On CP^1 and CP^2 Maps and Weierstrass Representations for Surfaces Immersed into Multi-dimensional Euclidean Spaces

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Abstract

An extension of the classic Enneper-Weierstrass representation for conformally parametrised surfaces in multi-dimensional spaces is performed.This is carried out by using low dimensional CP^1 and CP^2 sigma models which allow us to study constant mean curvature (CMC) surfaces immersed into Euclidean 3- and 8-dimensional spaces, respectively.Connections of Weierstrass type systems with these sigma models equations are established.In particular it is demonstrated that the generalised Weierstrass representation can admit different CMC-surfaces in R^3 which have globally the same Gauss map.A new procedure for constructing CMC-surfaces in R^n is proposed and illustrated by examples.

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Résumé

Nous effectuons une étude d'une généralisation du système classique de la représentation de Enneper-Weierstrass décrivant les surfaces conformément paramétrisées et plongées dans des espaces euclidiens multidimensionnels. Cette étude est inspirée des modèles sigma à basse dimension CP^1 et CP^2 permettant l'analyse des surfaces a courbure moyenne constante (CMC) plongées respectivement dans des espaces euclidiens à 3 et 8 dimensions. Les relations entre les systèmes du type de Weierstrass et les modèles sigma sont établies. En particulier, nous démontrons que la représentation généralisée de Weierstrass admet différentes surfaces-CMC en R^3 ayant la même transformation de Gauss globalement. Une nouvelle méthode de construction des surfaces-CMC en R^n est présentée ainsi que quelques exemples explicites. Ces exemples illustrent les considérations théoriques proposées dans ce papier.

1 Introduction

The objective of this paper is to study two-dimensional surfaces conformally immersed in multi-dimensional spaces with Euclidean metric. We look for explicit formulae for the position vector $X : \mathcal{D} \in C \to R^3$ of a surface for which X satisfies the Gauss-Weingarten and Gauss-Codacci equations identically. Such formulae describing minimal surfaces (i.e. zero mean curvature $H = 0$) imbedded in three-dimensional space were first formulated by Enneper [1] and Weierstrass [2] about one and half century ago.They consider two holomorphic functions $\psi(z)$ and $\phi(z)$ of complex variable $z \in C$ and introduced a three component complex vector valued functions $w = (w_1, w_2, w_3) : \mathcal{D} \to C^3$ defined by linear differential equations

$$w_1 = i(\psi^2 + \phi^2), \qquad w_2 = \psi^2 - \phi^2, \qquad w_3 = -2\psi\phi, \qquad \bar{\partial}\psi = 0 \qquad \bar{\partial}\phi = 0, \tag{1.1}$$

where the derivatives are abbreviated $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$.They show that the real vector valued functions

$$X = (Re \int_0^z i(\psi^2 + \phi^2)\, dz', Re \int_0^z (\psi^2 - \phi^2)\, dz', -2Re \int_0^z \psi\phi\, dz') \tag{1.2}$$

can be considered as a position vector of a parametrised surface immersed in R^3 with the conformal metric

$$ds^2 = (|\psi|^2 + |\phi|^2)\, dz\, d\bar{z}, \tag{1.3}$$

where z and $\bar{z}$ are local coordinates on $\mathcal{D}$. Since then this idea has been developed by many authors,see for a review of the subject eg. [3, 4, 5] and references therein. The theory of constant mean curvature (CMC)-surfaces play an essential role in several applications to problems appearing both in mathematics and in physics. In particular,many interesting applications to physics can be found to such diverse areas as in the fields of two-dimensional gravity [6, 7] ,string theory [8, 9], quantum field theory [10, 6], statistical physics [11, 12], fluid dynamics [13], theory of fluid membranes [14, 11]. It is worth mentioning an application of recent interest,namely the CanHam-Helfrich membrane model [15, 16], which is derived from microscopic models and making use of the generalised Weierstrass representations for arbitrary two-dimensional surfaces immersed into multi-dimensional Euclidean spaces [17]. This model allows to explain basic features and equilibrium shapes both for biological membranes and liquid interfaces [14]. Another relevant application of recent interest is the area of statistical mechanics. Any two-dimensional statistical system near a second-order phase transition can be described by a conformally invariant theory [18].

Our approach consists basically in modifying the original Enneper-Weierstrass representation (1.2) by adding to it a certain terms. For this purpose we show that it is convenient to use the connection between Weierstrass systems, CP^1 and CP^2 sigma model equations, and their Lax representations. Through these links we demonstrate that conformal immersion of CMC-surfaces into 3- and 8-dimensional spaces can be formulated. We show that a large classes of solutions of the Weierstrass system can be obtained and consequently can provide new classes of conformally parametrised CMC-surfaces in multi-dimensional spaces.

The paper is organized as follows. In Section 2,using the spinor representations,we rederive the classical Enneper-Weierstrass representation for minimal surfaces immersed into R^3. In Section 3 we describe in detail the generalised Weierstrass formulae for CMC-surfaces into R^3 in the context of the CP^1 sigma model and discuss some geometric aspects of CP^1 maps. Section 4 deals with CP^2 maps and the corresponding Weierstrass representation to conformally parametrised surfaces immersed into R^8 and some geometric characteristics of CMC-surfaces are presented. Some Propositions useful for the construction of CMC-surfaces based on Weierstrass representations are given in Sections 3

and 4. In Section 5 theoretical considerations are illustrated by examples and new interesting CMC-surfaces are found. Section 6 contains final remarks and possible future developments.

2 The Enneper-Weierstrass formulae for conformally parametrised surfaces in R^3

Let M^2 be a smooth orientable surface in 3-dimensional Euclidean space R^3 The surface M^2 is determined by a real vector-valued function

$$X = (X_1, X_2, X_3) : D \to R^3, \tag{2.1}$$

where D is a region in the complex plane C. The metric is assumed to be conformally flat

$$ds^2 = e^{2u}\, dz\, d\bar{z} \tag{2.2}$$

for any real valued function u of z and $\bar{z}$. The conformal parametrisation of a surface M^2 implies the following normalization of the position vector $X(z, \bar{z})$

$$(\partial X, \partial X) = 0, \qquad (\partial X, \bar{\partial} X) = \frac{1}{2} e^{2u}, \tag{2.3}$$

where the derivatives are abbreviated $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$. The bar denotes the complex conjugate and the brackets (,) denote the standard scalar product in R^3. The tangent vectors ∂X and $\bar{\partial} X$ and the real unit normal vector N on the surface M^2 satisfy the following scalar relations

$$(\partial X, N) = 0, \qquad (N, N) = 1, \tag{2.4}$$

where N can be calculated according to the formula

$$N = \frac{\partial X \times \bar{\partial} X}{|\partial X \times \bar{\partial} X|}. \tag{2.5}$$

Equations of a moving complex frame $\xi = (\partial X, \bar{\partial} X, N)^T$ satisfy the following Gauss-Weingarten equations

$$\partial \xi = U \xi, \qquad \bar{\partial} \xi = V \xi, \tag{2.6}$$

where 3 by 3 matrices U and V have the form

$$U = \begin{pmatrix} 2\partial u & 0 & J \\ 0 & 0 & \frac{1}{2} H e^{2u} \\ -H & -2e^{-2u} J & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 0 & \frac{1}{2} H e^{2u} \\ 0 & 2\bar{\partial} u & \bar{J} \\ -2e^{-2u} \bar{J} & -H & 0 \end{pmatrix}, \tag{2.7}$$

and the following notation has been introduced

$$J = (\partial^2 X, N), \qquad H = 2e^{-2u} (\partial \bar{\partial} X, N). \tag{2.8}$$

Formulae (2.6) are compatible with the scalar products (2.3) and (2.4). From (2.6) we can derive the equation for the unit normal vector N

$$\partial \bar{\partial} N + (\partial N, \bar{\partial} N) N + \bar{\partial} H \partial X + \partial H \bar{\partial} X = 0. \tag{2.9}$$

The corresponding Gauss-Codazzi equations of the conformally parametrised surface $M^2 \subset R^3$ are the compatibility conditions of equations (2.6) and have the following form

$$\partial\bar{\partial}u + \frac{1}{4}H^2 e^{2u} - 2|J|^2 e^{-2u} = 0 \tag{2.10}$$

$$\bar{\partial}J = \frac{1}{2}\partial H e^{2u}, \qquad \partial\bar{J} = \frac{1}{2}\bar{\partial}H e^{2u}. \tag{2.11}$$

The objective of this section is to rederive the original Enneper-Weierstrass formulae [1, 2] for inducing constant mean curvature (CMC) surfaces in R^3. We focus on constructing explicit formula for the position vector $X(z,\bar{z})$ of conformally parametrised surfaces into R^3 for which equations (2.3),(2.10) and (2.11) are fullfiled.

For computational purposes, it is useful to examine equations (2.3), (2.10) and (2.11) in terms of two-component spinor representation $\phi = (\psi_1, \psi_2)^T \in C^2$ of a surface in R^3. We show that we can determine by some quadratures the coordinates of the position vector $X(z,\bar{z})$ in terms of the components of the spinor ϕ satisfying equations (2.3) and (2.10-2.11).

Let us consider the complex vector $\vec{w}$ in C^3 equal to one of the tangent vector,say ∂X

$$\vec{w} = (w_1, w_2, w_3) = \partial X, \qquad w_i \in C, \qquad i = 1, 2, 3, \tag{2.12}$$

the 2 by 2 traceless matrix

$$w = \begin{pmatrix} w_3 & w_1 - iw_2 \\ w_1 + iw_2 & -w_3 \end{pmatrix}, \qquad trw = 0, \tag{2.13}$$

and the map

$$\vec{w} : C \to w = w_i\sigma_i \in sl(2, C), \tag{2.14}$$

where σ_i are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.15}$$

The map (2.14) satisfies

$$\vec{w}^2 = -\det w. \tag{2.16}$$

From (2.16), the determinant of the matrix w vanishes if and only if the vector $\vec{w}$ is null, which coincide with the first condition in (2.3). Hence, using (2.13), we can express uniquely the null vector $\vec{w}$ in terms of the complex two-component spinor ϕ as follows

$$w_1 = \frac{1}{2}(\psi_1^2 - \psi_2^2), \qquad w_2 = \frac{i}{2}(\psi_1^2 + \psi_2^2), \qquad w_3 = -\psi_1\psi_2 \tag{2.17}$$

From the assumption that the null vector $\vec{w}$ is equal to the tangent vector ∂X, we can express ∂X in terms of the spinor components ψ_1 and ψ_2 as follows

$$\partial X_1 = \frac{1}{2}(\psi_1^2 - \psi_2^2), \qquad \partial X_2 = \frac{i}{2}(\psi_1^2 + \psi_2^2), \qquad \partial X_3 = -\psi_1\psi_2. \tag{2.18}$$

The Enneper-Weierstrass representation for surfaces in R^3 can be obtained under an additional assumption that spinors ψ_1 and ψ_2 are two arbitrary holomorphic functions ψ_1 and ψ_2 of the complex variable $z \in C$. Then integrating equations (2.18) and taking into account the reality condition of the position vector

$$X(z,\bar{z}) = \bar{X}(z,\bar{z}), \tag{2.19}$$

we obtain the following representation [2]

$$X_1 = \frac{1}{2}\int_0^z (\psi_1^2 - \psi_2^2)\, dz' + \frac{1}{2}\int_0^{\bar z} (\bar\psi_1^2 - \bar\psi_2^2)\, dz',$$
$$X_2 = \frac{i}{2}\int_0^z (\psi_1^2 + \psi_2^2)\, dz' - \frac{i}{2}\int_0^{\bar z} (\bar\psi_1^2 + \bar\psi_2^2)\, d\bar z',$$
$$X_3 = -\int_0^z \psi_1\psi_2 dz' - \int_0^{\bar z} \bar\psi_1\bar\psi_2\, d\bar z. \tag{2.20}$$

Next, from (2.20) and invoking the second condition (2.3) implies

$$u = \ln(|\psi_1|^2 + |\psi_2|^2). \tag{2.21}$$

Substituting (2.21) into the Gauss-Codazzi equations (2.10-2.11), we obtain some restriction on the real function

$$p = |\psi_1|^2 + |\psi_2|^2, \tag{2.22}$$

of the form

$$\partial\bar\partial \ln p^2 + \frac{1}{2}H_0^2 p^2 - 2|J|^2 p^{-2} = 0, \tag{2.23}$$

where by virtue of (2.8), we obtain the conservation of the current

$$J = \psi_1\partial\psi_2 - \psi_2\partial\psi_1, \qquad \bar\partial J = 0, \tag{2.24}$$

and the constant mean curvature $H = H_0$. Note that the direction of the spinor $\phi = (\psi_1, \psi_2)^T$ is arbitrary, but its length is determined by equation (2.23). After the change of variable

$$\varphi = 2\ln p \tag{2.25}$$

equation (2.23) becomes Sh-Gordon type equation

$$\partial\bar\partial\varphi = -4H_0^2 \sinh\varphi - 2(H_0^2 - |J|^2)e^{-\varphi}, \qquad \bar\partial J = 0. \tag{2.26}$$

In [23] the system (2.26) had appeared within the spinor description of surfaces in R^3.

3 The generalised Weierstrass formulae for CMC-surfaces in R^3

The Wierstrass-Enneper formulae for inducing minimal surfaces has been studied for a long time by many authors (e.g. [3, 19, 22] and references therein). This subject most recently has been treated by B. Konopelchenko and I. Taimanov [24]. They establishes a direct connection between certain classes of CMC-surfaces and an integrable finite-dimensional Hamiltonian system. For a summary of their results, see Ref [25]. There was shown that with any spinor solutions $\phi = (\psi_1, \psi_2)^T$ of the Dirac type equations

$$\partial\psi_1 = p\psi_2, \quad \bar\partial\psi_2 = -p\psi_1, \qquad p = |\psi_1|^2 + |\psi_2|^2, \tag{3.1}$$

one can associate a CMC-surface immersed into R^3 with radius vector $X(z, \bar z)$ of the form (2.20)

$$X_1 = \int_\gamma (\psi_1^2 - \psi_2^2)dz' + (\bar\psi_1^2 - \bar\psi_2^2)d\bar z',$$
$$X_2 = \int_\gamma (\psi_1^2 + \psi_2^2)\, dz' - (\bar\psi_1^2 + \bar\psi_2^2)d\bar z',$$
$$X_3 = -\int_\gamma \psi_1\psi_2 dz' + \bar\psi_1\bar\psi_2, \tag{3.2}$$

where γ is an arbitrary curve, which does not depend on the trajectory but only on its endpoints z in C. These formulae are the starting point for analysis of CMC- surfaces in this paper, and according to [4], we will refer to system (3.1) as the generalised Weierstrass (GW) system.

In this paper, we examine certain aspects of CMC-surfaces in R^n in the context of low dimensional sigma models. In particular, we focus on constructing a Weierstrass representation for generic two-dimensional surfaces immersed in R^8, where the explicit form has not been known up to now. For convenience sake our investigation starts with a derivation of the position vector $X(z,\bar{z})$ of a surface in R^3 from the Lax pair for GW system (3.1). It has been shown [26] that GW system (3.1) is in one-to-one correspondence with the completely integrable two-dimensional Euclidean CP^1 sigma model

$$[\partial\bar{\partial}P, P] = 0, \tag{3.3}$$

where matrix P is a projector

$$P = \frac{1}{A}\begin{pmatrix} 1 & \bar{w} \\ w & |w|^2 \end{pmatrix}, \qquad A = 1 + |w|^2, \tag{3.4}$$

or equivalently,using the projector P, equation (3.3) takes the form

$$\partial\bar{\partial}w - \frac{2\bar{w}}{1+|w|^2}\partial w\bar{\partial}w = 0 \tag{3.5}$$

It has been shown [27] that if spinors ψ_1 and ψ_2 are solutions of GW system (3.1), then function w defined by

$$w = \frac{\psi_1}{\bar{\psi}_2}, \tag{3.6}$$

is a solution of CP^1 sigma model (3.5). The converse of this statmnet is also true. Namely, if w is a solution of CP^1 sigma model (3.5), then the spinor solutions ψ_1 and ψ_2 of GW system (3.1) have the form

$$\psi_1 = \epsilon w\frac{(\bar{\partial}\bar{w})^{1/2}}{1+|w|^2}, \qquad \psi_2 = \epsilon\frac{(\partial w)^{1/2}}{1+|w|^2}, \qquad p = \frac{|\partial w|}{1+|w|^2}, \qquad \epsilon = \pm 1. \tag{3.7}$$

Note that equation (2.9) with $H = 1$ for the unit normal vector $N = (n_1, n_2, n_3)$ to a CMC-surface adopts the well known form of the $SO(3)$ sigma model

$$\partial\bar{\partial}N + (\partial N, \bar{\partial}N)N = 0, \qquad (N, N) = 1. \tag{3.8}$$

Combining the map of the unit vector N onto the unit sphere S^2 with the stereographic projection, we obtain the Gauss map

$$w = \frac{n_1 + n_2}{1 + n_3} = \frac{\psi_1}{\bar{\psi}_2}. \tag{3.9}$$

which satisfies the CP^1 model equation (3.5). Hence expression (3.9) establishes a connection between the CP^1 and the $SO(3)$ sigma models.

As was shown by A. V. Mikhailov in [28] that equation (3.3) is a compatibility condition for the two linear spectral problems

$$\partial\Psi = \frac{2}{1+\lambda}[\partial P, P]\Psi, \qquad \bar{\partial}\Psi = \frac{2}{1-\lambda}[\bar{\partial}P, P]\Psi, \tag{3.10}$$

where $\lambda \in C$ represents the spectral parameter. The compatibility condition for (3.10) can be written in equivalent form of a conservation law

$$\partial K - \bar{\partial}K^{\dagger} = 0, \tag{3.11}$$

where the traceless 2 by 2 matrices K and $K^\dagger$ expressed in terms of w have the form

$$K = [\bar{\partial} P, P] = \frac{1}{A^2} \begin{pmatrix} \bar{w}\bar{\partial}w - w\bar{\partial}\bar{w} & \bar{\partial}\bar{w} + \bar{w}^2\partial w \\ -\bar{\partial}w - w^2\bar{\partial}\bar{w} & w\bar{\partial}\bar{w} - \bar{w}\bar{\partial}w \end{pmatrix}$$

$$-K^\dagger = [\partial P, P] = \frac{1}{A^2} \begin{pmatrix} \bar{w}\partial w - w\partial\bar{w} & \partial\bar{w} + \bar{w}^2\partial w \\ -\partial w - w^2\partial\bar{w} & w\partial\bar{w} - \bar{w}\partial w \end{pmatrix}, \tag{3.12}$$

and the Hermitian conjugate is denoted by $\dagger$.

Now, we derive the explicit form of matrices K and $K^\dagger$ in terms of spinors ψ_1 and ψ_2 in order to find the corresponding conservation laws for GW system (3.1). For computational purposes, it is useful to express the first derivatives of w in terms of ψ_1 and ψ_2 and the current (2.8) given by

$$J = \bar{\psi}_1 \partial \psi_2 - \psi_2 \partial \bar{\psi}_1, \tag{3.13}$$

which satisfies,

$$\bar{\partial} J = \bar{\partial}(\bar{\psi}_1 \partial \psi_2 - \psi_2 \partial \bar{\psi}_1) = -p\partial p + p\partial p = 0, \tag{3.14}$$

whenever (3.1) holds. Hence, J is any holomorphic function. Using equations (3.1), (3.6) and (3.13) it allows us to express the first derivatives of w in terms of ψ_1, ψ_2 and J

$$\partial w = A^2 \psi_2, \qquad \bar{\partial} w = -\bar{J}\bar{\psi}_2^{-2}, \tag{3.15}$$

where

$$A = 1 + \frac{|\psi_1|^2}{|\psi_2|^2}, \qquad \bar{J} = \psi_1 \bar{\partial}\bar{\psi}_2 - \bar{\psi}_2 \bar{\partial}\psi_1, \qquad \partial \bar{J} = 0. \tag{3.16}$$

As a consequence of (3.11) and (3.12) we find that the GW system possesses at least three more conservation laws

$$\begin{aligned} &\partial(\psi_1\bar{\psi}_2 + \bar{R}\bar{\psi}_1\psi_2) - \bar{\partial}(\bar{\psi}_1\psi_2 + R\psi_1\bar{\psi}_2) = 0, \\ &\partial(\psi_1^2 - \bar{R}\psi_2^2) + \bar{\partial}(\psi_2^2 - R\psi_1^2) = 0, \\ &\partial(\bar{\psi}_2^2 - \bar{R}\bar{\psi}_1^2) + \bar{\partial}(\bar{\psi}_1^2 - R\bar{\psi}_2^2) = 0, \end{aligned} \tag{3.17}$$

where the following notation has been introduced

$$R = \frac{J}{p^2}. \tag{3.18}$$

Note that formulae (3.17) differ from the conservation laws derived in [4] as they contain additional terms involving R. If we put $R = 0$ in equations (3.17) then we recover the expressions given in [4].

$$\partial(\psi_1\bar{\psi}_2) - \bar{\partial}(\bar{\psi}_1\psi_2) = 0, \qquad \partial(\psi_1^2) + \bar{\partial}(\psi_2^2) = 0 \qquad \partial(\bar{\psi}_2^2) + \bar{\partial}(\bar{\psi}_1^2) = 0. \tag{3.19}$$

As a result of conservation laws (3.17), we can introduce three real-valued functions $X_i(z, \bar{z})$ given by

$$\begin{aligned} X_1 &= \frac{i}{2}\int_\gamma [\bar{\psi}_1^2 + \psi_2^2 - R(\psi_1^2 + \bar{\psi}_2^2)]\, dz' - [\psi_1^2 + \bar{\psi}_2^2 - R(\bar{\psi}_1^2 + \psi_2^2)]\, d\bar{z}', \\ X_2 &= \frac{1}{2}\int_\gamma [\bar{\psi}_1^2 - \psi_2^2 + R(\psi_1^2 - \bar{\psi}_2^2]\, dz' + [\psi_1^2 - \bar{\psi}_2^2 + R(\bar{\psi}_1^2 - \psi_2^2)] d\bar{z}', \\ X_3 &= -\int_\gamma [\bar{\psi}_1\psi_2 + R\psi_1\bar{\psi}_2] dz' + [\psi_1\bar{\psi}_2 + \bar{R}\bar{\psi}_1\psi_2] d\bar{z}', \end{aligned} \tag{3.20}$$

where γ is any curve from a fixed point z in C. The functions X_i, $i = 1, 2, 3$ can be considered as components of a position vector of a locally parametrised surface by z and $\bar{z}$ immersed in R^3

$$X(z, \bar{z}) = (X_1(z, \bar{z}), X_2(z, \bar{z}), X_3(z, \bar{z})). \tag{3.21}$$

Using conformal changes of coordinates on the surface M^2 we can without loss of generality put $J = 1$. As a consequence it is ease to show that representation(3.20) with $R = 1/p^2$ can not be reduced by any transformation to the Weierstrass formulae (3.2).This means, as we will see latter,that the additional terms involving R play essential role in the construction of surfaces in R^3.

The tangent and normal unit vectors to the surface M^2 are

$$\begin{aligned} \partial X &= (i[\bar{\psi}_1^2 + \psi_2^2 - R(\psi_1^2 + \bar{\psi}_2^2)], [\bar{\psi}_1^2 - \psi_2^2 + R(\psi_1^2 - \bar{\psi}_2^2)], -2(\bar{\psi}_1\psi_2 + R\psi_1\bar{\psi}_2)), \\ \bar{\partial} X &= (-i[\psi_1^2 + \bar{\psi}_2^2 - \bar{R}(\bar{\psi}_1^2 + \psi_2^2)], [\psi_1^2 - \bar{\psi}_2^2 + \bar{R}(\bar{\psi}_1^2 - \psi_2^2)], -2(\psi_1\bar{\psi}_2 + \bar{R}\bar{\psi}_1\psi_2)), \end{aligned} \tag{3.22}$$

and

$$N = \frac{1}{p}(i(\bar{\psi}_1\bar{\psi}_2 - \psi_1\psi_2), \bar{\psi}_1\bar{\psi}_2 + \psi_1\psi_2, |\psi_1|^2 - |\psi_2|^2), \tag{3.23}$$

respectively. The first and second fundamental forms of the surface M^2 are given by

$$\begin{aligned} &I = (dX, dX) = 4(Jdz^2 + p^2(1 + |R|^2)dz\, d\bar{z} + \bar{J}d\bar{z}^2), \\ &II = (d^2X, N) = (4J + R + \bar{R})dz^2 + (2p + i(R - \bar{R}))dzd\bar{z} \\ &+(4\bar{J} - R - \bar{R})d\bar{z}^2. \end{aligned} \tag{3.24}$$

where

$$J = \frac{\partial w \partial \bar{w}}{(1 + |w|^2)^2}\,. \tag{3.25}$$

These quadratic forms contain the Hopf differential Jdz^2 and is invariant under any conformal changes of coordinates.The Gauss and mean curvature are

$$K = -p^{-2}\partial\bar{\partial}\ln p, \qquad H = 1, \tag{3.26}$$

respectively.

Remark 1. (**i**).In particular if $J = 0$ then the components of the fundamental forms become

$$g_{12} = 2p^2, \qquad g_{11} = g_{22} = 0, \qquad b_{12} = 2p, \qquad b_{11} = b_{22} = 0. \tag{3.27}$$

In this case the solutions of GW system (3.1) expressed in terms of w are represented by (3.7), where $w(z)$ is any holomorphic function. According to [29], the energy

$$E = \int\int_D \frac{\partial w \bar{\partial} w}{1 + |w|^2} dz \wedge d\bar{z}, \tag{3.28}$$

is finite when the function $w(z)$ is rational in z. Geometrically, such functions ψ_i parametrise an immersed sphere $S^2 \subset R^3$, since $J = 0$ implies the proportionality of fundamental forms I and II.

(**ii**). If in equations (3.24) we put $J \neq 0$ then there is no conformal immersion of surfaces in R^3. Hence, equations (3.22) and (3.23) imply that the representation (3.20) can admit different CMC-surfaces which globally have the same Gauss map (3.9). It is due to the fact that the tangent vectors ∂X and $\bar{\partial} X$ depend essentially on J while the unit normal vector N is independent of J. In [20, 21], using the isometric immersions, formulae similar to (3.22) and (3.23) for particular cases of isothermic surfaces have been discussed.

Let us now discuss the meaning of conservation laws (3.17). If J is a holomorphic function then according to (2.11) we deal with CMC-surfaces. If the CP^1 model is defined over S^2 then solutions w of (3.5) are any holomorphic or antiholomorphic function and so $J = 0$. However, if the CP^1 model is defined on R^2 then function w is not necesserely holomorphic or antiholomorphic and $J \neq 0$. Subtracting (3.19) from (3.17) and introducing new independent variables η and $\bar{\eta}$ according to

$$d\eta = J^{1/2}\, dz, \qquad d\bar{\eta} = \bar{J}^{1/2} d\bar{z}, \qquad \bar{\partial} J = 0, \tag{3.29}$$

we obtain the following system in factorised form

$$\begin{aligned} &|J|^2(\partial_\eta(\frac{\bar{\psi}_1\psi_2}{p^2}) - \bar{\partial}_{\bar{\eta}}(\frac{\psi_1\bar{\psi}_2}{p^2})) = 0, \\ &|J|^2(\partial_\eta(\frac{\psi_2^2}{p^2}) + \bar{\partial}_{\bar{\eta}}(\frac{\psi_1}{p^2})) = 0, \\ &|J|^2(\partial_\eta(\frac{\bar{\psi}_1^2}{p^2}) + \bar{\partial}_{\bar{\eta}}(\frac{\bar{\psi}_2^2}{p^2})) = 0, \end{aligned} \tag{3.30}$$

where the derivatives are abbreviated $\partial_\eta = \partial/\partial\eta$ and $\bar{\partial}\bar{\eta} = \partial/\partial\bar{\eta}$. Transformation (3.29) is well defined since J is a holomorphic function. Equations (3.30) imply to consider two separate cases, namely $J = 0$ which has been already treated in [4] and $j \neq 0$. In the latter case, under the change of variables (3.29) the GW system (3.1) adopts the form

$$\partial_\eta \psi_1 = \frac{p}{J}\psi_2, \qquad \bar{\partial}_{\bar{\eta}}\psi_2 = -\frac{p}{\bar{J}}\psi_1, \tag{3.31}$$

and the expression for the current J, given by (3.13), provides the following differential constraint (DC) on spinors ψ_1 and ψ_2

$$\bar{\psi}_1\partial_\eta\psi_2 - \psi_2\partial_\eta\bar{\psi}_1 = 1. \tag{3.32}$$

Keeping in mind that the complex coordinates η and $\bar{\eta}$ are is defined up to a conformal transformation, we can without loss of generality put $J = 1$. If $\psi_1 \neq 0$ then the system (3.31) subjected to DC (3.32) can be written in an equivalent form

$$\partial\psi_1 = p\psi_2, \quad \partial\psi_2 = \bar{\psi}^{-1}(1 + \psi_2\partial\bar{\psi}_1), \quad \bar{\partial}\psi_2 = -p\psi_1. \tag{3.33}$$

The compatibility condition for (3.33) does not imply any new DC on first order derivatives of ψ_1. Hence, system (3.33) is integrable and first derivatives $\bar{\partial}\psi_1$ and $\partial\psi_1$ are free variables.

The Gaussian curvature and mean curvature are

$$K = (|\psi_2|^2 - |\psi_1|^2)[|\bar{\partial}\psi_1|^2 + |\psi_2|^{-2}(1 + \psi_2\partial\bar{\psi}_1 + \bar{\psi}_2\bar{\partial}\psi_1)], \quad H = 1, \tag{3.34}$$

respectively. Equations of motion of the complex frame (2.6) are specified by DC (3.32) and are compatible with scalar products (2.3) and (2.4). After the change of dependent variables

$$p = e^{\varphi/2}, \tag{3.35}$$

the corresponding Gauss-Codazzi equations (2.10-2.11) take the form of elliptic Sh-Gordon equation

$$\partial\bar{\partial}\varphi + 4\sinh\varphi = 0. \tag{3.36}$$

Hence the CMC-surfaces are determined by formulae (3.20), where the spinor components ψ_1 and ψ_2 have to obey equations (3.33) with p determined by (3.35) and (3.36). In terms of arbitrary

conformal coordinates, we have proved that the spinors (ψ_1, ψ_2, p) can be viewed as the Weierstrass data of the CMC-surface M^2 in R^3.

Proposition 1. The generalised Weierstrass representation for immersion of a CMC- surface in R^3 is determined by formulae (3.20), where the spinors ψ_1 and ψ_2 obey the Dirac type equations (3.1).

According to [30], under the changes of independent variables (3.29) and dependent variables $S = 2p^2|J|^{-1}$, the system (3.1) is decoupled into a direct sum of the elliptic Sh-Gordon and Laplace equations

$$\partial_\eta \bar{\partial}_{\bar{\eta}} \ln S = S^{-1} - S, \qquad \bar{\partial}_{\bar{\eta}} J = 0, \qquad \partial_\eta \bar{J} = 0. \tag{3.37}$$

Hence, system (3.37) is completely integrable one. The Darboux and Bäcklund transformations for (3.1) have the form [31]

$$p = qy, \tag{3.38}$$

and the pseudopotential y satisfies Riccati equations

$$\begin{aligned} &\partial y = i\lambda^{1/2}(J\bar{J}^{1/2}q^{-2} - \bar{J}^{-1/2}q^2y^2), \qquad \bar{\partial} J = 0, \qquad \lambda \in C, \\ &\bar{\partial} y = i\bar{\lambda}^{1/2}\bar{J}^{1/2} + y\bar{\partial}\ln(\bar{J}^{1/2}q^{-2}) - i\bar{\lambda}^{1/2}\bar{J}^{1/2}y^2, \qquad |\lambda|^2 = 1, \end{aligned} \tag{3.39}$$

where functions p and q satisfy the Gauss-Codazzi equation (2.10) with $H_0 = 1$, J is any holomorphic function and λ is the Bäcklund parameter. Examples of N soliton solutions can be found in [32].

4 The CP^2 maps and the Weierstrass representation for surfaces in eight dimensional Euclidean spaces.

The objective of this section is to demonstrate a connection between the Generalised Weierstrass (GW) system [33]

$$\partial\psi_1 = (1 + \frac{|\psi_2|^2}{|\varphi_2|^2})\bar{\varphi}_1\bar{Q} - \frac{1}{2}[\frac{\psi_1\bar{\psi}_2}{\varphi_2} + \frac{|\psi_1|^2}{|\varphi_1|^2}\frac{\bar{\varphi}_2^2}{\bar{\varphi}_1}]\bar{P}, \tag{4.1}$$

$$\partial\psi_2 = (1 + \frac{|\psi_1|^2}{|\varphi_1|^2})\bar{\varphi}_2\bar{P} - \frac{1}{2}[\frac{\bar{\psi}_1\psi_2}{\varphi_1} + \frac{|\psi_2|^2}{|\varphi_2|^2}\frac{\bar{\varphi}_1^2}{\bar{\varphi}_2}]\bar{Q}, \tag{4.2}$$

$$\bar{\partial}\varphi_1 = -\frac{1}{2}[(\frac{\psi_2}{\bar{\varphi}_2}P + 2\frac{\psi_1}{\bar{\varphi}_1}Q)\varphi_1 + P\psi_1\frac{\varphi_2^2}{|\varphi_1|^2}], \tag{4.3}$$

$$\bar{\partial}\varphi_2 = -\frac{1}{2}[(\frac{\psi_1}{\bar{\varphi}_1}Q + 2\frac{\psi_2}{\bar{\varphi}_2}P)\varphi_2 + Q\psi_2\frac{\varphi_1^2}{|\varphi_2|^2}], \tag{4.4}$$

where the following notation has been introduced

$$\begin{aligned} P &= \frac{\psi_1\bar{\psi}_2\bar{\varphi}_1}{\varphi_2} + (|\varphi_2|^2 + |\psi_2|^2)\frac{\bar{\varphi}_2}{\varphi_2} \\ Q &= \frac{\bar{\psi}_1\psi_2\bar{\varphi}_2}{\varphi_1} + (|\varphi_1|^2 + |\psi_1|^2)\frac{\bar{\varphi}_1}{\varphi_1} \end{aligned} \tag{4.5}$$

and the CP^2 sigma model [29]

$$\partial\bar{\partial}w_1 - \frac{2\bar{w}_1}{A}\partial w_1\bar{\partial}w_1 - \frac{\bar{w}_2}{A}(\partial w_1\bar{\partial}w_2 + \bar{\partial}w_1\partial w_2) = 0, \tag{4.6}$$

$$\partial\bar{\partial}w_2 - \frac{2\bar{w}_2}{A}\partial w_2\bar{\partial}w_2 - \frac{\bar{w}_1}{A}(\partial w_1\bar{\partial}w_2 + \bar{\partial}w_1\partial w_2) = 0, \tag{4.7}$$

$$A = 1 + |w_1|^2 + |w_2|^2. \tag{4.8}$$

Next, we derive through this link the conservation laws for the GW system (4.1) - (4.4) in order to define real valued functions $X^i(z,\bar{z})$, $i=1,\cdots,8$ in terms of functions φ_α, ψ_α, $\alpha=1,2$ which are identified as the coordinates in 8-dim Euclidean space R^8. We willNote that in equations (4.1) - (4.4) eight of sixteen first derivatives of functions ψ_i and ϕ_i are known in terms of complex functions ψ_i and ϕ_i and their complex conjugates. show that there exists a conformal immersion of a surface M^2 into R^8 and some geometrical aspects of M^2 will be investigated. The formulae (4.1-4.4) and (4.6-4.8) are a starting point for our analysis. In this paper, when we refer to system (4.1-4.4), we mean the modified version of the original Weierstrass system (3.1).

Note that in equations (4.1-4.4) eight of sixteen first derivatives of functions ψ_i and φ_i are known in terms of complex functions ψ_i and φ_i and their complex conjugates. Note also that if the functions ψ_α tends to $\psi/\sqrt{2}$ and φ_α tends to φ, i.e. then system (4.1-4.4) is reduced to the Weierstrass formulae (3.1) for CMC-surfaces immersed in R^3

$$\begin{aligned}\partial\psi &= (|\psi|^2+|\varphi|^2)\varphi,\\ \bar{\partial}\varphi &= -(|\psi|^2+|\varphi|^2)\psi.\end{aligned}$$

In terms of w_i, $i=1,2$ the above limit adopts the form

$$w_i \to \frac{1}{\sqrt{2}}w, \qquad i=1,2, \tag{4.9}$$

and then the CP^2 sigma model (4.6-4.8) is reduced to the CP^1 sigma model (3.5). These limits characterise the properties of the solutions of systems (4.1-4.4) and (4.6-4.8).

First we show that there exists a one to one correspondence between GW system (4.1-4.4) and CP^2 sigma model (4.6-4.8). For this purpose, we define two new complex valued functions

$$w_1 = \frac{\psi_1}{\bar{\varphi}_1}, \qquad w_2 = \frac{\psi_2}{\bar{\varphi}_2} \tag{4.10}$$

and using GW system (4.1-4.4), we obtain

$$\begin{aligned}\partial w_1 &= A[w_1\bar{w}_2\varphi_2^2 + (1+|w_1|^2)\varphi_1^2],\\ \partial w_2 &= A[\bar{w}_1 w_2\varphi_1^2 + (1+|w_2|^2)\varphi_2^2].\end{aligned} \tag{4.11}$$

These relations generate the following transformation from the variables (w_1,w_2) and their derivatives to the set of variables $(\varphi_1,\varphi_2,\psi_1,\psi_2)$

$$\varphi_1 = \epsilon A^{-1}[(1+|w_2|^2)\partial w_1 - w_1\bar{w}_2\partial w_2]^{1/2}, \tag{4.12}$$
$$\varphi_2 = \epsilon A^{-1}[-\bar{w}_1 w_2\partial w_1 + (1+|w_1|^2)\partial w_2]^{1/2}, \tag{4.13}$$
$$\psi_1 = \epsilon w_1 A^{-1}[(1+|w_2|^2)\bar{\partial}\bar{w}_1 - \bar{w}_1 w_2\bar{\partial}\bar{w}_2]^{1/2}, \tag{4.14}$$
$$\psi_2 = \epsilon w_2 A^{-1}[-w_1\bar{w}_2\bar{\partial}\bar{w}_1 + (1+|w_1|^2)\bar{\partial}\bar{w}_2]^{1/2}. \tag{4.15}$$

Finally, one gets the following

Proposition 2. If the complex valued functions $(\varphi_1,\varphi_2,\psi_1,\psi_2)$ are solutions of GW system (4.1-4.4), then the rational functions (w_1,w_2) defined by (4.10) are solutions of CP^2 sigma model equations (4.6-4.8).

Conversely, if the complex valued functions (w_1,w_2) are solutions of the CP^2 sigma model equations (4.6-4.8), then the complex valued functions $(\varphi_1,\varphi_2,\psi_1,\psi_2)$ defined by (4.12-4.15) in terms of the functions (w_1,w_2) and their 1st derivatives satisfy the GW system (4.1-4.4).

Proof: Differentiation of equations (4.11) with repsect to z and $\bar{z}$, respectively yield

$$\partial\bar{\partial}w_1 = A[\bar{w}_2\varphi_2^2\bar{\partial}w_1 + w_1\varphi_2^2\bar{\partial}\bar{w}_2 + 2w_1\bar{w}_2\varphi_2\bar{\partial}\varphi_2 + 2(1+|w_1|^2)\varphi_1\bar{\partial}\varphi_1$$
$$+(\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1)\varphi_1^2] + [w_1\bar{w}_2\varphi_2^2 + (1+|w_1|^2)\varphi_1^2](\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1 + \bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2),$$

and

$$\partial\bar{\partial}w_2 = A[w_2\varphi_1^2\bar{\partial}\bar{w}_1 + \bar{w}_1\varphi_1^2\bar{\partial}w_2 + 2\bar{w}_1w_2\varphi_1\bar{\partial}\varphi_1 + 2(1+|w_2|^2)\varphi_2\bar{\partial}\varphi_2$$
$$+(\bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2)\varphi_2^2] + [\bar{w}_1w_2\varphi_1^2 + (1+|w_2|^2)\varphi_2^2](\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1 + \bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2)$$

and their respective complex conjugate equations.

Substituting (4.11) and into the left-hand side of the first equation (4.6), we obtain

$$\begin{aligned}
&\partial\bar{\partial}w_1 - \frac{2\bar{w}_1}{A}\partial w_1\bar{\partial}w_1 - \frac{\bar{w}_2}{A}(\partial w_1\bar{\partial}w_2 + \bar{\partial}w_1\partial w_2)\\
&= [Aw_1\varphi_2^2 + w_1|w_2|^2\varphi_2^2 + (1+|w_1|^2)w_2\varphi_1^2](w_1\bar{w}_2\bar{\varphi}_1^2 + (1+|w_2|^2)\bar{\varphi}_2^2)\\
&+[Aw_1\varphi_1^2 + w_1^2\bar{w}_2^2\varphi_2^2 + (1+|w_1|^2)w_1\varphi_1^2](\bar{w}_1w_2\bar{\varphi}_2^2 + (1+|w_1|^2)\bar{\varphi}_1^2)\\
&+2[w_1\bar{w}_2\varphi_2\bar{\partial}\varphi_2 + (1+|w_1|^2)\varphi_1\bar{\partial}\varphi_1].
\end{aligned} \tag{4.16}$$

Making use of equations of motion (4.1-4.4), we obtain that equation (4.16) is satisfied identically. Hence, equation (4.6) holds. An analogous result takes place for the second equation (4.7), since the CP^2 sigma model equations (4.6-4.8) are invariant under the following discrete transformation

$$w_i \to w_j, \qquad for \qquad i \neq j = 1, 2. \tag{4.17}$$

This observation implies that the left-hand side of (4.7) vanishes as well whenever (4.1-4.4) holds.

Conversely, differentiating (4.12) with respect to $\bar{z}$ and using (4.12), we get

$$\begin{aligned}
\bar{\partial}\varphi_1 &= \frac{1}{2A^2\varphi_1}[(\bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2)\partial w_1 + (1+|w_2|^2)\partial\bar{\partial}w_1\\
&-(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2)\partial w_2 - w_1\bar{w}_2\partial\bar{\partial}w_2]\\
&-\frac{\varphi_1}{A}[\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1 + \bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2].
\end{aligned} \tag{4.18}$$

Using equations (4.11) and (4.6-4.8), we can eliminate first and second derivatives of w_1 and w_2 in expression (4.18), we have

$$\begin{aligned}
\bar{\partial}\varphi_1 &= \frac{1}{2\varphi_1}\{w_1^2\bar{w}_2|w_2|^2\bar{\varphi}_1^2\varphi_2^2 + (1+|w_2|^2)w_1|w_2|^2|\varphi_2|^4 + (1+|w_1|^2)w_1|w_2|^2|\varphi_1|^4\\
&+(1+|w_1|^2)(1+|w_2|^2)w_2\varphi_1^2\bar{\varphi}_2^2 - w_1|w_1|^2|w_2|^2|\varphi_1|^4 - (1+|w_2|^2)|w_1|^2w_2\varphi_1^2\bar{\varphi}_2^2\\
&-(1+|w_2|^2)w_1^2\bar{w}_2\bar{\varphi}_1^2\varphi_2 - (1+|w_2|^2)^2w_1|\varphi_2|^4 - 2A\varphi_1^2(w_2\bar{\varphi}_2^2 + w_1\bar{\varphi}_1^2)\\
&+\frac{1}{A^2}(A[w_1\bar{w}_2^2\varphi_2^2 + \bar{w}_2(1+|w_1|^2)\varphi_1^2] + (1+|w_2|^2)[w_1\bar{w}_2^2\varphi_2^2 + \bar{w}_2(1+|w_1|^2)\varphi_1^2]\\
&-(A+|w_2|^2)|w_1|^2\bar{w}_2\varphi_1^2 - (A+1+|w_2|^2)w_1\bar{w}_2^2\varphi_2^2 - 2A\bar{w}_2\varphi_1^2)\bar{\partial}w_2\\
&+\frac{1}{A^2}((1+|w_2|^2)[(A+|w_1|^2)\bar{w}_2\varphi_2^2 + (A+1+|w_1|^2)\bar{w}_1\varphi_1^2]\\
&-A[\bar{w}_1|w_2|^2\varphi_1^2 + (1+|w_2|^2)\bar{w}_2\varphi_2^2] - [\bar{w}_1|w_1|^2|w_2|^2\varphi_1^2\\
&+(1+|w_2|^2)|w_1|^2\bar{w}_2\varphi_2^2] - 2A\bar{w}_1\varphi_1^2)\bar{\partial}w_1\}.
\end{aligned} \tag{4.19}$$

Collecting the coefficients with respect to the derivatives $\bar{\partial}w_1$ and $\bar{\partial}w_2$ in expression (4.19), we demonstrate that these coefficients vanish identically. In fact we have

$$\bar{\partial}w_1 : (A+1+|w_1|^2+(A+1+|w_1|^2)|w_2|^2-A|w_2|^2-|w_1|^2|w_2|^2-2A)\bar{w}_1\varphi_1^2$$
$$+(A+|w_1|^2+(A+|w_1|^2)|w_2|^2-A(1+|w_2|^2)-(1+|w_2|^2)|w_1|^2)\bar{w}_2\varphi_2^2 \equiv 0.$$

and

$$\bar{\partial}w_2 : (A(1+|w_1|^2)+A+|w_1|^2|w_2|^2-(A+|w_2|^2)|w_1|^2-2A)\bar{w}_2\varphi_1^2$$
$$(A+1+|w_2|^2-A-1-|w_2|^2)w_1\bar{w}_2\varphi_2^2 \equiv 0. \tag{4.20}$$

Hence, expression (4.20) becomes

$$\bar{\partial}\varphi_1 = -\frac{1}{2}\{\bar{w}_2 w_1^2\frac{\bar{\varphi}_1^2\varphi_2^2}{\varphi_1}+(1+|w_2|^2)w_1\frac{|\varphi_2|^4}{\varphi_1}+2Aw_2\varphi_1\bar{\varphi}_2^2+2Aw_1|\varphi_1|^2\varphi_1$$
$$-w_1|w_2|^2\frac{|\varphi_1|^4}{\varphi_1}-(1+|w_2|^2)w_2\varphi_1\bar{\varphi}_2\}$$

Using transformation (4.10) we obtain the first equation of motion (4.1), i.e.

$$\bar{\partial}\varphi_1 = -\frac{1}{2}\{\frac{\varphi_2}{\varphi_1}\bar{\psi}_2\psi_1^2+(1+|w_2|^2)\frac{|\varphi_2|^4}{|\varphi_1|^2}+(A+1+|w_1|^2)|\varphi_1|^2\psi_1+(A+|w_1|^2)\varphi_1\bar{\varphi}_2\psi_2\}. \tag{4.21}$$

Since the equations of motion (4.1-4.4) are invariant under the discrete transformation

$$\varphi_i \to \varphi_j, \quad \psi_i \to \psi_j, \quad for \quad i \neq j = 1,2, \tag{4.22}$$

an analogous result takes place for equation (4.2). Differentiation of (4.10) with respect to z gives

$$\partial\psi_1 = \bar{\varphi}_1\partial w_1 + w_1\partial\bar{\varphi}_1. \tag{4.23}$$

Substituting (4.11) and the complex conjugate equation of (4.1) into (4.23) we get (4.3). Making use of the discrete symmetry (4.22) into equation of motion (4.3), we obtain equation (4.4), which completes the proof. **QED**

An interesting property of the GW system (4.1-4.4) in the context of the CP^2 sigma model (4.6-4.8) is the existence of a gauge freedom in the definition of the variables w_1 and w_2 given by formula (4.10). It is due to the fact that numerator and denominator of (4.10) can be multiplied by any complex functions $f_i : C \to C$, $i = 1,2$. This means that if we introduce a new set of complex valued functions $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ which are related to functions $(\varphi_1, \varphi_2, \psi_1, \psi_2)$ in the following way

$$\varphi_i = f_i(z,\bar{z})\alpha_i, \quad \psi_i = \bar{f}(z,\bar{z})\beta_i, \quad i = 1,2, \tag{4.24}$$

then the transformation (4.24) leaves the functions w_1, w_2 invariant

$$w_1 = \frac{\beta_1}{\bar{\alpha}_1}, \quad w_2 = \frac{\beta_2}{\bar{\alpha}_2}. \tag{4.25}$$

Now, we can formulate the following

Proposition 3. If the complex valued functions w_1, w_2 are solutions of the CP^2 sigma model eqs (4.6-4.8), then for any two holomorphic functions f_i, $i = 1,2$ the complex functions $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ defined by

$$\alpha_1 = \epsilon f_1^{-1}A^{-1}[(1+|w_2|^2)\partial w_1 - w_1\bar{w}_2\partial w_2]^{1/2},$$
$$\alpha_2 = \epsilon f_2^{-1}A^{-1}[-\bar{w}_1 w_2\partial w_1 + (1+|w_1|^2)\partial w_2]^{1/2},$$
$$\beta_1 = \epsilon w_1\bar{f}_1^{-1}A^{-1}[(1+|w_2|^2)\bar{\partial}\bar{w}_1 - \bar{w}_1 w_2\bar{\partial}\bar{w}_2]^{1/2},$$
$$\beta_2 = \epsilon w_2\bar{f}_2^{-1}A^{-1}[-w_1\bar{w}_2\bar{\partial}\bar{w}_1 + (1+|w_1|^2)\bar{\partial}\bar{w}_2]^{1/2}, \quad \bar{\partial}f_i = 0, \tag{4.26}$$

satisfy the GW system (4.1-4.4).

Proof. The result is obtained directly by substituting (4.24) and (4.25) into CP^2 sigma model equations (4.6-4.8). We get differential constraints for the functions f_i and their first derivatives

$$(f_i^2 - 1)\bar{\partial} f_j = 0, \quad (\bar{f}_i^2 - 1)\partial \bar{f}_j = 0, \quad i, j = 1, 2. \tag{4.27}$$

Hence, the general solutions of this system are given by any holomorphic functions f_i such that

$$\bar{\partial} f_i = 0, \quad i = 1, 2 \tag{4.28}$$

hold. Next, invoking Proposition 1 implies that transformation (4.12-4.15) becomes the one given by (4.26). **QED**

Another interesting property in the context of CP^2 sigma model (4.6-4.8) and GW system (4.1-4.4) is the existence of the current [29]

$$J = A^{-2}\{\partial w_1 \partial \bar{w}_1 + \partial w_2 \partial \bar{w}_2 + (\bar{w}_1 \partial \bar{w}_2 - \bar{w}_2 \partial \bar{w}_1)(w_1 \partial w_2 - w_2 \partial w_1)\}. \tag{4.29}$$

The derivative of J with respect to z vanishes identically whenever eqs (4.6-4.8) are satisfied

$$\bar{\partial} J = 0. \tag{4.30}$$

This means that the current J, given by (4.29), is any holomorphic function.

If the functions $(\varphi_1, \varphi_2, \psi_1, \psi_2)$ are solutions of GW system (4.1-4.4), then the current J in terms of functions $(\varphi_1, \varphi_2, \psi_1, \psi_2)$ takes the form

$$J = \varphi_1 \partial \bar{\psi}_1 - \bar{\psi}_1 \partial \varphi_1 + \varphi_2 \partial \bar{\psi}_2 - \bar{\psi}_2 \partial \varphi_2, \tag{4.31}$$

The derivatives of (4.31) with respect to $\bar{z}$ vanishes

$$\bar{\partial} J = 0, \tag{4.32}$$

whenever eqs (4.1-4.4) hold. Furthermore particularly important is the density of the energy associated with the CP^2 sigma model (4.6-4.8) which is given by [29]

$$E = A^{-2}\{|\partial w_1|^2 + |\bar{\partial} w_1|^2 + |\partial w_2|^2 + |\bar{\partial} w_2|^2 + |w_2 \bar{\partial} w_1 - w_1 \bar{\partial} w_2|^2 + |w_2 \partial w_1 - w_1 \partial w_2|^2\} \tag{4.33}$$

In terms of complex functions $(\varphi_1, \varphi_2, \psi_1, \psi_2)$ the density of energy for GW system (4.1-4.4) adopts the form

$$\begin{aligned} E = {} & A^{-2}\{|\bar{\partial}\frac{\psi_1}{\bar{\varphi}_1}|^2 + |\bar{\partial}\frac{\psi_2}{\bar{\varphi}_2}|^2 + |\frac{\psi_2}{\bar{\varphi}_2}\bar{\partial}\frac{\psi_1}{\bar{\varphi}_1} - \frac{\psi_1}{\bar{\varphi}_1}\bar{\partial}\frac{\psi_2}{\bar{\varphi}_2}|^2\} + |\frac{\psi_1 \bar{\psi}_2 \varphi_2}{\bar{\varphi}_1} + (1 + \frac{|\psi_1|^2}{|\varphi_1|^2})\varphi_1^2|^2 \\ & + |\frac{\bar{\psi}_1 \psi_2 \varphi_1}{\bar{\varphi}_2} + (1 + \frac{|\psi_2^2|^2}{|\varphi_2|^2})\varphi_2^2|^2. \end{aligned} \tag{4.34}$$

Finally, a significant property of GW system (4.1-4.4) in the context of CP^2 sigma model (4.6-4.8) is the existence of a topological charge. Making use of the projector P given below by (4.37) and equations of motion (4.1-4.4) one finds that if the integral

$$I = \frac{i}{8\pi}\int_D tr(P[\partial P, \bar{\partial} P]\, dz \wedge d\bar{z} = \frac{i}{2\pi}\int_D \partial\bar{\partial} \ln A\, dz \wedge d\bar{z} \tag{4.35}$$

exists, it is an integer.

It has been demonstrated by A. Mikhailov [28] that the CP^2 sigma model equations (4.6-4.8) is a compatibility condition for two linear spectral problems

$$\partial\Phi = \frac{2}{1+\lambda}[\partial P, P]\Phi, \quad \bar{\partial}\Phi = \frac{2}{1-\lambda}[\bar{\partial} P, P]\Phi, \quad \lambda \in C \tag{4.36}$$

where 3 by 3 matrix P is a projector given by

$$P = A^{-1}M, \quad M = \begin{pmatrix} 1 & w_1 & w_2 \\ \bar{w}_1 & |w_1|^2 & \bar{w}_1 w_2 \\ \bar{w}_2 & w_1\bar{w}_2 & |w_2|^2 \end{pmatrix}, \tag{4.37}$$

and λ represents the spectral parameter. Using matrix P, the compatibility condition of eqs (4.36) implies

$$[\partial\bar{\partial}P, P] = 0, \tag{4.38}$$

whenever equations of motion (4.6-4.8) hold. Equivalently, formula (4.38) can be written in a divergent form

$$\partial[\bar{\partial}P, P] + \bar{\partial}[\partial P, P] = 0. \tag{4.39}$$

Hence, from equations (4.37) and (4.39) we obtain the explicit form of the local conservation laws for the CP^2 sigma model

$$\partial K + \bar{\partial} L = 0, \tag{4.40}$$

where we introduce the following notation for the tracelessness of matrices

$$K = \frac{1}{A^2}[\bar{\partial}M, M], \quad L = -K^{\dagger} = \frac{1}{A^2}[\partial M, M], \quad trK = trL = 0. \tag{4.41}$$

Here, the matrix elements of K and L are of the form

$$\begin{aligned}
&k_{11} = A^{-2}\{(\bar{w}_1\bar{\partial}w_1 + \bar{w}_2\bar{\partial}w_2) - (w_1\bar{\partial}\bar{w}_1 + w_2\bar{\partial}\bar{w}_2)\},\\
&k_{12} = A^{-2}\{|w_1|^2\bar{\partial}w_1 + w_1\bar{w}_2\bar{\partial}w_2 - (\bar{\partial}w_1 + w_1(\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1) + w_2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2))\},\\
&k_{13} = A^{-2}\{\bar{w}_1 w_2\bar{\partial}w_1 + |w_2|^2\bar{\partial}w_2 - (\bar{\partial}w_2 + w_1(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1) + w_2(\bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2))\},\\
&k_{21} = A^{-2}\{\bar{\partial}\bar{w}_1 + \bar{w}_1(\bar{w}_1(\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1) + \bar{w}_2(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1)\\
&-(|w_1|^2\bar{\partial}\bar{w}_1 + \bar{w}_1 w_2\bar{\partial}\bar{w}_2)\},\\
&k_{22} = A^{-2}\{w_1\bar{\partial}\bar{w}_1 + w_1\bar{w}_2(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1) - (\bar{w}_1\bar{\partial}w_1 + \bar{w}_1 w_2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2))\},\\
&k_{23} = A^{-2}\{w_2\bar{\partial}\bar{w}_1 + \bar{w}_1 w_2(\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1) + |w_2|^2(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1),\\
&-[\bar{w}_1\bar{\partial}w_2 + |w_1|^2(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1) + \bar{w}_1 w_2(\bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2)]\},\\
&k_{31} = A^{-2}\{\bar{\partial}\bar{w}_2 + \bar{w}_1(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2) + \bar{w}_2(\bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2)\\
&-(w_1\bar{w}_2\bar{\partial}\bar{w}_1 + |w_2|^2\bar{\partial}\bar{w}_2)\},\\
&k_{32} = A^{-2}\{w_1\bar{\partial}\bar{w}_2 + |w_1|^2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2) + w_1\bar{w}_2(\bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2)\\
&-[\bar{w}_2\bar{\partial}w_1 + w_1\bar{w}_2(\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1) + |w_2|^2(\bar{w}_2|^2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2)]\},\\
&k_{33} = A^{-2}\{w_2\bar{\partial}\bar{w}_2 + \bar{w}_1 w_2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2) - (\bar{w}_2\bar{\partial}w_2 + w_1\bar{w}_2(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1))\},
\end{aligned} \tag{4.42}$$

and

$$\begin{aligned}
&l_{11} = A^{-2}\{\bar{w}_1\partial w_1 + \bar{w}_2\partial w_2 - (w_1\partial\bar{w}_1 + w_2\partial\bar{w}_2)\},\\
&l_{12} = A^{-2}\{|w_1|^2\partial w_1 + w_1\bar{w}_2\partial w_2 - [\partial w_1 + w_1(\bar{w}_1\partial w_1 + w_1\partial\bar{w}_1) + w_2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2)]\},\\
&l_{13} = A^{-2}\{\bar{w}_1 w_2\partial w_1 + |w_2|^2\partial w_2 - [\partial w_2 + w_1(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1) + w_2(\bar{w}_2\partial w_2 + w_2\partial\bar{w}_2)]\},
\end{aligned}$$

$$
\begin{aligned}
l_{21} &= A^{-2}\{\partial\bar{w}_1 + \bar{w}_1(\bar{w}_1\partial w_1 + w_1\partial\bar{w}_1) + \bar{w}_2(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1) - (|w_1|^2\partial\bar{w}_1 + \bar{w}_1 w_2\partial\bar{w}_2)\},\\
l_{22} &= A^{-2}\{w_1\partial\bar{w}_1 + w_1\bar{w}_2(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1) - [\bar{w}_1\partial w_1 + \bar{w}_1 w_2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2)]\},\\
l_{23} &= A^{-2}\{w_2\partial\bar{w}_1 + \bar{w}_1 w_2(\bar{w}_1\partial w_1 + w_1\partial\bar{w}_1) + |w_2|^2(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1)\\
&-[\bar{w}_1\partial w_2 + |w_1|^2(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1) + \bar{w}_1 w_2(\bar{w}_2\partial w_2 + w_2\partial\bar{w}_2)]\},\\
l_{31} &= A^{-2}\{\partial\bar{w}_2 + \bar{w}_1(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2) + \bar{w}_2(\bar{w}_2\partial w_2 + w_2\partial\bar{w}_2) - (w_1\bar{w}_2\partial\bar{w}_1 + |w_2|^2\partial\bar{w}_2)\},\\
l_{32} &= A^{-2}\{w_1\partial\bar{w}_2 + |w_1|^2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2) + w_1\bar{w}_2(\bar{w}_2\partial w_2 + w_2\partial\bar{w}_2)\\
&-[\bar{w}_2\partial w_1 + w_1\bar{w}_2(\bar{w}_1\partial w_1 + w_1\partial\bar{w}_1) + |w_2|^2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2)]\},\\
l_{33} &= A^{-2}\{w_2\partial\bar{w}_2 + \bar{w}_1 w_2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2) - (\bar{w}_2\partial w_2 + w_1\bar{w}_2(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1))\},
\end{aligned}
\tag{4.43}
$$

respectively. Finally from eqs (4.40), (4.42) and (4.43) we show that there exists only five independent conservation laws for CP^2 sigma model (4.6-4.8). Namely we have the following independent conserved quantities

$$
\begin{aligned}
&\partial\{A^{-2}[\bar{w}_1\bar{\partial}w_1 + \bar{w}_2\bar{\partial}w_2 - (w_1\bar{\partial}\bar{w}_1 + w_2\bar{\partial}w_2)]\} + \bar{\partial}\{A^{-2}[\bar{w}_1\partial w_1 + \bar{w}_2\partial w_2\\
&-(w_1\partial\bar{w}_1 + w_2\partial\bar{w}_2)]\} = 0,\\
&\\
&\partial\{A^{-2}[w_1\bar{\partial}\bar{w}_1 + w_1\bar{w}_2(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1) - \bar{w}_1\bar{\partial}w_1 - \bar{w}_1 w_2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2)]\}\\
&+\bar{\partial}\{A^{-2}[w_1\partial\bar{w}_1 + w_1\bar{w}_2(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1) - \bar{w}_1\partial w_1 - \bar{w}_1 w_2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2)]\} = 0,\\
&\\
&\partial\{A^{-2}[|w_1|^2\bar{\partial}w_1 + w_1\bar{w}_2\bar{\partial}w_2 - \bar{\partial}w_1 - w_1(\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1) - w_2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2)]\}\\
&+\bar{\partial}\{A^{-2}[|w_1|^2\partial w_1 + w_1\bar{w}_2\partial w_2 - \partial w_1 - w_1(\bar{w}_1\partial w_1 + w_1\partial\bar{w}_1) - w_2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2)]\} = 0,\\
&\\
&\partial\{A^{-2}[\bar{w}_1 w_2\bar{\partial}w_1 - \bar{\partial}w_2 - w_1(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1) - w_2^2\bar{\partial}\bar{w}_2]\}\\
&+\bar{\partial}\{A^{-2}[\bar{w}_1 w_2\partial w_1 - \partial w_2 - w_1(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1) - w_2^2\partial\bar{w}_2]\} = 0,\\
&\\
&\partial\{A^{-2}[w_2\bar{\partial}\bar{w}_1 + \bar{w}_1^2 w_2\bar{\partial}w_1 + |w_2|^2 w_2\bar{\partial}\bar{w}_1] - A^{-2}[\bar{w}_1\bar{\partial}w_2 + |w_1|^2\bar{w}_1\bar{\partial}w_2 + \bar{w}_1 w_2^2\bar{\partial}\bar{w}_2]\}\\
&+\bar{\partial}\{A^{-2}[w_2\partial\bar{w}_1 + \bar{w}_1^2 w_2\partial w_1 + |w_2|^2 w_2\partial\bar{w}_1] - A^{-2}[\bar{w}_1\partial w_2 + |w_1|^2\bar{w}_1\partial w_2\\
&+\bar{w}_1 w_2^2\partial\bar{w}_2]\} = 0.
\end{aligned}
\tag{4.44}
$$

Consequently, as a result of the conservation laws (4.44) there exist eight real-valued functions $X^i(z,\bar{z})$ $i = 1,\cdots,8$ expressed in terms of functions (w_1, w_2), i.e.

$$
\begin{aligned}
&X^1 = \alpha\int_C A^{-2}\{-[\bar{w}_1\partial w_1 + \bar{w}_2\partial w_2 - (w_1\partial\bar{w}_1 + w_2\partial\bar{w}_2)]dz + [\bar{w}_1\bar{\partial}w_1 + \bar{w}_2\bar{\partial}w_2\\
&-(w_1\bar{\partial}\bar{w}_1 + w_2\bar{\partial}\bar{w}_2)]d\bar{z}],\\
&\\
&X^2 = \alpha\int_C A^{-2}\{[(1+|w_2|^2)(w_1\partial\bar{w}_1 - \bar{w}_1\partial w_1) + |w_1|^2(\bar{w}_2\partial w_2 - w_2\partial\bar{w}_2)]dz\\
&+[(1+|w_2|^2)(w_1\bar{\partial}\bar{w}_1 - \bar{w}_1\bar{\partial}w_1) + |w_1|^2(\bar{w}_2\bar{\partial}w_2 - w_2\bar{\partial}\bar{w}_2)]d\bar{z}\},\\
&\\
&X^3 = i\int_C -A^{-2}\{[-(1+\bar{w}_1^2+|w_2|^2)\partial w_1\\
&-(1+w_1^2+|w_2|^2)\partial\bar{w}_1 + \bar{w}_2(w_1-\bar{w}_1)\partial w_2 + w_2(\bar{w}_1 - w_1)\partial\bar{w}_2]dz\\
&+[-(1+\bar{w}_1^2+|w_2|^2)\bar{\partial}w_1 - (1+w_1^2+|w_2|^2)\bar{\partial}\bar{w}_1 + \bar{w}_2(w_1-\bar{w}_1)\bar{\partial}w_2 + w_2(\bar{w}_1 - w_1)\bar{\partial}\bar{w}_2]\,d\bar{z}\},
\end{aligned}
$$

$$
\begin{aligned}
X^4 = & \int_C A^{-2}\{[(1-\bar{w}_1^2+|w_2|^2)\partial w_1 \\
& +(-1+w_1^2-|w_2|^2)\partial\bar{w}_1-\bar{w}_2(w_1+\bar{w}_1)\partial w_2+w_2(w_1+\bar{w}_1)\partial\bar{w}_2]dz \\
& +[(-1+\bar{w}_1^2-|w_2|^2)\bar{\partial}w_1+(1-w_1^2+|w_2|^2)\bar{\partial}\bar{w}_1+\bar{w}_2(w_1+\bar{w}_1)\bar{\partial}w_2-w_2(w_1+\bar{w}_1)\bar{\partial}\bar{w}_2]d\bar{z}\},
\end{aligned}
$$

$$
\begin{aligned}
X^5 = i\int_C & -A^{-2}\{[\bar{w}_1(w_2-\bar{w}_2)\partial w_1+w_1(\bar{w}_2-w_2)\partial\bar{w}_1 \\
& -(1+|w_1|^2+\bar{w}_2^2)\partial w_2-(1+|w_1|^2+w_2^2)\partial\bar{w}_2]dz \\
& +[\bar{w}_1(w_2-\bar{w}_2)\bar{\partial}w_1+w_1(\bar{w}_2-w_2)\bar{\partial}\bar{w}_1-(1+|w_1|^2+\bar{w}_2^2)\bar{\partial}w_2-(1+|w_1|^2+w_2^2)\bar{\partial}\bar{w}_2]d\bar{z}\},
\end{aligned}
$$

$$
\begin{aligned}
X^6 = \int_C & A^{-2}\{[-\bar{w}_1(w_2+\bar{w}_2)\partial w_1+w_1(w_2+\bar{w}_2)\partial\bar{w}_1 \\
& +(1+|w_1|^2-\bar{w}_2^2)\partial w_2-(1+|w_1|^2-w_2^2)\partial\bar{w}_2]dz \\
& +[\bar{w}_1(w_2+\bar{w}_2)\bar{\partial}w_1-w_1(w_2+\bar{w}_2)\bar{\partial}\bar{w}_2-(1+|w_1|^2-\bar{w}_2^2)\bar{\partial}w_2+(1+|w_1|^2-w_2^2)\bar{\partial}\bar{w}_2]d\bar{z}\},
\end{aligned}
$$

$$
\begin{aligned}
X^7 = i\int_C & -A^{-2}\{[\bar{w}_2(1+|w_2|^2)+\bar{w}_1^2w_2]\partial w_1+[w_2(1+|w_2|^2)+w_1^2\bar{w}_2]\partial\bar{w}_1 \\
& -[\bar{w}_1(1+|w_1|^2)+w_1\bar{w}_2^2]\partial w_2-[w_1(1+|w_1|^2)+\bar{w}_1w_2^2]\partial\bar{w}_2\}dz \\
& +A^{-2}\{[\bar{w}_2(1+|w_2|^2)+\bar{w}_1^2w_2]\bar{\partial}w_1+[w_2(1+|w_2|^2)+w_1^2\bar{w}_2]\bar{\partial}\bar{w}_1 \\
& -[\bar{w}_1(1+|w_1|^2)+w_1\bar{w}_2^2]\bar{\partial}w_2-[w_1(1+|w_1|^2)+\bar{w}_1w_2^2]\bar{\partial}\bar{w}_2\}d\bar{z},
\end{aligned}
$$

$$
\begin{aligned}
X^8 = \int_C & A^{-2}\{[\bar{w}_2(1+|w_2|^2)-\bar{w}_1^2w_2]\partial w_1+[-w_2(1+|w_2|^2)+w_1^2\bar{w}_2]\partial\bar{w}_1 \\
& +[\bar{w}_1(1+|w_1|^2)-w_1\bar{w}_2^2]\partial w_2+[-w_1(1+|w_1|^2)+\bar{w}_1w_2^2]\partial\bar{w}_2\}dz \\
& +A^{-2}\{[-\bar{w}_2(1+|w_2|^2)+\bar{w}_1^2w_2]\bar{\partial}w_1+[w_2(1+|w_2|^2)-w_1^2\bar{w}_2^2]\bar{\partial}\bar{w}_1 \\
& +[-\bar{w}_1(1+|w_1|^2)+w_1\bar{w}_2^2]\bar{\partial}w_2+[w_1(1+|w_1|^2)-\bar{w}_1w_2^2]\bar{\partial}\bar{w}_2\}d\bar{z}. \quad \alpha\in Z
\end{aligned}
\tag{4.45}
$$

Note that by virtue of conservation laws (4.44) for the CP^2 sigma model (4.6-4.8) the r.h.s. in expression (4.45) do not depend on the choice of the contour C in complex plane. This is due to the fact that the differential of equations (4.45) are exact ones. We identify the functions $X^i(z,\bar{z})$ $i=1,\cdots,8$ with the coordinates of the radius vector

$$
\mathrm{X}(z,\bar{z}) = (X^1(z,\bar{z}),\cdots,X^8(z,\bar{z})), \tag{4.46}
$$

of a two-dimensional surface immersed into eight-dimensional Euclidean space E^8. Substituting (4.10) into (4.45) we can express the radius vector $\mathrm{X}(z,\bar{z})$ in terms of functions $(\varphi_1,\varphi_2,\psi_1,\psi_2)$ which adopts the form

$$
\begin{aligned}
X^1 &= 2\alpha\int_C(\bar{\psi}_1\varphi_1+\bar{\psi}_2\varphi_2)dz+(\psi_1\bar{\varphi}_1+\psi_2\bar{\varphi}_2)d\bar{z}, \\
X^2 &= 2\alpha\int_C\{\frac{\psi_1}{\varphi_1}[\varphi_1^2-\frac{\psi_1}{\bar{\varphi}_1}(\bar{\psi}_1\varphi_1+\bar{\psi}_2\varphi_2)-\frac{\psi_1}{\bar{\varphi}_1}\partial(A^{-1})] \\
&\quad -\frac{A\bar{\varphi}_2\psi_1}{\Omega}[\frac{J}{A^2}\frac{\psi_2}{\bar{\varphi}_2}+(\partial(A^{-1})+\bar{\psi}_1\varphi_1+\bar{\psi}_2\varphi_2)\varphi_2^2]\}dz
\end{aligned}
$$

$$
\begin{aligned}
&+\{\frac{\psi_1}{\bar\varphi_1}[\bar\varphi_1^2-\frac{\bar\psi_1}{\varphi_1}(\psi_1\bar\varphi_1+\psi_2\bar\varphi_2)-\frac{\bar\psi_1}{\varphi_1}\bar\partial(A^{-1})]\\
&-\frac{A\varphi_2\bar\psi_1}{\bar\Omega}[\frac{\bar J}{A^2}\frac{\bar\psi_2}{\varphi_2}+(\bar\partial(A^{-1})+\psi_1\bar\varphi_1+\psi_2\bar\varphi_2)\bar\varphi_2^2]\}d\bar z,
\end{aligned}
$$

$$
\begin{aligned}
X^3 \;=\;& i\int_C -\{-\frac{\bar\psi_1}{\varphi_1}[\partial(A^{-1})+2(\bar\psi_1\varphi_1+\bar\psi_2\varphi_2)]\\
&-\frac{A\bar\varphi_1\bar\varphi_2}{\Omega}[\frac{J}{A^2}\frac{\psi_2}{\bar\varphi_2}+(\partial(A^{-1})+\bar\psi_1\varphi_1+\bar\psi_2\varphi_2)\varphi_2^2]\\
&+[-\varphi_1^2+\frac{\psi_1}{\bar\varphi_1}(\bar\psi_1\varphi_1+\bar\psi_2\varphi_2)+\frac{\psi_1}{\bar\varphi_1}\partial(A^{-1})]\}dz\\
&+\{-\frac{\psi_1}{\bar\varphi_1}[\bar\partial(A^{-1})+2(\psi_1\bar\varphi_1+\psi_2\bar\varphi_2)]\\
&-\frac{A\varphi_1\varphi_2}{\bar\Omega}[\frac{\bar J}{A^2}\frac{\bar\psi_2}{\varphi_2}+(\bar\partial(A^{-1})+\psi_1\bar\varphi_1+\psi_2\varphi_2)\bar\varphi_2^2]\\
&+[-\bar\varphi_1^2+\frac{\bar\psi_1}{\varphi_1}(\psi_1\bar\varphi_1+\psi_2\bar\varphi_2)+\frac{\bar\psi_1}{\varphi_1}\bar\partial(A^{-1})]\}d\bar z,
\end{aligned}
$$

$$
\begin{aligned}
X^4 \;=\;& \int_C \{-\frac{\bar\psi_1}{\varphi_1}[\partial(A^{-1})+2(\bar\psi_1\varphi_1+\bar\psi_2\varphi_2)]\\
&-\frac{A\bar\varphi_1\bar\varphi_2}{\Omega}[\frac{J}{A^2}\frac{\psi_2}{\bar\varphi_2}+(\partial(A^{-1})+\bar\psi_1\varphi_1+\bar\psi_2\varphi_2)\varphi_2^2]\\
&-[-\varphi_1^2+\frac{\psi_1}{\bar\varphi_1}(\bar\psi_1\varphi_1+\bar\psi_2\varphi_2)+\frac{\psi_1}{\bar\varphi_1}\partial(A^{-1})]\}dz\\
&\{-\frac{\psi_1}{\bar\varphi_1}[\bar\partial(A^{-1})+2(\psi_1\bar\varphi_1+\psi_2\bar\varphi_2)]\\
&-\frac{A\varphi_1\varphi_2}{\bar\Omega}[\frac{\bar J}{A^2}\frac{\bar\psi_2}{\varphi_2}+(\bar\partial(A^{-1})+\psi_1\bar\varphi_1+\psi_2\bar\varphi_2)\bar\varphi_2^2]\\
&-[-\bar\varphi_1^2+\frac{\bar\psi_1}{\varphi_1}(\psi_1\bar\varphi_1+\psi_2\bar\varphi_2)+\frac{\bar\psi_1}{\varphi_1}\bar\partial(A^{-1})]\}d\bar z,
\end{aligned}
$$

$$
\begin{aligned}
X^5 \;=\;& i\int_C -\{-\frac{\bar\psi_2}{\varphi_2}[\partial(A^{-1})+2(\bar\psi_1\varphi_1+\bar\psi_2\varphi_2)]\\
&+\frac{A\bar\varphi_1\bar\varphi_2}{\Omega}[(\partial(A^{-1})+\bar\psi_1\varphi_1+\bar\psi_2\varphi_2)\varphi_1^2+\frac{J}{A^2}\frac{\psi_1}{\bar\varphi_1}]\\
&+[-\varphi_2^2+\frac{\psi_2}{\bar\varphi_2}(\bar\psi_1\varphi_1+\bar\psi_2\varphi_2)+\frac{\psi_2}{\bar\varphi_2}\partial(A^{-1})]\}dz\\
&+\{-\frac{\psi_2}{\bar\varphi_2}[\bar\partial(A^{-1})+2(\psi_1\bar\varphi_1+\psi_2\bar\varphi_2)]\\
&+\frac{A\varphi_1\varphi_2}{\bar\Omega}[(\bar\partial(A^{-1})+\psi_1\bar\varphi_1+\psi_2\bar\varphi_2)\bar\varphi_1^2+\frac{\bar J}{A^2}\frac{\bar\psi_1}{\varphi_1}]\\
&+[-\bar\varphi_2^2+\frac{\bar\psi_2}{\varphi_2}(\psi_1\bar\varphi_1+\psi_2\bar\varphi_2)+\frac{\bar\psi_2}{\varphi_2}\bar\partial(A^{-1})]\}d\bar z,
\end{aligned}
$$

$$
X^6 \;=\; \int_C \{-\frac{\bar\psi_2}{\varphi_2}[\partial(A^{-1})+2(\bar\psi_1\varphi_1+\bar\psi_2\varphi_2)]
$$

$$
\begin{aligned}
&+\frac{A\bar{\varphi}_1\bar{\varphi}_2}{\Omega}[(\partial(A^{-1})+\bar{\psi}_1\varphi_1+\bar{\psi}_2\varphi_2)\varphi_1^2+\frac{J}{A^2}\frac{\psi_1}{\bar{\varphi}_1}]\\
&-[-\varphi_2^2+\frac{\psi_2}{\bar{\varphi}_2}(\bar{\psi}_1\varphi_1+\bar{\psi}_2\varphi_2)+\frac{\psi_2}{\bar{\varphi}_2}\partial(A^{-1})]\}dz\\
&+\{-\frac{\psi_2}{\bar{\varphi}_2}[\bar{\partial}(A^{-1})+2(\psi_1\bar{\varphi}_1+\psi_2\bar{\varphi}_2)]\\
&+\frac{A\varphi_1\varphi_2}{\bar{\Omega}}[(\bar{\partial}(A^{-1})+\psi_1\bar{\varphi}_1+\psi_2\bar{\varphi}_2)\bar{\varphi}_1^2+\frac{\bar{J}}{A^2}\frac{\bar{\psi}_1}{\varphi_1}]\\
&-[-\bar{\varphi}_2^2+\frac{\bar{\psi}_2}{\varphi_2}(\psi_1\bar{\varphi}_1+\psi_2\bar{\varphi}_2)+\frac{\bar{\psi}_2}{\varphi_2}\bar{\partial}(A^{-1})]\}d\bar{z},
\end{aligned}
$$

$$
\begin{aligned}
X^7 \;=\;& i\int_C -\{\frac{\bar{\psi}_2}{\varphi_2}[\varphi_1^2-\frac{\psi_1}{\bar{\varphi}_1}(\bar{\psi}_1\varphi_1+\bar{\psi}_2\varphi_2)-\frac{\psi_1}{\bar{\varphi}_1}\partial(A^{-1})]\\
&+\frac{A\bar{\varphi}_2\psi_1}{\Omega}[(\partial(A^{-1})+\bar{\psi}_1\varphi_1+\bar{\psi}_2\varphi_2)\varphi_1^2+\frac{J}{A^2}\frac{\psi_1}{\bar{\varphi}_1}]\\
&+\frac{A\bar{\varphi}_1\psi_2}{\Omega}[\frac{J}{A^2}\frac{\psi_2}{\bar{\varphi}_2}+(\partial(A^{-1})+\bar{\psi}_1\varphi_1+\bar{\psi}_2\varphi_2)\varphi_2^2]\\
&+\frac{\bar{\psi}_1}{\varphi_1}[-\varphi_2^2+\frac{\psi_2}{\bar{\varphi}_2}(\bar{\psi}_1\varphi_1+\bar{\psi}_2\varphi_2)+\frac{\psi_2}{\bar{\varphi}_2}\partial(A^{-1})]\}dz\\
&+\{\frac{\psi_2}{\bar{\varphi}_2}[\bar{\varphi}_1^2-\frac{\bar{\psi}_1}{\varphi_1}(\psi_1\bar{\varphi}_1+\psi_2\bar{\varphi}_2)-\frac{\bar{\psi}_1}{\varphi_1}\bar{\partial}(A^{-1})]\\
&+\frac{A\varphi_2\bar{\psi}_1}{\Omega}[(\bar{\partial}(A^{-1})+\psi_1\bar{\varphi}_1+\psi_2\bar{\varphi}_2)\bar{\varphi}_1^2+\frac{\bar{J}}{A^2}\frac{\bar{\psi}_1}{\varphi_1}]\\
&+\frac{A\varphi_1\bar{\psi}_2}{\bar{\Omega}}[\frac{\bar{J}}{A^2}\frac{\bar{\psi}_2}{\varphi_2}+(\bar{\partial}(A^{-1})+\psi_1\bar{\varphi}_1+\psi_2\bar{\varphi}_2)\bar{\varphi}_2^2]\\
&+\frac{\psi_1}{\bar{\varphi}_1}[-\bar{\varphi}_2^2+\frac{\bar{\psi}_2}{\varphi_2}(\psi_1\bar{\varphi}_1+\psi_2\bar{\varphi}_2)+\frac{\bar{\psi}_2}{\varphi_2}\bar{\partial}(A^{-1})]\}d\bar{z},
\end{aligned}
$$

$$
\begin{aligned}
X^8 \;=\;& \int_C \{\frac{\bar{\psi}_2}{\varphi_2}[\varphi_1^2-\frac{\psi_1}{\bar{\varphi}_1}(\bar{\psi}_1\varphi_1+\bar{\psi}_2\varphi_2)-\frac{\psi_1}{\bar{\varphi}_1}\partial(A^{-1})]\\
&+\frac{A\bar{\varphi}_2\psi_1}{\Omega}[(\partial(A^{-1})+\bar{\psi}_1\varphi_1+\bar{\psi}_2\varphi_2)\varphi_1^2+\frac{J}{A^2}\frac{\psi_1}{\bar{\varphi}_1}]\\
&-\frac{A\bar{\varphi}_1\psi_2}{\Omega}[\frac{J}{A^2}\frac{\psi_2}{\bar{\varphi}_2}+(\partial(A^{-1})+\bar{\psi}_1\varphi_1+\bar{\psi}_2\varphi_2)\varphi_2^2]\\
&-\frac{\bar{\psi}_1}{\varphi_1}[-\varphi_2^2+\frac{\psi_2}{\bar{\varphi}_2}(\bar{\psi}_1\varphi_1+\bar{\psi}_2\varphi_2)+\frac{\psi_2}{\bar{\varphi}_2}\partial(A^{-1})]\}dz\\
&+\{\frac{\psi_2}{\bar{\varphi}_2}[\bar{\varphi}_1^2-\frac{\bar{\psi}_1}{\varphi_1}(\psi_1\bar{\varphi}_1+\psi_2\bar{\varphi}_2)-\frac{\bar{\psi}_1}{\varphi_1}\bar{\partial}(A^{-1})]\\
&+\frac{A\varphi_2\bar{\psi}_1}{\bar{\Omega}}[(\bar{\partial}(A^{-1})+\psi_1\bar{\varphi}_1+\psi_2\bar{\varphi}_2)\bar{\varphi}_1^2+\frac{\bar{J}}{A^2}\frac{\bar{\psi}_1}{\varphi_1}]\\
&-\frac{A\varphi_1\bar{\psi}_2}{\bar{\Omega}}[\frac{\bar{J}}{A^2}\frac{\bar{\psi}_2}{\varphi_2}+(\bar{\partial}(A^{-1})+\psi_1\bar{\varphi}_1+\psi_2\bar{\varphi}_2)\bar{\varphi}_2^2]\\
&-\frac{\psi_1}{\bar{\varphi}_1}[-\bar{\varphi}_2^2+\frac{\bar{\psi}_2}{\varphi_2}(\psi_1\bar{\varphi}_1+\psi_2\bar{\varphi}_2)+\frac{\bar{\psi}_2}{\varphi_2}\bar{\partial}(A^{-1})]\}d\bar{z},
\end{aligned}
\tag{4.47}
$$

where we introduce the following notation

$$\Omega = \varphi_1\psi_2|\varphi_1|^2 - \varphi_2\psi_1|\varphi_2|^2, \quad \bar{\Omega} = \bar{\varphi}_1\bar{\psi}_2|\varphi_1|^2 - \bar{\varphi}_2\bar{\psi}_1|\varphi_2|^2. \;\; \alpha \in Z \tag{4.48}$$

Under the assumption that the current J expressed in terms of w, relation (4.29),vanishes one can check that the position vector X, given by (4.45) with $\alpha = 6$, obeis the following relations

$$(\partial X, \partial X) = 0, \tag{4.49}$$

and

$$(\partial\bar{\partial}X, \partial\bar{\partial}X) = (\partial X, \bar{\partial}X)^2 \neq 0, \tag{4.50}$$

whenever the CP^2 sigma model (4.6-4.8) is satisfied. The explicit form of relation (4.50) in terms of w's or ψ's and ϕ's are quite complicated expression, so we shall not write it explicitly here. As a consequence of (4.49) and (4.50) the components of induced metric are

$$g_{zz} = g_{\bar{z}\bar{z}} = 0, \qquad g_{z\bar{z}} \neq 0 \tag{4.51}$$

and the norm of the mean curvature vector $\bar{H} = (g_{z\bar{z}})^{-1}\partial\bar{\partial}X$ is equal to one, i.e.$|\bar{H}|^2 = 1$. We would like to note that when w_i are holomorphic functions then $J = 0$ and formulae (4.45) define a surface on group $SU(3)$. Using expression in [35] we can calculate in a closed form for a given surface all geometric characteristics.

Having found the above we can conclude the following.

Proposition 4. The conformal immersion of CMC-surfaces into R^8 are determine by formulae (4.45) or (4.47), where the complex functions w_i have to obey the CP^2 sigma model equations (4.6-4.8),(or complex functions ψ_i and ϕ_i have to obey the first order system (4.1-4.4)) and the equation (4.29) (or (4.31)) for the current J equal to zero.

Note that the Weierstrass type representation (4.45) could be useful in the investigation of $N = 2$ superstring [34].

5 Examples and applications.

Now,based on the Proposition 4 we will construct certain classes of two-dimensional CMC-surfaces immersed into R^8.For this purpose we use the CP^2 sigma model defined over S^2. Note that for such model all solutions of the Euler-Lagrange equations (4.6-4.8) are well known [29]. Under the requirement of the finiteness of the action they split into three separate classes,i.e. analytic (i.e.$w_i = w_i(z)$),antianalytic (i.e.$w_i = w_i(\bar{z})$) and mixed one (i.e. the so called nonsplitting solutions of (4.6-4.8)). The latter one can be determined from either the holomorphic or antiholomorphic functions by the following procedure.

Consider three arbitrary holomorphic functions $f_i = f_i(z)$ and define for each pair the Wronskian

$$F_{ij} = f_i\partial f_j - f_j\partial f_i, \qquad \bar{\partial}f_i = 0 \qquad i, j = 1, 2, 3 \tag{5.1}$$

Next determine three complex valued functions

$$g_i = \sum_{k\neq i}^{3} \bar{f}_k(f_k\partial f_i - f_i\partial f_k) \qquad i = 1, 2, 3 \tag{5.2}$$

Then the solution w_i of CP^2 sigma model(4.6-4.8) can be determined as ratios of the components of g_i, i.e.

$$w_1 = \frac{g_1}{g_3}, \qquad w_2 = \frac{g_2}{g_3}, \qquad g_3 \neq 0 \tag{5.3}$$

Alternatively, similar class of solutions can be obtained when we consider three arbitrary antiholomorphic functions $f_i = f_i(\bar{z})$ and construct g_i in the same way as above,but using $\bar{\partial}$ instead of ∂ in the equations (5.2).

Now, let us discuss some classes of CMC-surfaces in R^n which can be obtained directly by applying the Weierstrass representation (4.45) and (5.3).

1. One of the simplest class of solutions which correspond to analytic choice of functions are

$$w_1 = z, \qquad w_2 = 1, \qquad A = 2 + |z|^2 \qquad J = 0. \tag{5.4}$$

Using CP^2 representation (4.45) we can find that the associated CMC-surface is immersed in R^3 and is given in a polar coordinates $(r = (x^2 + y^2)^{1/2}, \varphi)$ by

$$\begin{aligned} &X^1 = -2X^2 = 2X^6 = 2\sqrt{2}(2 + r^2)^{-1}, \quad X^3 = -X^7 = -2r(2 + r^2)^{-1} \sin\varphi \\ &X^5 = 0 \quad X^4 = X^8 = 2r(2 + r^2)^{-1} \cos\varphi \end{aligned} \tag{5.5}$$

The metric is conformally flat

$$I = \frac{4}{(2 + r^2)^2} (dr^2 + r^2 d\varphi^2) \tag{5.6}$$

This particular case corresponds to the immersion of CP^1 model into CP^2 model.

2. The class of two-soliton solutions of the CP^2 model (4.6-4.8) is determined for example by two analytic functions

$$w_1 = z^2, \qquad w_2 = \sqrt{2}z, \qquad A = (1 + |z|^2), \qquad J = 0 \tag{5.7}$$

Integrating formulae (4.45) we obtain the associated CMC-surface which can be written in a polar coordinates as follows

$$\begin{aligned} &X^1 = 2(1 + r^2)^{-2}, \qquad X^2 = -2(1 + 2r^2)(1 + r^2)^{-2}, \qquad X^3 = -2r^2(1 + r^2)^{-2} \sin 2\varphi, \\ &X^4 = 2r^2(1 + r^2)^{-2} \cos 2\varphi, \qquad X^5 = -2\sqrt{2}r(1 + r^2)^{-2} \sin\varphi, \\ &X^6 = 2\sqrt{2}r(1 + r^2)^{-2} \cos\varphi, \qquad X^7 = \sqrt{2}(2r^2 - 3)r(1 + r^2)^{-2} \sin\varphi, \\ &X^8 = \sqrt{2}(2r^2 - 3)r(1 + r^2)^{-2} \cos\varphi, \end{aligned} \tag{5.8}$$

The corresponding first fundamental form is conformal

$$I = \frac{2}{(1 + r^2)^2} (dr^2 + r^2 d\varphi^2) \tag{5.9}$$

3. A class of nonanalytic solutions of the CP^2 model (4.2) is provided by

$$\begin{aligned} &w_1 = \frac{\bar{z} + z}{1 - |z|^2}, \quad w_2 = \frac{\bar{z} - z}{1 - |z|^2}, \\ &A = (\frac{1 + |z|^2}{1 - |z|^2})^2, \quad J = 0. \end{aligned} \tag{5.10}$$

From (4.32) and using (5.2) we obtain the expression for the associated surface which can be written in polar coordinates as follows

$$\begin{aligned} &X^1 = X^2 = X^4 = X^5 = X^7 = 0, \\ &X^3 = \frac{-4r}{1 + r^2} \sin\varphi, \quad X^6 = \frac{-4r}{1 + r^2} \cos\varphi, \quad X^8 = \frac{4}{1 + r^2} \end{aligned} \tag{5.11}$$

Hence the CMC-surface is immersed in R^3. The metric is conformal

$$I = \frac{16}{(1 + r^2)^2} (dr^2 + r^2 d\varphi^2) \tag{5.12}$$

This case corresponds to the immersion of the CP^1 model into the CP^2 model.

6 Final remarks and future developments

We have demonstrated links between the CP^1 and CP^2 sigma models and Wieirstrass representations for two-dimensional surfaces immersed into Euclidean spaces R^3 and R^8, respectively. These links enabled us to propose an algorithm for the construction of CMC-surfaces immersed into R^n. This new approach is tested in Section 5. It proved to be effective, as we were able to reproduce easily the known results which were obtained usually by much more complicated procedure. Its potential for providing new meaningful results was shown in the case of nonsplitting solutions of CP^2 sigma model leading to new interesting surfaces in R^8.

The analytic method of construction CMC-surfaces immersed into R^n presented here is limited by several assumptions. For example we study only low dimensional sigma models CP^1 and CP^2 defined over S^2. The question arises whether our approach can be extended to higher dimensional CP^N sigma models and to Weierstrass systems describing surfaces immersed in multi-dimensional Euclidean and pseudo-Riemannian spaces. If this may provide new classes of solutions and consequently new classes of surfaces in these multi-dimensional spaces. Other requirement of the proposed method, worth investigating is the CP^N models involving maps from R^2 (not necessarly S^2) if we want to have nontrivial topology to CP^N. We can expect that relaxing it can broaden the applicability of our approach.

Finally, it is worth noticing that the CMC-surfaces can be used "in reverse" to address certain physical problems. Namely, we sometimes know the analytical description of CMC-surfaces in the physical systems for which analytic models are not fully developed. Using our approach we can select appropriate sigma model corresponding to the given Weierstrass representation and characterise the class of equations describing the physical phenomena in question. This was attempted successfully for Weierstrass representation for CMC-surfaces in 3-dimensional Euclidean space [36], but not to our knowledge for multi-dimensional spaces. These and other questions will be addressed in future work.

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