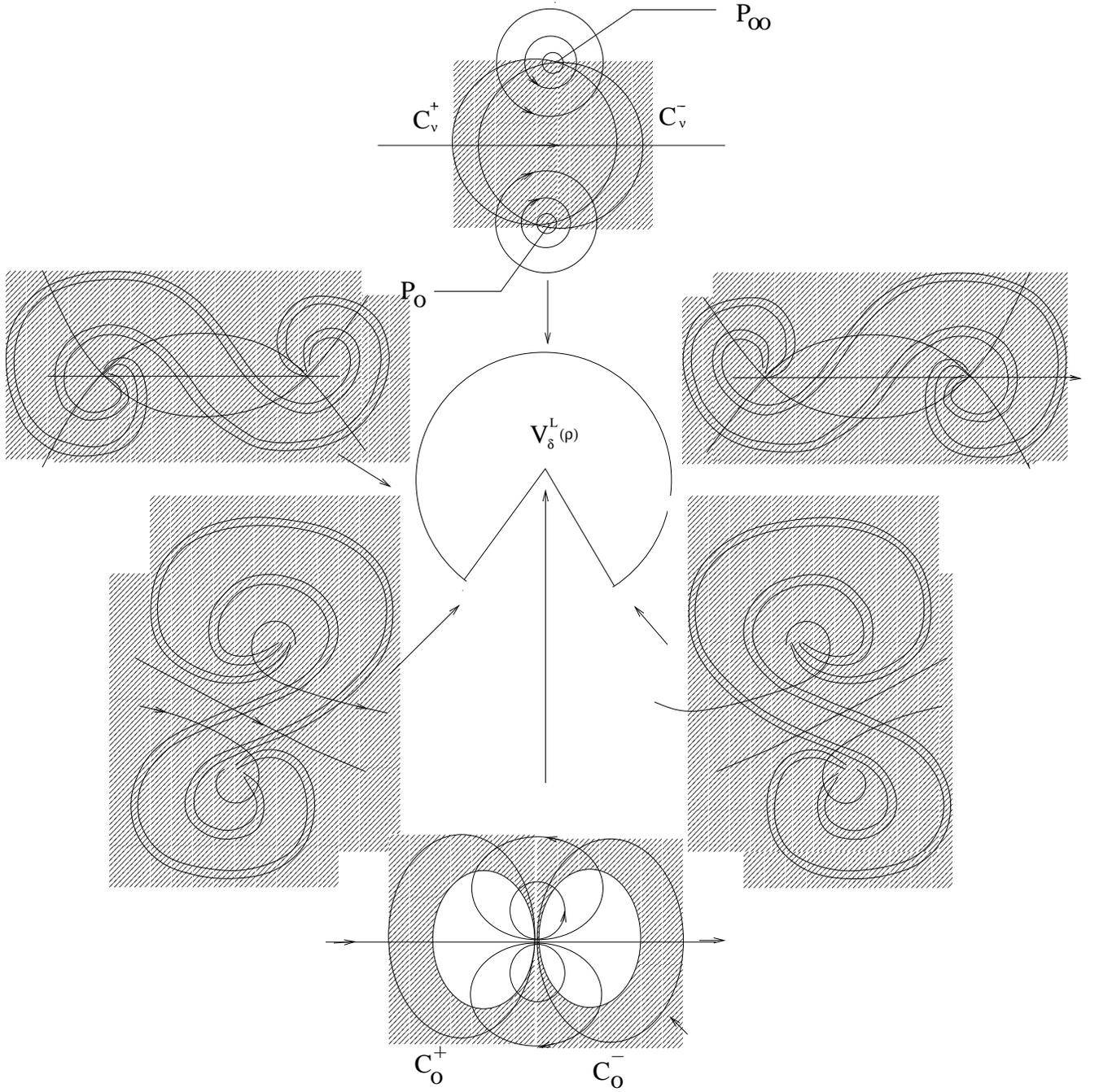


FIGURE 8. The crescents  $C_\nu^\pm$  for  $\nu \in V_\delta^L(\rho)$

- and  $Q_\nu^-/G_\nu$ , which is precisely the map  $L_\nu = \Phi_\nu^- \circ T_{-\alpha(\nu)} \circ (\Phi_\nu^+)^{-1}$  in the parametrizations  $\Phi_\nu^+$  and  $\Phi_\nu^-$  of  $Q_\nu^+$  and  $Q_\nu^-$  respectively. Projecting through  $p_\nu$  and  $E$ , it gives (4.16).
- (2) A transition between  $C_\nu^-$  and  $C_\nu^+$  is defined in neighborhoods of the ends of the strip  $C_\nu^-$ , corresponding to the  $G_\nu$ -orbits of points in  $Q_\nu^- \cap Q_\nu^+$ . This transition induces the map  $\Psi_\nu = \Phi_\nu^+ \circ (\Phi_\nu^-)^{-1}$  between  $Q_\nu^-/G_\nu$  and  $Q_\nu^+/G_\nu$  in the parametrizations  $\Phi_\nu^-$  and  $\Phi_\nu^+$  respectively. This map induces through  $E$  the map  $\psi_\nu = \phi_\nu^+ \circ (\phi_\nu^-)^{-1}$ .
  - (3) From Proposition 4.10 we know that  $k_\nu^{0,\infty}$  lift to the restrictions of the map  $\Phi_\nu^- \circ T_{-\alpha(\nu)} \circ (\Phi_\nu^-)^{-1}$  in neighborhoods of  $0, \infty \in \mathbb{CP}^1$ . On the other hand,  $l_\nu$  lifts into  $L_\nu$  and  $\phi_\nu^{0,\infty}$  lift as restrictions of  $\Psi_\nu = \Phi_\nu^+ \circ (\Phi_\nu^-)^{-1}$ . We soon verify that  $\Phi_\nu^- \circ T_{-\alpha(\nu)} \circ (\Phi_\nu^-)^{-1} = L_\nu \circ \Psi_\nu$ , which induces (4.18). As  $\Phi_\nu^\pm$  and  $T_{-\alpha(\nu)}$  depend holomorphically on  $\nu$ , it is the same for  $L_\nu$ , and  $\Psi_\nu$ . Now, as these functions project onto  $l_\nu$  and  $\phi_\nu^{0,\infty}$  through  $E$ , these last functions depend also holomorphically on  $\nu \neq 0$  as

well as their composition  $k_\nu^{0,\infty}$ .

□

We thank Christian Bonatti for the proof of the following proposition.

**Proposition 4.13.** *The pair  $(k_\nu^0, k_\nu^\infty)$  defined in (4.17) is a representative of the modulus.*

*The first derivatives  $(k_\nu^0)'(0)$  and  $(k_\nu^\infty)'(\infty)$  are analytic invariants. They have the following expression :*

$$(k_\nu^\infty)'(\infty) = e^{4\pi^2/\mu_\infty(\nu)}, \quad (k_\nu^0)'(0) = e^{4\pi^2/\mu_0(\nu)}. \quad (4.20)$$

*Proof.* As the map  $l_\nu$  is a global holomorphic diffeomorphism of  $\mathbb{CP}^1$ , sending 0 (resp.  $\infty$ ) to 0 (resp.  $\infty$ ) it is a linear map. Then as a consequence of (4.18), the pair  $(k_\nu^0, k_\nu^\infty)$  is equivalent to the pair  $(\psi_\nu^0, \psi_\nu^\infty)$ , in the sense of definition (4.6), and is a representative of the modulus.

As proved in Corollary 2.3 the parameter  $\nu$  is not changed by an analytic equivalence. Then to prove that  $(k_\nu^0)'(0)$  and  $(k_\nu^\infty)'(\infty)$  are analytic invariants, it suffices to prove the formulas (4.20). We have seen in Proposition 4.12 that  $(k_\nu^0, k_\nu^\infty)$  depend holomorphically on  $\nu$ . Then, it suffices to prove (4.20) on some non empty open subset of  $V_\delta^L$ , let us say for  $\nu$  such that  $\mu_\nu^0$  and  $\mu_\nu^\infty \notin i\mathbb{R}$  (i.e., for values of  $\nu$  where the two fixed points  $P_0$  and  $P_\infty$  are hyperbolic). Also, we shall just consider the point  $P_0$  for instance, the other case being completely similar. As it is hyperbolic at  $P_0$ , the diffeomorphism  $g_\nu$  is linearizable in a neighborhood  $U$  of  $P_0$ . Then there exists a holomorphic universal covering map  $\pi : \Omega \rightarrow U \setminus \{P_0\}$ , where  $\Omega = \{Z = X + iY | Y < Y_0\}$ , for some  $Y_0$ , whose covering transformation is the translation  $T_1$  and such that  $g_\nu$  lifts as the translation  $T_{\mu_0(\nu)/2\pi i}$ . We have seen above for the universal covering map  $p_\nu$ , that the first return  $k_\nu^0$  is holomorphically conjugate to the map induced by the inverse of the covering transformation in the quotient space  $Q/G_\nu$ . This is true for any universal covering map. Then,  $k_\nu^0$  is holomorphically conjugate to the map induced by  $T_{-1}$  in the quotient space  $\Omega/T_{\mu_0(\nu)/2\pi i}$ . The pair of translations  $(T_{-1}, T_{\mu_0(\nu)/2\pi i})$  is linearly conjugate to the pair  $(T_{-2\pi i/\mu_0(\nu)}, T_1)$ . This implies that  $k_\nu^0$  is holomorphically conjugate to the linear map  $w \rightarrow e^{4\pi^2/\mu_0(\nu)}w$  and then that  $(k_\nu^0)'(0) = e^{4\pi^2/\mu_0(\nu)}$ . □

*Remark.*

- (1) The presentation of the modulus through the pair  $(k_\nu^0, k_\nu^\infty)$  has no limit at  $\nu = 0$ , while there exist continuous representatives for  $\psi_\nu^0, \psi_\nu^\infty$ . On the other hand one translation domain and one Fatou coordinate is enough.
- (2) The presentation of the modulus of analytic classification as the equivalence class of pairs  $(k_\nu^0, k_\nu^\infty)$  allows to study the dynamics of the singular points  $P_0$  and  $P_\infty$  when their multipliers are on the unit circle, i.e. the map  $g_\nu$  may not be linearizable in their neighborhood [S].

## 4.5 Interpretation of the modulus in terms of Lavaurs vector fields.

**Proposition 4.14.** *We consider a pair of admissible Lavaurs translation domains, each with a global Fatou coordinate  $\Phi^\pm = (\Phi_\nu^\pm)$  with base points  $Z_0^\pm(\nu)$  such that  $z_0^\pm(\nu) = p_\nu(Z_0^\pm(\nu))$ . Let  $\xi_\nu^L$  be the unique multivaluated Lavaurs vector field described in Proposition 3.14. Then*

- (1) *The Lavaurs phase  $\sigma(\nu)$  is characterized by  $(\xi_\nu^L)^{\sigma(\nu)}(z_0^+(\nu)) = z_0^-(\nu)$ , where  $(\xi_\nu^L)^W(z)$  is the flow of  $\xi_\nu^L$  (This flow is multivaluated and we have to choose a determination of it).*
- (2) *Considering a crescent  $C_\nu$  we have two determinations of the Lavaurs vector field:  $\xi_\nu^L$  and its analytic extensions  $\xi_\nu^{L,0}$  (resp.  $\xi_\nu^{L,\infty}$ ) obtained in the neighborhoods of  $P_0$  (resp.  $P_\infty$ ) after the first return. This yields respective vector fields on  $S_\nu$  or domains of  $S_\nu$ :  $\tilde{\xi}_\nu^L, \tilde{\xi}_\nu^{L,0}$  and  $\tilde{\xi}_\nu^{L,\infty}$ . Then*

$$\begin{aligned} (\hat{k}_\nu^0)_*(\tilde{\xi}_\nu^L) &= \tilde{\xi}_\nu^{L,0} \\ (\hat{k}_\nu^\infty)_*(\tilde{\xi}_\nu^L) &= \tilde{\xi}_\nu^{L,\infty} \end{aligned} \quad (4.21)$$

*Proof.*

- (1) Let us consider the two points  $T_{-\alpha(\nu)}(Z_0^+(\nu))$  and  $Z_0^-(\nu)$  in the translation domain  $Q_\nu^-$ . The time needed to pass from the first one to the second one using the flow of the Lavaurs vector field  $\hat{\xi}_\nu^-$  on  $Q_\nu^-$ , is equal to  $\sigma(\nu)$ , as it follows from its definition (4.9). By projection by  $p_\nu$  we obtain the desired formula  $(\xi_\nu^L)^{\sigma(\nu)}(z_0^+(\nu)) = z_0^-(\nu)$ .
- (2) Two consecutive determinations of the Lavaurs vector fields differ by a the time-map  $(\xi_\nu^L)^\tau$ , where  $\tau$  is the time to pass from  $T_{-\alpha(\nu)}(Z_0^+(\nu))$  and  $Z_0^+(\nu)$  by the flow of  $\hat{\xi}_\nu^-$ . As the map  $T_{-\alpha(\nu)}$  induces the diffeomorphisms  $\hat{k}_\nu^{0,\infty}$  on  $S_\nu$ , near the points  $P_{0,\infty}$ , we obtain the formulas (4.21).

*Remark.* Let  $g_{i,\nu}$  be two prepared families. We construct a (multivalued) conjugation between the two Lavaurs vector fields. Then sending the base point  $z_{0,1}(\nu)$  to  $z_{0,2}(\nu)$  this yields a (multivalued) conjugation  $H_\nu$  between the families. The conjugation is univalued if and only if the two families have the same invariant  $(\psi_\nu^0, \psi_\nu^\infty)$ .

## 5. MODULUS OF ANALYTIC CLASSIFICATION IN THE GLUTSYUK POINT OF VIEW

In this section we limit ourselves to  $\nu \in V_\delta^G$ . We have an admissible pair of Glutsyuk translation domains  $Q_\nu^{G,0,\infty}$ , together with its admissible pair of Fatou coordinates  $\Phi_\nu^{G,0,\infty}$ , both depending continuously on  $\nu$ .

**Definition 5.1.**

- (1) Let

$$U_\nu^{0,\infty} = p_\nu(Q_\nu^{G,0,\infty}). \quad (5.1)$$

We call these domains *Glutsyuk normalization domains*. They are the domains of definition of the Glutsyuk vector fields  $\xi_\nu^{G,0,\infty}$ .

- (2) The quotient of  $U_\nu^{0,\infty}$  by  $g_\nu$  yields a torus  $\mathfrak{T}^0$  (resp.  $\mathfrak{T}^\infty$ ) of modulus  $\alpha_0(\nu)$  (resp.  $\alpha_\infty(\nu)$ ).
- (3) For  $\nu \neq 0$  the Fatou coordinate induces a holomorphic isomorphism  $\phi_\nu^{0,\infty} : \mathfrak{T}_\nu^{0,\infty} \rightarrow \mathbb{T}_\nu^{0,\infty}$ , where  $\mathbb{T}_\nu^{0,\infty}$  are the canonical tori  $\mathbb{C}^2/(\mathbb{Z} \times \alpha_{0,\infty} \mathbb{Z})$ . When  $\nu \rightarrow 0$   $\alpha_{0,\infty} \rightarrow \infty$  and the tori tend to cylinders.

**Proposition 5.2.** We consider an admissible pair of Fatou coordinates  $(\Phi_\nu^0, \Phi_\nu^\infty)$  and the associated map

$$\Psi_\nu^G = \Phi_\nu^0 \circ (\Phi_\nu^\infty)^{-1}. \quad (5.2)$$

- (1)  $\Psi_\nu^G$  commutes with  $T_1$ . Hence it passes to the quotient, yielding a map  $\hat{\Psi}_\nu^G$  defined on an open subset  $\Phi_\nu^0(Q_\nu^0 \cap Q_\nu^\infty)/\mathbb{Z}$  of the cylinder  $\mathbb{C}/\mathbb{Z}$  with values in the cylinder.
- (2) For  $\nu \neq 0$  the domain of  $\hat{\Psi}_\nu^G$  on  $\mathbb{C}/\mathbb{Z}$  contains a countable union of annuli corresponding to horizontal strips in  $Q_\nu^0$  and  $Q_\nu^\infty$ . Then the map  $\Psi_\nu^G$  verifies

$$\Psi_\nu^G \circ T_{\alpha_\infty(\nu)} = T_{\alpha_0(\nu)} \circ \Psi_\nu^G. \quad (5.3)$$

- (3) This induces a holomorphic diffeomorphism

$$\psi_\nu^G : A_\nu^\infty \rightarrow A_\nu^0 \quad (5.4)$$

between two annuli located respectively on  $\mathbb{T}_\nu^\infty$  and  $\mathbb{T}_\nu^0$  for  $\nu \neq 0$

- (4) In the limit case  $\nu = 0$ ,  $\psi_0^G$  is defined on the two ends of the cylinder. Conjugated with the exponential function this is the pair  $(\psi_0^0, \psi_0^\infty)$ .

The following theorem proved by Glutsyuk has a simple proof as in Theorem 4.7, using (5.3).

**Theorem 5.3** [G].

- (1) We define an equivalence relation on families  $\Psi_\nu^G$  by the action of the group of translations in the source and target space. The equivalence class of the family  $\Psi_\nu^G$  is an analytic invariant for the weak equivalence on  $V_\delta^G$ . It depends analytically on  $\nu \neq 0$ . In the limit the map  $\Psi_\nu^G$  tends to  $\Psi_0$  which is equivalent to the Ecalle-Voronin modulus.
- (2) Another interpretation of the same invariant is by means of the comparison of the times of the Glutsyuk vector fields:

$$(\xi_{\nu,0}^G)^W(z_0(\nu)) = (\xi_{\nu,\infty}^G)^{\Psi_\nu^G(W)}(z_\infty(\nu)). \quad (5.5)$$

**Definition 5.4.** We call Glutsyuk invariant the family of equivalence classes of  $\psi_\nu^G$  with respect to composition with translations in the source and target space.

*Remark.* Another presentation of the same invariant is by means of the diffeomorphism  $\tilde{\psi}_\nu^G$  which compares the two Glutsyuk vector fields  $\xi_{\nu,0}^G, \xi_{\nu,\infty}^G$  on the intersection of their domains. This is also equivalent to the diffeomorphism  $\psi_\nu^G$ .

**Proposition 5.5.** We consider a family  $g_\nu$  unfolding a singular point as in (2.13). If for  $\nu = 0$  the Ecalle-Voronin modulus  $(\psi^0, \psi^\infty)$  is not trivial i.e. one of the maps  $\psi^0$  or  $\psi^\infty$  is nonlinear then the corresponding unfolded map  $\Psi_\nu^G$  is not a translation for small  $\nu$ .

*Proof.* If for  $\nu = 0$  the Ecalle-Voronin modulus  $(\psi^0, \psi^\infty)$  is not trivial then one of the  $\Psi_0^{0,\infty}$  is not a translation, yielding that by continuity that  $\Psi_\nu^G$  is not a translation for small  $\nu$ .  $\square$

Here we show directly how the Fourier coefficients of the Glutsyuk modulus  $\Psi_\nu^G$  are linked to the coefficients of  $(\Psi_0^0, \Psi_0^\infty)$ , for  $\nu \in V_\delta^G$ .

We take Fatou coordinates depending continuously on  $\nu$  for  $\nu \in V_\delta^G$ . The domain of  $\Psi_\nu^G$  contains a union of two horizontal strips  $A_\nu^\infty$  (resp  $A_\nu^0$ ), located just above (resp. below)  $B_\nu$ . As the functions  $\Psi_\nu^G$  satisfy  $\Psi_\nu^G(Z+1) = \Psi_\nu^G(Z) + 1$  we can write  $\Psi_\nu^G - Z$  in  $A_\nu^\infty$  as a Fourier series:

$$(\Psi_\nu^G(Z) - Z)|_{A_\nu^\infty} = \sum_{n \in \mathbb{Z}} c_n(\nu) \exp(2i\pi n Z) \quad (5.6)$$

If  $I^\infty$  is any horizontal segment of length 1 in  $A_\nu^\infty$ , then for  $\nu \neq 0$

$$c_n(\nu) = \int_{I^\infty} (\Psi_\nu^G(Z) - Z) \exp(-2i\pi n Z) dZ. \quad (5.7)$$

This yields a Fourier series for  $\Psi_\nu^G(Z) - Z$  in  $A_\nu^0$  using (5.3). We can write:

$$(\Psi_\nu^G(Z) - Z)|_{A_\nu^0} = \sum_{n \in \mathbb{Z}} d_n(\nu) \exp(2i\pi n Z) \quad (5.8)$$

where

$$d_n(\nu) = \begin{cases} c_0(\nu) - \alpha^\infty(\nu) - \alpha^0(\nu) & n = 0 \\ c_n(\nu) \exp(2\pi i n \alpha^0(\nu)) & n \neq 0 \end{cases} \quad (5.9)$$

We also have for  $\nu \neq 0$

$$d_n(\nu) = \int_{I^0} (\Psi_\nu^G(Z) - Z) \exp(-2i\pi n Z) dZ. \quad (5.10)$$

where  $I^0$  is any horizontal segment of length 1 in  $A_\nu^0$ . The idea is to take segments  $I^{0,\infty}$  which pass to the limit. For that we consider the case  $\nu = 0$ . Let  $\Psi_0^0$  and  $\Psi_0^\infty$  be the liftings of  $\psi^0$  and  $\psi^\infty$ . They are defined on disjoint domains. They contain horizontal strips  $A_0^0$  and  $A_0^\infty$  and we have [I]

$$\begin{aligned} (\Psi_0^\infty(Z) - Z)|_{A_0^\infty} &= c_0(0) + \sum_{n \in \mathbb{N}} c_{-n}(0) \exp(-2i\pi nZ) \\ (\Psi_0^0(Z) - Z)|_{A_0^0} &= d_0(0) + \sum_{n \in \mathbb{N}} d_n(0) \exp(2i\pi nZ). \end{aligned} \quad (5.11)$$

It is possible to choose the segments  $I_0$  (resp.  $I_\infty$ ) of length one belonging respectively to all  $A_\nu^0$  (resp.  $A_\nu^\infty$ ) for all  $\nu \in V_\delta^G$ . Then we have

$$\begin{aligned} c_n(0) &= \int_{I_\infty} (\Psi_0^\infty(Z) - Z) \exp(-2i\pi nZ) dZ \\ d_{-n} &= \int_{I_0} (\Psi_0^0(Z) - Z) \exp(2i\pi nZ) dZ. \end{aligned} \quad (5.12)$$

Hence the  $c_n$  and  $d_{-n}$  depend continuously in  $\nu$ .

Note that the hypothesis that the map  $\psi^\infty$  (resp.  $\psi^0$ ) is nonlinear is equivalent to  $c_n(0) \neq 0$  (resp.  $d_{-n} \neq 0$ ) for some  $n \neq 0$ . This implies  $c_n(\nu) \neq 0$  (resp.  $d_{-n} \neq 0$ ) for small  $\nu$ . Moreover, from (5.9)  $d_{-n}(\nu) \neq 0$  implies  $c_{-n}(\nu) \neq 0$ . Hence  $\Psi_\nu^G$  is not a translation.

**Corollary 5.6.** *With the notations above we have*

$$d_0(\nu) - c_0(\nu) = \begin{cases} -2\pi ia(0) & \nu = 0 \\ -\alpha^\infty(\nu) - \alpha^0(\nu) = -2\pi ia(\nu) & \nu \neq 0. \end{cases} \quad (5.13)$$

*In particular this quantity is an analytic invariant of the system. It measures the shift between the two singular points. As it is determined by  $\Psi_\nu^P$  it follows that, from the knowledge of  $\Psi_\nu^P$  and the multiplier of one fixed point, we recover the multiplier of the other fixed point.*

## 6. COMPARING THE LAVAURS AND GLUTSYUK POINTS OF VIEW

We limit ourselves to values of  $\nu \in V_\delta^G$ . We describe the passage from the modulus in the Lavaurs point of view to the modulus of the Glutsyuk point of view for a fixed value of  $\nu$ . In practice we can compute the Lavaurs invariant in terms of the Glutsyuk vector fields.

Considering a crescent  $C_\nu$ , we have introduced the sphere  $S_\nu = C_\nu/g_\nu = Q_\nu/G_\nu$ , which we call the Lavaurs sphere. The first return maps on  $C_\nu$  induces a diffeomorphism  $\hat{k}_\nu^0$  (resp.  $\hat{k}_\nu^\infty$ ) in the neighborhood of  $P_0$  (resp.  $P_\infty$ ) on  $S_\nu$ . As it is proved in Proposition 4.11, the diffeomorphism  $\hat{k}_\nu^0$  (resp.  $\hat{k}_\nu^\infty$ ) of  $Q_\nu/G_\nu$  is induced by  $T_{-\alpha(\nu)}$ . We recall that the diffeomorphisms  $\hat{k}_\nu^{0,\infty}$  are conjugate to the diffeomorphisms  $k_\nu^{0,\infty}$  defined in neighborhoods of  $0, \infty$  on  $\mathbb{CP}^1$ , with eigenvalues at these fixed points given by the formulas (4.20). Then we have also that  $(\hat{k}_\nu^{0,\infty})'(P_{0,\infty}) = e^{4\pi^2/\mu_{0,\infty}(\nu)} = e^{2\pi i \gamma_{\nu,0,\infty}}$ , with

$$\gamma_{\nu,0,\infty} = -2\pi i / \mu_{0,\infty}(\nu) \quad (6.1)$$

**Definition 6.1.** *As the Glutsyuk vector field  $\xi_{\nu,0,\infty}^G$  commutes with  $g_\nu$  it induces vector fields  $\tilde{\xi}_{\nu,0,\infty}^G$  on the neighborhoods  $\tilde{U}_{\nu,0,\infty} = C_\nu \cap U_{\nu,0,\infty}/g_\nu$  of  $P_0$  (resp.  $P_\infty$ ) on the Lavaurs sphere  $S_\nu$ .*

**Theorem 6.2.** *The first return diffeomorphisms  $\hat{k}_\nu^{0,\infty}$  are time-diffeomorphisms of the flow  $(\tilde{\xi}_{\nu,0,\infty}^G)^W$  of the vector fields induced by the Glutsyuk vector fields :*

$$\begin{aligned}\hat{k}_\nu^0 &= (\tilde{\xi}_{\nu,0}^G)^{\gamma_{\nu,0}} \\ \hat{k}_\nu^\infty &= (\tilde{\xi}_{\nu,\infty}^G)^{\gamma_{\nu,\infty}}.\end{aligned}\tag{6.2}$$

*Proof.*  $G_\nu$  and  $T_{\alpha(\nu)}$  are two diffeomorphisms of the flow of  $\hat{\xi}_{\nu,0}^G$  for two different times. Passing to the quotient on  $S_\nu$  we get that the first return map is the diffeomorphism of the flow  $\tilde{\xi}_{\nu,0}^G$  for some time  $t$ . As  $\tilde{\xi}_{\nu,0}^G$  is of period 1, this time is  $t = \gamma_{\nu,0}$ . Similarly near  $P_\infty$ .  $\square$

*Remark.*

The invariants of analytic classification in the Lavaurs and in the Glutsyuk point of view are the two sides of the same phenomenon namely an obstruction to embed the family (2.13) into the model (2.14). In the Glutsyuk point of view we use the fact that the two singular points are linearizable for  $\nu \in V_\delta^P$ . Hence we embed the family (2.13) into the model family in the neighborhoods of  $P_0$  and  $P_\infty$  and we read the obstruction in the intersection of the two neighborhoods. In the Lavaurs point of view, we embed the family (2.13) into the model in a crescent-like region between the two singular points  $P_0$  and  $P_\infty$ . We take the analytic extension of the embedding. This extension is ramified at  $P_0$  and  $P_\infty$  and the ramification, given by the diffeomorphisms  $\hat{k}_\nu^{0,\infty}$ , is the obstruction to embed the full family into the model.

## 7. GLUEING LAVAURS AND GLUTSYUK POINTS OF VIEW ALONG A SEGMENT THROUGH THE ORIGIN IN $\epsilon$ SPACE

In this section we return to the original parameter  $\epsilon$  and we show that the Lavaurs and the Glutsyuk point of view glue together naturally, for  $\epsilon$  varying in a segment.

Let  $F_\epsilon$  be the lifting of the unfolding  $f_\epsilon$  of a parabolic germ. Given  $\eta \in (-\pi + 2\delta, \pi - 2\delta)$ , introduce the segments  $I(\eta) = \{0\} \cup \{\epsilon \mid \sqrt{\epsilon} \in V_\delta, \arg(\pm\epsilon) = \eta\}$ ,  $I^P(\eta) = \{\epsilon \in I(\eta) : \arg \epsilon = \eta\}$ ,  $I^L(\eta) = \{\epsilon \in I(\eta) : \arg \epsilon = \eta + \pi\}$ .

*Remark.* Note that studying the family  $F_\epsilon$ , for  $\epsilon$  varying in  $I(\eta) = I^P(\eta) \cup I^L(\eta) \cup \{0\}$  corresponds to studying the family  $G_\nu$ , for  $\nu$  varying in the union of two segments,  $\sqrt{I^P(\eta)}$  and  $\sqrt{I^L(\eta)}$ , forming a straight angle at the origin.

**Lemma 7.2.** *There exist lines  $\ell_\epsilon^{0,\infty}$  in  $\mathbb{C}$  directed by  $\frac{i}{\sqrt{\epsilon}}$ , which are admissible lines for  $F_\epsilon$ ,  $\epsilon \in I(\eta)$ .*

- (1) *They are admissible lines of Glutsyuk type (associated to  $P_0$  and  $P_\infty$ ), for  $\epsilon \in I^P(\eta)$  and are admissible lines of Lavaurs type located on opposite sides of the fundamental hole, for  $\epsilon \in I^L(\eta)$ .*
- (2) *Let  $\hat{C}_\epsilon^{0,\infty}$  be the fundamental domains of  $F_\epsilon$ ,  $\epsilon \in I(\eta)$  limited by the line  $\ell_\epsilon^{0,\infty}$  and  $F_\epsilon(\ell_\epsilon^{0,\infty})$ . There exists a point  $Z_0^{0,\infty} \in \bigcap_{\epsilon \in I(\eta)} \text{Int} \hat{C}_\epsilon^{0,\infty}$ .*
- (3) *Let  $\Phi_\epsilon^{0,\infty}$ ,  $\epsilon \in I(\eta)$ , be the family of Fatou coordinates of  $F_\epsilon$ , all having the same point  $Z_0^{0,\infty}$  as base point i.e.  $\Phi_\epsilon^{0,\infty}(Z_0^{0,\infty}) = 0$ . Then the family  $\Phi_\epsilon^{0,\infty}$ ,  $\epsilon \in I(\eta)$ , defines a continuous map on the set  $Q(\eta)^{0,\infty} = \bigcup_{\epsilon \in I(\eta)} \{\epsilon\} \times \hat{C}_\epsilon^{0,\infty}$ .*

*Proof.* The proof follows from Propositions 3.6, 3.7, Theorem 3.8 and 3.14, using Remark 7.1.  $\square$

Of course, the family  $\Phi_\epsilon^{0,\infty}$  is defined on the bigger set given by the translation domains, but it will be more convenient to restrict ourselves to the fundamental domains in order to be able to return to the initial variable  $z$ .

Let  $C_\epsilon^{0,\infty} \subset \mathbb{C}$  be the set in the  $z$ -plane given by  $C_\epsilon^{0,\infty} = p_\epsilon(\hat{C}_\epsilon^{0,\infty})$ ,  $\epsilon \in I(\eta)$ . Note that  $C_\epsilon^{0,\infty}$ ,  $\epsilon \in I^G(\eta)$  is an annulus,  $C_\epsilon^{0,\infty}$ ,  $\epsilon \in I^L(\eta)$  is a simply connected domain and  $C_\epsilon^{0,\infty}$ ,  $\epsilon = 0$ , is a crescent whose two ends coincide. We study the real 3-dimensional set

$$N(\eta) = \bigcup_{\epsilon \in I(\eta)} \{\epsilon\} \times C_\epsilon \tag{7.1}$$

and its quotient by  $f_\epsilon$ .

Define two model sets  $M^0(\eta)$  (resp.  $M^\infty(\eta)$ ) as the set of equivalence classes of couples  $(\epsilon, Z) \subset I(\eta) \times \bar{\mathbb{C}}$ , with respect to the equivalence relation

$$(\epsilon, Z) \sim (\epsilon', Z'), \text{ if } \begin{cases} \epsilon = \epsilon' \text{ and } Z \equiv Z' \pmod{1} \\ \epsilon = \epsilon' \in I^G(\eta) \text{ and } Z \equiv Z' \pmod{\alpha_{0,\infty}(\sqrt{\epsilon})} \\ \epsilon = \epsilon' = 0 \text{ and } Z \text{ and } Z' \text{ are points at infinity in } \hat{C}_\epsilon \end{cases} \quad (7.2)$$

Here  $\alpha_{0,\infty}(\sqrt{\epsilon}) = 2i\pi/\mu_{0,\infty}(\sqrt{\epsilon})$ , where  $\mu_{0,\infty}(\sqrt{\epsilon})$  is the multiplier of the fixed point of  $f_\epsilon$  corresponding to the chosen Glutsyuk translation domain.

*Remark 7.3.* The model set  $M^{0,\infty}(\eta)$  is a three-dimensional open set. It is fibered by the first coordinate  $\epsilon \in I(\eta)$ . For any value of  $\epsilon \in I(\eta) \setminus \{0\}$ , the fiber is a holomorphic curve. It is a torus of modulus  $\mu_{0,\infty}(\sqrt{\epsilon})$ , for  $\epsilon \in I^G(\eta)$ , a sphere, for  $\epsilon \in I^L(\eta)$  and a sphere with two points identified, for  $\epsilon = 0$  (cf. Figure 9).

**Lemma 7.4.** *The family of Fatou coordinates  $\Phi_\epsilon$ ,  $\epsilon \in I(\eta)$ , induces a mapping  $\phi : N(\eta)/f_\epsilon \rightarrow M(\eta)$ . That is, given a class of an element  $(\epsilon, z)$  in  $N(\eta)/f_\epsilon$ , with  $z = p_\epsilon(Z)$ .*

*Then its image by  $\phi$  is given by the class of  $(\epsilon, \Phi_\epsilon(Z)) \in M(\eta)$ . The mapping  $\phi : N(\eta)/f_\epsilon \rightarrow M(\eta)$  is a homeomorphism respecting the fibers and its restriction to any fiber is a holomorphic map.*

*Proof.* The proof follows from relations (3.7) verified for Fatou coordinates and (3.8), verified for Fatou coordinates in a Glutsyuk domain.  $\square$

On the other hand, the comparison of the Fatou coordinates gives a family of maps

$$\Psi_\epsilon = \Phi_\epsilon^0 \circ (\Phi_\epsilon^\infty)^{-1}, \quad \epsilon \in I(\eta). \quad (7.3)$$

Conjugating with the covering maps  $p_\epsilon$ , we obtain a family of maps  $\psi_\epsilon$ ,  $\epsilon \in I(\eta)$ . This family defines a mapping from the model  $M(\eta)^\infty$  to the model  $M(\eta)^0$ .

**Proposition 7.5.** *The family of mappings  $\psi_\epsilon$ ,  $\epsilon \in I(\eta)$ , is continuous, defined in a full neighborhood of the point  $\hat{0}$ . It preserves the fibers. For  $\epsilon = 0$ , the function  $\psi_\epsilon$  is the Ecalle-Voronin modulus (with the points 0 and  $\infty$  of the sphere identified). For  $\epsilon \in I^L(\eta)$ , the functions  $\psi_\epsilon$  are just the unfoldings of the Ecalle-Voronin moduli in the Lavaurs point of view as in Proposition 4.1. For  $\epsilon \in I^G(\eta)$ , the functions  $\psi_\epsilon$  are just the unfoldings of the Ecalle-Voronin moduli in the Glutsyuk point of view as in (5.4).*

*Proof.* The proof is as the proof of Proposition 3.16 and 3.17 using Remark 7.1.  $\square$

## 8. NONLINEARIZABILITY OF NEIGHBORING FIXED POINTS

**Theorem 8.1.** *We consider a fixed point of a germ of analytic diffeomorphism  $f(z) = z + z^2 + o(z)$  and for which  $(\psi^0, \psi^\infty)$  is the Ecalle-Voronin modulus. Then*

$$\psi^{0,\infty}(w_{0,\infty}) = \sum_{j=1}^{\infty} a_j^{0,\infty} w_{0,\infty}^j \quad (8.1)$$

with  $a_1^{0,\infty} \neq 0$ . We consider an unfolding  $f_\epsilon(z) = z + z^2 - \epsilon + (z^2 - \epsilon)O(z)$  of  $f$  which has the two singular points  $P_0 = -\sqrt{\epsilon}$  and  $P_\infty = \sqrt{\epsilon}$ . Then

- (1) *If  $a_q^0 \neq 0$  (resp.  $a_q^\infty \neq 0$ ) for  $q > 1$  and  $a_j^0 = 0$  (resp.  $a_j^\infty = 0$ ) for  $j = 2, \dots, q-1$ , then there exists  $N \in \mathbb{N}$  such that for  $n > N$  and all  $\epsilon$  satisfying  $\lambda_0(\epsilon) = \exp(-2\pi i \frac{q-1}{n})$  (resp.  $\lambda_\infty(\epsilon) = \exp(+2\pi i \frac{q-1}{n})$ ), then the corresponding diffeomorphism  $f_\epsilon$  is not linearizable at  $P_0$  (resp.  $P_\infty$ ). In particular the nonlinearity of  $\psi_0$  (resp.  $\psi_\infty$ ) implies that  $f_\epsilon$  is non linearizable at  $P_0$  (resp.  $P_\infty$ ) as*

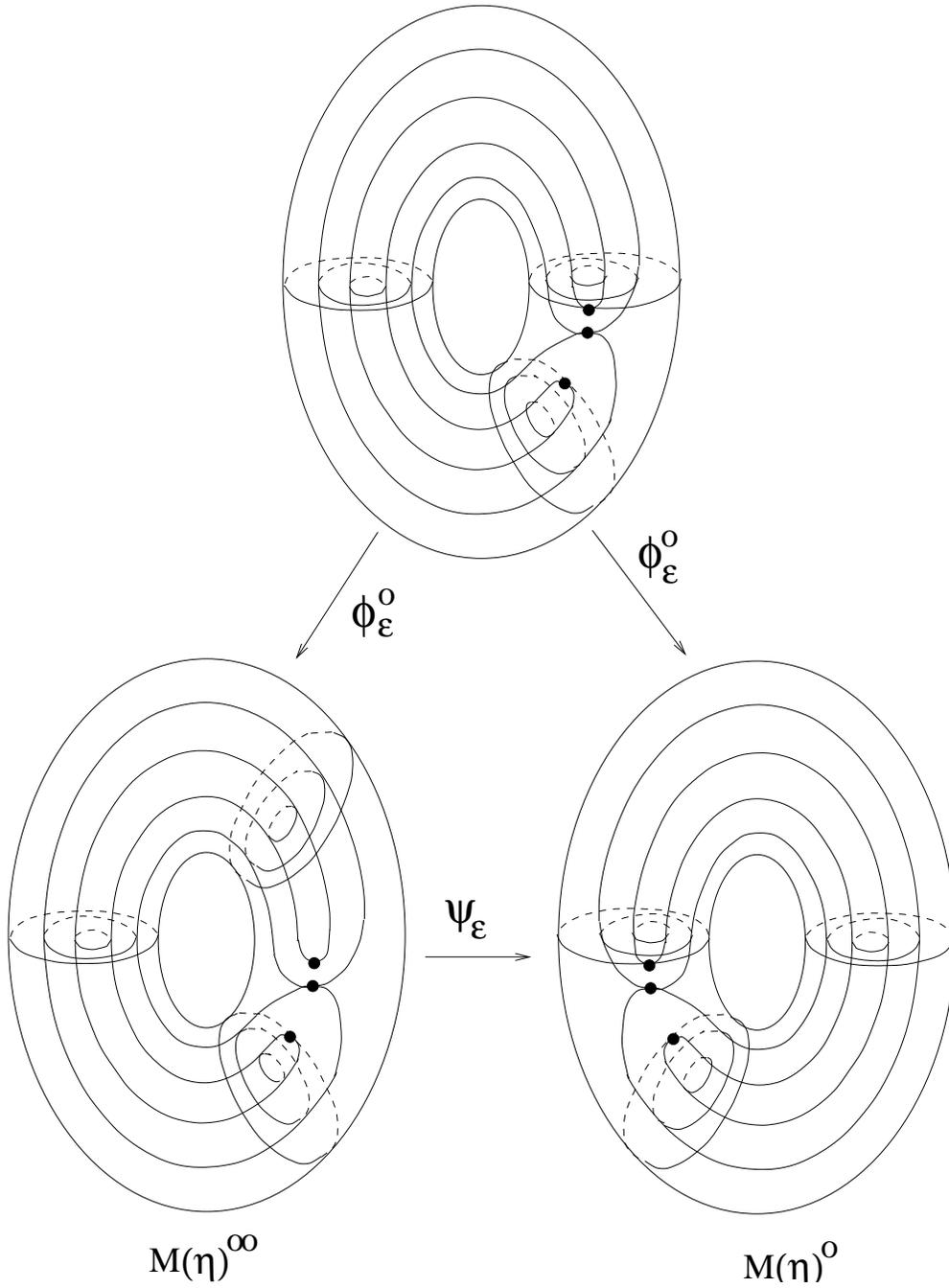


FIGURE 9. Global organization: Glutsyuk and Lavaurs fibers and moduli

soon as  $\epsilon$  is sufficiently small and  $\lambda_0(\epsilon) = \exp(-\frac{2\pi i}{n})$  (resp.  $\lambda_\infty(\epsilon) = \exp(\frac{2\pi i}{n})$ ) for  $n$  sufficiently large.

(2) Let

$$h^{0,\infty}(\tau, w_{0,\infty}) = \tau \sum_{j=1}^{\infty} a_j^{0,\infty} w_{0,\infty}^j. \quad (8.2)$$

If the coefficients  $a_n^{0,\infty}$  are such that the first coefficient of the normal form of  $h^{0,\infty}(\tau, w_{0,\infty})$  is nonzero when  $\tau a_1^0 = \exp(\frac{2\pi i m}{p})$  (resp.  $\tau a_1^\infty = \exp(-\frac{2\pi i m}{p})$ ), with  $p \in \mathbb{N}$  and  $m \in \{1, \dots, p-1\}$  then there exists  $\epsilon_0 > 0$  such that for all  $\epsilon$  satisfying  $|\epsilon| < \epsilon_0$  and  $\lambda_0(\epsilon) = \exp(-2\pi i \frac{p}{n})$  (resp.  $\lambda_\infty(\epsilon) = \exp(+2\pi i \frac{p}{n})$ ) with  $n \equiv m \pmod{p}$  then the corresponding diffeomorphism  $f_\epsilon$  is not linearizable at  $P_0$  (resp.  $P_\infty$ ).

*Proof.* The whole proof relies on the fact that the functions  $\psi_\epsilon^0$  and  $\psi_\epsilon^\infty$  can be defined continuously in  $\epsilon$  at least for  $\epsilon$  sufficiently small:  $\psi_\epsilon^{0,\infty}(w_{0,\infty}) = \sum_{i=1}^{\infty} a_i^{0,\infty}(\epsilon) w_{0,\infty}^i$ .

We make the proof near  $P_0$ .

- (1) There is a neighborhood of the origin contained in the domain of all  $\psi_\epsilon^0$  for all  $\epsilon$  sufficiently small. It contains a circle  $C(0, r_1)$ . From

$$a_n^0(\epsilon) = \frac{1}{2\pi i} \int_{C(0, r_1)} \frac{\psi_\epsilon^0(\zeta) d\zeta}{\zeta^{n+1}} \quad (8.3)$$

we deduce that  $a_n^0(\epsilon)$  is continuous in  $\epsilon$ .

We consider the fundamental neighborhood as a sphere with two distinguished points corresponding to  $P_0$  and  $P_\infty$ . The first return map defined locally in a neighborhood of  $P_0$  is given by  $k_\epsilon^0(w) = \tau(\epsilon) \sum_{i=1}^{\infty} a_i^0(\epsilon) w^i$ . If  $a_1^0(\epsilon) \tau(\epsilon) = \exp\left(-2\pi i \frac{n}{q-1}\right)$ , then the map  $k_\epsilon^0$  will be nonlinearizable if its normal form is nontrivial. This will be equivalent to  $f_\epsilon$  non linearizable at the origin as  $k_\epsilon^0$  is the first ‘‘renormalization’’ of  $f_\epsilon$  in the neighborhood of  $P_0$ . The fact that  $k_\epsilon^0$  has a multiplier  $\exp\left(-\frac{2\pi i n}{q-1}\right)$  corresponds to an initial multiplier  $\exp\left(-\frac{2\pi i (q-1)}{n}\right)$  for the fixed point  $P_0$  of  $f_\epsilon$  by (4.20).

Let us first look at the case  $q = 2$ . Then the term in  $w^2$  is resonant, yielding the nonlinearizability of  $k_\epsilon^0$ .

We then consider the case  $q > 2$ . We consider the two cases where  $q - 1$  is relatively prime with  $n$  and  $(q - 1, n) = d > 1$ . In the first (resp. second) case we will find  $N_1$  (resp.  $N_d$  for each  $d|(q - 1)$ ) such that for  $n > N_1$  (resp.  $n > N_d$ ) the conclusion is true. The  $N$  we look for is  $N = \max_{d|(q-1)} \{N_d\}$ . To bring the function to normal form we look for a change of coordinate  $w_1 = w + o(w)$ .

When  $q - 1$  is relatively prime with  $n$  then the normalizing change of coordinate has the form  $w_1 = w + \sum_{i=2}^{q-1} c_i(\epsilon) w^i + o(w^q)$  and the monomial  $w^q$  is the first resonant monomial. Because  $a_i^0(\epsilon) = O(\epsilon)$  for  $i \leq q - 1$  we have that  $c_i(\epsilon) = O(\epsilon)$ . Hence the function  $k_\epsilon^0$  in the new coordinate  $w_1$  has the form  $\bar{k}_\epsilon^0(w_1) = a_1^0(\epsilon) \tau(\epsilon) w_1 + C(\epsilon) w_1^q$ , where  $C(\epsilon) = \tau(\epsilon) a_q^0(\epsilon) + O(\epsilon) \neq 0$  for  $\epsilon$  sufficiently small, i.e.  $n$  sufficiently large.

When  $(q - 1, n) = d > 1$  then  $q - 1 = dm$  and the monomials  $w^{m+1}, w^{2m+1}, \dots, w^{dm+1}$  are resonant. The change of coordinate we are looking for is of the form  $w_1 = w + \sum_{\substack{2 \leq i \leq q-1 \\ i \not\equiv 1 \pmod{m}}} c_i(\epsilon) w^i + o(w^q)$ . As before, for each  $i$ ,  $c_i(\epsilon) = O(\epsilon)$  yielding that the normal form is  $\bar{k}_\epsilon^0(w_1) = a_1^0(\epsilon) \tau(\epsilon) w_1 + \sum_{y=1}^d D_y(\epsilon) w_1^{ym+1}$  where  $D_i(\epsilon) = O(\epsilon)$  for  $i < d$  and  $D_d(\epsilon) = \tau(\epsilon) a_q^0(\epsilon) + O(\epsilon) \neq 0$  for  $\epsilon$  sufficiently small, i.e.  $n$  sufficiently large.

- (2) The proof is similar to that of (1): indeed the hypothesis in (1) guarantees that the first coefficient of the normal form of  $g_{0,\infty}(\tau, w)$  does not vanish when  $\lambda_{0,\infty}(\epsilon)$  is a  $p$ -th root of unity.  $\square$

**Corollary 8.2.** *We consider  $h^{0,\infty}(\tau, w_{0,\infty})$  as in (8.2). If the coefficients  $a_n^{0,\infty}$  are such that the first coefficient of the normal form of  $h^{0,\infty}(\tau, w_{0,\infty})$  is nonzero for all  $\tau a_1^{0,\infty} = \exp\left(\frac{2\pi i q}{p}\right)$  with  $p \in \mathbb{N}$ , then for any  $M \in \mathbb{N}$  there exists  $N \in \mathbb{N}$  such that for all  $\epsilon$  satisfying and  $\lambda_0(\epsilon) = \exp\left(-2\pi i \frac{p}{n}\right)$  (resp.  $\lambda_\infty(\epsilon) = \exp\left(2\pi i \frac{p}{n}\right)$ ) with  $p \leq M$  and  $n > N$  then the corresponding diffeomorphism  $f_\epsilon$  is not linearizable at  $P_0$  (resp.  $P_\infty$ ).*

*Proof.* The proof is similar to that of Theorem 8.1 (2): indeed we find for each  $1 \leq p \leq M$  values  $\epsilon_{0,p,j} > 0$ ,  $j \in \{1, \dots, p-1\}$  where the conclusion holds when  $\lambda_0(\epsilon) = \exp\left(-2\pi i \frac{p}{n}\right)$  (resp.  $\lambda_\infty(\epsilon) = \exp\left(-2\pi i \frac{p}{n}\right)$ ) with  $n \equiv j \pmod{p}$  and  $|\epsilon| < \epsilon_{0,p,j}$  which yields  $n > N(p, j)$ . We take  $N = \max\{N(p, j) | 1 \leq p \leq M, 1 \leq j \leq p\}$ .  $\square$

*Remark.*

- i) Yoccoz proved [Y] that the quadratic polynomial  $P(z) = \tau z + z^2$  is non linearizable except when  $\tau = \exp(2\pi i \alpha)$  with  $\alpha$  an irrational Bryuno number. Hence if  $\psi_0(w)$  (resp.  $\psi_\infty(w)$ ) is a quadratic polynomial it satisfies the hypothesis of Corollary 8.2.

ii) In general it seems impossible to realize any constant family  $(\psi^0, \psi^\infty)$  unless  $\psi^0 = \psi^\infty = \psi$  (because  $\sqrt{\epsilon}$  and  $-\sqrt{\epsilon}$  are exchanged when we make one turn in  $\epsilon$  and their dynamics are characterized by the continuous unfoldings of  $\psi^\infty$  (resp.  $\psi^0$ ) when their multipliers are on the unit circle. In the latter case we could expect to realize a constant family  $\psi_\epsilon^0 = \psi_\epsilon^\infty \equiv \psi$ . Taking for instance  $\psi(w) = w + w^2$  we could conclude to the nonlinearizability of  $\pm\sqrt{\epsilon}$  as soon as their multipliers would not be on the unit circle or would be different from  $\exp(2\pi i\alpha)$  with  $\alpha$  an irrational Bryuno number.

## 9. THE CASE OF A SADDLE-NODE OF A VECTOR FIELD

We address briefly in this section the particular case of a saddle-node of codimension 1 as we will use it to discuss the examples of Section 10. The saddle-node will be discussed in more details in a forthcoming paper. An analytic system with a saddle-node of multiplicity 2 at the origin can be brought by an analytic change of coordinate to the prenormal form

$$\begin{aligned}\dot{x} &= x^2 \\ \dot{y} &= y(1 + ax) + x^2 R(x, y),\end{aligned}\tag{9.1}$$

with  $R(x, y)$  analytic in a neighborhood of the origin

**9.1. The Martinet-Ramis invariants of analytic classification for a saddle-node.** The description of the analytic invariants of a saddle-node, called Martinet-Ramis invariants makes use of first integrals defined in sectorial neighborhoods of the saddle-node ([MR1] and [I]). Moreover it is shown in [MR1] that the analytic class of a saddle-node is characterized by the analytic class of the holonomy of its strong separatrix (see also [I]). Indeed the first integral is a tool to describe the space of orbits of the vector field (leaves of the foliation), the later coinciding (up to isolated elements) with the space of orbits of the holonomy map.

For the holonomy map the space of orbits is described by two spheres (fundamental domains for sectorial neighborhoods  $U^\pm$  of the origin) which are glued in the neighborhoods of zero and infinity by the Ecalle-Voronin moduli  $(\psi^0, \psi^\infty)$ . When unfolding a saddle-node in the Siegel direction we will get essentially a saddle and a node (“essentially” because if  $a \notin \mathbb{Z}$  there may be a small shift between the values of  $\epsilon$  where one point is a saddle and the other is a node). We choose to call  $\infty$  (resp. 0) the point of the sphere which will be attached to the node (resp. saddle). The coordinates on the spheres are uniquely determined up to linear transformations of each sphere. The holonomy map is defined for a section  $\{y = y_0\}$  with  $y_0$  small: call it  $h_{y_0}$ . All leaves of the foliation intersect this section except possibly one (the center manifold). The leaves intersect any one of the fundamental domains exactly once. Hence, it is natural to take the spherical coordinate as a first integral. Then the Ecalle-Voronin modulus represents exactly the transitions between the two first integrals  $H_{\pm,0}$  defined on  $U^\pm \times W$  where  $W$  is a neighborhood of the origin in  $y$  space. These two first integrals are the “canonical” first integrals

$$H(x, Y) = Yx^{-a}e^{\frac{1}{x}}\tag{9.2}$$

for the model

$$\begin{aligned}\dot{x} &= x^2 \\ \dot{Y} &= Y(1 + ax),\end{aligned}\tag{9.3}$$

where  $Y$  has to be thought of as the normalizing coordinate on the two domains  $U^\pm \times W$ . (See for instance [I]).

## 9.2. The answer to a question of Martinet and Ramis.

In [MR1] we find the following sentence (the equation (2) to which they refer is our equation (9.1)): “*Un phénomène qui reste un peu surprenant à nos yeux est que les holonomies produites par les équations (2) ne sont pas arbitraires: on obtient seulement une “petite partie du module d’Ecalte”. (Nous nous proposons de montrer dans un article ultérieur qu’il n’en est plus de même dans le cas des équations résonantes “non*

dégénérées" ( $\lambda = -p/q \neq 0$ ): le module des classes d'équivalence analytiques d'équations différentielles s'identifie complètement au "module d'Ecalte".

When unfolding the saddle-node in the Siegel direction we find a saddle and a node. The node has the property that it is integrable (its holonomy is linearizable) as soon as it is non resonant (there are no small divisor problems nor convergence problems). Let  $(\psi_0^0, \psi_0^\infty)$  be the modulus of classification of the saddle-node, where  $\psi_0^\infty$  is the part of the modulus attached to the node. If we unfold the saddle-node in a generic family depending on a parameter  $\epsilon$  we know from Section 4 that the first return map in the neighborhood of the node has the form  $k_\epsilon^\infty = l_\epsilon \circ \psi_\epsilon^\infty$ . Moreover, from Theorem 8.1 we need that the first coefficient of the normal form of  $\tau\psi_0^\infty$  must vanish as soon as  $\tau = \exp(2\pi ip/q)$  with  $(p, q) = 1$  and  $q > 1$ . If this were not the case, then by Theorem 8.1 the holonomy of the node should be non linearizable when the eigenvalues would be of the form  $(1, p/q)$ . Moreover if  $\psi_0^\infty$  is not linear then the node should be nonlinearizable as soon as it is resonant. Let us show that the function  $\psi_0^\infty(w) = aw + c$ ,  $a, c \neq 0$ , in the neighborhood of infinity is the only function with this property. Using action of linear transformations in the source and target space we can rather consider the function  $\psi_0^\infty(w) = w - 1$ . If we localize at 0 by means of  $w_1 = 1/w$  this induces the Möbius transformation  $\bar{\psi}_0^\infty(w_1) = \frac{w_1}{1-w_1}$ .

**Proposition 9.1.** *The only germ of analytic function  $F(z) = z + z^2 + \sum_{n>2} c_n z^n$  having the property that the first coefficient of the normal form of  $\tau F(z)$  vanishes as soon as  $\tau = \exp(2\pi ip/q)$ ,  $(p, q) = 1$ ,  $q > 1$  is the Möbius transformation  $F(z) = \frac{z}{1-z}$ .*

*Proof.* Let  $c_2 = 1$ . Let  $c_n$  be given the weight  $n - 1$ . It is well known that the first coefficient  $N(p, q)$  of the normal form of  $\tau F(z)$ , where  $\tau = \exp(2\pi ip/q)$  is a quasi-homogeneous polynomial in the  $c_n$  of degree  $q$ . We have

$$N(p, q) = c_{q-1} + P_{p,q}(c_2, \dots, c_{q-2}), \quad (9.4)$$

where  $P_{p,q}$  is a polynomial in  $(c_2, \dots, c_{q-2})$ . We solve by induction the equations  $N(1, 1) = 1$ ,  $N(1, 2) = 0$ ,  $\dots$ ,  $N(1, q) = 0$  for  $q > 1$  and find  $c_n$  for all  $n$ . Each of these equations has a unique solution  $c_{q-1}$  in terms of the previous  $c_i$ ,  $i < q - 1$ . As the Möbius function is already a solution of the problem we find recursively  $c_{q-1} = 1$ .  $\square$

## 10. EXAMPLES

Let us start with a few definitions.

**Definition 10.1.** *We consider a system with a saddle point at the origin*

$$\begin{aligned} \dot{x} &= x + P(x, y) = x + o(x, y) \\ \dot{y} &= -\lambda y + Q(x, y) = -\lambda y + o(x, y) \end{aligned} \quad (10.1)$$

with  $\lambda \in \mathbb{R}^-$ .

- (1) *The origin is integrable if and only if the holonomy of any separatrix is linearizable.*
- (2) *Let  $\lambda = 1$ . The origin is normalizable if the holonomy has a parabolic point with trivial Ecalle-Voronin modulus ( $\psi^0$  and  $\psi^\infty$  are linear). The holonomy is iterable in Ecalle's terminology [E].*
- (3) *Let  $\lambda = 1$ . The origin is half-normalizable if the holonomy is half-iterable in Ecalle's terminology, i.e. it has a parabolic point with either  $\psi^0$  or  $\psi^\infty$  linear.*

**Proposition 10.2.** *The Lotka-Volterra system*

$$\begin{aligned} \dot{x} &= x(1 - x + y) \\ \dot{y} &= y(-\lambda + x + dy). \end{aligned} \quad (10.2)$$

*has a half normalizable point at the origin and the corresponding  $\psi^\infty$  is a Möbius function when  $d = 1$  and  $\lambda = 1$ . It cannot be approached by integrable saddles with  $\lambda = 1 + \frac{1}{n}$ . On the other hand it can be*

approached with integrable saddles when  $\lambda < 1$  as it lies on  $d = \frac{\lambda}{2\lambda-1}$ , all points of which are integrable except when  $\lambda = 1 + \frac{1}{n}$ .

*Proof.* The idea (introduced in [CR]) is to look at the monodromy group of the projective line  $x = 0$ . There are 3 singular points. The monodromy of the point  $P = (0, 1)$  is the identity. Indeed the Jacobian has the eigenvalues 1 and 2. Moreover the node has two analytic curves tangent to the eigenspaces yielding that it is linearizable. As the multiplier of the monodromy around  $P$  is  $\exp(4\pi i)$  then the monodromy is the identity. Hence the monodromy at the origin is the inverse of the monodromy around the point  $\infty$ . Let us study the monodromy around  $\infty$ . This point is a saddle-node with an analytic center manifold. Hence it is at least half-normalizable as  $\psi_0^\infty$  is linear. Let us put it to normal form. The change of coordinates  $(v, z) = (\frac{1}{y}, \frac{x}{y})$  brings (10.2) (after multiplication by  $z$ ) to

$$\begin{aligned}\dot{v} &= -2v^2 + 2vz \\ \dot{z} &= -z - vz + z^2.\end{aligned}\tag{10.3}$$

We make the change of coordinate  $V = -\frac{2v}{(1-z)^2}$ . This brings the system to the form

$$\begin{aligned}\dot{V} &= V^2(1 - z + z^2) \\ \dot{z} &= -z + \frac{1}{2}Vz(1 - z)^2 + z^2\end{aligned}\tag{10.4}$$

Scaling of time yields the system

$$\begin{aligned}\dot{V} &= V^2 \\ \dot{z} &= -z(1 - \frac{1}{2}V) - Vz^2 + o(z^2).\end{aligned}\tag{10.5}$$

The origin is not orbitally normalizable (the change of coordinate  $z = Z + f(V)Z^2$  removing the terms in  $z^2$  is divergent). Hence the other part of the modulus is of the form  $\psi_0^0(w) = Aw + Bw^2 + o(w^2)$  with  $A, B \neq 0$ . By Theorem 8.1 this implies the non-linearizability of the monodromy of the bifurcating saddle points with hyperbolicity ratios  $\frac{1}{n}$  with  $n$  sufficiently large. This implies the non-linearizability of the monodromy map when its multiplier is of the form  $\exp(-\frac{2\pi i}{n})$ . Hence the monodromy map of the origin is half-normalizable and the non-linear part of its analytic invariant is the one which controls the non-linearizability of its monodromy map when the multiplier is of the form  $\exp(+\frac{2\pi i}{n})$ . The latter corresponds to the non-integrability of the saddle point at the origin when  $\lambda = 1 + \frac{1}{n}$ . If we call  $(\psi_1^0, \psi_1^\infty)$  its modulus then we have that  $\psi_1^0$  is linear and  $\psi_1^\infty$  is nonlinear.

We can be more precise and see that  $\psi_1^\infty$  is an affine map. Indeed, taking  $u = xy$ , (10.2) yields the system:

$$\begin{aligned}\dot{u} &= u(1 - \lambda + (1 + d)y) \\ \dot{y} &= -\lambda y + u + dy^2.\end{aligned}\tag{10.6}$$

It has a node precisely when  $\lambda > 1$  and the node is resonant as soon as  $\lambda = 1 + \frac{1}{n}$ . This means that for this system we have  $\psi_2^0$  linear and  $\psi_2^\infty$  affine. The conclusion follows as the holonomy of the  $x = 0$  and  $u = 0$  on a section  $y = y_0$  are conjugate by a linear map.

We must now show that for all points of the curve  $d = \frac{\lambda}{2\lambda-1}$  the origin is integrable except when  $\lambda = 1 + \frac{1}{n}$  where it is only orbitally normalizable. Indeed the system has the following invariant conic which was first found by Chavarriga:

$$F(x, y) = \left(1 + \frac{y}{1 - 2\lambda}\right)^2 - \frac{2xy}{(1 - \lambda)(1 - 2\lambda)} = 0,\tag{10.7}$$

with cofactor  $K(x, y) = \frac{2\lambda y}{2\lambda-1}$  This conic yields an integrating factor

$$V(x, y) = x^{\frac{2\lambda-1}{\lambda-1}} y^{\frac{\lambda}{\lambda-1}} F^{-\frac{\lambda+1}{2(\lambda-1)}}.\tag{10.8}$$

As proved in [CMR] this yields the integrability of the origin except when the two exponents of the factors  $x$  and  $y$  in  $V(x, y)$  are both integers greater than 1, in which case the point is only orbitally normalizable. This is the case precisely when  $\lambda = 1 + \frac{1}{n}$ .  $\square$

## 11. CONCLUSION AND PERSPECTIVES

The present work opens many avenues of research. Let us mention a few:

- (1) The theorems presented here are only continuous in  $\nu$  at  $\nu = 0$ . Is it possible to do better? Surely it is not possible to get analyticity in  $\nu$ . Indeed a drastic phenomenon occurs at the limit of a Lavaurs sectorial domains: the two points are exchanged while the functions  $\psi_\nu^0$  and  $\psi_\nu^\infty$  determine the dynamics near  $P_0$  and  $P_\infty$ . This cannot pass in general to the limit as  $\delta \rightarrow 0$ . We conjecture that the modulus is 1-summable in  $\sqrt{\epsilon}$ . It seems however that a necessary condition for the modulus to depend holomorphically on  $\sqrt{\epsilon}$  is that  $\psi_0^0 = \psi_0^\infty = \psi$  (when localized at the same point). In this particular case is it possible to realize the constant modulus  $\psi_\nu^0 = \psi_\nu^\infty = \psi$ ?
- (2) Although we have given complete moduli of analytic classification for analytic families (2.12) unfolding a parabolic point, we have not identified the space of moduli for such families. That is, the realization problem remains open. The solution of this problem goes through an answer to the question of identifying the exact dependence of the moduli of analytic classification on  $\nu$ .
- (3) A similar question is raised for the space of moduli for families unfolding a generic saddle-node.
- (4) If we could improve the dependence in  $\nu$  and derive results for the theory of real vector fields, we could hope for applications of the theory to the finite cyclicity of graphics for analytic families of vector fields.
- (5) A better understanding of the dependence of the modulus of analytic classification on the parameter could also allow for generic Ecalle-Voronin modulus to draw conclusions on the non linearizability of neighboring fixed points: for instance the nonlinearizability of all fixed points with multipliers being roots of unity or of the form  $\exp(2\pi i\alpha)$  with  $\alpha$  an irrational non Bryuno number.
- (6) A natural question is to generalize the results obtained here for generic  $q$ -parameter unfoldings of a parabolic fixed point of a map  $f_0(z) = z + z^{q+1} + o(z^{q+1})$ . A treatment for 1-parameter families and particular sectors of the parameter space has already been done by Oudkerk [O] and allows to draw conclusions as in our Theorem 8.1 when all our singular points have multipliers close to the unit circle.
- (7) A simpler question is to consider the generic bifurcation of a fixed point with multiplier being a root of unity  $f_0(z) = \exp(2i\pi \frac{p}{q})z + z^{q+1} + o(z^{q+1})$ . Indeed this bifurcation is of codimension 1. Moreover, the functions  $\psi_i^{0,\infty}$ ,  $i = 1, \dots, q$ , of the Ecalle-Voronin modulus of the map  $f_0^q$  commute with the map  $w \mapsto \exp(2i\pi/q)w$ . A natural question to ask is if the unfolding of the modulus has the same property.
- (8) We have explained for a generic saddle-node why the Martinet-Ramis modulus is not the full Ecalle-Voronin modulus. This is because one half of the modulus controls the dynamics of the node which is much simpler than the dynamics of a generic point with a multiplier on the unit circle. A natural question is to find a similar explanation for a saddle-node of higher multiplicity.

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