Geometric Aspects of CP^N Harmonic Maps

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CRM-2853

August 2002

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Abstract

We introduce a Weierstrass-like system of equations corresponding to CP^N fields which generalise the systems, previously constructed, for CP^1 and CP^2. We use a set of conserved quantities for the CP^N model to suggest a possible geometrical interpretation of such maps.

Résumé

Dans ce papier on effectue une étude du système des équations de type de Weierstrass correspondant aux transformations harmoniques CP^N. Celui-ci généralise les systèmes correspondant aux champs CP^1 et CP^2. Une série des quantités conservées sont dérivées à partir du problème spectral linéaire du modèle CP^N. Nous servons de ces quantités pour construire un modèle généralisé de la représentation de Weierstrass décrivant des surfaces paramétrisées conformément plongées dans des espaces euclidiens multidimensionnels. Nous présentons une interprétation géométrique du système de Weierstrass proposé.

1 Introduction

Sigma models in two spatial dimensions have been studied for variety of reasons. They are low dimensional analogues of four-dimensional Yang-Mills theories which play a pivotal role in particle physics, they arise in some areas of condensed-matter physics *etc* and ... they are also interesting from a purely mathematical point of view.

Of course, there are many classes of σ models; amongst them particularly important are the so-called CP^{N-1} σ models. These models, all, possess topological properties and, as such, lead to the appearance of "topological solitons".

The models are a generalisation of the, perhaps the simplest, σ model, namely the S^2 model - also called the vector $O(3)$ model. The CP^{N-1} models involve maps from R^2, or S^2 if one wants to have topology, to CP^{N-1}. It is easiest to define them in terms of the Lagrangian density[1][3]

$$L\,=\,\frac{1}{4}(D_\mu z)^\dagger\cdot D_\mu z, \tag{1}$$

where z is a vector field of N components, which satisfies

$$z^\dagger\cdot z\,=\,1 \tag{2}$$

and where

$$D_\mu\,=\,\partial_\mu\,-\,z\,(z^\dagger\cdot\partial_\mu z). \tag{3}$$

Here $\mu=1,2$, of course, and denotes x and y.

The total Lagrangian is then given by

$$\mathcal{L}\,=\,\int L\,dx\,dy \tag{4}$$

and if the model is defined over S^2 we require that L is finite.

Defining

$$z\,=\,\frac{f}{|f|} \tag{5}$$

it is easy to check that the Euler Lagrange equations for f are

$$\left(1\,-\,\frac{f f^\dagger}{|f|^2}\right)\left[\partial\bar\partial f\,-\,\partial f\,\frac{(f^\dagger\cdot\bar\partial f)}{|f|^2}\,-\,\bar\partial f\,\frac{(f^\dagger\cdot\partial f)}{|f|^2}\right]\,=\,0, \tag{6}$$

where we have introduced holomorphic derivatives; *ie* where

$$\partial=\frac{\partial}{\partial(x+iy)}\,=\,\frac{\partial}{\partial\zeta} \tag{7}$$

and where $\bar\partial$, of course, is the derivative with respect to $\bar\zeta=x-iy$.

As is well known [4] equations (6) can be written as a compatibility condition for a set of two linear spectral equations for a N component auxiliary vector Ψ

$$\partial\Psi\,=\,\frac{2}{1+\lambda}\,[\partial P,\,P]\,\Psi$$
$$\bar\partial\Psi\,=\,\frac{2}{1-\lambda}\,[\bar\partial P,\,P]\,\Psi, \tag{8}$$

where the matrix P is the projector given by

$$P\,=\,\frac{1}{A}\,f\,f^\dagger, \tag{9}$$

[3]Summation over the repeated indices is assumed throughout this paper

where $A = f^\dagger \cdot f$.

The compatibility conditions for (8) are, clearly,

$$[\partial\bar{\partial}P,\, P] \,=\, 0 \tag{10}$$

which, as can be easily checked, are equivalent to (6).

Note that (10) can be written in the form of a conservation law

$$\partial\,[\bar{\partial}P,\, P] \,+\, \bar{\partial}\,[\partial P,\, P] \,=\, 0 \tag{11}$$

i.e.

$$\partial K \,+\, \bar{\partial} M \,=\, 0, \tag{12}$$

where the matrices K and L are given by

$$K \,=\, [\bar{\partial}P,\, P] \,=\, \frac{\bar{\partial}f\, f^\dagger \,-\, f\,\bar{\partial}f^\dagger}{|f|^2} \,+\, \frac{f\, f^\dagger}{|f|^4}\left[(\bar{\partial}f^\dagger \cdot f) - (f^\dagger \cdot \bar{\partial}f)\right] \tag{13}$$

and

$$M \,=\, [\partial P,\, P] \,=\, \frac{\partial f\, f^\dagger \,-\, f\,\partial f^\dagger}{|f|^2} \,+\, \frac{f\, f^\dagger}{|f|^4}\left[(\partial f^\dagger \cdot f) - (f^\dagger \cdot \partial f)\right]. \tag{14}$$

Note that we due to the invariance of the Lagrangian L under

$$z \,\rightarrow z' \,=\, z\, e^{i\varphi} \tag{15}$$

we can set one of the components of f, say f_1 to 1. Then, in the CP^1 case, all quantities are expressible through $W = \frac{f_2}{f_1} = f_2$.

Recently, there has been a lot of interest in relating CP^1 maps to the solutions of the Weierstrass problem.[2, 5]

In this case one considers a set of first order equations for complex fields φ and ψ given by:

$$\partial\,\psi \,=\, p\,\varphi, \qquad \bar{\partial}\varphi \,=\, -p\,\psi. \tag{16}$$

Here $p \,=\, |\varphi|^2 + |\psi|^2$. In [5] it was shown that the relation between CP^1 maps and the φ, ψ fields of the Weierstarss problem is given by (up to an overall multiplication of φ and ψ by -1)

$$\psi \,= W\,\frac{(\bar{\partial}\bar{W})^{\frac{1}{2}}}{1+|W|^2}, \qquad \varphi \,=\, \frac{(\partial W)^{\frac{1}{2}}}{1+|W|^2}, \tag{17}$$

where

$$W \,=\, \frac{\psi}{\bar{\varphi}}, \tag{18}$$

and, of course, their respective complex conjugates.

Then, from the Weierstrass system one can construct a system of 3 real variables X_i, $i = 1, 2, 3$ and, treating $X_i(x, y)$ as a map of R^2 into R^3 discuss the geometry of the original maps.

In a recent paper we have generalised this construction to the CP^2 case.

In this paper we present a further generalisation to the CP^N case.

2 CP^N model

To present our generalisation let us look first at the general elements of the matrices K and M.

Thus

$$K_{ij} \,=\, \frac{1}{A^2}\left[\bar{f}_k\, f_k\, \bar{\partial}f_i\, \bar{f}_j \,-\, \bar{f}_k\, f_k\, f_i\, \bar{\partial}\bar{f}_j \,+\, f_i\, \bar{f}_j\, \bar{\partial}\bar{f}_k\, f_k \,-\, f_i\, \bar{f}_j\, \bar{f}_k\, \bar{\partial}f_k\right] \tag{19}$$

and

$$M_{ij} \,=\, \frac{1}{A^2}\left[\bar{f}_k\, f_k\, \partial f_i\, \bar{f}_j \,-\, \bar{f}_k\, f_k\, f_i\, \partial \bar{f}_j \,+\, f_i\, \bar{f}_j\, \partial \bar{f}_k\, f_k \,-\, f_i\, \bar{f}_j\, \bar{f}_k\, \partial f_k\right]. \tag{20}$$

Let us introduce

$$F_{ij} \,=\, f_i\, \partial f_j \,-\, f_j \partial f_i \tag{21}$$

and

$$\tilde{F}_{ij} \,=\, f_i\, \bar{\partial} f_j \,-\, f_j \bar{\partial} f_i. \tag{22}$$

Then

$$K_{ij} \,=\, \bar{f}_j \bar{\Phi}_i^2 \,-\, f_i \bar{\varphi}_j^2 \tag{23}$$

and

$$M_{ij} \,=\, \bar{f}_j \varphi_i^2 \,-\, f_i \Phi_j^2, \tag{24}$$

where we have defined

$$\varphi_i^2 \,=\, \frac{1}{A^2}\, \bar{f}_k\, F_{ki} \tag{25}$$

and

$$\Phi_i^2 \,=\, \frac{1}{A^2}\, f_k\, \bar{\tilde{F}}_{ki}. \tag{26}$$

Note that we have two constraints; namely,

$$\bar{f}_k\, \varphi_k^2 \,=\, 0, \qquad f_k\, \Phi_k^2 \,=\, 0 \tag{27}$$

which tell us that we have only $N-1$ independent φ_i's *ie* in our further discussion we can take as independent $\varphi_2, ... \varphi_N$ (and similarily for Φ_i). At the same time, using symmetry (15), we can set, say, $f_1 = 1$ and so we end up with

$$\varphi_i^2 \,=\, \frac{1}{A^2}\left[(1 + f_k \bar{f}_k)\, \partial\, f_i \,-\, f_i(\bar{f}_k\, \partial f_k)\right], \tag{28}$$

where $A = 1 + |f_2|^2 + |f_3|^2 ... + |f_N|^2$ and all the sums over repeated indices run over $k = 2...N$. Note that the $k = i$ term cancels between two contributions in (28) leaving just ∂f_i.

This allows us to invert (28) and so express all ∂f_i in terms of φ_l's. This way we find

$$\partial f_i \,=\, A\, \left[\varphi_i^2 \,+\, f_i\, \bar{f}_k \varphi_k^2\right]. \tag{29}$$

Thus in the CP^1 case we have

$$\partial f_2 \,=\, A(1 + |f_2|^2)\varphi_2^2, \tag{30}$$

and f_2 is often denoted by W, while in the CP^2 case we have

$$\begin{aligned} \partial f_2 \,&=\, A\left[(1 + |f_2|^2)\varphi_2^2 \,+\, f_2 \bar{f}_3\, \varphi_3^2\right] \\ \partial f_3 \,&=\, A\left[(1 + |f_3|^2)\varphi_3^2 \,+\, f_3 \bar{f}_2\, \varphi_2^2\right] \end{aligned} \tag{31}$$

with corresponding appropriate A factors. Note that in [3] f_2 and f_3 are denoted by W_1 and W_2.

All this discussion can be repeated for Φ_i's with the straightforward changes $\partial \to \bar{\partial}$ and complex conjugation.

3 Generalised Weierstrass System

To introduce a generalised Weierstrass system we need a set of φ_i and ψ_i which generalise the φ and ψ of the CP^1 case and φ_i, ψ_i, $i=1,2$ of the CP^2 case.

Note that our φ_i $i=2,..N$, defined in (25) provide such a choice as (30) agrees thr definition of φ in (17). So what should we use for ψ_i? Clearly (18) suggests that we put

$$\psi_i \,=\, f_i\,\bar\varphi_i. \qquad \text{no} \quad \text{summation} \tag{32}$$

Then to complete the generalisation of the Weierstrass system we need analogues of (16), *i.e.* we need to prescribe $\bar\partial\varphi_i$ and $\partial\psi_i$. Note that as (no summation)

$$\partial\psi_i \,=\, \partial\,(f_i\,\bar\varphi_i) \,=\, \partial f_i\,\bar\varphi_i \,+\, f_i\,\left(\bar\partial\varphi_i\right)^\star \tag{33}$$

where *, like $^-$, also denotes complex conjugation. So we need only to specify $\bar\partial\varphi_i$.

To do this note that as

$$\varphi_i^2 \,=\, \frac{1}{A}\,\partial f_i \,-\, f_i\,\frac{\bar f\cdot\partial f}{A^2} \tag{34}$$

we have

$$\bar\partial\varphi_i^2 \,=\, -2\frac{\varphi_i^2}{A^3}\,\left(\bar f_k\,\bar\partial f_k \,+\, f_k\,\bar\partial\bar f_k\right)$$
$$+\,\frac{1}{A^2}\left[(1+|f|^2)\,\partial\bar\partial f_i \,+\, (\bar f_k\bar\partial f_k)\,\partial f_i \,-\, \bar\partial f_i\,(\bar f_k\partial f_k) \,-\, f_i\,(\bar f_k\,\partial\bar\partial f_k)\right] \tag{35}$$

However, the equations for f_i (6) give us

$$\partial\bar\partial f_i \,=\, f_i\,\frac{(\bar f_k\,\partial\bar\partial f_k)}{A} \,+\, \partial f_i\frac{(\bar f_k\,\bar\partial f_k)}{A} \,+\, \bar\partial f_i\frac{(\bar f_k\,\partial f_k)}{A} \,-\, 2f_i\frac{(\bar f_k\,\partial f_k)(\bar f_l\,\bar\partial f_l)}{A^2} \tag{36}$$

we note that all the terms involving $\bar\partial f$ and $\partial\bar f$ in (35) cancel and we end up with

$$\bar\partial\varphi_i \,=\, -\frac{\varphi_i}{2A}\,(f_k\,\bar\partial\bar f_k). \tag{37}$$

However, from (29) we note that

$$\bar\partial\bar f_k \,=\, A\,\left[\bar\varphi_k^2 \,+\, \bar f_k\,f_l\bar\varphi_l^2\right] \tag{38}$$

and so we have

$$\bar\partial\varphi_i \,=\, -\varphi_i\,A\,(f_k\bar\varphi_k^2) \,=\, -\varphi_i\,A\,(\psi\cdot\bar\varphi). \tag{39}$$

The equations for ψ_i then follow from (33); we get

$$\partial\psi_i \,=\, A\,\varphi_i^2\,\bar\varphi_i \,=\, \varphi_i\,A\,|\varphi_i|^2, \qquad \text{no summation} \tag{40}$$

Thus, our **modified Weierstrass system** is a set of equations for φ_i, ψ_i, where $i=2,3...N$ given by

$$\bar\partial\varphi_i \,=\, -\varphi_i\,A\,(\psi\cdot\bar\varphi), \qquad \partial\psi_i \,=\, \varphi_i\,A\,|\varphi_i|^2 \qquad \text{no summation,} \tag{41}$$

where

$$A = 1 \,+\, \sum_{k=1}^{N}\frac{|\psi_k|^2}{|\varphi_k|^2}. \tag{42}$$

From our construction it is clear that this system of equations is equivalent to the equations of the CP^{N-1} model.

Moreover, it is easy to check that our system of equations, for $N=1,2$ reduces to the equations studied before [7].

4 Geometrical Connections

Next we define a set of real variables X_i, constructed out of our ψ_i's and φ_i's treating $X_i(x,y)$ as a map of R^2 into R^M, for some M so that we can discuss the geometry of these surfaces. This will generalise the discussion given by Konopelchenko and colaborators[6].

To construct such coordinates it is convenient to exploit the conservation laws for our system of equations.

To do this we look at (12) and note that K and M are given by (23) and (24). However, we note that we drop the Φ terms in (23) and (24) and we still have our conservation laws; namely, we define

$$K'_{ij} = -f_i\,\bar{\varphi}_j^2 \tag{43}$$

$$M'_{ij} = \varphi_i^2\,\bar{f}_j \tag{44}$$

and then note that we still have

$$\partial K' + \bar{\partial} M' = 0. \tag{45}$$

This is easy to check the validity of (45) by using formulae (37-39).

Note that as our conservation laws do not involve Φ_i then they can be written entirely in terms of Weierstrass variables φ_i and ψ_i.

Next we consider ($l = 1, ...N$ - no summation)

$$X_{ll} = \int_\gamma \bar{f}_l \varphi_l^2\, d\zeta + \int_\gamma f_l\, \bar{\varphi}_l^2\, d\bar{\zeta} = \int_\gamma \bar{\psi}_l \varphi_l\, d\zeta + \int_\gamma \psi_l\, \bar{\varphi}_l\, d\bar{\zeta}. \tag{46}$$

These quantities have been constructed from the diagonal entries of matrices M' and K'. From the off-diagonal entries we construct

$$\begin{aligned} X_{lk} + iY_{lk} &= \int_\gamma (\alpha \bar{f}_l \varphi_k^2 + \bar{\alpha} \bar{f}_k \varphi_l^2) d\zeta + \int_\gamma (\bar{\alpha} f_l\, \bar{\varphi}_k^2 + \alpha f_k\, \bar{\varphi}_l^2) d\bar{\zeta} \\ &= \int_\gamma (\alpha \bar{\psi}_l \varphi_k + \bar{\alpha} \bar{\psi}_k \varphi_l)\, d\zeta + \int_\gamma (\bar{\alpha} \psi_l\, \bar{\varphi}_k + \alpha \psi_k\, \bar{\varphi}_l) d\bar{\zeta}. \end{aligned} \tag{47}$$

Note that the transposition of the indices $lk \leftrightarrow kl$ corresponds to the interchange $\alpha \leftrightarrow \bar{\alpha}$. So instead of thinking of Y_{kl} we can consider X_{kl} with a different choice of α or, in turn, we can treat as different X_{kl} and X_{lk}. In our expression we take all $l, k = 1...N$ - and for $k = 1$ or $l = 1$ we can use our constraints to rewrite all our expressions in terms of independent φ_i and ψ_i, $i = 2, ...N$. Note that the conservation laws (45) guarantee that X_i do not depend on the choice of the contour γ (but only its endpoints).

This is because X's can be rewritten as

$$X = \int_\Gamma F(\zeta, \bar{\zeta})\, d\zeta + \bar{F}(\zeta, \bar{\zeta})\, d\bar{\zeta}, \tag{48}$$

where F satisfy

$$\bar{\partial} F = \partial \bar{F}, \tag{49}$$

which shows that the integrands are total derivatives.

Looking at the diagonal terms we note that $\sum_l X_{ll} = 0$; this follows from the tracelessness of matrices K and M.

Note that our expressions for X_i, all, involve products of φ_i and $\bar{\psi}_i$ and their complex conjugate. However, both ψ_1 and φ_1 should be eliminated using the constraint (first expression in (27)) and $f_1 = 1$. It is easy to check that this elimination, in the $CP1$ case leads to

$$X_{12} = \int (\bar{\psi}_2^2 - \varphi_2^2)\, d\zeta + \int (\psi_2^2 - \bar{\varphi}_2^2)\, d\bar{\zeta} \tag{50}$$

and

$$Y_{12} \,=\, i \int (\bar{\psi}_2^2 + \varphi_2^2)\, d\zeta \,-\, i \int (\psi_2^2 + \bar{\varphi}_2^2)\, d\bar{\zeta} \tag{51}$$

which appear in [2] and [5].

Next, following [2, 5] we introduce the metric

$$g_{\alpha,\beta} \,=\, \sum_{lk} \frac{\partial X_{lk}}{\partial \alpha} \frac{\partial X_{lk}}{\partial \beta}, \tag{52}$$

where α and β are ζ or $\bar{\zeta}$. To do this we have to make a choice for the normalisation of off-diagonal variables X_{kl} and Y_{kl}. We make the natural choice $\alpha = \frac{1}{2}$ and $\alpha = \frac{i}{2}$. We find that

$$g_{\zeta\zeta} \,=\, \left(\sum_i \bar{f}_i\, \varphi_i^2 \right)^2 \,=\, 0 \tag{53}$$

as this expression is exactly our constraint (27). Similarly

$$g_{\bar{\zeta}\bar{\zeta}} \,=\, 0. \tag{54}$$

The only nonzero term of the metric is

$$g_{\zeta\bar{\zeta}} \,=\, \left(1 + |f_2|^2 + |f_3|^2 + ... + |f_N|^2\right) \left[|\sum_k \bar{f}_k \varphi_k^2|^2 + |\varphi_2|^4 + |\varphi_3|^4 \,+ ... +\, |\varphi_N|^4 \right]. \tag{55}$$

Of course, we can rewrite this expression to involve generalised Weierstrass variables φ_i and ψ_i by using (32). Note, however, that expressing all quantities in terms of f_i and ∂f_i, through (28), our expressions simplify further and we obtain

$$g_{\zeta\bar{\zeta}} \,=\, |Dz|^2 \tag{56}$$

where D denotes the D_μ derivative (3) but evaluated with respect to ζ.

In the special case of the CP^1 maps $g_{\zeta\bar{\zeta}}$ takes a particularly simple form; it is given by

$$g_{\zeta\bar{\zeta}} \,=\, \left(1 + \frac{|\psi_2|^2}{|\varphi_2|^2}\right) \left[|\varphi_2|^4 + |\psi_2|^2 |\varphi_2|^2\right] \,=\, \left[|\psi_2|^2 + |\varphi_2|^2\right]^2 \,=\, \frac{|\partial f_2|^2}{(|f_2|^2+1)^2} \tag{57}$$

which is, of course $|Dz|^2$.

5 Conclusions and Further work

In this paper we have managed to generalize the Weierstrass system to the CP^{N-1} case. Thus we have found a set of $2N$ complex variables ψ_i and φ_i which satisfy first order equations (41) - which are equivalent to the full equations of the CP^N model *ie* (6).

We have also introduced a set of $N^2 - 1$ real quantities X's, which can be treated as coordinates in R^{N*N-1} and have shown that the metric of our map satisfies

$$g_{\zeta\zeta} \,=\, 0, \qquad g_{\zeta\bar{\zeta}} \,=\, |Dz|^2, \qquad g_{\bar{\zeta}\bar{\zeta}} \,=\, 0. \tag{58}$$

Our work here, generalises the approach of Konopelchenko et al, and we intend to use it to look at other aspects of their work when generalised to CP^{N-1}.

Acknowledgments

This work was started when one of us (WJZ) visited the CRM, Université de Montreal in March 2000. He would like to thank the LMS and P. Winternitz for the support of his stay at the CRM.

The work was continued when WJZ spent a sabbatical term at the Center for Theoretical Physics, MIT and when MG visited the University of Durham. WJZ would like to thank the Center for the support of his stay at MIT and its hospitality. MG would like to thank the University of Durham for the award of Alan Richards fellowship to him that allowed him to spend a term in Durham.

The work was finished when MG was in France sampling its wines.

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