Differential systems for biorthogonal polynomials appearing in 2-matrix models and the associated Riemann-Hilbert problem[*]

M. Bertola[†‡§] B. Eynard[†¶] J. Harnad[†‡‖]

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[†]Centre de recherches mathématiques, Université de Montréal, C.P. 6128, succ. Centre-ville, Montréal, Québec, H3C 3J7, Canada

[‡]Department of Mathematics and Statistics, Concordia University, 7141 Sherbrooke W., Montréal, Québec, H4B 1R6, Canada

[§]`bertola@crm.umontreal.ca`

[¶]Service de Physique Théorique, CEA/Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France; `eynard@spht.saclay.cea.fr`

[‖]`harnad@crm.umontreal.ca`

Abstract

We consider biorthogonal polynomials that arise in the study of a generalization of two-matrix Hermitean models with two arbitrary polynomial potentials $V_1(x)$, $V_2(y)$ of any degree, with arbitrary complex coefficients. A compatible sequence of fundamental systems is constructed for the system of ODEs satisfied by consecutive subsequences ("windows"), of lengths equal to the degrees of the potentials, within the dual sequences of biorthogonal polynomials. The (Stokes) sectorial asymptotics of the fundamental systems are derived through saddle-point integration and the Riemann-Hilbert problem characterizing the differential and recursion equations is deduced.

1 Introduction

In [4, 5] the differential systems satisfied by sequences of biorthogonal polynomials associated to 2-matrix models were studied, together with the deformations induced by changes in the coefficients of the potentials determining the orthogonality measure. For ensembles consisting of pairs of $N \times N$ hermitian matrices M_1 and M_2, the $U(N)$ invariant probability measure is taken to be of the form:

$$\frac{1}{\tau_N} d\mu(M_1, M_2) := \frac{1}{\tau_N} \exp \frac{1}{\hbar} \mathrm{tr}\ (-V_1(M_1) - V_2(M_2) + M_1 M_2) dM_1 dM_2\ . \tag{1-1}$$

where $dM_1 dM_2$ is the standard Lebesgue measure for pairs of Hermitian matrices and the *potentials* V_1 and V_2 are chosen to be polynomials of degrees $d_1 + 1$, $d_2 + 1$ respectively. The overall positive small parameter $\hbar$ in the exponential is taken of order N^{-1} when considering the large N limit, but in the present context it will just play the role of Planck's constant in the string equation. Using the Harish-Chandra-Itzykson-Zuber's formula, one can reduce the computation of the corresponding partition function to an integral over only the eigenvalues of the two matrices

$$\tau_N := \int\int d\mu(M_1, M_2) \propto \int \prod_{i=1}^{N} \mathrm{d}x_i \mathrm{d}y_i \Delta(x)\Delta(y) \mathrm{e}^{-\frac{1}{\hbar}\sum_{j=1}^{N} V_1(x_j) + V_2(y_j) - x_j y_j}\ , \tag{1-2}$$

and then express all spectral statistics in terms of the associated biorthogonal polynomials, in the same spirit as orthogonal polynomials are used in the spectral statistics of one-matrix models. In this context, what is meant by biorthogonal polynomials is a pair of sequences of monic polynomials

$$\pi_n(x) = x^n + \cdots, \qquad \sigma_n(y) = y^n + \cdots, \qquad n \in \mathbb{N} \tag{1-3}$$

which are mutually dual with respect to the associated coupled measure

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{d}y\ \pi_n(x)\sigma_m(y) \mathrm{e}^{-\frac{1}{\hbar}(V_1(x) + V_2(y) - xy)} = h_n \delta_{mn}, \tag{1-4}$$

on the product space.

In this work, we use essentially the same definition of orthogonality, but extend it to the case of polynomials V_1 and V_2 with arbitrary (possibly complex) coefficients, and the contours of integration are no longer restricted to the real axis, but may be chosen as curves in the complex plane starting and ending at ∞, chosen so that the integrals are convergent. The orthogonality relations determine the two families uniquely, if they exist [15, 5].

It was shown in [4, 5] that the finite consecutive subsequences of lengths $d_2 + 1$ and $d_1 + 1$ respectively, within the sequences of dual quasi-polynomials:

$$\psi_n(x) = \frac{1}{\sqrt{h_n}} \pi_n(x) \mathrm{e}^{-\frac{1}{\hbar}V_1(x)}\ ;\ \ \phi_n(y) = \frac{1}{\sqrt{h_n}} \sigma_n(y) \mathrm{e}^{-\frac{1}{\hbar}V_2(y)}\ , \tag{1-5}$$

beginning (or ending) at the points $n = N$, satisfy compatible overdetermined systems of first order differential equations with polynomial coefficients of degrees d_1 and d_2, respectively, as well as recursion relations relating consecutuve values of N. In fact, certain quadruples of Differential–Deformation–Difference equations (DDD for short) were derived for these "windows" as well as for their Fourier Laplace transforms, in which the deformation parameters were taken to be the coefficients of the potentials V_1 and V_2. It was also shown in [4, 5] that these systems are Frobenius compatible and hence admit joint fundamental systems of solutions.

In the present work we explicitly construct such fundamental systems in terms of certain integral transforms applied to the biorthogonal polynomials. The main purpose is to derive the Riemann–Hilbert problem characterizing the sectorial asymptotic behaviour at $x = \infty$ or $y = \infty$.

The ultimate purpose of this analysis is to deduce in a rigorous way the double–scaling limits $N \to \infty$, $\hbar N = \mathcal{O}(1)$ of the partition function and spectral statistics, for which the corresponding large N asymptotics of the biorthogonal polynomials are required. (See [14, 25] and references therein for further background on 2-matrix models, and [11, 16, 17, 18, 15] for other more recent developments.) The study of the large N limit of matrix integrals is of considerable interest in physics, because many physical systems with a large number of strongly correlated degrees of freedom (quantum chaos, mesoscopic conductors, ...) seem to have the same statistical properties as the spectra of random matrices. Also, the large N expansion of a random matrix integral (if it exists) is expected to be the generating fuctional of discretized surfaces, thus random matrices provide a powerful tool for studying statistical physics on a random surface. (The 2-matrix model was first introduced in this context, as the Ising model on a random surface [25]).

It has been understood for some time that the 1-matrix model is not general enough, since it cannot represent all models of statistical physics (e.g., it contains only the $(p,2)$ conformal minimal models). In order to recover the missing conformal models ((p,q) with p and q integers), it is necessary to introduce at least a two-matrix model [11]. The 1-matrix model is actually strictly included in the 2-matrix model, since if one takes $d_2 = 1$, and integrates the gaussian matrix M_2 out, one sees that the 1-matrix model follows, and hence may be seen as a particular case.

It should also be mentioned that most of the results about the 2-matrix model (in particular those derived in the present work) can be easily extended to multi-matrix models (see appendix of [4]) without major modifications. Indeed the multi-matrix model is not expected to be very different from the 2-matrix case [11] (in particular, it contains the same conformal models).

The present paper is organized as follows: in Section 2, we set up the necessary formalism for biorthogonal polynomials, beginning with the systems of differential and recursion relations they satisfy, recalling the main definitions and results of [4]. We then derive the fundamental systems of solutions to the overdetermined systems for the "windows" of biorthogonal polynomials in two ways; one by exploiting the recursion matrices Q, P for the biorthogonal polynomials, which satisfy the string equation, and another by giving explicit integral formulas for solutions and showing their independence when taken over a suitably defined homology basis of inequivalent integration paths.

In Section 3 we use saddle-point integration methods to deduce the asymptotic form of these fundamental systems of solutions within the various Stokes sectors. and from these, to deduce the Stokes matriuces and jump disconinuities at ∞. The full formulation of the matrix Riemann-Hilbert problem characterizing these solutions is given in Theorem 3.1.

Publication remark: This work was completed substantially in its present form in the winter of 2001-2002, and the results were presented, in preliminary version, at the AMS regional meeting in Montreal, May 2002, as well as at the meeting on Random Matrices at the Courant Institute, June 2002. While completing editorial corrections to the the present version, we received the preprint [24], where results along similar lines are obtained for the case of cubic potentials. In the interests of timely dissemination, we have chosen to post the present version in electronic data base form, although further editorial revisions may still follow before final publication.

2 Setting

The notation and setting follows essentially [4], with some minor modifications that we will point out in due course. In order to mantain the paper as self–contained as possible we recall the main points of [4].
Let us fix two polynomials which we will refer to as the "potentials",

$$V_1(x) = u_0 + \sum_{K=1}^{d_1+1} \frac{u_K}{K} x^K \ , \qquad V_2(y) = v_0 + \sum_{J=1}^{d_2+1} \frac{v_J}{J} y^J \ . \tag{2-1}$$

Using these potentials we define a *bimoment functional* i.e. a pairing between polynomials of x and y by means of the following formula

$$(\pi, \sigma) := \int\int \mathrm{d}x\mathrm{d}y\, \mathrm{e}^{-\frac{1}{\hbar}(V_1(x)+V_2(y)-xy)} \pi(x)\sigma(y) \ . \tag{2-2}$$

The contours of integration in the x and y plane have not been specified yet: in order to have convergent integrals we must define two suitable contours of integration Γ_x, Γ_y in the x and y complex plane respectively. In fact there are precisely d_1 (homological) independent choices of for the contours Γ_x and d_2 for the contours Γ_y as follows from the specialization to polynomial potentials of [7]. The necessary and sufficient condition for the convergence of these integrals is that the contours approach ∞ in such a way that

$$\Re(V_1(x)) \underset{x\to\infty}{\longrightarrow} \infty \underset{y\to\infty}{\longleftarrow} \Re(V_2(y)) \tag{2-3}$$

Let us define the sectors

$$\mathcal{S}_k^{(y)} := \left\{ y \in \mathbb{C}\ ,\ \arg(y) \in \left(\vartheta_y + \frac{(2k-1)\pi}{2(d_2+1)}, \vartheta_y + \frac{(2k+1)\pi}{2(d_2+1)} \right) \right\}, \\ \vartheta_y := -\arg(v_{d_2+1})/(d_2+1)\ ,\quad k = 0, \dots, 2d_2+1\ . \tag{2-4}$$

$$\mathcal{S}_k^{(x)} := \left\{ x \in \mathbb{C}\ ,\ \arg(x) \in \left(\vartheta_x + \frac{(2k-1)\pi}{2(d_1+1)}, \vartheta_x + \frac{(2k+1)\pi}{2(d_1+1)} \right) \right\}, \\ \vartheta_x := -\arg(u_{d_1+1})/(d_1+1)\ ,\quad k = 0, \dots, 2d_1+1\ . \tag{2-5}$$

We define the contours $\Gamma_y^{(k)}$ coming from ∞ within the sector $\mathcal{S}_{2k-2}^{(y)}$ and returning to infinity in the sector $\mathcal{S}_{2k}^{(y)}$ (similar definition for the $\Gamma_x^{(k)}$ contours). Note that, since there are no singularities in the finite region of the y-plane for the differentials we are considering, we have

$$\sum_{k=0}^{d_2} \Gamma_y^{(k)} = 0 = \sum_{k=0}^{d_1} \Gamma_x^{(k)} \ , \text{ homologically} \ . \tag{2-6}$$

Therefore there are only d_2 (homologically) linearly independent contours $\Gamma_y^{(k)}$ (and d_1 contours $\Gamma_x^{(k)}$).

We define two sequences of monic polynomials $\pi_n(x), \sigma_n(y)$ of degree n such that they are biorthogonal w.r.t. the pairing

$$\big(\pi_n, \sigma_m\big)_{\varkappa} := \sum_{\substack{i=1,\dots,d_1 \\ j=1,\dots,d_2}} \varkappa^{ij} \int_{\Gamma_x^{(i)} \times \Gamma_y^{(j)}} \mathrm{d}x \wedge \mathrm{d}y \, \pi_n(x) \mathrm{e}^{-\frac{1}{\hbar}(V_1(x)+V_2(y)-xy)} \sigma_m(y) = h_n \delta_{nm} \ , \tag{2-7}$$

$$\pi_n(x) = x^n + \dots \ ; \sigma_n(y) = y^n + \dots \ , \tag{2-8}$$

where the $d_1 \times d_2$ matrix $\varkappa^{ij}$ is generic in such a way that all the principal minors of the bimoment matrix are nondegenerate. We will denote the integral operator as follows for brevity

$$\sum_{\substack{i=1,\dots,d_1 \\ j=1,\dots,d_2}} \varkappa^{ij} \int_{\Gamma_x^{(i)} \times \Gamma_y^{(j)}} := \int_{\varkappa\Gamma} \ . \tag{2-9}$$

The aforementioned nondegeneracy condition that ensures the existence of the biorthogonal polynomials is given by

$$\det \left[\big(x^i, y^j\big)_{\varkappa} \right]_{i,j=0,\dots,N-1} =: \Delta_N(\varkappa) \neq 0 \qquad \forall N \in \mathbb{N} \ . \tag{2-10}$$

Since all $\Delta_N(\varkappa)$ are homogeneous polynomials in the $\varkappa_{ij}$s of degree $N+1$, they are all simultaneously nonzero for generic values of the parameters $\varkappa$.

We introduce the quasipolynomials and the corresponding wave-vectors

$$\begin{aligned} &\psi_n(x) = \frac{1}{\sqrt{h_n}} \pi_n(x) \mathrm{e}^{-\frac{1}{\hbar}V_1(x)} \ ; \quad \phi_n(y) = \frac{1}{\sqrt{h_n}} \sigma_n(y) \mathrm{e}^{-\frac{1}{\hbar}V_2(y)} \ ; \\ &\Psi_\infty(x) := [\psi_0(x), \dots, \psi_n(x), \dots]^t \ ; \qquad \Phi_\infty(y) := [\phi_0(y), \dots, \phi_n(y), \dots]^t \ . \end{aligned} \tag{2-11}$$

In matrix notation the biorthogonality reads

$$\int_{\varkappa\Gamma} \mathrm{d}x \wedge \mathrm{d}y \, \mathrm{e}^{\frac{xy}{\hbar}} \, \Psi_\infty(x) {\Phi_\infty}^t(y) = \mathbf{1} \tag{2-12}$$

where $\mathbf{1}$ denotes the semiinfinite unit matrix.

We denote the Fourier-Laplace transforms of one or the other semiinfinite wave-vectors along the relevant paths by

$$\underline{\Psi}_\infty{}^{(j)}(y) := \int_{\Gamma_x^{(j)}} \mathrm{d}x \, \mathrm{e}^{\frac{xy}{\hbar}} \, {\Psi_\infty}^t(x) \ , \quad j = 1, \dots, d_1 \tag{2-13}$$

$$\underline{\Phi}_\infty{}^{(k)}(x) := \int_{\Gamma_y^{(k)}} \mathrm{d}y \, \mathrm{e}^{\frac{xy}{\hbar}} \, {\Phi_\infty}^t(y) \ , \quad k = 1, \dots, d_2. \tag{2-14}$$

The recurrence relations for the biorthogonal polynomials are encompassed by the matrix equations

$$x\Psi_\infty(x) = Q\Psi_\infty(x) \ ; \qquad \hbar\partial_x \Psi_\infty(x) = -P\Psi_\infty(x) \ ; \tag{2-15}$$

$$y\Phi_\infty(y) = P^t\Phi_\infty(y) \ ; \qquad \hbar\partial_y \Phi_\infty(y) = -Q^t\Phi_\infty(y) \ . \tag{2-16}$$

where the two matrices P and Q are the finite band size matrices (see [14, 16, 4] for a simple proof of this)

$$Q := \begin{bmatrix} \alpha_0(0) & \gamma(0) & 0 & 0 & \cdots \\ \alpha_1(1) & \alpha_0(1) & \gamma(1) & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \alpha_{d_2}(d_2) & \alpha_{d_2-1}(d_2) & \cdots & \alpha_0(d_2) & \gamma(d_2) \\ 0 & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{2-17}$$

$$P := \begin{bmatrix} \beta_0(0) & \beta_1(1) & \cdots & \beta_{d_1}(d_1) & \cdots \\ \gamma(0) & \beta_0(1) & \beta_1(2) & \ddots & \beta_{d_1}(d_1+1) \\ 0 & \gamma(1) & \beta_0(2) & \ddots & \ddots \\ 0 & 0 & \gamma(2) & \beta_0(3) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{2-18}$$

satisfying the string equation

$$[P, Q] = \hbar 1 \ . \tag{2-19}$$

For the dual sequences of Fourier–Laplace transforms, a simple integration by parts of Eqs. 2-15, 2-16 gives

$$x\underline{\Phi}_\infty{}^{(j)}(x) = \underline{\Phi}_\infty{}^{(j)}(x)Q \ ; \qquad \hbar\partial_x\underline{\Phi}_\infty{}^{(j)}(x) = \underline{\Phi}_\infty{}^{(j)}(x)P \ ; \quad j = 1, \ldots, d_2 \tag{2-20}$$

$$y\underline{\Psi}_\infty{}^{(k)}(y) = \underline{\Psi}_\infty{}^{(k)}(y)P^t \ ; \qquad \hbar\partial_y\underline{\Psi}_\infty{}^{(k)}(y) = \underline{\Psi}_\infty{}^{(k)}(y)Q^t \ ; \quad k = 1, \ldots, d_1 \ . \tag{2-21}$$

Notice that integration by parts is allowed due to the exponential decay of the integrand along the chosen contours.

We recall the definition of dual windows

Definition 2.1 *We call a* **window of size** $\mathrm{d}_1 + 1$ **or** $\mathrm{d}_2 + 1$ *any subset of* $d_1 + 1$ *or* $d_2 + 1$ *consecutive elements of type* ψ_n, $\underline{\phi}_n$, ϕ_n *or* $\underline{\psi}_n$, *with the notations*

$$\Psi_N := [\psi_{N-d_2}, \ldots, \psi_N]^t \ , \quad N \geq d_2, \qquad \Phi_N := [\phi_{N-d_1}, \ldots, \phi_N]^t \ , \quad N \geq d_1 \tag{2-22}$$

$$\Psi^N := [\psi_{N-1}, \ldots, \psi_{N+d_1-1}]^t \ , \quad N \geq 0, \qquad \Phi^N := [\phi_{N-1}, \ldots, \phi_{N+d_2-1}]^t \ , \qquad N \geq 1 \tag{2-23}$$

$$\underline{\Psi}_N{}^{(j)} := [\underline{\psi}^{(j)}_{N-d_2}, \ldots, \underline{\psi}^{(j)}_N] \quad N \geq d_2, \qquad \underline{\Phi}_N{}^{(k)} := [\underline{\phi}^{(k)}_{N-d_1}, \ldots, \underline{\phi}^{(k)}_N] \ , \quad N \geq d_1 \tag{2-24}$$

$$\underline{\Psi}^{N\,(j)} := [\underline{\psi}^{(j)}_{N-1}, \ldots, \underline{\psi}^{(j)}_{N+d_1-1}] \ , \quad N \geq 1, \qquad \underline{\Phi}^{N\,(k)} := [\underline{\phi}^{(k)}_{N-1}, \ldots, \underline{\phi}^{(k)}_{N+d_2-1}] \ , \qquad N \geq 1 \ . \tag{2-25}$$

Notice the difference in the positioning of the windows for the vectors constructed from the ψ_n's and the ϕ_n's (and the different notation we are using as opposed to the one in [4], where $\underline{\Phi}^N$ now would be rather $\underline{\Phi}^{N+1}$) and the fact that the barred quantities are defined to be row vectors while the unbarred ones are column vectors.

Definition 2.2 *The two pairs of windows* (Ψ_N, $\underline{\Phi}^N$) *and* (Φ_N, $\underline{\Psi}^N$) *of dimensions* $d_2 + 1$ *and* $d_1 + 1$ *respectively, will be called* **dual windows** *and are defined for* $N \geq d_2$ *and* $N \geq d_1$ *respectively.*

We also recall that by using the finite-band recurrence relations in Eqs. 2-15,2-16 , and Eqs. 2-20, 2-21 one can define the notions of *folding* on the finite windows above [4]. This way one can express four polynomial ODEs for the two pairs of dual windows which are obtained as the folding of the differential recurrence relation

$$\hbar\partial_x\Psi_N(x) = -D_1^N(x)\Psi_N(x) \ ; \qquad \hbar\partial_x\underline{\Phi}^{N(k)}(x) = \underline{\Phi}^{N(k)}(x)\underline{D}_1^N(x) \ , \quad k = 1, \ldots, d_2, \quad N > d_2; \tag{2-26}$$

$$\hbar\partial_y\Phi_N(y) = -D_2^N(y)\Phi_N(y) \ ; \qquad \hbar\partial_y\underline{\Psi}^{N(j)}(y) = \underline{\Psi}^{N(j)}(y)\underline{D}_2^N(y) \ , \quad j = 1, \ldots, d_1, \quad N > d_1. \tag{2-27}$$

We point out that the windows $\underline{\Phi}^{N(k)}$ and $\underline{\Psi}^{N(j)}$ for different k, j satisfy all the same ODEs (2-26, 2-27) because they all satisfy the same x and ∂_x recurrence relations (Eqs. 2-20, 2-21).

In [4] it was shown that there exists a natural nondegenerate pairing between any pair of solutions $\Psi_N(x)$ and $\underline{\Phi}^N(x)$ to Eqs. 2-26 (similarly for Eqs. 2-27);

$$\left(\underline{\Phi}^N, \Psi_N\right)_N := \underline{\Phi}^N(x) \overset{N}{\mathbb{A}} \Psi_N(x) \ , \tag{2-28}$$

which is a *constant* in x. In other words the matrices D_1^N and $\underline{D}_1^N$ are conjugate to each other by means of the matrix $\overset{N}{\mathbb{A}}$ (Theorem 4.1 in [4])

$$\overset{N}{\mathbb{A}} D_1^N(x) = \underline{D}_1^N(x) \overset{N}{\mathbb{A}} \ , \quad \overset{N}{\mathbb{A}} := \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & -\gamma(N-1) \\ \hline \alpha_{d_2}(N) & \cdots & \alpha_2(N) & \alpha_1(N) & 0 \\ 0 & \alpha_{d_2}(N+1) & \cdots & \alpha_2(N+1) & 0 \\ 0 & 0 & \alpha_{d_2}(N+2) & \cdots & 0 \\ 0 & 0 & 0 & \alpha_{d_2}(N+d_2-1) & 0 \end{array}\right] . \tag{2-29}$$

The $(d_2 + 1) \times (d_2 + 1)$ matrix $\overset{N}{\mathbb{A}}$ is the only nonzero block in the commutator $\left[\Pi_0^{N-1}, Q\right]$, where Π_0^{N-1} denotes the semiinfinite sparse matrix equal to the identity in the principal minor of dimension N (a "canonical" projector). In the following it is convenient to use the same notation $\overset{N}{\mathbb{A}}$ both for the finite matrix or the semiinfinite one.

It was proven also in [4] (Thm. 4.1) that one can choose the windows Ψ_N and $\underline{\Phi}^N$ as joint solution also of the PDEs arising from the infinitesimal change of the coefficients of the two potentials (deformation equations): this way the pairing becomes independent of all deformation parameters and of N.

2.1 Fundamental solutions of the D_1 and $\underline{D}_1$ systems

In this section we will explicitly construct solutions of the pair of dual ODEs defined by the matrices D_1 and $\underline{D}_1$: we leave to the reader the very simple formulation of the following statements for the other pair.

It is clear that if we have d_2+1 linearly independent solutions of the recurrence relations (2-15, 2-20) we obtain a fundamental system for each of the ODEs in Eqs. (2-26) by taking suitable windows; it will be shown in a moment that this is not possible, inasmuch as the relations (2-20) have precisely only d_2 solutions while relations (2-15) have only one. However we can find the "missing" solutions by a small "perturbation" of the initial conditions in the recurrence relations[7]; the corresponding solutions will satisfy the same recursions relations for n big enough, thus providing us with the desired solutions of each of the systems (2-26).

Proposition 2.1 The semiinfinite systems

$$\left\{\begin{array}{l} x\Psi_\infty(x) = Q\Psi_\infty(x) \\ \hbar\partial_x\Psi_\infty(x) = -P\Psi_\infty(x) \end{array}\right. \tag{2-30}$$

$$\left\{\begin{array}{l} x\underline{\Phi}_\infty(x) = \underline{\Phi}_\infty(x)Q \\ \hbar\partial_x\underline{\Phi}_\infty(x) = \underline{\Phi}_\infty(x)P \end{array}\right. \tag{2-31}$$

have 1 and d_2 solutions respectively (similar statement for Φ_∞, $\underline{\Psi}_\infty$, Eqs. 2-16 and 2-21).

Proof. The compatibility is guaranteed by the string equation $[P,Q]=\hbar$. Recalling that

$$\Big(P - V_1'(Q)\Big)_{\geq 0} = \Big(Q - V_2'(P)\Big)_{\leq 0} = 0 \tag{2-32}$$

we have

$$0 = [(P - V_1'(Q))\,\underline{\Psi}_\infty]_0 = -\hbar\psi_0'(x) - V_1'(x)\psi_0(x) \tag{2-33}$$

$$0 = [\underline{\Phi}_\infty\,(Q - V_2'(P))]_0 = x\underline{\phi}_0(x) - V_2'(\hbar\partial_x)\underline{\phi}_0(x)\ . \tag{2-34}$$

Thus we have only one solution of the first equation and d_2 solutions for the second. Using the x recursion relation for the ψ sequence we can build all the rest of the sequence starting from the (unique) ground state $\psi_0(x)$. Using the ∂_x recursion relation we can build the rest of the $\underline{\phi}_n$ sequence starting from $\underline{\phi}_0$. Q.E.D.

Up to multiplicative constants the ψ solution is exactly the solution given by the quasipolynomials, while the different $\underline{\phi}$ solutions are precisely the d_2 different Fourier–Laplace transforms of the quasipolynomials $\phi_n(y)$; indeed $\underline{\phi}_0(x)$ can be expressed by

$$\underline{\phi}_0(x) \propto \int_\Gamma \mathrm{d}y\, \mathrm{e}^{\frac{1}{\hbar}(xy - V_2(y))}\ , \tag{2-35}$$

where Γ is any of the contours $\Gamma_y^{(k)}$ or a linear combination of them.

We now consider a modified system in order to find the other solutions

Proposition 2.2 The semiinfinite systems

$$\left\{\begin{array}{l} x\Psi_\infty(x) = Q\Psi_\infty(x) - W_2(-\hbar\partial_x)F(x) \\ \hbar\partial_x\Psi_\infty(x) = -P\Psi_\infty(x) + F(x) \end{array}\right. \tag{2-36}$$

$$\left\{\begin{array}{l} x\underline{\Phi}_\infty(x) = \underline{\Phi}_\infty(x)Q + U(x) \\ \hbar\partial_x\underline{\Phi}_\infty(x) = \underline{\Phi}_\infty(x)P + U(x)W_1(x) \end{array}\right. \tag{2-37}$$

$$W_1(x) := \frac{V_1'(Q) - V_1'(x)}{Q - x}\ ; \qquad W_2(\hbar\partial_x) := \frac{V_2'(P) - V_2'(\hbar\partial_x)}{P - \hbar\partial_x}\ ; \tag{2-38}$$

$$F(x) := [f(x), 0, 0, \cdots]^t\ ; \qquad U(x) := [u(x), 0, 0, \cdots] \tag{2-39}$$

have both d_2+1 solutions for the unknowns $(f(x),\ \Psi_\infty(x))$ and $(u(x),\ \underline{\Phi}_\infty(x))$

[7] This is not new in the context of orthogonal polynomials, where there exist the orthogonal polynomials of the second kind, see e.g. [10, 27]

Proof. The compatibility of these systems is not obvious; we have

$$\hbar\Psi_\infty = [\hbar\partial_x, x]\Psi_\infty = \hbar\partial_x\left(Q\Psi_\infty - W_2(-\hbar\partial_x)F\right) - x\left(-P\Psi_\infty + F\right) = \tag{2-40}$$
$$= Q\left(-P\Psi_\infty + F\right) - \partial_x W_2(-\hbar\partial_x)F + P\left(Q\Psi_\infty - W_2(-\hbar\partial_x)F\right) + xF = \tag{2-41}$$
$$= [P,Q]\Psi_\infty + QF - (\hbar\partial_x + P)\frac{V_2'(P) - V_2'(-\hbar\partial_x)}{P + \hbar\partial_x}F(x) + xF = \tag{2-42}$$
$$= \hbar\Psi_\infty + \Big(Q - V_2'(P)\Big)F + \Big(x - V_2'(-\hbar\partial_x)\Big)F \tag{2-43}$$

Given that only the first entry of F is nonzero and that $Q - V_2'(P)$ is strictly upper-triangular, the second term vanishes. Last term gives the following ODE for the first entry of $F(x)$

$$V_2'(-\hbar\partial_x)f(x) = xf(x). \tag{2-44}$$

Each solution of Eq. 2-44 gives a compatible system. We label such solutions $f^{(\alpha)}$ with $\alpha = 0,\ldots,d_2$, $f^{(0)} \equiv 0$ being the trivial solution.
For each such solution we can now solve for Ψ. First of all one can prove by induction that

$$Q^k\Psi_\infty = x^k\Psi_\infty + \frac{Q^k - x^k}{Q - x}W_2(-\hbar\partial_x)F. \tag{2-45}$$

Next we compute as for the previous proposition

$$0 = \left[\left(P - V_1'(Q)\right)\Psi_\infty\right]_0 = \left[-\hbar\partial_x\Psi_\infty + F - V_1'(x)\Psi_\infty - \frac{V_1'(Q) - V_1'(x)}{Q - x}W_2(-\hbar\partial_x)F\right]_0 = \tag{2-46}$$
$$= -\hbar\psi_0'(x) - V_1'(x)\psi_0(x) + \left[1 - W_1(x)W_2(-\hbar\partial_x)\right]_{00} f(x) \tag{2-47}$$

That is $\psi_0(x)$ must solve that first order ODE's: considering the fact that there are d_2+1 choices for the function $f(x) = f^{(\alpha)}(x)$ we correspondingly obtain d_2+1 independent solutions $\Psi^{(\alpha)}(x)$ to the system.

We consider now the second system (2-38). The compatibility gives

$$\hbar\underline{\Phi}_\infty = [\hbar\partial_x, x]\underline{\Phi}_\infty = \hbar\partial_x\left(\underline{\Phi}_\infty Q + U\right) - x\left(\underline{\Phi}_\infty P + UW_1(x)\right) = \tag{2-48}$$
$$= \left(\underline{\Phi}_\infty P + UW_1(x)\right)Q + \hbar\partial_x U - \left(\underline{\Phi}_\infty Q + U\right)P - UW_1(x)x = \tag{2-49}$$
$$= \underline{\Phi}_\infty[P,Q] - UP + U\frac{V_1'(Q) - V_1'(x)}{Q - x}(Q - x) + \hbar\partial_x U = \tag{2-50}$$
$$= \hbar\underline{\Phi}_\infty - U\Big(P - V_1'(Q)\Big) + \Big(\hbar\partial_x - V_1'(x)\Big)U(x) \tag{2-51}$$

Given that only the first entry of U is nonzero and that $P - V_1'(Q)$ is strictly lower-triangular, the second term vanishes. The last term gives the following ODE for the first entry of $U(x)$, that is

$$\hbar u'(x) = V_1'(x)u(x) \quad\Rightarrow\quad u(x) = c\,e^{\frac{1}{\hbar}V_1(x)}\ . \tag{2-52}$$

Of course the compatibility is guaranteed also if we take the trivial solution.
We next consider the solutions $\underline{\Phi}_\infty$; by a computation similar to the previous we have

$$0 = \left[\underline{\Phi}_\infty\Big(Q - V_2'(P)\Big)\right]_0 = \left[x\underline{\Phi}_\infty - U - V_2'(\hbar\partial_x)\underline{\Phi}_\infty + UW_1(x)\frac{V_2'(P) - V_2'(\hbar\overleftarrow{\partial}_x)}{P - \hbar\overleftarrow{\partial}_x}\right]_0 = \tag{2-53}$$
$$= \Big(x - V_2'(\hbar\partial_x)\Big)\phi_0(x) - \left[1 - W_1(x)W_2(\hbar\partial_x)\right]_{00} u(x) \tag{2-54}$$

Therefore we have the d_2 solutions corresponding to the case $u(x) \equiv 0$ (which give the solutions of the "unmodified" recurrence relations, i.e. the Fourier–Laplace transforms of the quasipolynomials) and the extra solutions corresponding to the nontrivial solution $u(x) = ce^{V_1(x)}$. We denote this last by $\underline{\Phi}_\infty{}^{(0)}$ and in general $\underline{\Phi}_\infty{}^{(\alpha)}$. Q.E.D.

We will give explicit integral representations for the d_2+1 solutions in the next section.
With the solutions $\Psi_\infty{}^{(\alpha)}$, $\underline{\Phi}_\infty{}^{(\beta)}$, $\alpha,\beta = 0,\ldots,d_2$ we can construct the $(d_2+1)\times(d_2+1)$ modified kernels

$$K^{N(\alpha,\beta)}_{11}(x,x') := \sum_{n=0}^{N-1}\underline{\phi}_n^{(\alpha)}(x)\psi_n^{(\beta)}(x') = \underline{\Phi}_\infty{}^{(\alpha)}(x)\Pi_0^{N-1}\Psi_\infty{}^{(\beta)}(x')\ . \tag{2-55}$$

We can obtain also the following modified Christoffel–Darboux formulæ

Proposition 2.3 [Christoffel–Darboux Kernels]

$$(x-x')K^{N(\alpha,\beta)}_{11}(x,x') = \tag{2-56}$$
$$= \left(\underline{\Phi}_\infty{}^{(\alpha)}(x)Q + \delta_{\alpha 0}U(x)\right)\Pi_0^{N-1}\Psi_\infty{}^{(\beta)}(x') - \underline{\Phi}_\infty{}^{(\alpha)}(x)\Pi_0^{N-1}\left(Q\Psi_\infty{}^{(\beta)}(x') - W_2(-\hbar\partial'_x)F^{(\beta)}(x')\right) = \tag{2-57}$$
$$= -\underline{\Phi}_\infty{}^{(\alpha)}(x)\left[\Pi_0^{N-1}, Q\right]\Psi_\infty{}^{(\beta)}(x') + \delta_{\alpha 0}u(x)\psi_0^{(\beta)}(x') + \underline{\Phi}_\infty{}^{(\alpha)}(x)\Pi_0^{N-1}W_2(-\hbar\partial'_x)F^{(\beta)}(x') \tag{2-58}$$

We recall that the commutator of the finite band size matrix Q with the projector Π_0^{N-1} gives a sparse semiinfinite matrix which correspond to the Christoffel–Darboux kernel matrix $\overset{N}{\mathbb{A}}$. As a corollary we obtain the pairing between the solutions of the systems (2-26) by setting $x = x'$

Corollary 2.1

$$\left(\underline{\Phi}^{N(\alpha)}, \Psi_N{}^{(\beta)}\right)_N := \underline{\Phi}^{N(\alpha)}(x)\,\overset{N}{\mathbb{A}}\,\Psi_N{}^{(\beta)}(x) = \underline{\Phi}_\infty{}^{(\alpha)}(x)\left[\Pi_0^{N-1}, Q\right]\Psi_\infty{}^{(\beta)}(x) =$$
$$= \delta_{\alpha 0}u(x)\psi_0^{(\beta)}(x) + \underline{\Phi}_\infty{}^{(\alpha)}(x)\Pi_0^{N-1}W_2(-\hbar\partial_x)F^{(\beta)}(x) \tag{2-59}$$

We know that this is a constant (in x) if $N > d_2$ [4]; but if $N > d_2$ the projector in the second term is irrelevant because the vector $W_2(-\hbar\partial_x)F^{(\beta)}(x)$ has only the first d_2+1 nonzero entries and hence we have

$$\left(\underline{\Phi}^{N(\alpha)}, \Psi_N{}^{(\beta)}\right)_N = \delta_{\alpha 0}u(x)\psi_0^{(\beta)}(x) + \underline{\Phi}_\infty{}^{(\alpha)}(x)\frac{V_2'(P) - V_2'(-\hbar\partial_x)}{P + \hbar\partial_x}F^{(\beta)}(x)\ , \qquad N > d_2 \tag{2-60}$$

If $\alpha \neq 0$ then $\underline{\Phi}_\infty{}^{(\alpha)}(x)P = \hbar\partial_x\underline{\Phi}_\infty{}^{(\alpha)}(x)$ and hence we have

$$\underline{\Phi}_\infty{}^{(\alpha)}(x)\,\overset{N}{\mathbb{A}}\,\Psi_\infty{}^{(\beta)}(x) = \underline{\Phi}_\infty{}^{(\alpha)}(x)\frac{V_2'(\hbar\overleftarrow{\partial}_x) - V_2'(-\hbar\overrightarrow{\partial}_x)}{\hbar\overleftarrow{\partial}_x + \hbar\overrightarrow{\partial}_x}F^{(\beta)}(x) = \tag{2-61}$$
$$= \frac{V_2'(\hbar\partial_{x'}) - V_2'(-\hbar\partial_x)}{\hbar\partial_{x'} + \hbar\partial_x}\underline{\phi}_0^{(\alpha)}(x')f^{(\beta)}(x)\bigg|_{x'=x}\ , \qquad N > d_2, \alpha \neq 0. \tag{2-62}$$

In this expression $\underline{\phi}_0$ and f are kernel solutions of a pair of adjoint differential equations

$$(V_2'(\hbar\partial_x) - x)\underline{\phi}_0^{(\alpha)}(x) = 0\ , \quad (V_2'(-\hbar\partial_x) - x)f^{(\beta)}(x) = 0\ , \tag{2-63}$$

and the last expression in Eq. 2-62 is nothing but the bilinear concomitant of the pair (which is a constant in our case).

For $\beta = 0$ then we have

$$\underline{\Phi}_\infty{}^{(\alpha)}(x)\,\overset{N}{\mathbb{A}}\,\Psi_\infty{}^{(0)}(x) = \delta_{\alpha 0}u(x)\psi_0^{(0)}(x) = \delta_{\alpha 0}c\,\mathrm{e}^{\frac{1}{\hbar}V_1(x)}\frac{1}{\sqrt{h_0}}\mathrm{e}^{-\frac{1}{\hbar}V_1(x)} = \delta_{\alpha 0}\frac{c}{\sqrt{h_0}}\ . \tag{2-64}$$

2.2 Explicit Integral representations

Proposition 2.4 The d_2+1 semiinfinite wave-vectors $\underline{\Phi}_\infty{}^{(j)}$ and $\Psi_\infty{}^{(j)}$ with components defined by

$$\underline{\phi}_n^{(0)}(x) = \mathrm{e}^{\frac{1}{\hbar}V_1(x)}\int_{\varkappa\Gamma}\mathrm{d}s\wedge\mathrm{d}y\,\frac{\mathrm{e}^{-\frac{1}{\hbar}(V_1(s)-sy)}\phi_n(y)}{x-s}\ , \tag{2-65}$$

$$\underline{\phi}_n^{(k)}(x) = \int_{\Gamma_y^{(k)}}\mathrm{d}y\,\mathrm{e}^{\frac{xy}{\hbar}}\phi_n(y)\ , \qquad k = 1,\dots,d_2 \tag{2-66}$$

$$\psi_n^{(k)}(x) = \int_{\widetilde{\Gamma}_y^{(k)}}\mathrm{d}y\,\mathrm{e}^{\frac{1}{\hbar}(V_2(y)-yx)}\int_{\varkappa\Gamma}\mathrm{d}z\wedge\mathrm{d}t\,\frac{\mathrm{e}^{-\frac{1}{\hbar}(V_2(t)-zt)}\psi_n(z)}{t-y}\ , \qquad k = 1,\dots,d_2 \tag{2-67}$$

$$\psi_n^{(0)}(x) = \psi_n(x) = \frac{1}{\sqrt{h_0}}\pi_n(x)\mathrm{e}^{-\frac{1}{\hbar}V_1(x)}\ , \tag{2-68}$$
$$n = 0, 1, \dots, \infty\ ,$$

are the solutions of the modified systems (2-36, 2-38).

In this proposition the contours $\widetilde{\Gamma}_y^{(k)}$ are defined similarly to the contours $\Gamma_y^{(k)}$ except for the requirement that $\Re(V_2(y)) \to -\infty$ as $y \to \infty,\ y \in \widetilde{\Gamma}_y^{(k)}$.

Remark 2.1 We point out that the we would have solutions even by choosing any admissible contour (or linear combination of) in Eqs. 2-66, 2-67; we will use this arbitrariness later.

Remark 2.2 The functions $\underline{\phi}_n^{(0)}(x)$ are piecewise analytic functions in each connected component $\mathbb{C}_x \setminus \bigcup_{j=1}^{d_1} \Gamma_x^{(j)}$.

Proof.
For each of the four kind of sequences we define the semiinfinite wave-vectors

$$\underline{\Phi}_\infty{}^{(j)} := [\underline{\phi}_0^{(j)}, \underline{\phi}_1^{(j)}, \dots] \ , \qquad j = 0, \dots, d_2 \ ; \qquad \Psi_\infty{}^{(k)} := [\psi_0^{(k)}, \psi_1^{(k)}, \dots]^t \qquad k = 0, \dots, d_2 \ . \tag{2-69}$$

There is nothing to prove for $\underline{\Phi}_\infty{}^{(j)}$ for $j > 0$ and for $\Psi_\infty{}^{(0)}$ as these are the Fourier-Laplace transforms and the quasipolynomials respectively and satisfy the corresponding unmodified systems (2-30, 2-31).
Let us consider $\underline{\Phi}_\infty{}^{(0)}$: we first check the x recurrence relation

$$x\underline{\phi}_n^{(0)}(x) = \mathrm{e}^{\frac{1}{\hbar}V_1(x)} \int_{\varkappa\Gamma} \mathrm{d}s \wedge \mathrm{d}y \left[\frac{(x-s)\mathrm{e}^{-\frac{1}{\hbar}(V_1(s)-sy)}\phi_n(y)}{x-s} + \frac{\mathrm{e}^{-\frac{1}{\hbar}(V_1(s)-sy)} s\, \phi_n(y)}{x-s} \right] =$$
$$= \delta_{n0}\sqrt{h_0}\mathrm{e}^{\frac{1}{\hbar}V_1(x)} - \mathrm{e}^{\frac{1}{\hbar}V_1(x)} \int_{\varkappa\Gamma} \mathrm{d}s \wedge \mathrm{d}y \, \frac{\mathrm{e}^{-\frac{1}{\hbar}(V_1(s)+sy)}\, \phi_n'(y)}{x-s} =$$
$$= \delta_{n0}\sqrt{h_0}\mathrm{e}^{\frac{1}{\hbar}V_1(x)} + \sum_{l=-1}^{d_2} \alpha_j(n+j)\underline{\phi}_{n+j}^{(0)}(x) \ .$$

In matricial form we have

$$x\underline{\Phi}_\infty{}^{(0)} = [\sqrt{h_0}\mathrm{e}^{\frac{1}{\hbar}V_1(x)}, 0, \dots] + \underline{\Phi}_\infty{}^{(0)}Q =: U + \underline{\Phi}_\infty{}^{(0)}Q \ . \tag{2-70}$$

It is easy to prove by induction that

$$x^k\underline{\Phi}_\infty{}^{(0)} = U\frac{x^k - Q^k}{x-Q} + \underline{\Phi}_\infty{}^{(0)}Q^k \ . \tag{2-71}$$

Let us now look at the ∂_x differential recurrence; after shifting the derivative on s inside the integral, integrating by parts and using Eq. 2-71, we get

$$\hbar\partial_x\underline{\Phi}_\infty{}^{(0)}(x) = U\frac{V_1'(x) - V_1'(Q)}{x-Q} + \underline{\Phi}_\infty{}^{(0)}(x)P \ . \tag{2-72}$$

This proves that the given integral expression is indeed the extra solution to Eq. 2-38.

As for the $\psi_n^{(k)}$ solutions we compute

$$\hbar\partial_x\psi_n^{(k)}(x) = -\int_{\widetilde{\Gamma}_y^{(k)}} \mathrm{d}y\ y\,\mathrm{e}^{\frac{1}{\hbar}(V_2(y)-yx)} \int_{\varkappa\Gamma} \mathrm{d}z \wedge \mathrm{d}t\, \frac{\mathrm{e}^{-\frac{1}{\hbar}(V_2(t)-zt)}\psi_n(z)}{t-y} = \tag{2-73}$$
$$= \int_{\widetilde{\Gamma}_y^{(k)}} \mathrm{d}y\,\mathrm{e}^{\frac{1}{\hbar}(V_2(y)-yx)} \int_{\varkappa\Gamma} \mathrm{d}z \wedge \mathrm{d}t\,\mathrm{e}^{-N(V_2(t)-zt)}\psi_n(z) + \tag{2-74}$$
$$- \int_{\widetilde{\Gamma}_y^{(k)}} \mathrm{d}y\,\mathrm{e}^{\frac{1}{\hbar}(V_2(y)-yx)} \int_{\varkappa\Gamma} \mathrm{d}z \wedge \mathrm{d}t\, \frac{\mathrm{e}^{-\frac{1}{\hbar}(V_2(t)-zt)}\, t\,\psi_n(z)}{t-y} = \tag{2-75}$$
$$= \delta_{n0}f^{(k)}(x) + \int_{\widetilde{\Gamma}_y^{(k)}} \mathrm{d}y\,\mathrm{e}^{\frac{1}{\hbar}(V_2(y)-yx)} \int_{\varkappa\Gamma} \mathrm{d}z \wedge \mathrm{d}t\, \frac{\mathrm{e}^{-\frac{1}{\hbar}(V_2(t)-zt)}\psi_n'(z)}{t-y} = \tag{2-76}$$
$$= \delta_{n0}f^{(k)}(x) - \sum_{j=-1}^{d_1} \beta_j(n+j)\psi_{n+j}^{(k)}(x) \ , \tag{2-77}$$
$$f^{(k)}(x) := \sqrt{h_0}\int_{\widetilde{\Gamma}_y^{(k)}} \mathrm{d}y\,\mathrm{e}^{\frac{1}{\hbar}(V_2(y)-xy)} \ . \tag{2-78}$$

Which, in matrix notation reads

$$\hbar\partial_x\Psi_\infty{}^{(k)}(x) = -P\Psi_\infty{}^{(k)}(x) + F^{(k)}(x) \ . \tag{2-79}$$

We leave to the reader the check that the x-multiplication equation holds as in Eq. 2-36. Q. E. D.

For completeness we recall that the matrices P, Q satisfy the following deformation equations (in [4] the matrix that now is denoted by P was denoted by $-P$)

$$\partial_{u_K} Q = -\left[Q, \mathrm{U}^K\right] \quad , \quad \partial_{v_J} Q = \left[Q, \mathrm{V}^J\right] \tag{2-80}$$

$$\partial_{u_K} P = -\left[P, \mathrm{U}^K\right] \quad , \quad \partial_{v_J} P = \left[P, \mathrm{V}^J\right] \tag{2-81}$$

where

$$\mathrm{U}^K := -\frac{1}{K}\left\{\left[Q^K\right]_{>0} + \frac{1}{2}\left[Q^K\right]_0\right\} \ , \ \mathrm{V}^J := -\frac{1}{J}\left\{\left[P^J\right]_{<0} + \frac{1}{2}\left[P^J\right]_0\right\} \ . \tag{2-82}$$

Here the subscript $<$ means the part of the matrix below the main diagonal and the subscript 0 denotes the diagonal. These equations have the following consequences on the wave-vectors

Lemma 2.1 *The wave-vectors ${\Psi_\infty}^{(0)}(x)$ (i.e. the quasipolynomials) and ${\underline{\Phi}_\infty}^{(k)}(x)$, $k = 1, \ldots, d_2$ (i.e. the Fourier–Laplace transform of the quasipolynomials $\phi_n(y)$) satisfy the following deformation equations*

$$\begin{aligned}
\partial_{u_K} {\Psi_\infty}^{(0)}(x) &= \mathrm{U}^K {\Psi_\infty}^{(0)}(x) \ , & (2\text{-}83)\\
\partial_{v_J} {\Psi_\infty}^{(0)}(x) &= -\mathrm{V}^J {\Psi_\infty}^{(0)}(x) \ , & (2\text{-}84)\\
\partial_{u_K} {\underline{\Phi}_\infty}^{(k)}(x) &= -{\underline{\Phi}_\infty}^{(k)}(x)\mathrm{U}^K \ , & (2\text{-}85)\\
\partial_{v_J} {\underline{\Phi}_\infty}^{(k)}(x) &= {\underline{\Phi}_\infty}^{(k)}(x)\mathrm{V}^J \ , & (2\text{-}86)
\end{aligned}$$

Proof. It follows straigforwardly from the two equations (2-80, 2-81) together with the fact that the Fourier–Laplace transform commutes with the differentiation w. r. t. the deformation parameters Q.E.D.

We also remark that the given integral expressions satisfy the following modified deformation equations

Proposition 2.5 The wave-vectors defined componentwise by Prop. 2.4 satisfy the following deformation equations

$$\partial_{u_K} {\underline{\Phi}_\infty}^{(0)}(x) = -{\underline{\Phi}_\infty}^{(0)}(x)\mathrm{U}^K + \frac{1}{K} U(x) \frac{x^K - Q^K}{x - Q} \tag{2-87}$$

$$\partial_{v_J} {\underline{\Phi}_\infty}^{(0)}(x) = {\underline{\Phi}_\infty}^{(0)}(x)\mathrm{V}^J \tag{2-88}$$

$$\partial_{u_K} {\Psi_\infty}^{(k)}(x) = \mathrm{U}^K {\Psi_\infty}^{(k)}(x) \tag{2-89}$$

$$\partial_{v_J} {\Psi_\infty}^{(k)}(x) = -\mathrm{V}^J {\Psi_\infty}^{(k)}(x) - \frac{1}{J} \frac{P^J - (-\hbar\partial_x)^J}{P + \hbar\partial_x} F^{(k)} \ , \tag{2-90}$$

where the semiinfinite vector $U(x)$ (not to be confused with the deformation matrix U^K) is defined in eq. 2-70 and the semiinfinite vectors $F^{(k)}(x)$, $k = 1, \ldots, d_2$ are defined in eqs. (2-78, 2-79).

Remark 2.3 In particular the sequences of functions defined in Prop. 2.4 satisfy identical deformation equations for N big enough ($N > d_1$ or $N > d_2$ according to which system is under consideration), and hence the relevant windows satisfy the full DDD equations specified in [4].

Proof. We recall that by induction one can prove that

$$x^K {\underline{\Phi}_\infty}^{(0)}(x) = {\underline{\Phi}_\infty}^{(0)}(x) Q^K + U(x) \frac{Q^K - x^K}{Q - x} \tag{2-91}$$

$$(-\hbar\partial_x)^J {\Psi_\infty}^{(k)}(x) = P^J {\Psi_\infty}^{(k)}(x) - \frac{P^J - (-\hbar\partial_x)^J}{P + \hbar\partial_x} F^{(k)}(x) \ , \tag{2-92}$$

where $F^{(k)}(x)$ is the same as in Eqs. (2-78, 2-79). As for the v_J deformation equations for ${\underline{\Phi}_\infty}^{(0)}$ the proof follows from Lemma 2.1 and from the fact that the integral transform defining this wave-vector does not depend on the v_J's. On the contrary the u_K deformations give (using again Lemma 2.1)

$$\partial_{u_K} {\underline{\Phi}_\infty}^{(0)}(x) = -{\underline{\Phi}_\infty}^{(0)}(x)\mathrm{U}^K + \frac{1}{K} {\underline{\Phi}_\infty}^{(0)}(x) \left(x^K - Q^K\right) \ , \tag{2-93}$$

from which the proof follows using eq. (2-91). Similarly there is nothing to prove for the u_J deformation of the wave-vectors ${\Psi_\infty}^{(k)}$, while for the v_J deformations (using an integration by parts together with Lemma 2.1) one obtains

$$\partial_{v_J} {\Psi_\infty}^{(k)}(x) = -\mathrm{V}^J {\Psi_\infty}^{(k)}(x) + \frac{1}{J}\left((-\hbar\partial_x)^J - P^J\right) {\Psi_\infty}^{(k)}(x) \ , \tag{2-94}$$

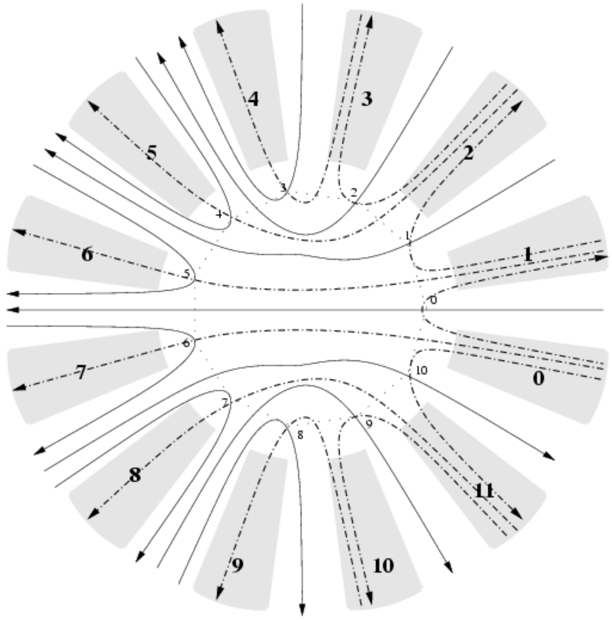

Figure 1: The SDC γ(solid) and $\tilde{\gamma}$ (line-dot-line) for a potential of degree 11.

and the proof now follows from eq. (2-92). Q.E.D.

As we were pointing out earlier, the choice of contours of integration in Eqs. (2-66, 2-67) is largely arbitrary. In particular the choice of contours that diagonalizes the pairing (2-28) is linked to the notion of dual steepest descent-ascent contours given hereafter.

Definition 2.3 *The steepest descent contours (SDCs) and the dual steepest ascent contours (SACs) for integrals of the form*

$$\int_\Gamma \mathrm{d}y\, \mathrm{e}^{-\frac{1}{\hbar}(V_2(y)-xy)} H(y) \ , \qquad \int_{\tilde{\Gamma}} \mathrm{d}y\, \mathrm{e}^{\frac{1}{\hbar}(V_2(y)-xy)} H(y) \ , \tag{2-95}$$

with $H(y)$ at most of exponential type, are the contours γ_k and $\tilde{\gamma}_k$ respectively uniquely defined (as $x \to \infty$ within a suitable sector) as follows

$$\gamma_k := \left\{ y \in \mathbb{C};\ \Im(V_2(y) - xy) = \Im\left(V_2(y_k(x)) - xy_k(x)\right) \ , \Re(V_2(y)) \underset{\substack{y\,\to\,\infty \\ y\,\in\,\gamma_k}}{\longrightarrow} +\infty \ . \right\} \ , \tag{2-96}$$

$$\widetilde{\gamma}_k := \left\{ y \in \mathbb{C};\ \Im(V_2(y) - xy) = \Im\left(V_2(y_k(x)) - xy_k(x)\right) \ , \Re(V_2(y)) \underset{\substack{y\,\to\,\infty \\ y\,\in\,\widetilde{\gamma}_k}}{\longrightarrow} -\infty \ . \right\} \ , \tag{2-97}$$

where $y_k(x)$ are the d_2 branches of the solution to

$$V_2'(y) = x \ , \tag{2-98}$$

which behave as $(v_{d_2+1})^{-\frac{1}{d_2}} x^{\frac{1}{d_2}}$ as $x \to \infty$ in the sector, for the different determinations of the roots of x. Their homology class is constant as $x \to \infty$ within the sector (see Fig. 1)

With this choice of contours of integration these particular solutions to the systems (2-36, 2-38) have the property in the following proposition

Proposition 2.6 If we choose dual steepest descent-ascent contours defined as in Def. 2.3, $\tilde{\gamma}_k$ and γ_k for the integrals in Eqs. 2-66, 2-67, then we have

$$C^{\alpha\beta} := \underline{\Phi}^{N\alpha}(x) \overset{N}{\mathbb{A}} \Psi_N{}^{\beta}(x) = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 2i\pi\hbar \mathbf{1}_{d_2} \end{array}\right] \ , \qquad N > d_2 \ . \tag{2-99}$$

Proof.
For $\beta = 0$ the statement follows from Eq. 2-64 (where the constant $c = \sqrt{\hbar_0}$).
For $\alpha = 0 \neq \beta$ by inspection of the exponential asymptotic behavior of the integrals (2-65–2-68) it is easy to conclude that $C^{0\beta} = 0,\ \beta \neq 0$.
Finally for $k := \alpha \neq 0 \neq \beta =: j$, C^{kj} equals the bilinear concomitant of the corresponding functions (Eq. 2-62). Let us consider a fixed sector in the x plane and choose a basis of steepest descent contours $\widetilde{\gamma}_k$ and $\widetilde{\gamma}_k$. We use formula (2-62) with

$$f^{(j)}(x) := \sqrt{\hbar_0}\int_{\widetilde{\gamma}_j}\mathrm{d}y\, \mathrm{e}^{\frac{1}{\hbar}(V_2(y)-xy)}\ ,\quad \underline{\varphi}_0^{(k)}(x) := \frac{1}{\sqrt{\hbar_0}}\int_{\gamma_k}\mathrm{d}y\, \mathrm{e}^{-\frac{1}{\hbar}(V_2(y)-xy)}\ . \tag{2-100}$$

The respective asymptotic behaviors, computed by the saddle–point method, are

$$f^{(j)}(x) \simeq \sqrt{\hbar_0}\mathrm{e}^{\frac{1}{\hbar}(V_2(y_j(x))-xy_j(x))}\sqrt{\frac{-2\pi\hbar}{V_2''(y_j(x))}}\big(1+\mathcal{O}(\lambda^{-1})\big) \tag{2-101}$$

$$\underline{\varphi}_0^{(k)}(x) \simeq \frac{1}{\sqrt{\hbar_0}}\mathrm{e}^{-\frac{1}{\hbar}(V_2(y_k(x))-xy_k(x))}\sqrt{\frac{2\pi\hbar}{V_2''(y_k(x))}}\big(1+\mathcal{O}(\lambda^{-1})\big)\ , \tag{2-102}$$

where $y_k(x)$ are the d_2 solutions of $V_2'(y) = x$, which behave as the d_2 roots of x for $x \to \infty$ (within a specified sector which is not relevant here). It is clear that the bilinear concomitant being a constant, it must vanish for $j \neq k$ since the exponential parts of the asymptotic behavior in Eqs. 2-101, 2-102 can not give a nonzero constant when multiplied together.
For $j = k$ then the bilinear concomitant is given by the integral

$$\frac{V_2'(\hbar\partial_{x'}) - V_2'(-\hbar\partial_x)}{\hbar\partial_{x'} + \hbar\partial_x}\underline{\varphi}_0^{(k)}(x')f^{(k)}(x)\bigg|_{x'=x} = \tag{2-103}$$

$$= \int_{\widetilde{\gamma}_k\times\gamma_k}\mathrm{d}y\wedge\mathrm{d}y'\,\mathrm{e}^{\frac{1}{\hbar}(V_2(y)-V_2(y')-x(y-y'))}\,\frac{V_2'(y)-V_2(y')}{y-y'} \simeq \tag{2-104}$$

$$\simeq \mathrm{e}^{\frac{1}{\hbar}(V_2(y_k(x))-xy_k(x))}\sqrt{\frac{2\pi\hbar}{V_2''(y_k(x))}}\ \cdot \tag{2-105}$$

$$\cdot\mathrm{e}^{-\frac{1}{\hbar}(V_2(y_k(x))-xy_k(x))}\sqrt{\frac{-2\pi\hbar}{V_2''(y_k(x))}}\big(1+\mathcal{O}(\lambda^{-1})\big)V_2''(y_k(x)) = 2i\pi\hbar \tag{2-106}$$

This concludes the proof. Q.E.D.

Had we chosen the contours $\widetilde{\Gamma}_y^{(k)}$ and $\Gamma_y^{(k)}$ rather than the steepest descent contours, then we would have had a block–diagonal constant matrix for $C^{\alpha\beta}$. Notice that the pairing of these integrals is independent also of the deformation parameters of $V_1,\ V_2$. As we know from [4], this can always be accomplished, but here we have explicitly seen this occurring for the particular normalizations chosen in the integrals.

3 Asymptotic behavior at infinity and Riemann–Hilbert problem

Given the duality between the D_1 and $\underline{D}_1$ ODEs

$$\overset{N}{\mathbb{A}}\, D_1^N(x) \equiv \underline{D}_1^N(x)\,\overset{N}{\mathbb{A}}\ , \tag{3-1}$$

it is clear that the Stokes matrices around the irregular singularity at $x = \infty$ can be computed for either one of these systems. Since we have explicit integral representations of the fundamental solutions we can read the asymptotic behavior off the solution itself. The solutions for the $\underline{D}_1$ system are simpler to analyze but in principle one could consider the asymptotic behavior of the solutions to the D_1 system given by taking the suitable windows in the sequences (2-67, 2-68).

Proposition 3.1 [Formal Asymptotics] The system

$$\frac{d}{dx}\Phi(x) = \Phi(x)\underline{D}_1^N(x)\ , \tag{3-2}$$

possesses a solution with a formal asymptotic behavior at $x = \infty$ of the form

$$\Phi(x) \sim \mathrm{e}^{\frac{1}{\hbar}T(x)}W\,x^G Y(x^{\frac{1}{d}}) \tag{3-3}$$

where $Y = Y_0 + \mathcal{O}(x^{-1/d_2})$ is a matrix–valued function analytic at infinity (Y_0 is a diagonal invertible matrix specified in the proof) and

$$T(x) := \sum_{j=0}^{d_2} \frac{d_2 t_j}{d_2 - j + 1} x^{\frac{d_2+1-j}{d_2}} \Omega^{d_2+1-j} + V_1(x) E \tag{3-4}$$

$$W := \left[\begin{array}{c|cccc} 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & \omega & \omega^2 & \cdots & \omega^{d_2} \\ 0 & \omega^2 & \omega^4 & \cdots & \omega^{2d_2} \\ \vdots & & & \vdots & \\ 0 & \omega^{d_2} & \omega^{2d_2} & \cdots & \omega^{d_2^2} \end{array}\right] \tag{3-5}$$

$$G := \mathrm{diag}\left(-N, \frac{N + \frac{1}{2} - \frac{d_2}{2}}{d_2}, \frac{N + \frac{3}{2} - \frac{d_2}{2}}{d_2}, \ldots, \frac{N - \frac{1}{2} + \frac{d_2}{2}}{d_2}\right) \tag{3-6}$$

$$\Omega := \mathrm{diag}(0, \omega, \omega^2, \ldots, \omega^{d_2-1}, \omega^{d_2}) \ , \quad \omega := \mathrm{e}^{\frac{2i\pi}{d_2}} \tag{3-7}$$

$$E := \mathrm{diag}(1, 0, \ldots, 0) \tag{3-8}$$

$$t_0 := (v_{d_2+1})^{-\frac{1}{d_2}} \ , \tag{3-9}$$

$$t_1 := -\frac{1}{d_2} \frac{v_{d_2}}{v_{d_2+1}} \ , \tag{3-10}$$

$$t_j := \frac{1}{j-1} \mathop{\mathrm{res}}_{y=\infty} \left(V_2'(y)\right)^{\frac{j-1}{d_2}} \mathrm{d}y \ . \ \ j = 2, \ldots, d_2. \tag{3-11}$$

Such formal asymptotics is the asymptotics of a real solution within the sectors $\mathcal{S}_k$

$$\mathcal{S}_k := \left\{ x \in \mathbb{C} \ , \ \arg(x) \in \left(\vartheta + \frac{(2k-1)\pi}{2(d_2+1)}, \vartheta + \frac{(2k+1)\pi}{2(d_2+1)}\right)\right\}, \quad \text{for } d_2 \text{ odd}$$

$$\mathcal{S}_k := \left\{ x \in \mathbb{C} \ , \ \arg(x) \in \left(\vartheta + \frac{k\pi}{(d_2+1)}, \vartheta + \frac{(k+1)\pi}{(d_2+1)}\right)\right\}, \quad \text{for } d_2 \text{ even} \tag{3-12}$$

$$\vartheta := -\frac{\arg(v_{d_2+1})}{d_2} \ , \quad k = 0, \ldots, 2d_2 + 1 \ , \tag{3-13}$$

minus the contours $\bigcup_k \Gamma_x^{(k)}$ used to define the pairing.

Proof.
In any given sector $\mathcal{S}_k$ of Eq. (3-13) we can choose a basis of steepest descent contours $\gamma_y^{(k)}$, $k = 1, \ldots, d_2$; the reason why these are the Stokes sectors and the proper construction of the steepest descent contours is delayed to the discussion about the Stokes' matrices in Sect. 3.1.1.
We Fourier-Laplace transform the quasipolynomials $\phi_n(y)$ along these contours in order to obtain the functions $\underline{\varphi}_n(x)$. Notice that they are not necessarily the same as the previously introduced $\underline{\phi}_n^{(k)}(x)$ inasmuch as the steepest descent contours do not all coincide with the contours $\Gamma_y^{(k)}$ defined previously. However they are suitable linear combinations with integer coefficients of such $\underline{\phi}_n^{(k)}(x)$ since the choice of the steepest descent contours is just a different basis in the "homology" of the y-plane.
We then consider the asymptotic expansions (we set $s_2 := d_2 + 1$) for

$$\underline{\varphi}_n^{(k)}(x) := \int_{\gamma_y^{(k)}} \mathrm{d}y \, \mathrm{e}^{-\hbar^{-1}(V_2(y) - xy)} \phi_n(y) \ . \tag{3-14}$$

Asymptotically the main contribution is given at the critical point of the exponent $V_2(y) - xy$ corresponding to the steepest descent contour $\gamma_y^{(k)}$. That is we need to compute

$$V_2(y) - xy\big|_{V_2'(y) - x = 0} \tag{3-15}$$

asymptotically as $x \to \infty$ within the specified sector. Let us solve the relation $V_2'(y) - x = 0$ in series expansion in the local parameter at ∞ given by one determination of the d_2-th root of x, $\lambda := x^{\frac{1}{d_2}}$.

$$v_{s_2} y^{d_2} + v_{d_2} y^{d_2-1} + \ldots = V_2'(y) = \lambda^{d_2} \tag{3-16}$$

$$y(\lambda) = \lambda \sum_{j=0}^{\infty} t_j \lambda^{-j} \ . \tag{3-17}$$

Then we have the formulas (recalling $\lambda = (V_2'(y))^{\frac{1}{d_2}}$)

$$
\begin{aligned}
t_0 &= (v_{s_2})^{-\frac{1}{d_2}} \ , \\
t_1 &= \mathop{\rm res}_{\lambda=\infty} y(\lambda)\frac{{\rm d}\lambda}{\lambda} = \frac{1}{d_2}\mathop{\rm res}_{y=\infty} y\frac{V_2''(y)}{V_2'(y)}{\rm d}y = -\frac{1}{d_2}\frac{v_{d_2}}{v_{s_2}} \\
t_j &= \mathop{\rm res}_{\lambda=\infty} \frac{\lambda^{j-1}}{j-1} y'(\lambda){\rm d}\lambda = \frac{1}{j-1}\mathop{\rm res}_{y=\infty} \left(V_2'(y)\right)^{\frac{j-1}{d_2}}{\rm d}y \ .
\end{aligned} \tag{3-18}
$$

Let us denote by $y_k(x)$ the d_2 solutions of the equation $V_2'(y) = x$ which have been solved by series in the d_2-th root of x in Eq. 3-17 and 3-18. We then have, up to an additive constant and in a neighborhood of $x = \infty$

$$
V_2(y_k(x)) - xy_k(x) = -\int^x y_k(x'){\rm d}x' = -d_2\int^{\lambda_k} y(\lambda')\lambda'^{d_2-1}{\rm d}\lambda' = -\sum_{j=0}^{\infty}\frac{d_2 t_j}{s_2-j}\lambda_k^{s_2-j} \ . \tag{3-19}
$$

This formula is proven by taking the derivative (w.r.t. x) of both sides and using the defining equation for $y_k(x)$. Notice that there is no logarithmic contribution since $t_{s_2} = 0$. The different saddle critical points are computed by replacing λ with $\omega^k\lambda$. Inserting into the integral representation of the functions $\underline{\varphi}_n^{(k)}(x)$

$$
\underline{\varphi}_n^{(k)}(x) := \frac{1}{\sqrt{h_n}}\int_{\gamma_k^{(y)}}{\rm d}y\, {\rm e}^{-N(V_2(y)-xy)}\sigma_n(y) \simeq \tag{3-20}
$$

$$
\simeq \frac{C}{\sqrt{h_n}}{\rm e}^{\frac{1}{\hbar}\int^x y_k(x'){\rm d}x'}\sigma_n(y(\lambda_k))\int_{\mathbb{R}}{\rm e}^{-\frac{1}{2\hbar}V_2''(y(\lambda_k))t^2}{\rm d}t = \tag{3-21}
$$

$$
= \frac{C}{\sqrt{h_n}}{\rm e}^{\int^x y_k(x'){\rm d}x'}\sigma_n(y(\lambda_k))\sqrt{\frac{2\pi\hbar}{V_2''(y(\lambda_k))}} \tag{3-22}
$$

Differentiating $V_2'(y(\lambda_k)) = {\lambda_k}^{d_2}$ we obtain the relation

$$
V_2''(y(\lambda_k)) = \frac{d_2{\lambda_k}^{d_2-1}}{y'(\lambda_k)} \tag{3-23}
$$

where $y'(\lambda)$ means differentiation w.r.t. λ.
Therefore we obtain

$$
\underline{\varphi}_n^{(k)}(x) \sim {\rm e}^{-\hbar^{-1}(V_2(y_k(x))-xy_k(x))}\overbrace{\frac{\sigma_n(y_k(x))}{\sqrt{h_n}}\sqrt{\frac{2\pi\hbar}{V_2''(y_k(x))}}}^{\simeq\sqrt{\frac{2\pi\hbar}{h_n}}{\lambda_k}^{n-\frac{d_2-1}{2}}v_{s_2}^{-\frac{2n-1}{2d_2}}} \sim
$$

$$
\sim \sqrt{\frac{2\pi\hbar}{d_2h_n}}{\rm e}^{-\hbar^{-1}(V_2(y_k(x))-xy_k(x))}{\lambda_k}^{n-\frac{d_2-1}{2}}v_{s_2}^{-\frac{2n-1}{2d_2}} = \tag{3-24}
$$

$$
= \frac{C}{\sqrt{h_n}}{\rm e}^{\frac{1}{\hbar}\int^x y_k(x'){\rm d}x'}\sigma_n(y(\lambda_k))\sqrt{\frac{2\pi\hbar y'(\lambda_k)}{d_2{\lambda_k}^{d_2-1}}} = \tag{3-25}
$$

$$
= \tilde{C}\frac{v_{s_2}^{-\frac{n+1}{d_2}}}{\sqrt{h_n}}{\lambda_k}^{n+\frac{1-d_2}{2}}\exp\left[\frac{1}{\hbar}\sum_{j=0}^{d_2}\frac{d_2t_j}{s_2-j}{\lambda_k}^{s_2-j}\right]\left(1+\mathcal{O}\left(\lambda^{-1}\right)\right) \ , \tag{3-26}
$$

where the constant $\tilde{C}$ does not depend on n, λ (depends only and universally on the coefficients of $V_2(y)$) and $\lambda_k := \omega^k\lambda$.
The functions $\phi_n^{(0)}(x)$ have the following asymptotic expansion as $x \to \infty$ in $\mathbb{C}\setminus\bigcup_{k=1}^{d_1}\Gamma_x^{(k)}$

$$
\begin{aligned}
\phi_n^{(0)}(x) &:= {\rm e}^{\frac{1}{\hbar}V_1(x)}\int_{\varkappa\Gamma}{\rm d}s\wedge{\rm d}y\,\frac{{\rm e}^{-\hbar^{-1}(V_1(s)-sy)}\phi_n(y)}{(x-s)} \simeq \\
&\simeq {\rm e}^{\frac{1}{\hbar}V_1(x)}\sum_{k=0}^{\infty}x^{-k-1}\int_{\varkappa\Gamma}{\rm d}s\wedge{\rm d}y\, s^k{\rm e}^{-\hbar^{-1}V_1(s)}\underline{\phi}_n(y) = \\
&= \sqrt{h_n}x^{-n-1}{\rm e}^{\hbar^{-1}V_1(x)}\left(1+\mathcal{O}\left(x^{-1}\right)\right) \ .
\end{aligned} \tag{3-27}
$$

Now, the rows of the fundamental matrix $\Phi(x)$ are given by $[\underline{\varphi}^{(k)}_{N-1}(x),\dots,\underline{\varphi}^{(k)}_{N+d_2-1}(x)]$, $k=1,\dots,d_2$ while first row is given by $[\phi^{(0)}_{N-1},\dots,\phi^{(0)}_{N+d_2-1}]$. Therefore the matrix of leading terms is given by

$$
\begin{array}{c}
\mathrm{e}^{\frac{1}{\hbar}T(x)}\mathrm{diag}\left(\tilde{C},1,\cdots\right)\left[\begin{array}{ccc}
h_{N-1}v_{s_2}^{\frac{N}{d_2}}x^{-N} & \cdots & h_{N+d_2-1}v_{s_2}^{\frac{N+d_2}{d_2}}x^{-N-d_2}\\
{\lambda_1}^{N-\frac{1}{2}-\frac{d_2}{2}} & \cdots & {\lambda_1}^{N-\frac{1}{2}+\frac{d_2}{2}}\\
{\lambda_2}^{N-\frac{1}{2}-\frac{d_2}{2}} & \cdots & {\lambda_2}^{N-\frac{1}{2}+\frac{d_2}{2}}\\
\vdots & \cdots & \vdots\\
{\lambda_{d_2}}^{N-\frac{1}{2}-\frac{d_2}{2}} & \cdots & {\lambda_{d_2}}^{N-\frac{1}{2}+\frac{d_2}{2}}
\end{array}\right].\\
\cdot\mathrm{diag}\left(\frac{\tilde{C}v_{s_2}^{-\frac{N}{d_2}}}{\sqrt{h_{N-1}}},\dots,\frac{\tilde{C}v_{s_2}^{-\frac{N+d_2}{d_2}}}{\sqrt{h_{N+d_2-1}}}\right).
\end{array}
\tag{3-28}
$$

Notice that the determinant of the Vandermonde–like matrix is actully very simple: by computing it along the first row one realizes that only the first and last minor are not zero. Indeed for all other minor the coeersponding submatrix has the first and last column proportional. The first minor is a constant in x while the last is of order x^{-d_2}. Therefore we can write

$$
\left[\begin{array}{ccc}
h_{N-1}v_{s_2}^{\frac{N}{d_2}}x^{-N} & \cdots & h_{N+d_2-1}v_{s_2}^{\frac{N+d_2}{d_2}}x^{-N-d_2}\\
{\lambda_1}^{N-\frac{1}{2}-\frac{d_2}{2}} & \cdots & {\lambda_1}^{N-\frac{1}{2}+\frac{d_2}{2}}\\
{\lambda_2}^{N-\frac{1}{2}-\frac{d_2}{2}} & \cdots & {\lambda_2}^{N-\frac{1}{2}+\frac{d_2}{2}}\\
\vdots & \cdots & \vdots\\
{\lambda_{d_2}}^{N-\frac{1}{2}-\frac{d_2}{2}} & \cdots & {\lambda_{d_2}}^{N-\frac{1}{2}+\frac{d_2}{2}}
\end{array}\right]=
$$

$$
=\left[\begin{array}{cccc}
h_{N-1}v_{s_2}^{\frac{N}{d_2}}x^{-N} & 0 & \cdots & 0\\
0 & {\lambda_1}^{N+\frac{1}{2}-\frac{d_2}{2}} & \cdots & {\lambda_1}^{N-\frac{1}{2}+\frac{d_2}{2}}\\
0 & {\lambda_2}^{N-\frac{1}{2}-\frac{d_2}{2}} & \cdots & {\lambda_2}^{N-\frac{1}{2}+\frac{d_2}{2}}\\
\vdots & & \cdots & \vdots\\
0 & {\lambda_{d_2}}^{N-\frac{1}{2}-\frac{d_2}{2}} & \cdots & {\lambda_{d_2}}^{N-\frac{1}{2}+\frac{d_2}{2}}
\end{array}\right](1+\mathcal{O}(x^{-1}))= \tag{3-29}
$$

$$
=\left[\begin{array}{cccc}
h_{N-1}v_{s_2}^{\frac{N}{d_2}}x^{-N} & 0 & \cdots & 0\\
0 & {\lambda_1}^{N+\frac{1}{2}-\frac{d_2}{2}} & \cdots & {\lambda_1}^{N-\frac{1}{2}+\frac{d_2}{2}}\\
0 & {\lambda_2}^{N-\frac{1}{2}-\frac{d_2}{2}} & \cdots & {\lambda_2}^{N-\frac{1}{2}+\frac{d_2}{2}}\\
\vdots & & \cdots & \vdots\\
0 & {\lambda_{d_2}}^{N-\frac{1}{2}-\frac{d_2}{2}} & \cdots & {\lambda_{d_2}}^{N-\frac{1}{2}+\frac{d_2}{2}}
\end{array}\right](1+\mathcal{O}(x^{-1}))= \tag{3-30}
$$

$$
\begin{array}{c}
=\left[\begin{array}{cccc}
1 & 0 & \cdots & 0\\
0 & \omega^{N+\frac{1}{2}-\frac{d_2}{2}} & \cdots & \omega^{N-\frac{1}{2}+\frac{d_2}{2}}\\
0 & (\omega^2)^{N+\frac{1}{2}-\frac{d_2}{2}} & \cdots & (\omega^2)^{N-\frac{1}{2}+\frac{d_2}{2}}\\
\vdots & & \cdots & \vdots\\
0 & 1 & \cdots & 1
\end{array}\right].\\
\cdot\mathrm{diag}\left(h_{N-1}v_{s_2}^{\frac{N}{d_2}}x^{-N},\lambda^{N+\frac{1}{2}-\frac{d_2}{2}},\dots,\lambda^{N-\frac{1}{2}+\frac{d_2}{2}}\right)(1+\mathcal{O}(x^{-1}))
\end{array}
\tag{3-31}
$$

When inserting this into the asymptotics we see that, up to factoring the constant invertible (diagonal) matrix on the left

$$
\mathrm{diag}(1,\tilde{C}\omega^{N+\frac{1}{2}-\frac{d_2}{2}},\tilde{C}\omega^{2(N+1/2-d_2/2)},\cdots,\tilde{C}) \tag{3-32}
$$

which is irrelevant for the asymptotics and depends on N in a rather trivial manner, we obtain a solution with an asymptotic behavior

$$
\Phi(x)\simeq\exp\frac{1}{\hbar}\left(EV_1(x)+\sum_{j=0}^{d_2}\frac{d_2t_j\lambda^{s_2-j}}{d_2-j+1}\Omega^{s_2-j}\right)\cdot W\cdot
$$

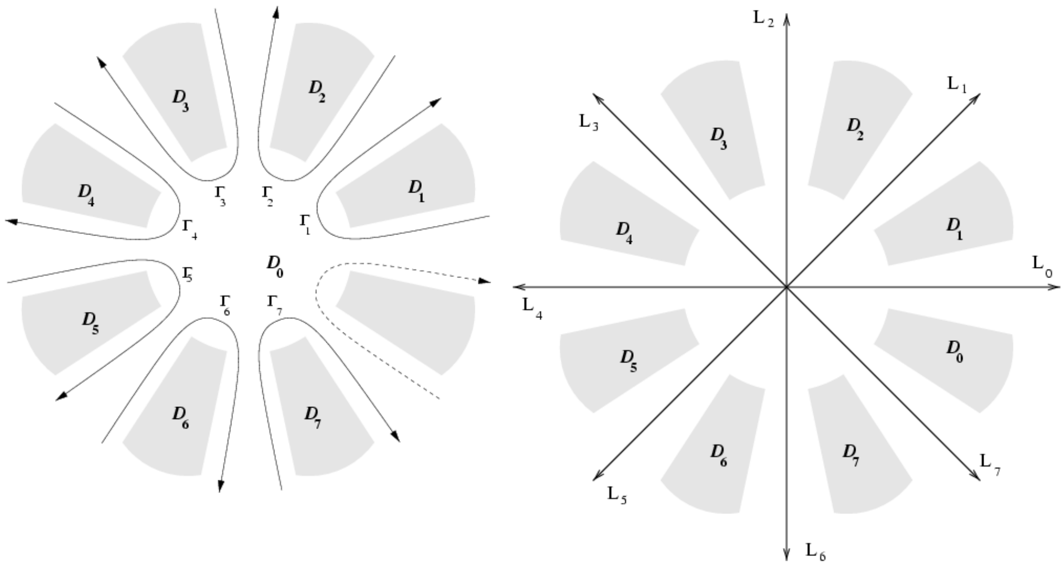

Figure 2: The two possible choices for the domains of definition of $\underline{\phi}_n^{(0)}$ in the case $d_1 = 7$

$$
\cdot \mathrm{diag}\left(h_{N-1}{v_{s_2}}^{\frac{N}{d_2}}x^{-N}, \lambda^{N+\frac{1}{2}-\frac{d_2}{2}}, \dots, \lambda^{N-\frac{1}{2}+\frac{d_2}{2}}\right)\mathrm{diag}\left(\frac{{v_{s_2}}^{-\frac{N}{d_2}}}{\sqrt{h_{N-1}}}, \dots, \frac{{v_{s_2}}^{-\frac{N+d_2}{d_2}}}{\sqrt{h_{N+d_2-1}}}\right)(1+\mathcal{O}(\lambda^{-1})) =
$$

$$
= \mathrm{e}^{\frac{1}{\hbar}T(x)} \cdot W \cdot \mathrm{diag}\left(\sqrt{h_{N-1}}x^{-N}, \frac{{v_{s_2}}^{-\frac{N+1}{d_2}}\lambda^{N+\frac{1}{2}-\frac{d_2}{2}}}{\sqrt{h_N}}, \dots, \frac{{v_{s_2}}^{-\frac{N+d_2}{d_2}}\lambda^{N-\frac{1}{2}+\frac{d_2}{2}}}{\sqrt{h_{N+d_2-1}}}\right)(1+\mathcal{O}(\lambda^{-1})) =
$$

$$
= \exp\left(\frac{1}{\hbar}WT(x)W^{-1}\right)x^G \mathrm{diag}\left(\sqrt{h_{N-1}}, \frac{{v_{s_2}}^{-\frac{N+1}{d_2}}}{\sqrt{h_N}}, \dots, \frac{{v_{s_2}}^{-\frac{N+d_2}{d_2}}}{\sqrt{h_{N+d_2-1}}}\right)(1+\mathcal{O}(\lambda^{-1})) \tag{3-33}
$$

where W is the block-diagonal matrix

$$
W = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & \omega^2 & \cdots & \omega^{d_2} \\ 0 & \omega^2 & \omega^4 & \cdots & \omega^{2d_2} \\ \vdots & & & & \vdots \\ 0 & 1 & \cdots & & 1 \end{pmatrix} \tag{3-34}
$$

Notice that $W^{-1}\Omega W$ is the permutation matrix (in the subblock)

3.1 Riemann Hilbert problem

In this section we analyze the Riemann–Hilbert problem that arises naturally for the solutions of the $\underline{D}_1$ system. Recalling the expression for the sequence $\{\underline{\phi}_n^{(0)}(x)\}_{n\in\mathbb{N}}$ (Eq. 2-65), it is clear that they are piecewise analytic functions in each of the d_1+1 connected domains of the x-plane minus the contours $\Gamma_x^{(j)}$.

Remark 3.1 The sequence $\underline{\phi}_n^{(0)}(x)$, $\mu = 0, \dots, d_1$ can be analytically continued to entire functions, since the contours $\Gamma_x^{(j)}$ can be deformed arbitrarily in the finite part of the x-plane. Therefore the "discontinuities" in the definition of the Hilbert integral are just apparent and have an intrinsic meaning only when studying the asymptotic behavior at infinity.

We denote by $\mathcal{D}_\mu$ the connected domain to the right of the contours $\Gamma_x^{(\mu)}$ for $\mu = 1, \dots, d_1$ while $\mathcal{D}_0$ is the domain to the left of all contours. By retracting the contours to rays L_μ from the origin, the domains $\mathcal{D}_\mu$ become sectors in the x–plane (see Fig. 2).

Let us denote by $\underline{\Phi}^{\mathrm{N}}$ the piecewise analytic invertible matrix

$$
\underline{\Phi}^{\mathrm{N}}(x) := \left[\underline{\Phi}^{N(0)}, \underline{\Phi}^{N(1)}, \dots, \underline{\Phi}^{N(d_2)}\right]^t . \tag{3-35}
$$

The fact that this is an invertible matrix follows from the fact that the various $\underline{\Phi}^{N(k)}$ are linearly independent and $\underline{\Phi}^{N(0)}$ is independent from them because it satisfies the same recurrence relation but with different initial (in n)

conditions.
These matrices satisfy the Riemann-Hilbert problem

$$\underline{\Phi}^{\rm N}{}_+(x) = \begin{bmatrix} 1 & 2i\pi\varkappa_{\mu,1} & 2i\pi\varkappa_{\mu,2} & \cdots & 2i\pi\varkappa_{\mu,d_2} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \underline{\Phi}^{\rm N}{}_-(x)\ , \quad x \in \Gamma_x^{(\mu)},\ \mu = 1, \dots, d_1\ , \tag{3-36}$$

where the subscripts $_+$, $_-$ denote the limiting value from the right or from the left w.r.t. the orientation of the contour. Equivalently we can shrink $\mathcal{D}_0$ by retracting the d_1 contours to the origin in such a way that the d_1+1 regions become sectors. Denoting by L_μ the ray which separates $\mathcal{D}_\mu$ and $\mathcal{D}_{\mu+1}$, the corresponding RH problem then reads:

$$\underline{\Phi}^{\rm N}{}_+(x)\Big|_{x\in L_\mu} = G_\mu \underline{\Phi}^{\rm N}{}_-(x) := \begin{bmatrix} 1 & 2i\pi J_{\mu,1} & 2i\pi J_{\mu,2} & \cdots & 2i\pi J_{\mu,d_2} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \underline{\Phi}^{\rm N}{}_-(x) \tag{3-37}$$

$$J_{\mu,j} := \varkappa_{\mu,j} - \varkappa_{\mu+1,j}\ ; \mu = 0, \dots, d_1,\ j = 1, \dots, d_2,\ \varkappa_{0,j} := \varkappa_{d_1+1,j} := 0 \tag{3-38}$$

In order to formulate the complete RH problem we need to supplement the discontinuity data with the formal asymptotic around the irregular singularity $x = \infty$ and the Stokes matrices. In doing so one should be careful that the lines L_μ for which the discontinuities are defined do not coincide with any of the Stokes' lines. We can always avoid this occurrence by perturbing the lines L_μ.

3.1.1 Stokes Matrices

The fundamental solution of the system $\underline{D}_1$ is constituted by d_2 Fourier-Laplace tranforms $\underline{\phi}_n^{(k)}(x),\ k = 1, \dots d_2$ and one Hilbert-Fourier-Laplace tranform $\underline{\phi}_n^{(0)}(x)$. The asymptotic behavior of the d_2 F-L transforms is analyzed by means of the steepest descent method in each of the sectors $\mathcal{S}_k$, $k = 0, \dots 2d_2+1$ (3-13).

The computation is achieved by expressing the change of homology basis from the $\Gamma_y^{(1)}, \dots, \Gamma_y^{(d_2)}$ contours to the steepest descent contours[8]. In order to simplify the analysis of the Stokes matrices we point out that there is no essential loss of generality in assuming $V_2(y) = y^{d_2+1}\frac{1}{d_2+1}$; indeed, since we are concerned about topological structures of the SDCs, as $x \to \infty$ the d_2 solutions of the equation $V_2'(y) = x$ entering Def. 2.3 are distinct and asymptotic to the d_2 roots of x (up to a nonzero factor). The particular choice of the leading coefficient is also only for practical purposes; should we choose a different coefficient we should appropriately rotate by $\vartheta = -\arg(v_{d_2+1})/d_2$ counterclockwise the pictures we are going to draw, but without any essential difference.

With this assumption, the Stokes phenomenon can be studied directly on the integrals

$$\int {\rm d}y\, {\rm e}^{-\frac{1}{\hbar}\left(\frac{y^{d+1}}{d+1} - xy\right)} = |x|^{\frac{1}{d}} \int {\rm d}z\, \exp\left[-\Lambda\left(\frac{z^{d+1}}{d+1} - {\rm e}^{i\alpha}z\right)\right]\ ;\ \ \Lambda := \frac{1}{\hbar}|x|^{\frac{d+1}{d}}\ ,\ \ \alpha := \arg(x)\ , \tag{3-39}$$

where, in order to avoid too many subscripts in the formulas to come, we have set $d = d_2$. The integrals in (3-39) can be written as

$$\int {\rm d}z\, \exp\left[-\Lambda\left(\frac{z^{d+1}}{d+1} - {\rm e}^{i\alpha}z\right)\right] = \int {\rm d}s\, {\rm e}^{-\Lambda s}\frac{{\rm d}z}{{\rm d}s}, \tag{3-40}$$

where $z = Z(s)$ is the map inverse to

$$s = S(z) := \frac{z^{d+1}}{d+1} - {\rm e}^{i\alpha}z\ ,\quad z = Z(s). \tag{3-41}$$

The map $z = Z(s)$ is a $(d+1)$-fold covering of the s-plane branching around points (z, s) whose projection on the s-plane are the d critical values

$$s_{cr}^{(j)} = s_{cr}^{(j)}(\alpha) = -\frac{d}{d+1}{\rm e}^{i\frac{(d+1)\alpha}{d}}\omega^j\ ,\ \ \omega := {\rm e}^{\frac{2i\pi}{d}}\ ,\quad j = 0, \dots, d-1. \tag{3-42}$$

[8] we suppress the subscript y from the steepest descent contours.

In realizing this $d+1$-fold covering, we take the branch-cuts on the s-plane to be the rays $\Im(s) = \Im(s_{cr}^{(j)})$ =const, extending to $\Re(s) = +\infty$. As $\Lambda \to +\infty$ the integrals (3-39) have leading asymptotic behavior that depends only on the critical values of the map $s(z)$ and on the homology class of the contour.
We now come back to the computation of the Stokes' lines; by the definition of the SDC's γ_k (Def. 2.3, with now $V_2(y) = y^{d+1}/(d+1)$) their image in the s-plane are contours which come from $\Re(s) = +\infty$ on one side of the branch-cut (and on the appropriate sheet) and go back to $\Re(s) = +\infty$ on the other side of the branch-cut (and on the same sheet)[9]. The cuts on the s-plane may overlap only for those values of $\alpha = \arg(x)$ for which the imaginary parts of two different critical values $s_{cr}^{(i)}(\alpha)$ and $s_{cr}^{(j)}(\alpha)$ coincide: a straightforward computation yields the lines separating the sectors $\mathcal{S}_k$ defined in Eqs. (3-13), which –under our inessential simplifying assumptions on $V_2(y)$– now reads with d_2 replaced by d and $\vartheta = 0$.

We need also the following

Definition 3.1 *For a given sector $\mathcal{S}$ centered around a ray $\arg(x) = \alpha_0$ with width $A < \pi$, the* **dual sector** $\mathcal{S}^\vee$ *is the sector centered around the ray $\arg(x) = \alpha_0 + \pi$ and with width $\pi - A$.*

For $x \to \infty$ in each sector the SDCs are a constant integral linear combination of the contours $\Gamma_y^{(k)}$'s; when x crosses the Stokes line between two adjacent sectors the homology of the SDC's changes discontinuously.
We denote the SDCs relative to the sector $\mathcal{S}_k$ by $\gamma_j^{(k)}$, $j = 0..d-1$ and the matrix of change of basis with C_k. In matrix form we have

$$\vec{\gamma}^{(k)} = C_k \vec{\Gamma} \ , C_k \in GL(d, \mathbb{Z}) \ . \tag{3-43}$$

Our first objective is to compute these matrices C_k.

For each fixed *generic* (i.e. away from the Stokes' lines) α we can construct a diagram (essentially a Hurwitz diagram) which describes the sheet structure of the inverse map $z = Z(s)$. We draw $d+1$ identical ordered d-gons each representing a copy of the s-plane and whose (labeled) vertices represent the projections of the d critical values $s_{cr}^{(j)}$. Two vertices with the same label of two different d-gons are joined by a segment if the two sheets are glued together along a horizontal branch-cut originating at the corresponding crtical value and going to $\Re(s) = +\infty$. Since all the branch-points of the inverse map are of order 2 there are at most two sheets glued along each cut.
Furthermore we give an orientation to the segments (represented by an arrow) with the understanding that this gives an orientation to the corresponding SDC. The convention is that an arrow going from sheet j to sheet k means that the SDC runs on sheet j coming from $\Re(s) = +\infty$ below the cut and going back above the same cut (or, which is "homologically" the same, a contour running on sheet k coming from $+\infty$ above and returning to $+\infty$ below the cut).

The diagram can be biunivocally associated to a matrix of size $d \times (d+1)$ in which each row correspond to a SDC and each column to a sheet. This matrix has a -1 in the (k, j) entry if the k-th SDC points to the j-th sheet, a 1 if the k-th SDC originates on the j-th sheet or a 0 otherwise. Hence each line of the matrix has only one $+1$ and one -1. The matrix corresponding to the diagram in the figure is

$$Q_0 := \begin{array}{c} \\ \left[\begin{array}{cccccccccccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \end{array} \begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \ . \tag{3-44}$$

As $\alpha = \arg(x)$ ranges within a fixed sector $\mathcal{S}_k$ the diagram does not change topology and the corresponding matrix Q_k remains unchanged.

We can now describe how a given diagram changes when $\alpha = \arg(x)$ crosses the line between two adjacent sectors $\mathcal{S}_k$ and $\mathcal{S}_{k+1}$ (counterclockwise); these lines correspond precisely to the values of α for which two distinct critical values have the same imaginary parts (so that the cuts may overlap if they are on the same sheet); we leave to the reader the simple check that these lines are precisely the boundaries of the Stokes sectors $\mathcal{S}_k$.
As α *increases* by $\pi/(d+1)$ from $\mathcal{S}_k$ to $\mathcal{S}_{k+1}$ the d-gons rotate by π/d. In this process the connections between the sheets change according to the following rule; if a the branch-point P_j on sheet r crosses the cut originating from a

[9] For the SAC's we should perform cuts extending to $\Re(s) = -\infty$ instead.

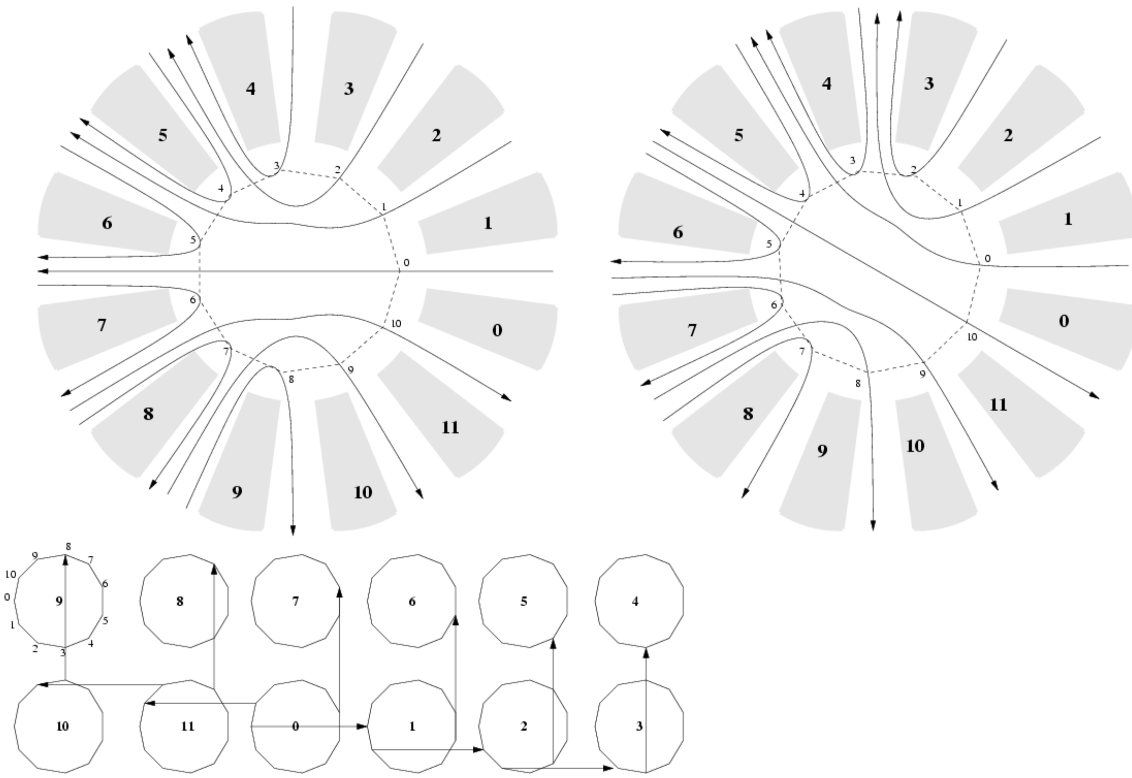

Figure 3: Sheet structure in the case $d = 11$, $\alpha = 0$. The contours depicted in figure are the SDC in the z plane. The picture on the right represents the contours after incrementing α by $2\pi/12$. Notice the labeling the contours.

different branch-point P_h on the same sheet (on the left of P_j on sheet r, and hence P_j crosses the cut from below as it moves upwards) then the P_j (and its cut) jumps on the sheet s which is glued to sheet r along the cut originating at P_h (see Fig. 4).
Diagrammatically the tip (or the tail) of the corresponding arrow moves from one d-gon to the one connected along the vertex h.
In terms of the matrix Q_k the j-th row reflects along the hyperplane orthogonal to the h-th row.
The corresponding SDCs $\gamma_j^{(k)}$ and $\gamma_h^{(k)}$ are related to the SDCs $\gamma_j^{(k+1)}$ and $\gamma_h^{(k+1)}$ by the following relation

$$\begin{aligned} \gamma_j^{(k+1)} &= \gamma_j^{(k)} \\ \gamma_h^{(k+1)} &= \gamma_h^{(k)} + \epsilon_{hj}\gamma_j^{(k)} \end{aligned} \tag{3-45}$$

where the *incidence number* $\epsilon_{hj} = \epsilon_{jh}$ is 1 if the SDCs $\gamma_j^{(k)}, \gamma_h^{(k)}$ have the opposite orientation and -1 if they have the same orientation. Alternatively the incidence number is just the (standard) inner product of the corresponding rows h, j of the matrix Q_k.

We denote by M_k the $d \times d$ matrix which expresses this change in the SDCs. One can then check that the matrices Q_k and Q_{k+1} are related by

$$Q_{k+1} = M_k^{t^{-1}} Q_k \ . \tag{3-46}$$

Therefore we can reconstruct the matrices M_k once we have an initial diagram representing the sheet structure.

Before constructing the initial diagram we want to point out that when α increases by $2\pi/(d+1)$ (i.e. we cross *two* Stokes lines) the steepest descent (unoriented) contours are the same as the original ones but rotated by the same amount *clockwise*.
That is the *unoriented* (i.e. forgetting the orientation of the SDCs) diagram is the same up to permuting cyclically the labels of the $(d+1)$ sheets and labels of the d cuts: notice that the critical points rotate by $\frac{2\pi}{d(d+1)}$ and the critical values by $\frac{2\pi}{d}$ *counterclockwise*. Indeed we have

$$s\left(z; \alpha + \frac{\pi}{d+1}\right) = \frac{z^{d+1}}{d+1} - e^{i\alpha} e^{i\frac{\pi}{d+1}} z = s(e^{i\frac{\pi}{d+1}} z; \alpha) \ . \tag{3-47}$$

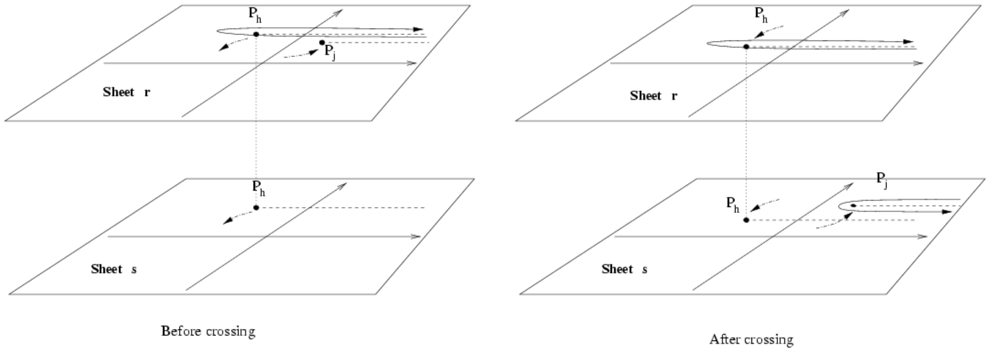

Figure 4: The swapping of two critical values and the corresponding cuts.

As for the orientation of the SDC's relative to the new labeling the d-th SDC passing through $z_{cr}^{(d-1)}$ reverses its orientation relative to the (oriented) SDCs obtained by just rotating the initial SDCs (see Figure 3).
This implies that the matrices Q_k representing the diagrams in the various sectors and the Stokes' matrices M_k satisfy the recurrence relation

$$\mathfrak{S}_{d+1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathrm{GL}(d+1,\mathbb{Z}) \ , \quad p := \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathrm{GL}(d,\mathbb{Z}) \tag{3-48}$$

$$Q_{k+2} = p^{-1} Q_k \mathfrak{S}_{d+1} \ , \qquad M_{k+2} = p \cdot M_k \cdot p^{-1} \ . \tag{3-49}$$

Therefore it is necessary to compute just Q_0, Q_1 and M_0, M_1.
Notice that the wedge contours $\Gamma_y^{(j)}$, $j = 0, \dots, d$ are in one to one correspondence with the sheets of the map $Z(s)$, therefore the same argument proves that

$$P := \begin{bmatrix} -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathrm{GL}(d,\mathbb{Z}) \tag{3-50}$$

$$C_{k+2} = p\, C_k\, P \ , \tag{3-51}$$

Where both the matrices $\mathfrak{S}_{d+1}$ and P implement the cyclic group $\mathbb{Z}_{d+1}$ (although P is of size $d \times d$): indeed the matrix P is exactly the matrix implementing the generator of $\mathbb{Z}_{d+1}$ on the "wedge" contours, i.e. with the additional constraint $\sum_{i=0}^{d} \Gamma_i = 0$.
We now describe how to obtain the initial diagram, for example how we got Figure 3.
We start by noticing that the "wedge" contours $\Gamma_y^{(k)}$ enclose the sector $\mathcal{S}_{2k-1}$ of width $\pi/(d+1)$. It is an easy estimate that the corresponding integrals are (more than) exponentially decreasing in the dual sector. Indeed

$$\left| \int_{\Gamma_k} \mathrm{e}^{-\frac{1}{\hbar}\left(y^{d+1}/(d+1) - xy\right)} \mathrm{d}y \right| \leq \exp\left(|x| M\right) \int_{\Gamma_k} \left| \mathrm{e}^{-\frac{1}{\hbar} y^{d+1}/(d+1)} \right| \mathrm{d}y \tag{3-52}$$

where M is the supremum of $\Re(xy)$ as y goes along $\Gamma_y^{(k)}$. Now the contour $\Gamma_y^{(k)}$ can be deformed as to "skim" the sector $\mathcal{S}_{2k-1}$ as close as we wish. Then the constant M is finite and negative if x lies within the dual sector $\mathcal{S}_k^\vee$ and in the region outside $\Gamma_y^{(k)}$.
We now consider the case of odd d, leaving the easy generalization to even d's to the interested reader. The main (only) difference is that for odd d, $\alpha = \arg(x) = 0$ is not a Stokes' line. Had we to study the case d even, then we should take a convenient initial value of α (e.g. $\alpha = \epsilon << 1$ or $\alpha = \pi/2(d+1)$, which correspond to an anti–Stokes line).

Let us focus on any of the SDCs attached to a critical value lying in the right s-plane. Since the real part of those critical values is positive, the corresponding integrals decrease as $\exp(-\frac{d}{d+1}\omega^j \Lambda)$, ($\Lambda := \frac{1}{\hbar}|x|^{\frac{d+1}{d}}$) that is are (more

than) exponentially suppressed on the line $x \in \mathbb{R}_+$.
As we increase α the d-gons rotate by $\frac{d+1}{d}\alpha$. It should be clear from the previous description of the change of homology of the SDCs that the SDCs attached to such critical values do not change homology class while they remain in the right s-plane: this is so because there are no branchpoints to their right.
Therefore the corresponding integrals are exponentially suppressed as long as α ranges in a corresponding sector of width $\frac{d}{d+1}\pi = \pi - \frac{1}{d+1}\pi$. By careful inspection of the anti-Stokes lines (i.e. the lines along which the integrals are most decreasing) for these integrals one concludes that they coincide with appropriate $\Gamma_y^{(k)}$ in the left z-plane.
This argument proves that all "wedge" contours $\Gamma_y^{(k)}$ lying in the left plane are homological to SDCs (these are the SDCs in the left z-plane in Figure 3).

We then take the first critical value lying in the right s-plane (in Figure 3, the SDC number 10). As we move α so as to move this critical value to the right s-plane, the corresponding SDC can acquire a contribution only from the first SDC in the left plane (number 9 in our example). As a consequence the corresponding integral is exponentially suppressed in a sector of width $\pi - 2\frac{\pi}{d+1}$. Considering its anti-Stokes line and its linear independence from the previously identified SDCs, we conclude that it must enclose two odd-numbered sectors S_{2k+1} and S_{2k+3} (in our example this is contour 10). Proceeding this way we can easily identify the homology classes of all SDCs for $\alpha = 0$. The labelling of the sheets and the SDCs is largely arbitrary and the choice we have made in the example is just for "aesthetic" reasons.

With these notation the basis of contours $\Gamma_y^{(k)}$, $k = 1, \dots d$ and the basis $\gamma_j^{(0)}$, $j = 0, \dots d-1$ are related by

$$\vec{\gamma}^{(k)} = C_k \vec{\Gamma} = M_{k-1} \cdots M_0 C_0 \vec{\Gamma} \tag{3-53}$$
$$\vec{\gamma}^{(0)} = C_0 \vec{\Gamma} \tag{3-54}$$

$$C_0 := \left[\begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 1 & 0 & \overset{\downarrow 1}{0} & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]_{\uparrow 1} \quad \begin{array}{l} \downarrow \left[\frac{d}{2}\right]+1 \\ \\ \leftarrow \left[\frac{d}{4}\right] \\ \\ \\ \\ \\ \leftarrow \left[\frac{3d}{4}\right]+1 \\ \uparrow_{\left[\frac{d}{2}\right]+3} \quad \uparrow_d \end{array} \tag{3-55}$$

It is not difficult to give an explicit description of the matrices Q_0, Q_1 and M_0, M_1 but we will not give it here for brevity: it is a lengthy but straightforward calculation which gives the matrices in the next Table (reported for $d = 2, \dots, 11$) where one can clearly extrapolate the correct rule.

We now turn our attention to the Stokes matrices.
Consider the subblock of the fundamental system of solutions of the $\underline{D}_1$ ODE relative to the F–L. transforms; if we denote by $Y(x)$ the $d_2 \times d_2$ such block whose rows are the integrals of $\varphi^N(y)$ on the contours $\Gamma_y^{(k)}$ and by $Y_k(x)$ the analog matrix obtained by integrating over the SDCs, we have

$$Y_k(x) = C_k Y(x). \tag{3-56}$$

The Stokes matrices are then given by

$$S_k := Y_{k+1} Y_k{}^{-1} = C_{k+1} C_k{}^{-1} = M_k \ . \tag{3-57}$$

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} M_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad ; d=2 \tag{3-58}$$

$$M_0 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad ; d=3 \tag{3-59}$$

$$M_0 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad ; d=4 \tag{3-60}$$

$$M_0=\begin{bmatrix}1&0&-1&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&1&1\end{bmatrix} M_1=\begin{bmatrix}1&-1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&1&0&1\end{bmatrix} \quad ;d=5 \tag{3-61}$$

$$M_0=\begin{bmatrix}1&0&-1&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&1&0&1\end{bmatrix} M_1=\begin{bmatrix}1&-1&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&1&1&0\\0&0&1&0&0&1\end{bmatrix} \quad ;d=6 \tag{3-62}$$

$$M_0=\begin{bmatrix}1&0&0&-1&0&0&0\\0&1&-1&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&1&0&1\end{bmatrix} M_1=\begin{bmatrix}1&0&-1&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&1&1&0\\0&0&0&1&0&0&1\end{bmatrix} \quad ;d=7 \tag{3-63}$$

$$M_0=\begin{bmatrix}1&0&0&-1&0&0&0&0\\0&1&-1&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&1&1&0\\0&0&0&0&1&0&0&1\end{bmatrix} M_1=\begin{bmatrix}1&0&-1&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&1&0&1&0\\0&0&0&1&0&0&0&1\end{bmatrix} \quad ;d=8 \tag{3-64}$$

$$M_0=\begin{bmatrix}1&0&0&0&-1&0&0&0&0\\0&1&0&-1&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&1&0\\0&0&0&0&0&1&0&0&1\end{bmatrix} M_1=\begin{bmatrix}1&0&0&-1&0&0&0&0&0\\0&1&-1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&1&0\\0&0&0&0&1&0&0&0&1\end{bmatrix} \quad ;d=9 \tag{3-65}$$

$$M_0=\begin{bmatrix}1&0&0&0&-1&0&0&0&0&0\\0&1&0&-1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0&1&0\\0&0&0&0&0&1&0&0&0&1\end{bmatrix} M_1=\begin{bmatrix}1&0&0&-1&0&0&0&0&0&0\\0&1&-1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&1&0&0\\0&0&0&0&0&1&0&0&1&0\\0&0&0&0&1&0&0&0&0&1\end{bmatrix} \quad ;d=10 \tag{3-66}$$

$$M_0=\begin{bmatrix}1&0&0&0&0&-1&0&0&0&0&0\\0&1&0&0&-1&0&0&0&0&0&0\\0&0&1&-1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0&1&0\\0&0&0&0&0&0&1&0&0&0&1\end{bmatrix} M_1=\begin{bmatrix}1&0&0&0&-1&0&0&0&0&0&0\\0&1&0&-1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1&1&0&0\\0&0&0&0&0&0&1&0&0&1&0\\0&0&0&0&0&1&0&0&0&0&1\end{bmatrix} \quad ;d=11 . \tag{3-67}$$

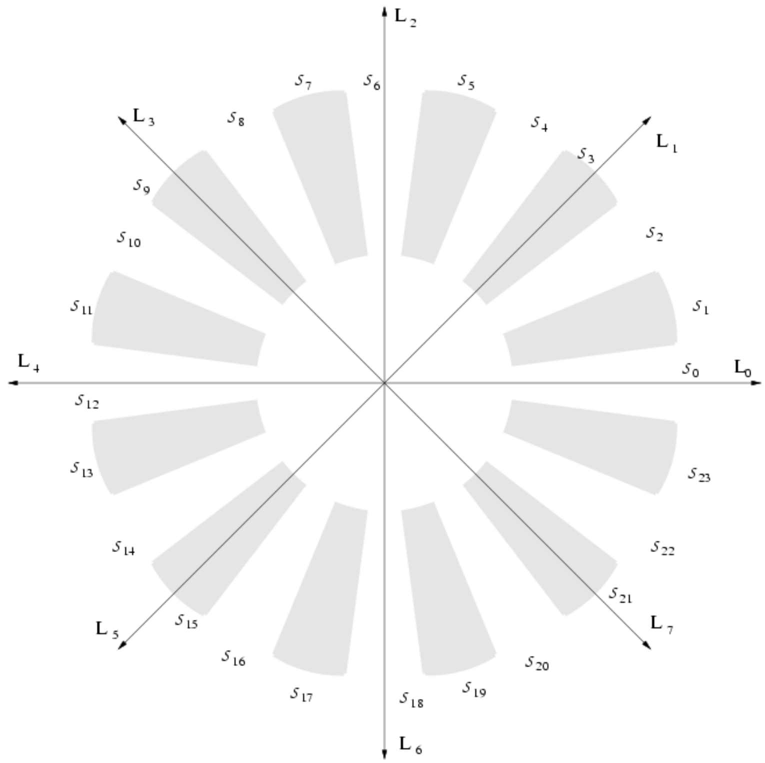

Figure 5: The structure of Stokes sectors and discontinuities for the case $d_1 = 7$ and $d_2 = 11$ and both leading coefficients for the potential real and positive.

In order to complete the description of the Riemann–Hilbert problem we need to consider also the extra solution obtained by Hilbert–Fourier-Laplace transform: in doing so we extend the previously computed Stokes matrices M_k to the full fundamental system of $d_2 + 1$ solutions by means of

$$\widehat{M}_k := \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & M_k \end{array}\right] . \tag{3-68}$$

Summarizing the whole discussion, we have proved the

Theorem 3.1 [Riemann–Hilbert Problem] Given the Stokes' sectors $\mathcal{S}_k$, $k = 0, \dots, 2d_2 + 1$ around $x = \infty$ defined in Eq. (3-13) and the $d_1 + 1$ rays L_μ lying in the sectors $\mathcal{S}^{(x)}_{2\mu}$, $\mu = 0, \dots, d_1$ and chosen as to avoid the Stokes lines (see, e.g., Fig. 5), we formulate the following Riemann–Hilbert problem for the GL_{d_2+1}-valued piecewise analytic function $\Phi(x)$

$$\Phi_+(x) = G_\mu \, \Phi_-(x) \ , \qquad x \in L_\mu \ , \tag{3-69}$$

with the Stokes matrices $\widehat{M}_k$ constructed in this paragraph and the formal asymptotic specified in Prop. 3.1.

4 Summary and comments on large N asymptotics in multi-matrix models

A complete formulation of the Riemann–Hilbert problem characterizing fundamental systems of solutions to the differential - recursion equations satisfied by biorthogonal polynomials associated to 2-matrix models with polynomial potentials is provided in Theorem 3.1.The approach derived here can be straightforwardlly extended to the case of biorthogonal polynomials associated to a finite chain of coupled matrices with polynomial potentials following the lines outlined in the appendix of [4].

The Riemann-Hibert data, consisting essentially of the Stokes matrices at ∞, are independent of both the integer parameter N corresponding to the matrix size and the deformation parameters determining the potentials; that is, the fundamental solutions constructed are solutions simultaneously to a differential-difference generalized isomonodromic deformation problem. This provides a first main step towards a rigorous analysis of the $N \to \infty$, $\hbar N = \mathcal{O}(1)$ limit of the partition function and the spectral statistics of coupled random matrices (as well as the question of universality in the resulting Fredholm kernels, and hence the spectral statistics of 2-matrix models). Such an analysis should follow similar lines to the methods that were previously successfully applied to ordinary orthogonal polynomials in the 1-matrix case [8, 19, 22, 23, 12, 13]. The main difference in the 2-(or more) matrix cases is that in the double–scaling limit the functional dependence of the free energy on the eigenvalue distributions is not as explicit as in the 1-matrix models [26, 21]. It is also clear that the hyperelliptic spectral curve that arises in the solution of the one-matrix model has to be replaced by a different algebraic curve, which arises naturally in the spectral duality of the spectral curves of [4], as was pointed out in [16].

In order determine the large N asymptotics with the help of the the RH problem, one should begin with an ansatz that can be checked *a posteriori* against the given case. In the 1-matrix case [12, 13], this was provided by means of hyperelliptic Θ-functions. The physical heuristics and the basic tools for generating such ansatz were also given in [9, 16]. Much of of the heuristics can be extended to the 2-matrix model [6], and this will be the subject of a subsequent work [3].

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