Generalized coherent and squeezed states based on the $h(1) \oplus \mathrm{su}(2)$ algebra

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Abstract

States which minimize the Schrödinger–Robertson uncertainty relation are constructed as eigenstates of an operator which is a element of the $h(1) \oplus \mathrm{su}(2)$ algebra. The relations with supercoherent and supersqueezed states of the supersymmetric harmonic oscillator are given. Moreover, we are able to compute gneneral Hamiltonians which behave like the harmonic oscillator Hamiltonian or are related to the Jaynes–Cummings Hamiltonian.

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1 Introduction

Minimum uncertainty states (MUS) are usually understood through the minimization of the Heisenberg uncertainty relation (HUR). These states are well-known [1] since long and associated with the so-called Coherent States (CS) [2] and Squeezed States (SS) [3]. But, it has been observed [4, 5, 6] that a more accurate uncertainty relation may be used to construct generalized CS and SS. Indeed, this relation known as the Schrödinger–Robertson uncertainty relation (SRUR) [7] can be minimized and gives rise to new classes of CS and SS which have received different names in the litterature, such as correlated states [4] or intelligent states [5]. There are two main reasons to consider such last states. First, when the two hermitian operators entering in the SRUR are non canonical operators, i.e. their commutator is not a multiple of the identity, the HUR could be redundant while the SRUR not. Second, the MUS that minimize the SRUR are shown to be eigenstates of a linear combination of the two hermitian operators entering in the SRUR.

Recently [8] a connection has been made with the CS and SS based on group theoretic approches [9] and the concept of Algebra Eigenstates (AES). In particular, AES have been constructed for the algebras su(2) and su(1, 1). This concept constitute a unification of different definitions of CS and SS.

In this paper, we give a general construction of AES based on the direct sum $h(1) \oplus \mathrm{su}(2)$. The Heisenberg algebra $h(1)$ being relevant for the problem of the harmonic oscillator and the algebra su(2) for particles with spin, we have a procedure to find general CS and SS for supersymmetric systems, for example. These are clearly MUS for which the dispersions of corresponding operators may be calculated easily. We show finally how to use these states in the construction of particularly relevant Hamiltonians and in the calculation of their dispersions.

In the Section 2, we put the emphasis on the SRUR and its relevancy with respect to the determination of MUS. The application to the position and momentum operators MUS leads to the well-known CS and SS of the harmonic oscillator while when the angular momentum operators MUS are considered we have in mind the su(2) CS and SS. These particular applications are given to bring a new light on these states and also to facilitate the treatement of the $h(1) \oplus \mathrm{su}(2)$ CS and SS. In Section 3, we construct the AES based on the $h(1) \oplus \mathrm{su}(2)$ algebra and show how this gives CS and SS which generalize the supercoherent and supersqueezed states obtained in other approaches [10, 11]. Finally, in Section 4, we construct general Hamiltonians similar to the one of the harmonic oscillator but where the so-called annihilation operator is now an element of the algebra $h(1) \oplus \mathrm{su}(2)$. This permits us to use our CS and SS to compute the mean value and the dispersions of the corresponding energies. We show also how the well-known Jaynes–Cummings Hamiltonian enters in this scheme.

2 Coherent and squeezed states as minimum uncertainty states

This section will be concerned by the general definition and properties of MUS (§2.1). They are explicitly constructed when the usual position and momentum operators are considered (§2.2) as well as when the angular momentum operators are taken (§2.3). The connection is made with already known results.

2.1 Minimum uncertainty relation

It is well-known [7] that, for two hermitian operators A and B such that the commutator is

$$[A, B] = iC, \quad C \neq 0, \tag{2.1}$$

the HUR

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{\langle C \rangle^2}{4} \tag{2.2}$$

is satisfied. The mean value and dispersion of a given operator X are defined, as usual, by

$$\langle X \rangle = \langle \psi | X | \psi \rangle, \quad (\Delta X)^2 = \langle X^2 \rangle - \langle X \rangle^2, \tag{2.3}$$

for a normalized state $|\psi\rangle$ describing the evolution of a quantum system. As observed by Puri [6], for noncanonical operators, i.e. such that C is not a multiple of the identity I, we can have $\langle C \rangle = 0$ and the relation (2.2) is then redundant. The SRUR [1, 7] is never redundant and writes:

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} \big(\langle C \rangle^2 + \langle F \rangle^2 \big), \tag{2.4}$$

where $\langle F \rangle$ is a measure of the correlation between A and B. The operator F is hermitian and given by

$$F = \big\{ A - \langle A \rangle I, B - \langle B \rangle I \big\}, \tag{2.5}$$

where $\{\ ,\ \}$ denotes the anti-commutator. If there is no correlation between the operators A and B, i.e. if $\langle F\rangle = 0$, the SRUR reduces to the usual HUR.

We are interested here in the description of states which minimize the SRUR (2.4). A necessary and sufficient condition to get them is to solve the eigenvalues equation:

$$[A + i\lambda B]|\psi\rangle = \beta|\psi\rangle, \tag{2.6}$$

where

$$\beta = \big[\langle A\rangle + i\lambda\langle B\rangle\big], \quad \lambda \in \mathbb{C},\ \lambda \neq 0. \tag{2.7}$$

Note that, if $\mathrm{Re}\,\lambda \neq 0$, once we know the value of β, this last relation may be inverted to give the mean values

$$\langle A\rangle = \mathrm{Re}\,\beta + \frac{\mathrm{Im}\,\lambda}{\mathrm{Re}\,\lambda}\mathrm{Im}\,\beta, \quad \langle B\rangle = \frac{\mathrm{Im}\,\beta}{\mathrm{Re}\,\lambda} \tag{2.8}$$

and, if $\mathrm{Re}\,\lambda = 0$, we get

$$\langle A\rangle = \mathrm{Re}\,\beta + \mathrm{Im}\,\lambda\langle B\rangle. \tag{2.9}$$

As a consequence of (2.6), one has

$$(\Delta A)^2 = |\lambda|\Delta, \quad (\Delta B)^2 = \frac{1}{|\lambda|}\Delta, \tag{2.10}$$

with

$$\Delta = \frac{1}{2}\sqrt{\langle C\rangle^2 + \langle F\rangle^2}. \tag{2.11}$$

So the states $|\psi\rangle$ satisfying (2.6) with $|\lambda| = 1$ will be called **coherent** because they satisfy

$$(\Delta A)^2 = (\Delta B)^2 = \Delta, \tag{2.12}$$

i.e. the dispersions in A and B are the same and minimized in the sense of SRUR. The states $|\psi\rangle$ satisfying (2.6) with $|\lambda| \neq 1$ will be called **squeezed** because if $|\lambda| < 1$, we have $(\Delta A)^2 < \Delta < (\Delta B)^2$ and if $|\lambda| > 1$, we have $(\Delta B)^2 < \Delta < (\Delta A)^2$.

Some other relations are also useful for our considerations. The direct computation of $(\Delta A)^2$ and $(\Delta B)^2$ is usually complicated but in the MUS that satisfy (2.6), we can write

$$(\Delta A)^2 = \frac{1}{2}\big|\mathrm{Re}\,\lambda\langle C\rangle + \mathrm{Im}\,\lambda\langle F\rangle\big|, \tag{2.13}$$

$$(\Delta B)^2 = \frac{1}{2|\lambda|^2}\big|\mathrm{Re}\,\lambda\langle C\rangle + \mathrm{Im}\,\lambda\langle F\rangle\big|, \tag{2.14}$$

with

$$\mathrm{Im}\,\lambda\langle C\rangle = \mathrm{Re}\,\lambda\langle F\rangle. \tag{2.15}$$

For $\mathrm{Re}\,\lambda = 0$, we have $\langle C\rangle = 0$, which corresponds to the case where the HUR is redundant. The MUS satisfy the minimum SRUR (MSRUR)

$$(\Delta A)^2(\Delta B)^2 = \Delta^2, \tag{2.16}$$

with

$$(\Delta A)^2 = \frac{1}{2}\big|\mathrm{Im}\,\lambda\langle F\rangle\big|, \quad (\Delta B)^2 = \frac{1}{2}\left|\frac{\langle F\rangle}{\mathrm{Im}\,\lambda}\right| \tag{2.17}$$

and

$$\Delta = \frac{1}{2}\big|\langle F\rangle\big|. \tag{2.18}$$

For $\mathrm{Re}\,\lambda \neq 0$, from (2.15), we have

$$\langle F\rangle = \frac{\mathrm{Im}\,\lambda}{\mathrm{Re}\,\lambda}\langle C\rangle. \tag{2.19}$$

Moreover, from (2.13) and (2.14), we get

$$(\Delta A)^2 = \left|\frac{|\lambda|^2}{2\mathrm{Re}\,\lambda}\langle C\rangle\right|, \quad (\Delta B)^2 = \left|\frac{1}{2\mathrm{Re}\,\lambda}\langle C\rangle\right| \tag{2.20}$$

and, then,

$$\Delta = \left|\frac{|\lambda|}{2\mathrm{Re}\,\lambda}\langle C\rangle\right|. \tag{2.21}$$

In this case, it is sufficient to compute the mean value of C to deduce that of F and the dispersions. The particular case where $\mathrm{Im}\,\lambda = 0$ corresponds to the fact that the MSUR coïncides with the minimum HUR (MHUR).

2.2 Position and momentum coherent and squeezed states

Let us apply the preceding considerations to the special case of the usual position x and momentum p operators of a given quantum system. The canonical commutation relation (if $\hbar = 1$) being

$$[x, p] = iI, \tag{2.22}$$

the SRUR writes:

$$(\Delta x)^2 (\Delta p)^2 \geq \frac{1}{4}\big(1 + \langle F \rangle^2\big). \tag{2.23}$$

The MUS $|\psi, \lambda, \beta\rangle$ satisfy the eigenvalues equation:

$$[x + i\lambda p]|\psi, \lambda, \beta\rangle = \beta|\psi, \lambda, \beta\rangle. \tag{2.24}$$

If we introduce the usual creation $a^\dagger$ and annihilation a operators

$$a^\dagger = \frac{x - ip}{\sqrt{2}}, \quad a = \frac{x + ip}{\sqrt{2}}, \tag{2.25}$$

such that $[a, a^\dagger] = I$, the equation (2.24) becomes

$$\frac{1}{\sqrt{2}}\big[(1 - \lambda)a^\dagger + (1 + \lambda)a\big]|\psi, \lambda, \beta\rangle = \beta|\psi, \lambda, \beta\rangle. \tag{2.26}$$

The general resolution of Eq. (2.26) is obtained by expressing the state $|\psi, \lambda, \beta\rangle$ as a superposition of the energy eigenstates $\big\{|n\rangle, n = 0, 1, 2, \ldots\big\}$ of the usual harmonic oscillator Hamiltonian

$$H_0 = w\left(a^\dagger a + \frac{1}{2}\right). \tag{2.27}$$

Let us recall that these eigenstates satisfy

$$a|n\rangle = \sqrt{n}|n - 1\rangle, \quad a^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle \tag{2.28}$$

and we can write them as

$$|n\rangle = \frac{{a^\dagger}^n}{\sqrt{n!}}|0\rangle, \quad n = 0, 1, 2, \ldots. \tag{2.29}$$

So if we insert

$$|\psi, \lambda, \beta\rangle = \sum_{n=0}^{\infty} C_{\lambda,\beta,n}|n\rangle, \quad C_{\lambda,\beta,n} \in \mathbb{C}, \tag{2.30}$$

in Eq. (2.26), using the expressions (2.28), we get the recurrence system

$$\frac{1}{\sqrt{2}}\big[\sqrt{n}(1 - \lambda)C_{\lambda,\beta,n-1} + \sqrt{n + 1}(1 + \lambda)C_{\lambda,\beta,n+1}\big] = \beta C_{\lambda,\beta,n}, \quad n = 1, 2, 3, \ldots,$$

$$\frac{(1 + \lambda)}{\sqrt{2}}C_{\lambda,\beta,1} = \beta C_{\lambda,\beta,0}. \tag{2.31}$$

The case $\lambda = -1$ does not give any solution and must be eliminated. If we set

$$\left(\frac{1 - \lambda}{1 + \lambda}\right) = \delta e^{i\phi}, \quad \delta \in \mathrm{R}_+, \phi \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right[, \tag{2.32}$$

the resolution of the recurrence system (2.31) leads to the general solution of Eq. (2.26):

$$|\psi, \lambda, \beta\rangle = C_{\lambda,\beta,0} \exp\left(-\delta e^{i\phi}\frac{{a^\dagger}^2}{2}\right) \exp\left(\frac{\beta}{\sqrt{2}}(1 + \delta e^{i\phi})a^\dagger\right)|0\rangle. \tag{2.33}$$

The special case $\lambda = 1$ corresponds to $\delta = 0$ and gives rise to the usual expression of the CS of the harmonic oscillator. These states (2.33) can also be obtained as the action of two unitary operators on the fundamental state. The first one [9] is the usual displacement operator D associated with an irreducible representation of the Heisenberg–Weyl group $H(1)$ with algebra $h(1) = \{a, a^\dagger, I\}$. The second one is the squeezed operator S associated with an irreducible representation of SU(1,1) with algebra su(1,1) = $\{a^2, (a^\dagger)^2, aa^\dagger + a^\dagger a\}$. This is a known fact [12] when squeezed states of the harmonic oscillator are studied. We have explicitly

$$|\psi, \lambda, \beta\rangle = S\big(\chi(\delta, \phi)\big) D(\eta)|0\rangle, \tag{2.34}$$

where

$$D(\eta) = \exp\,(\eta a^\dagger - \bar{\eta} a) \quad \text{and} \quad S(\chi) = \exp\left(\chi \frac{{a^\dagger}^2}{2} - \bar{\chi}\frac{a^2}{2}\right) \tag{2.35}$$

with

$$\eta = \frac{\beta}{\sqrt{2}} \frac{(1+\delta e^{i\phi})}{\sqrt{1-\delta^2}} \quad \text{and} \quad \chi(\delta, \phi) = -\tanh^{-1}(\delta) e^{i\phi}. \tag{2.36}$$

The condition for having normalizable states is that $0 \le \delta < 1$. Let us insist here on the fact that these SS already obtained in the literature as eigenstates of a linear combination of a and $a^\dagger$ are also MUS such that $(\Delta x)^2 (\Delta p)^2 = \Delta^2 = \big(1 + \langle F\rangle^2\big)/4$. From Eq. (2.19) and the fact that $\langle C\rangle = 1$, we get

$$\langle F\rangle = \frac{\operatorname{Im}\lambda}{\operatorname{Re}\lambda} = \frac{-2\delta \sin\phi}{(1-\delta^2)} \tag{2.37}$$

and the factor Δ is

$$\Delta(\delta, \phi) = \sqrt{\frac{1}{4}\big(1 + \langle F\rangle^2\big)} = \sqrt{\frac{1}{4} + \frac{\delta^2 \sin^2\phi}{(1-\delta^2)^2}}. \tag{2.38}$$

Moreover, from (2.13) and (2.14), the dispersions are

$$(\Delta x)^2 = \frac{|\lambda|^2}{2|\operatorname{Re}\lambda|} = \frac{(1 - 2\delta\cos\phi + \delta^2)}{2(1-\delta^2)} \tag{2.39}$$

and

$$(\Delta p)^2 = \frac{1}{2|\operatorname{Re}\lambda|} = \frac{(1 + 2\delta\cos\phi + \delta^2)}{2(1-\delta^2)}. \tag{2.40}$$

Let us recall now that the CS are not only the one for $\lambda = 1$ but also all the states where $|\lambda| = 1$. From the relation (2.32), we deduce that

$$\lambda = \frac{1 - \delta e^{i\phi}}{1 + \delta e^{i\phi}} = \frac{(1-\delta^2) - 2i\delta\sin\phi}{(1 + 2\delta\cos\phi + \delta^2)} \tag{2.41}$$

and then

$$|\lambda|^2 = \frac{1 - 2\delta\cos\phi + \delta^2}{1 + 2\delta\cos\phi + \delta^2}. \tag{2.42}$$

This means that CS occur also for $\phi = -\pi/2$ or $\phi = \pi/2$ and $\delta \neq 0$. The other values of λ describe x-squeezed states when $\phi \in]-\pi/2, \pi/2[$ and p-squeezed states when $\phi \in]\pi/2, 3\pi/2[$. On the other hand, for fixed values of ϕ the expression (2.38) attains its minimum value 1/2 when $\delta = 0$ and when $\phi = 0$ and $\phi = \pi$ for fixed values of δ. In the first of these cases, we have $\lambda = 1$ and we are in the standard CS of the harmonic oscillator, i.e. eigenstates of the a operator. In the second case, λ is a positive real quantity equal to $(1-\delta)/(1+\delta) \le 1$ if $\phi = 0$ and to $(1+\delta)/(1-\delta) \ge 1$ if $\phi = \pi$. We are in the special SS states that are eigenstates of the $(a + \delta a^\dagger)$ and $(a - \delta a^\dagger)$ operators respectively.

Fig. 1 shows the behaviour of $(\Delta x)^2$, $(\Delta p)^2$ and Δ as functions of δ for $\phi = \pi/6$. In this region $(\Delta x)^2 \big((\Delta p)^2\big)$ is always less (greater) than Δ, as expected. For $\delta = 0$, the three curves coïncide, the intersection point corresponds to the CS $|\psi, 1, \beta\rangle$. The value of $\Delta =$ (2.38) when $\delta = 0$ is also the minimum value 1/2 which corresponds to the MHUR. Fig. 2 shows the behaviour of the same quantities as functions of ϕ for $\delta = 0.5$. The points where the three curves intersect are the CS.

2.3 Angular momentum coherent and squeezed states

Let us now take the angular momentum operators J_k for $k = 1, 2, 3$, which satisfy the usual su(2) commutations relations

$$[J_k, J_l] = i\varepsilon_{klm} J_m, \quad k, l, m = 1, 2, 3. \tag{2.43}$$

Here we want to solve the eigenvalues equation

$$(J_1 + i\lambda J_2)|\psi, \lambda, \beta\rangle = \beta|\psi, \lambda, \beta\rangle, \tag{2.44}$$

where $\beta = \big[\langle J_1\rangle + i\lambda\langle J_2\rangle\big]$. On the contrary of the preceding example where the HUR is never redundant (because x and p are canonical), here the commutator of J_1 and J_2 is not a multiple of the identity and then $\langle J_3\rangle$ may be equal to zero for some special cases. Some of these cases have been discussed elsewhere [6, 13, 14, 15]. Here we give the general solution of the equation (2.44), for all possible values of λ and β.

It would be better to work with the operators $J_\pm = J_1 \pm iJ_2$ instead of J_1 and J_2. So that the equation (2.44) becomes

$$\frac{1}{2}\big[(1+\lambda)J_+ + (1-\lambda)J_-\big]|\psi,\lambda,\beta\rangle = \beta|\psi,\lambda,\beta\rangle. \tag{2.45}$$

Using the usual complete set of angular momentum states $\big\{|j,r\rangle\big\}$, j integer or half-odd integer and $r \in \big\{-j, -(j-1), \ldots, j-1, j\big\}$, we know that

$$J^2|j,r\rangle = (J_1^2 + J_2^2 + J_3^2)|j,r\rangle = j(j+1)|j,r\rangle, \tag{2.46}$$

$$J_3|j,r\rangle = r|j,r\rangle \tag{2.47}$$

and

$$J_\pm|j,r\rangle = \sqrt{(j\mp r)(j\pm r+1)}|j,r\pm 1\rangle. \tag{2.48}$$

This means that for each j fixed, the eigenstates $|\psi,\lambda,\beta\rangle^j$ of Eq. (2.45) may be written as

$$|\psi,\lambda,\beta\rangle^j = \sum_{r=-j}^{j} C^j_{\lambda,\beta,r}|j,r\rangle, \quad C^j_{\lambda,\beta,r} \in \mathbb{C}, \tag{2.49}$$

where the coefficients $C^j_{\lambda,\beta,r}$ satisfy a recurrence system of the form

$$(1+\lambda)\sqrt{(j+r)(j-r+1)}C^j_{\lambda,\beta,r-1} + (1-\lambda)\sqrt{(j-r)(j+r+1)}C^j_{\lambda,\beta,r+1} = 2\beta C^j_{\lambda,\beta,r}, \tag{2.50}$$

for $r = -j, \ldots, j$ and $C^j_{\lambda,\beta,j+1} = C^j_{\lambda,\beta,-(j+1)} = 0$.

For $\lambda = \pm 1$, the unique eigenstates are $|\psi,\pm 1,0\rangle^j = |j,\pm j\rangle$. For $\lambda \neq \pm 1$ and $\beta = 0$, the recurrence relation (2.50) is solved to give

$$|\psi,\lambda,0\rangle^j = C^j_{\lambda,0,j}e^{i(j\phi/2)}\sum_{k=0}^{j}(-1)^k\frac{\binom{j}{k}}{\sqrt{\binom{2j}{2k}}}\delta^k e^{-i(j-2k)\phi/2}|j,j-2k\rangle, \quad j \text{ integer}, \tag{2.51}$$

where we have used the formula (2.32) to express λ in terms of the δ and ϕ. It is again possible to express such a state from the action of unitary operators associated with an irreducible representation of a group which is here SU(2). Indeed, we have

$$|\psi,\lambda,0\rangle^j = C^j_{\lambda,0}\exp\left[-\frac{1}{2}\ln(\delta)J_3\right]U|j,0\rangle, \tag{2.52}$$

where

$$U = \exp\left(-\frac{\pi}{4}(e^{-i\phi/2}J_+ - e^{i\phi/2}J_-)\right). \tag{2.53}$$

For the general case $\lambda \neq \pm 1$, the analysis of the system (2.50) shows that for each j, there exists $(2j+1)$ possible values for the eigenvalue β, which are

$$\beta^j_m = m\sqrt{1-\lambda^2}, \quad m = -j, \ldots, j. \tag{2.54}$$

If we use the relation

$$[J_1 + i\lambda J_2]\left[\exp\left(-\frac{1}{2}\ln(\delta)J_3\right)U\right] = \left[\exp\left(-\frac{1}{2}\ln(\delta)J_3\right)U\right][\sqrt{1-\lambda^2}J_3], \tag{2.55}$$

we see immediately that the corresponding eigenstate $|\psi,\lambda,\beta^j_m\rangle^j$ is

$$|\psi,\lambda,\beta^j_m\rangle^j \equiv |\psi,\lambda,m\rangle^j = C^j_{\lambda,m}\exp\left[-\frac{1}{2}\ln(\delta)J_3\right]U|j,m\rangle, \quad m = -j, \ldots, j, \tag{2.56}$$

where $U \equiv$ (2.53). They can be written in terms of the Jacobi polynomials as

$$|\psi, \lambda, m\rangle^j = C^j_{\lambda,m} \exp\left(-\frac{1}{2}\ln(\delta) J_3\right) e^{im\phi/2} e^{-i(\phi/2)J_3} \sum_{r=-j}^{j} 2^r \sqrt{\frac{(j+r)!(j-r)!}{(j-m)!(j+m)!}} P_{j+r}^{-r+m,-r-m}(0)|j,r\rangle. \tag{2.57}$$

In these last states, we want to compute now the mean values and dispersions of some operators in order to exhibit their behaviour in the CS and SS.

If $\operatorname{Re}\lambda \neq 0$, the mean values of J_1 and J_2 in the states (2.57) are obtained using (2.8) and (2.54). In terms of δ and ϕ as defined by (2.32), we get

$$\langle J_1\rangle^j_m = 2m\frac{\delta^{1/2}}{(\delta+1)}\cos\left(\frac{\phi}{2}\right), \quad \langle J_2\rangle^j_m = 2m\frac{\delta^{1/2}}{(\delta+1)}\sin\left(\frac{\phi}{2}\right). \tag{2.58}$$

The relations (2.19)–(2.21) applied to our case tell us that $(\Delta J_1)^2$, $(\Delta J_2)^2$, Δ and $\langle F\rangle$ are all obtained from the mean value of J_3, i.e.

$$\left((\Delta J_1)^2\right)^j_m = \frac{|\lambda|^2}{2\operatorname{Re}\lambda}\langle J_3\rangle^j_m, \quad \left((\Delta J_2)^2\right)^j_m = \frac{1}{2\operatorname{Re}\lambda}\langle J_3\rangle^j_m$$

$$\Delta^j_m = \frac{|\lambda|}{2\operatorname{Re}\lambda}\langle J_3\rangle^j_m, \quad \langle F\rangle^j_m = \frac{\operatorname{Im}\lambda}{\operatorname{Re}\lambda}\langle J_3\rangle^j_m. \tag{2.59}$$

The mean values of J_3 in the states (2.57) or equivalently in the states (2.56) are given by

$$\langle J_3\rangle^j_m = -\frac{\partial}{\partial q}\ln\left(\langle j,m|U^\dagger e^{-qj_3}U|j,m\rangle\right), \tag{2.60}$$

where $q = \ln\delta$. After some computations, we get

$$\langle J_3\rangle^j_m = -|m|\tanh\left(\frac{q}{2}\right) - \frac{1}{2}\sinh(q)\left(j+|m|+1\right)\frac{P^{1,1+2|m|}_{j-|m|-1}(\cosh q)}{P^{0,2|m|}_{j-|m|}(\cosh q)}. \tag{2.61}$$

Inserting (2.61) into the expression (2.59), we get

$$\left((\Delta J_1)^2\right)^j_m = (1-2\delta\cos\phi+\delta^2)\Lambda^j_m(\delta), \quad \left((\Delta J_2)^2\right)^j_m = (1+2\delta\cos\phi+\delta^2)\Lambda^j_m(\delta), \tag{2.62a}$$

$$(\Delta)^j_m = \sqrt{1-2\delta^2\cos(2\phi)+\delta^4}\,\Lambda^j_m(\delta), \quad \langle F\rangle^j_m = -4\delta\sin\phi\Lambda^j_m(\delta), \tag{2.62b}$$

where

$$\Lambda^j_m(\delta) = \left[\frac{|m|}{2(1+\delta)^2} + \frac{(j+|m|+1)}{8\delta}\frac{P^{1,1+2|m|}_{j-|m|-1}((1+\delta^2)/2\delta)}{P^{0,2|m|}_{j-|m|}((1+\delta^2)/2\delta)}\right]. \tag{2.63}$$

The case $\operatorname{Re}\lambda = 0$ may be obtained as the limit case of the preceding one by taking $\delta = 1$ in the expressions (2.62a), (2.62b) and (2.63). Let us recall that it corresponds to $\langle J_3\rangle = 0$ and $\lambda = -i\tan\phi/2$. We get

$$\left((\Delta J_1)^2\right)^j_m = \frac{1}{2}\left[j(j+1)-m^2\right]\sin^2\left(\frac{\phi}{2}\right), \quad \left((\Delta J_2)^2\right)^j_m = \frac{1}{2}\left[j(j+1)-m^2\right]\cos^2\left(\frac{\phi}{2}\right), \tag{2.64a}$$

$$(\Delta)^j_m = \frac{1}{4}\left[j(j+1)-m^2\right]|\sin\phi| \quad \text{and} \quad \langle F\rangle^j_m = -\frac{1}{2}\left[j(j+1)-m^2\right]\sin\phi, \tag{2.64b}$$

using the fact that

$$P^{\alpha,\beta}_n(1) = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!}. \tag{2.65}$$

These are exactly the results given by Puri [6].

To illustrate these considerations by a concrete example, let us take the "spin–1/2" case, i.e. $j = 1/2$. The expressions (2.62a), (2.62b) thus reduce to

$$\left((\Delta J_1)^2\right)_\pm = \frac{(1-2\delta\cos\phi+\delta^2)}{4(1+\delta)^2}, \quad \left((\Delta J_2)^2\right)_\pm = \frac{(1+2\delta\cos\phi+\delta^2)}{4(1+\delta)^2} \tag{2.66}$$

and

$$\Delta_{\pm}(\delta,\phi) = \frac{1}{4}\sqrt{1+4\left(\frac{\delta^2\sin^2\phi - \delta(1+\delta)^2}{(1+\delta)^4}\right)}, \tag{2.67}$$

where we have used the $\pm$ sign for the values of $m = \pm 1/2$. The MSRUR thus writes

$$\left((\Delta J_1)^2\right)_{\pm}\left((\Delta J_2)^2\right)_{\pm} = (\Delta_{\pm})^2(\delta,\phi) = \frac{1}{16}\left[1+4\left(\frac{\delta^2\sin^2\phi - \delta(1+\delta)^2}{(1+\delta)^4}\right)\right]. \tag{2.68}$$

For fixed values of $\phi \neq 0$ and π, the expression (2.67) attains its minimum value $|\sin\phi|/8$ when $\delta = 1$. On the other hand, for fixed values of δ such that $\delta \in [0,1[\cup]1,\infty]$, the minimum of (2.67) is $(1/4)\sqrt{[1-(4\delta)/(1+\delta)^2]}$ when $\phi = 0$ or $\phi = \pi$. In the first case we have $\lambda = -i(\sin\phi)/(1+\cos\phi)$, which means that we have some special classes of SS from which we recognize CS with $\lambda = -i$ (eigenstates of the $J_1 + J_2$ operator) and with $\lambda = i$ (eigenstates of the $J_1 - J_2$ operator). In the second case, we have $\lambda = (1-\delta)/(1+\delta) \leq 1$ if $\phi = 0$ and $\lambda = (1+\delta)/(1-\delta) \geq 1$ if $\phi = \pi$, i.e. the minimum $\Delta_{\pm}(\delta,0) = \Delta_{\pm}(\delta,\pi)$ values occur for the special states which are eigenstates of the operators $(J_+ + \delta J_-)$ and $(J_+ - \delta J_-)$ respectively. Let us recall that the CS with $\lambda = 1$ occur when $\delta = 0$ and those with $\lambda = -1$ when $\delta \mapsto \infty$. They correspond to the eigenstates of J_+ and J_- operators respectively. For such states, according to equation (2.68), we have $\left((\Delta J_1)^2\right)_{\pm} = \left((\Delta J_2)^2\right)_{\pm} = \left(\Delta_{\pm}(0,\phi)\right)^2 = \lim_{\delta\mapsto\infty}\left(\Delta_{\pm}(\delta,\phi)\right)^2 = 1/4$.

Fig. 3 shows the behaviour of the dispersions $\left((\Delta J_1)^2\right)$, $\left((\Delta J_2)^2\right)$ and Δ as functions of δ for $\phi = \pi/6$ and $j = 1/2$. The minimum value of $\Delta_{\pm}$ is here 0,0625. In Fig. 4, we see that the graphs as a function of ϕ are very similar to ones for the preceding example of x and p.

3 Algebra Eigenstates Associated to $h(1) \oplus \mathrm{su}(2)$

This section begins (§3.1) with a review of the SUSY harmonic oscillator and its Super-Coherent States (SCS) studied by Aragone and Zypman [10]. We follow (§3.2) by the general construction of AES based on the algebra $h(1) \oplus \mathrm{su}(2)$. These states are defined as eigenstates of an arbitrary linear combination of the generators of the considered algebra [8]. Then we consider special solutions to CS and SS for the so-called super-position and super-momentum operators (§3.3).

3.1 The SUSY harmonic oscillator and its super-coherent states

Let us recall that the quantum SUSY harmonic oscillator is defined as a combination of a bosonic and a fermionic oscillators. Its Hamiltonian is given by

$$H_{\mathrm{SUSY}} = w(a^\dagger a - f^\dagger f), \tag{3.1}$$

where the bosonic creation and annihilation operators $a^\dagger$ and a are defined as in (2.25) and the corresponding fermionic operators $f^\dagger$ and f are defined as

$$f^\dagger = \sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2), \quad f = \sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2), \tag{3.2}$$

(the σ_i, $i = 1,2$ being the usual Pauli matrices) for the spin 1/2 fermion. We can thus write

$$H_{\mathrm{SUSY}} = w\left(a^\dagger a - \frac{1}{2}\right) - \frac{w}{2}\sigma_3. \tag{3.3}$$

The representation space, we are working with in this context, is nothing else than the direct product

$$\begin{aligned}\mathcal{F} = \mathcal{F}_b \otimes \mathcal{F}_f &= \{|n\rangle, n = 0,1,2,\ldots\} \otimes \left\{\left|\frac{1}{2},\frac{1}{2}\right\rangle = |+\rangle, \left|\frac{1}{2},\frac{-1}{2}\right\rangle = |-\rangle\right\} \\ &= \{|n,+\rangle, |n,-\rangle, n = 0,1,2,\ldots\}.\end{aligned} \tag{3.4}$$

Following Aragone and Zypman [10], SCS may be constructed as eigenstates of a SUSY annihilation operators $\left(\sqrt{2}(a+\sigma_+)\right)$. They are shown to be given as a linear combination of the following normalized pure states

$$|\psi\rangle_+ = D\left(\frac{z}{\sqrt{2}}\right)|0,+\rangle \tag{3.5}$$

and

$$|\psi\rangle_- = D\left(\frac{z}{\sqrt{2}}\right)\frac{[a^\dagger|0,+\rangle - |0,-\rangle]}{\sqrt{2}}, \tag{3.6}$$

in terms of the displacement operator D given in (2.35) and where we recognize in (3.5), the usual CS of the harmonic oscillator. A discussion [10, 11] of the properties of such states has led to the observation that, except for the state $|\psi_+\rangle \equiv$ (3.5), no other linear combination of (3.5) and (3.6) will minimize the usual HUR. This means that these states satisfy $(\Delta x)^2(\Delta p)^2 \geq 1/4$, the equality between the position x and the momentum p being realized only for $|\psi_+\rangle \equiv$ (3.5).

Such a fact can be clarified from our discussion of Section 2.1. The SCS (3.5) and (3.6) are in fact MUS for the SRUR (2.4) with

$$A = \frac{1}{\sqrt{2}}\left[(a^\dagger + a) + \sigma_1\right] = \left[x + \frac{\sigma_1}{\sqrt{2}}\right] \quad \text{and} \quad B = \frac{1}{\sqrt{2}}\left[i(a^\dagger - a) + \sigma_2\right] = \left[p + \frac{\sigma_2}{\sqrt{2}}\right], \tag{3.7}$$

these operators being different from x and p. The SCS are coherent in the sense that they satisfy the Eq. (2.6) with $\lambda = 1$.

Clearly, in such a context, through the group theory level, we are combining the information coming from both the Heisenberg–Weyl $h(1)$ and the su(2) algebras realized in terms of the Pauli matrices in the spin 1/2 case. Its is then natural to ask the questions of determining general CS and SS for the direct sum $h(1) \oplus \text{su}(2)$ which will indeed include the special SCS we just discussed.

3.2 Algebra eigenstates

We are working with the $h(1) \oplus \text{su}(2)$ algebra generated by $\{a, a^\dagger, I; J_+, J_-, J_3\}$ as defined in the preceding sections. AES [8] for this algebra are defined as eigenstates corresponding to a complex combination of the associated generators. A general hermitian operator A constructed from a combination of these generators is

$$A = A_1 a + \bar{A}_1 a^\dagger + A_2 I + A_3 J_+ + \bar{A}_3 J_- + A_4 J_3, \quad A_2, A_4 \in \mathbb{R},\ A_1, A_3 \in \mathbb{C}. \tag{3.8}$$

Two such operators, called A and B, satisfy the commutation relation (2.1) with

$$C = \left[i(\bar{A}_1 B_1 - A_1 \overline{B}_1)I + 2i(B_3\bar{A}_3 - \overline{B}_3 A_3)J_3 + i(A_3B_4 - A_4B_3)J_+ + i(A_4\overline{B}_3 - \bar{A}_3B_4)J_-\right]. \tag{3.9}$$

Once we search for states satisfying (2.6), i.e. for eigenstates of $A + i\lambda B$ ($\lambda \in \mathbb{C}, \lambda \neq 0$), we are in fact considering AES and we know from Section 2.1 that they minimize the SRUR (2.4). Let us then study the solutions of such a general eigenstate equation (2.6) for A and B on the form (3.8).

It is convenient to rewrite this equation as

$$[\alpha_- a + \alpha_+ a^\dagger + \alpha_3 I + \beta_- J_+ + \beta_+ J_- + \beta_3 J_3]|\psi\rangle = z|\psi\rangle, \tag{3.10}$$

where

$$\begin{aligned} \alpha_- &= A_1 + i\lambda B_1, \quad \alpha_+ = \bar{A}_1 + i\lambda\overline{B}_1, \quad \alpha_3 = A_2 + i\lambda B_2, \\ \beta_- &= A_3 + i\lambda B_3, \quad \beta_+ = \bar{A}_3 + i\lambda\overline{B}_3, \quad \beta_3 = A_4 + i\lambda B_4. \end{aligned} \tag{3.11}$$

To solve (3.10), we express $|\psi\rangle$ as a superposition of fundamental states $|n; j, m\rangle$ which constitute a generalization of the Fock space (3.4) for spin j. We write

$$|\psi\rangle^j = \sum_{m=-j}^{j}\sum_{n=0}^{\infty} C^j_{n,m}|n; j, m\rangle, \tag{3.12}$$

for fixed j, integer or half-odd integer. Let us recall that we have

$$\begin{aligned} a|n; j, m\rangle &= \sqrt{n}|n-1; j, m\rangle, \\ a^\dagger|n; j, m\rangle &= \sqrt{n+1}|n+1; j, m\rangle, \\ J_\pm|n; j, m\rangle &= \sqrt{(j \mp m)(j \pm m + 1)}|n; j, m \pm 1\rangle, \end{aligned} \tag{3.13}$$

with

$$\langle n; j, m \mid l; j, r\rangle = \delta_{nl}\delta_{mr}. \tag{3.14}$$

Inserting (3.12) into (3.10) and taking into account the relations (3.13) and (3.14), we get a recurrence system which becomes more and more complicated as j increases. We also notice that the case where $\alpha_- = 0$ with $\alpha_+ \neq 0$ does not give any solution and must be eliminated. Here two ways of solving it completely are presented. The first one uses the results obtained in Section 2.2 and Appendix Appendix A where AES of su(2) are explicitly constructed. It is described explicitly in this section using operators acting on a fundamental state. The second one is based on the method of resolution of a first order system of linear differential equations and is described in the Appendix Appendix B.

With respect to the discussion in Appendix Appendix A, we have mainly two types of eigenvalues for z. The first type is given by

$$z = \rho_m^j + \alpha_3 + mb, \quad \rho_m^j \in \mathbb{C}, \tag{3.15}$$

for fixed j and where $m = -j, \ldots, j$ and

$$b = \sqrt{4\beta_+\beta_- + \beta_3^2} \neq 0. \tag{3.16}$$

If we compare the equations (2.26) and (A.5) and their respective solutions (2.33) and (A.15), we find the set of solutions

$$|\psi\rangle_m^j = (C_m^j)^{-1/2} \exp\left[-\frac{\alpha_+}{2\alpha_-}a^{\dagger 2} + \frac{\rho_m^j}{\alpha_-}a^\dagger\right] T_{\text{eff}}|0; j, m\rangle, \tag{3.17}$$

when $\alpha_- \neq 0$. Here T_{eff} is given by (A.14) when $\{\beta_+ \neq 0, \beta_- \neq 0\}$, (A.18) when $\{\beta_+ = 0, \beta_3 \neq 0\}$, (A.20) when $\{\beta_- = 0, \beta_3 \neq 0\}$ and finally the identity when $\{\beta_- = \beta_+ = 0, \beta_3 \neq 0\}$.

The second type corresponds to the so-called degenerate case ($b = 0$) where $z = \rho + \alpha_3$. The sets of independent solutions are now given by

$$|\psi\rangle_m^j = (C_m^j)^{-1/2} \exp\left[-\frac{\alpha_+}{2\alpha_-}a^{\dagger 2} + \frac{\rho}{\alpha_-}a^\dagger\right] \times \sum_{k=0}^{j-m} (-1)^k \binom{j-m}{k} \frac{(2j-k)!}{(2j)!} (a^\dagger)^{j-m-k} \left(\frac{\alpha_- J_-}{\beta_-}\right)^k |0; j, j\rangle, \tag{3.18}$$

when $\beta_+ = \beta_3 = 0$,

$$|\psi\rangle_m^j = (C_m^j)^{-1/2} \exp\left[-\frac{\alpha_+}{2\alpha_-}a^{\dagger 2} + \frac{\rho}{\alpha_-}a^\dagger\right] \times \sum_{k=0}^{j-m} (-1)^k \binom{j-m}{k} \frac{(2j-k)!}{(2j)!} (a^\dagger)^{j-m-k} \left(\frac{\alpha_- J_+}{\beta_+}\right)^k |0; j, -j\rangle, \tag{3.19}$$

when $\beta_- = \beta_3 = 0$ and

$$|\psi\rangle_m^j = (C_m^j)^{-1/2} \exp\left[-\frac{\alpha_+}{2\alpha_-}a^{\dagger 2} + \frac{\rho}{\alpha_-}a^\dagger\right] \times \left[\sum_{k=0}^{j-m} (-1)^k \binom{j-m}{k} \frac{(2j-k)!}{(2j)!} (a^\dagger)^{j-m-k} \left(\frac{\alpha_-}{\beta_+}\right)^k \frac{d^k e^{\vartheta J_+}}{d\vartheta^k}\right] |0; j, -j\rangle, \tag{3.20}$$

when β_+, β_- and β_3 are different from zero and for $\vartheta = \beta_3/(2\beta_+) = -2\beta_-/\beta_3$.

3.3 Coherent and squeezed states for the super-position and super-momentum operators

Let us consider the eigenstates of equation (3.10) corresponding to the following special values of the parameters

$$A_4 = B_4 = A_2 = B_2 = 0, \quad A_1 = iB_1 = \frac{\mu}{\sqrt{2}}, \quad (\mu \neq 0), \quad A_3 = iB_3 = \frac{\tau}{\sqrt{2}}, \tag{3.21}$$

so that A will be called the super-position operator denoted by X and B the super-momentum operator denoted by P. We have

$$X = \frac{1}{\sqrt{2}}\left[(\mu a + \bar{\mu} a^\dagger) + (\tau J_+ + \bar{\tau} J_-)\right], \quad P = \frac{i}{\sqrt{2}}\left[(\bar{\mu} a^\dagger - \mu a) + (\bar{\tau} J_- - \tau J_+)\right]. \tag{3.22}$$

We see that the operators (3.7) associated to the SCS are then a special case where $\mu = \bar{\mu} = \tau = \bar{\tau} = 1$ in the spin-1/2 case.

The eigenstates equation (3.10) now writes

$$[X + i\lambda P]|\psi\rangle = z|\psi\rangle \tag{3.23}$$

and the operator C in (3.9) is diagonal and takes the form

$$C = |\mu|^2 I + 2|\tau|^2 J_3. \tag{3.24}$$

Since, we have

$$\begin{aligned} \alpha_- &= \frac{\mu(1+\lambda)}{\sqrt{2}}, \quad \alpha_+ = \frac{\bar{\mu}(1-\lambda)}{\sqrt{2}}, \quad \alpha_3 = 0, \\ \beta_- &= \frac{\tau(1+\lambda)}{\sqrt{2}}, \quad \beta_+ = \frac{\bar{\tau}(1-\lambda)}{\sqrt{2}}, \quad \beta_3 = 0 \end{aligned} \tag{3.25}$$

and finally

$$b = \sqrt{2}|\tau|\sqrt{1-\lambda^2}, \tag{3.26}$$

we can use the preceding solutions to give all the solutions of equation (3.23).

For $\lambda = 1$, we have $\alpha_+ = \beta_+ = b = 0$ and the eigenstate equation is

$$[\mu a + \tau J_+]|\psi\rangle = \frac{z}{\sqrt{2}}|\psi\rangle. \tag{3.27}$$

The normalized solutions are obtained from (3.18) and take the form

$$|\psi\rangle^j_m = \left(C^j_m(\mu,\tau)\right)^{-1/2} D\left(\frac{z}{\mu\sqrt{2}}\right)\left[\sum_{k=0}^{j-m}(-1)^k \binom{j-m}{k}\frac{(2j-k)!}{(2j)!}(a^\dagger)^{j-m-k}\left(\frac{\mu J_-}{\tau}\right)^k\right]|0;j,j\rangle, \tag{3.28}$$

where the normalization constant is given by

$$C^j_m(\mu,\tau) = (j-m)!\left[\sum_{k=0}^{j-m}\binom{j-m}{k}\frac{(2j-k)!}{(2j)!}\left(\frac{|\mu|^2}{|\tau|^2}\right)^k\right]. \tag{3.29}$$

Let us recall that in this case we have CS for which

$$(\Delta X) = (\Delta P) = \Delta = \frac{1}{2}\langle C\rangle. \tag{3.30}$$

The mean value of C is easy to compute and we have

$$\langle C\rangle^j_m = |\mu|^2 + 2|\tau|^2\left[j + |\tau|^2 \frac{\partial}{\partial|\tau|^2}\ln\left(C^j_m(\mu,\tau)\right)\right]. \tag{3.31}$$

In the special case $j = 1/2$, we find the normalized and orthogonal states

$$|\psi\rangle^+ = D\left(\frac{z}{\mu\sqrt{2}}\right)|0;+\rangle, \quad |\psi\rangle^- = D\left(\frac{z}{\mu\sqrt{2}}\right)\frac{|\tau|}{\sqrt{|\mu|^2+|\tau|^2}}\left[a^\dagger|0;+\rangle - \frac{\mu}{\tau}|0;-\rangle\right], \tag{3.32}$$

where D is again given by (2.35). In those states, we have

$$\langle C\rangle^+ = |\mu|^2 + |\tau|^2, \quad \langle C\rangle^- = \left[(|\mu|^2+|\tau|^2) - \frac{2|\mu|^2|\tau|^2}{(|\mu|^2+|\tau|^2)}\right]. \tag{3.33}$$

This is clearly a generalisation of SCS considered by Aragone and Zypman [10] and recalled in (**??**) and (**??**).

From (3.33), we see that the dispersions of ΔX and ΔP given by (3.30) computed in the CS $|\psi\rangle^-$ are smaller than in the states $|\psi\rangle^+$. The states $|\psi\rangle^-$ thus are the closest to classical states for the SUSY harmonic oscillators (this means with respect to the super-position and the super-momentum) while $|\psi\rangle^+$ are indeed the ones closest to classical states of the standard harmonic oscillator (i.e. they minimize the HUR for X and P). Let us mention that if we take $\mu = 1$, we see that $\langle C\rangle^+$ has its minimum value equal to 1 for $\tau \mapsto 0$ and in this case $X = x$ and $P = p$. For the same value of μ, we see that $\langle C\rangle^-$ takes the form

$$\langle C\rangle^- = \frac{1+|\tau|^4}{1+|\tau|^2}, \tag{3.34}$$

which has a minimum value $\langle C\rangle^-_{\min} = 2(\sqrt{2}-1) < 1$ for $|\tau|^2 = \sqrt{2}-1$.

For $\lambda \neq \pm 1$, from equation (3.17) and $T_{\rm eff} \equiv$ (A.13)), using also (2.35) and (2.36), we get the states

$$|\psi\rangle^j_m = (C^j_m)^{-1/2} S\big(\chi(\delta, \phi - 2\phi_u)\big) D\big(\eta_m(z,\delta,\phi,\mu,\tau)\big) \times \exp\left(\frac{-\tau\delta^{-1/2}e^{-i\phi/2}}{|\tau|} J_+\right) \exp\left(\frac{\bar{\tau}\delta^{1/2}e^{i\phi/2}}{2|\tau|} J_-\right)|0; j, m\rangle, \tag{3.35}$$

where

$$\eta_m(z,\delta,\phi,\mu,\tau) = \frac{1}{\mu}\left\{\frac{z(1+\delta e^{i\phi})}{\sqrt{2}} - 2m|\tau|\delta^{1/2}e^{i\phi/2}\right\}, \quad \mu = |\mu|e^{i\phi_u} \tag{3.36}$$

and where we have used instead of λ the parameters δ and ϕ as given in (2.32). Let us mention that this general expression (3.35) clearly shows the presence of the unitary operators D and S associated with $h(1)$ and su$(1,1)$ respectively which is the contribution of the bosonic part of our SUSY model. Moreover, the fermionic contribution appears through the action of a unitary operator associated with su(2).

Now these states satisfy the MUR

$$(\Delta X)^j_m (\Delta P)^j_m = \Delta^j_m = \frac{1}{2}\sqrt{1 + \frac{4\delta^2 \sin^2\phi}{(1-\delta^2)^2}}\,|\langle C\rangle^j_m|. \tag{3.37}$$

The mean value of C is

$$\langle C\rangle^j_m = |\mu|^2 + 2|\tau|^2 \frac{(1-\delta)}{(1+\delta)}\left(j - \frac{4(j+|m|)\delta}{(1+\delta)^2}\Omega\right), \tag{3.38}$$

where Ω is expressed in terms of Jacobi polynomials (see Appendix Appendix A),

$$\Omega = \frac{P^{(-2j,1)}_{j-|m|-1}(1-(8\delta/(1+\delta)^2))}{P^{(-2j-1,0)}_{j-|m|}(1-(8\delta/(1+\delta)^2))}, \tag{3.39}$$

for $m = -j+1, \ldots, j-1$ and $\Omega = 0$ for $m = \pm j$. In fact, we see that in these last cases, we have

$$\langle C\rangle^j_{\pm j} = |\mu|^2 + 2j|\tau|^2 \frac{(1-\delta)}{(1+\delta)}. \tag{3.40}$$

Its is now interesting to examine the behaviour of the dispersions ΔX and ΔP in these states for the spin 1/2 case. Using (2.20) with (3.40) for $j = 1/2$, we get

$$(\Delta X^2)_\pm = \frac{(1 - 2\delta\cos\phi + \delta^2)}{2(1-\delta^2)}\left[|\mu|^2 + |\tau|^2\frac{(1-\delta)}{(1+\delta)}\right],$$
$$(\Delta P^2)_\pm = \frac{(1 + 2\delta\cos\phi + \delta^2)}{2(1-\delta^2)}\left[|\mu|^2 + |\tau|^2\frac{(1-\delta)}{(1+\delta)}\right]. \tag{3.41}$$

with

$$\Delta_\pm = \frac{\sqrt{(1-\delta^2)^2 + 4\delta^2\sin^2\phi}}{2(1-\delta^2)}\left[|\mu|^2 + |\tau|^2\frac{(1-\delta)}{(1+\delta)}\right]. \tag{3.42}$$

If we take $\delta = 0$ (i.e. $\lambda = 1$) in these last expressions, we find only the values of the dispersions of X and P in the usual coherent states $|\psi\rangle^+$ as given by (3.32) and not the ones in the CS $|\langle\psi\rangle^-$, that is the reason why that case has been treated separately.

Fig. 5 and Fig. 6 show the behaviour of $\big((\Delta X)^2\big)_\pm$ and $\big((\Delta P)^2\big)_\pm$ and $\Delta_\pm$ as functions of δ for $\phi = \pi/6$ and as functions of ϕ for $\delta = 0.5$ respectively. We notice a similar behaviour as for the position and momentum operators.

4 Construction of $h(1) \oplus$ su(2) Hamiltonians

An application of our CS and SS based on the algebra $h(1)\oplus$su(2) will be the study of possible Hamiltonians which can be written as $\mathcal{H} = w\mathcal{A}^\dagger\mathcal{A}$, where $\mathcal{A}$ is a linear combination of the generators of $h(1)\oplus$su(2). It is clear that the usual harmonic oscillator Hamiltonian will enter in the scheme as a special case (§4.1) but also the Jaynes–Cummings [16] one in the strong coupling limit (§4.2) and (§4.3).

Moreover, since the CS and SS already constructed in the preceding section are in fact eigenstates of the operator $\mathcal{A}$, we would be able to find easily some properties of the mean value and the dispersion of the associated energies in those states.

4.1 Isospectral $\boldsymbol{h}(1) \oplus \mathrm{su}(2)$ harmonic oscillator Hamiltonians

We are interested in systems for which the Hamiltonian is expressed in the form

$$\mathcal{H} = w\mathcal{A}^\dagger \mathcal{A}, \tag{4.1}$$

where

$$\mathcal{A} = \alpha_- a + \alpha_+ a^\dagger + \alpha_3 I + \beta_- J_+ + \beta_+ J_- + \beta_3 J_3, \quad \alpha_- \neq 0, \tag{4.2}$$

is an element of the $h(1) \oplus \mathrm{su}(2)$ algebra. The commutator of the operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ is

$$[\mathcal{A}, \mathcal{A}^\dagger] = \left(|\alpha_-|^2 - |\alpha_+|^2\right) I + \left(|\beta_-|^2 - |\beta_+|^2\right) J_3 + (\beta_3\bar{\beta}_+ - \bar{\beta}_3\beta_-) J_+ + (\bar{\beta}_3\beta_+ - \beta_3\bar{\beta}_-) J_-. \tag{4.3}$$

If $|Z\rangle$ is an eigenstate of the operator $\mathcal{A}$ with eigenvalue z, i.e.

$$\mathcal{A}|Z\rangle = z|Z\rangle, \tag{4.4}$$

then the mean value of the energy in this state will always be given by

$$\langle Z|\mathcal{H}|Z\rangle = w|z|^2 \tag{4.5}$$

and the dispersion by

$$(\Delta\mathcal{H})^2 = w^2|z|^2\langle Z|[\mathcal{A}, \mathcal{A}^\dagger]|Z\rangle. \tag{4.6}$$

Firstly, let us consider the special case where

$$[\mathcal{A}, \mathcal{A}^\dagger] = I. \tag{4.7}$$

This imposes the following conditions on the parameters:

$$|\alpha_-|^2 - |\alpha_+|^2 = 1, \quad |\beta_-| = |\beta_+| \quad \text{and} \quad \beta_3\bar{\beta}_+ - \bar{\beta}_3\beta_- = 0, \tag{4.8}$$

i.e.

$$\alpha_- = \cosh\alpha e^{i\theta_-}, \quad \alpha_+ = \sinh\alpha e^{i\theta_+}, \quad \beta_\pm = \beta e^{i\varphi_\pm} \tag{4.9}$$

and

$$\beta_3 = \begin{cases} re^{i(\varphi_+ + \varphi_-)/2}, \ r \in \mathbb{R}_+ \cup \{0\} & \text{if } \beta \neq 0 \\ re^{i\varphi_3}, \ r \in \mathbb{R}_+ \cup \{0\} & \text{if } \beta = 0. \end{cases} \tag{4.10}$$

When $\beta \neq 0$, the operator $\mathcal{A}$ then takes the form

$$\mathcal{A} = \cosh\alpha e^{i\theta_-} a + \sinh\alpha e^{i\theta_+} a^\dagger + \alpha_3 I + \beta(e^{i\varphi_-} J_+ + e^{i\varphi_+} J_-) + re^{i(\varphi_+ + \varphi_-)/2} J_3. \tag{4.11}$$

The parameter b given in (3.16) becomes $b = \sqrt{4\beta^2 + r^2}e^{i(\varphi_+ + \varphi_-)/2}$ and is different from zero. Therefore in this case, according to the equation (3.17), the normalized solutions of the eigenstates equation (4.4) are given by

$$|Z\rangle_m^j = S(\Lambda) D\big(\zeta_m(\alpha_3, 1)\big) T D(ze^{-i\theta_-})|0; j, m\rangle, \tag{4.12}$$

where

$$\Lambda = -\alpha e^{i(\theta_+ - \theta_-)}, \quad \zeta_m(\alpha_3, \epsilon) = -[\alpha_3 + \epsilon m\sqrt{4\beta^2 + r^2}e^{i(\varphi_+ + \varphi_-)/2}]e^{-i\theta_-} \tag{4.13}$$

and

$$T = \exp\left(-\frac{\tilde{\theta}}{2}[e^{-i(\varphi_+ - \varphi_-)/2} J_+ - e^{i(\varphi_+ - \varphi_-)/2} J_-]\right), \tag{4.14}$$

with

$$\frac{\tilde{\theta}}{2} = \tan^{-1}\left(\sqrt{1 - \frac{r}{2\beta^2}(\sqrt{4\beta^2 + r^2} - r)}\right). \tag{4.15}$$

This means that T is an unitary operator.

We remark that, if we define the new operator

$$\begin{aligned} \mathcal{A}_0 &= D^\dagger(-\alpha_3 e^{-i\theta_-}) S^\dagger(\Lambda) \mathcal{A} S(\Lambda) D(-\alpha_3 e^{-i\theta_-}) \\ &= e^{i\theta_-} a + \beta(e^{i\varphi_-} J_+ + e^{i\varphi_+} J_-) + re^{i(\varphi_+ + \varphi_-)/2} J_3, \end{aligned} \tag{4.16}$$

which is simpler than the original $\mathcal{A}$, then the new Hamiltonian $\mathcal{H}_0 = w\mathcal{A}_0^\dagger\mathcal{A}_0$ is isospectral to the Hamiltonian $\mathcal{H} \equiv$ (4.1).

The dispersion of $\mathcal{H}$ calculated on the states (4.12) is, from (4.6) and (4.7), given by $(\Delta\mathcal{H})^2 = w^2|z|^2$ and is the same as the one of $\mathcal{H}_0$ calculated on the states $D\big(\zeta_m(-z,1)\big)T|0;j,m\rangle$. This value is exactly the dispersion of the harmonic oscillator in the usual CS.

On the other hand, due to (4.7) we have $[\mathcal{H},\mathcal{A}] = -w\mathcal{A}$, so we have a complete analogy with the harmonic oscillator. The CS associated to the Hamiltonian $\mathcal{H}$, called generalized harmonic oscillator, are those given by the equation (4.12) and thus, one can write them in the form

$$|Z\rangle_m^j = \mathcal{D}(z)|\tilde{0}\rangle_m^j, \text{ where } \mathcal{D}(z) = \exp(z\mathcal{A}^\dagger - \bar{z}\mathcal{A}) \tag{4.17}$$

and $|\tilde{0}\rangle_m^j$, $m = -j,\ldots,j$, are the fundamental states of the system $\mathcal{H}$, that is the eigenstates of $\mathcal{H}$ corresponding to the $(2j+1)$ degenerate eigenvalue 0. They are also eigenstates of $\mathcal{A}$ corresponding to the eigenvalue 0. So they can be written

$$|\tilde{0}\rangle_m^j = S(\Lambda)D\big(\zeta_m(\alpha_3,1)\big)T|0;j,m\rangle. \tag{4.18}$$

Furthermore, the SS associated with $\mathcal{H}$, are given by

$$|\tilde{\psi}\rangle_m^j = \mathcal{S}(\chi)\mathcal{D}(z)|\tilde{0}\rangle_m^j, \tag{4.19}$$

where the supersqueezed operator $\mathcal{S}(\chi)$ is given by $\exp(\chi{\mathcal{A}^\dagger}^2/2 - \overline{\chi}\mathcal{A}^2/2)$ and the superdisplacement operator $\mathcal{D}(z)$ is given in (4.17). If we define $\mathcal{X} = (\mathcal{A}+\mathcal{A}^\dagger)/\sqrt{2}$ and $\mathcal{P} = i(\mathcal{A}^\dagger - \mathcal{A})/\sqrt{2}$, these states (4.19) minimize the SRUR $(\Delta\mathcal{X})^2(\Delta\mathcal{P})^2 = \big(1+\langle F\rangle^2\big)/4$, i.e. they are solutions of the eigenstate equation $\big[(1-\lambda)\mathcal{A}^\dagger + (1+\lambda)\mathcal{A}\big]|\psi\rangle = \sqrt{2}|\psi\rangle$.

The eigenstates of $\mathcal{H}$ corresponding to the $(2j+1)$ degenerate energy eigenvalue $E_n = nw$ are now given by

$$|\tilde{n}\rangle_m^j = \frac{{\mathcal{A}^\dagger}^n}{\sqrt{n!}}|\tilde{0}\rangle_m^j. \tag{4.20}$$

These states may be obtained as the action of an unitary operator on the states $|n;j,m\rangle$. Indeed, if we introduce the unitary operator

$$U_n^m = e^{-in\theta_-}S(\Lambda)D\big(\zeta_m(\alpha_3,1)\big)T, \tag{4.21}$$

we see that, from (4.20), we have

$$\begin{aligned}
|\tilde{n}\rangle_m^j &= \frac{e^{in\theta_-}}{\sqrt{n!}}(\mathcal{A}^\dagger)^n U_n^m|0;j,m\rangle,\\
&= \frac{e^{in\theta_-}}{\sqrt{n!}}U_n^m\big((U_n^m)^\dagger\mathcal{A}^\dagger U_n^m\big)^n|0;j,m\rangle,\\
&= \frac{e^{in\theta_-}}{\sqrt{n!}}U_n^m\big(e^{-i\theta_-}a^\dagger + \sqrt{4\beta^2+r^2}e^{-i(\varphi_+ + \varphi_-)/2}(J_3-m)\big)^n|0;j,m\rangle.
\end{aligned} \tag{4.22}$$

Since we have $(J_3-m)|0;j,m\rangle = 0$, we finally find

$$|\tilde{n}\rangle_m^j = U_m^n|n;j,m\rangle. \tag{4.23}$$

In the case $\beta = 0$, the operator $\mathcal{A}$ is given by

$$\mathcal{A} = \cosh\alpha e^{i\theta_-}a + \sinh\alpha e^{i\theta_+}a^\dagger + \alpha_3 I + re^{i\varphi_3}J_3. \tag{4.24}$$

Then, if $r \neq 0$, one has the same results as above, except that it is necessary to replace T by I and b by $\beta_3 = re^{i\varphi_3}$. If $r = 0$, $\mathcal{A}$ is an element of the algebra $h(1)$ and then the results are the ones obtained in Section 2.1 for the standard harmonic oscillator after applying the unitary transformation $S(\Lambda)D(-\alpha_3e^{-i\theta_-})$.

4.2 Strong-coupling limit of the Jaynes–Cummings Hamiltonian as limit of $h(1)\oplus\mathrm{su}(2)$ Hamiltonians

We are going to consider now the case where

$$[\mathcal{A},\mathcal{A}^\dagger] = I + 2xJ_3, \quad x \in \mathbb{R}. \tag{4.25}$$

This imposes the following conditions on the parameters:

$$|\alpha_-|^2 - |\alpha_+|^2 = 1, \quad |\beta_-|^2 - |\beta_+|^2 = x \quad \text{and} \quad \beta_3\bar{\beta}_+ - \bar{\beta}_3\beta_- = 0. \tag{4.26}$$

We already know the results when $x = 0$. When $x \neq 0$, the conditions (4.26) imply

$$\alpha_- = \cosh\alpha e^{i\theta_-}, \quad \alpha_+ = \sinh\alpha e^{i\theta_+}, \quad \beta_3 = 0 \tag{4.27}$$

and

$$\beta_- = \begin{cases} x^{1/2}\cosh\beta e^{i\varphi_-}, & \text{if } x > 0 \\ |x|^{1/2}\sinh\beta e^{i\varphi_-}, & \text{if } x < 0, \end{cases} \tag{4.28}$$

$$\beta_+ = \begin{cases} x^{1/2}\sinh\beta e^{i\varphi_+}, & \text{if } x > 0 \\ |x|^{1/2}\cosh\beta e^{i\varphi_+}, & \text{if } x < 0. \end{cases} \tag{4.29}$$

The parameter $b \equiv$ (3.16) becomes $b = |x|^{1/2}\sqrt{2\sinh(2\beta)}e^{i(\varphi_+ + \varphi_-)/2}$, this means that $b = 0$ if and only if $\beta = 0$.

In the case $\beta \neq 0$, according to the equations (3.17), (A.7), (A.11), and (A.12), the normalized eigenstates of the operator $\mathcal{A}$ are given by

$$\begin{aligned} |Z(x)\rangle_m^j &= \left(C_m^j(x)\right)^{-1/2} S(\Lambda)D(-\alpha_3 e^{-i\theta_-})D(\eta_m(z,x)) \\ &\qquad \times \exp\left[-\frac{x}{2|x|}\ln(\tanh\beta)J_3\right]U|0;j,m\rangle, \end{aligned} \tag{4.30}$$

where

$$\eta_m(z,x) = \left[z - m|x|^{1/2}\sqrt{2\sinh(2\beta)}e^{i(\varphi_+ + \varphi_-)/2}\right]e^{-i\theta_-}, \tag{4.31}$$

$$U = \exp\left[-\frac{\pi}{4}\left(e^{-i(\varphi_+ - \varphi_-)/2}J_+ - e^{i(\varphi_+ - \varphi_-)/2}J_-\right)\right] \tag{4.32}$$

and

$$\begin{aligned} C_m^j(x) &= \langle j,m|U^\dagger \exp\left[-\frac{x}{|x|}\ln(\tanh\beta)J_3\right]U|j,m\rangle \\ &= \left(\frac{1+\tanh\beta}{2\sqrt{\tanh\beta}}\right)^{\mp 2m} P_{j\pm m}^{0;\mp 2m}\left(\frac{1+\tanh^2\beta}{2\tanh\beta}\right). \end{aligned} \tag{4.33}$$

From (4.6) and (4.25), the dispersion of the $\mathcal{H} \equiv$ (4.1) in the states (4.30) can be calculated explicitly. We get

$$\left((\Delta\mathcal{H})^2\right)_m^j = w^2|z|^2\left(1 + 2x\,{}_m^j\langle Z(x)|J_3|Z(x)\rangle_m^j\right). \tag{4.34}$$

In the last expression, the mean value of J_3 is obtained in a similar way then to get (2.61). The result is

$${}_m^j\langle Z(x)|J_3|Z(x)\rangle_m^j = \frac{x}{|x|}\left\{|m|e^{-2\beta} + \frac{(j+|m|+1)}{2\sinh(2\beta)}\frac{P_{j-|m|-1}^{1;2|m|}(\coth(2\beta))}{P_{j-|m|}^{0;2|m|}(\coth(2\beta))}\right\}. \tag{4.35}$$

If we take $m = \pm j$, the dispersion of $\mathcal{H}$ is

$$\left((\Delta\mathcal{H})^2\right)_{\pm j}^j = w^2|z|^2\left(1 + 2j|x|e^{-2\beta}\right) \tag{4.36}$$

and, in particular, when $j = 1/2$, we get

$$\left((\Delta\mathcal{H})^2\right)_\pm = w^2|z|^2\left(1 + |x|e^{-2\beta}\right). \tag{4.37}$$

Fig. 7 shows the graphs of $\left((\Delta\mathcal{H})^2\right)_\pm$ as functions of β for different values of $|x|$ when $w^2|z|^2$ is taken equal to 1.

Let us compute the new operator $\mathcal{A}_0$ defined as (4.16). We get

$$\mathcal{A}_0 = \begin{cases} e^{i\theta_-}a + x^{1/2}\cosh\beta e^{i\varphi_-}J_+ + x^{1/2}\sinh\beta e^{i\varphi_+}J_-, & \text{if } x > 0 \\ e^{i\theta_-}a + |x|^{1/2}\sinh\beta e^{i\varphi_-}J_+ + |x|^{1/2}\cosh\beta e^{i\varphi_+}J_-, & \text{if } x < 0, \end{cases} \tag{4.38}$$

and a new Hamiltonian $\mathcal{H}_0 = w\mathcal{A}_0^\dagger \mathcal{A}_0$ isospectral to the Hamiltonian $\mathcal{H}$ which takes the form

$$\begin{aligned}\mathcal{H}_0 \;=\;& w\{a^\dagger a + |x|\big[\sinh^2(\beta)J_-J_+ + \cosh^2(\beta)J_+J_-\big] \\ &+|x|^{1/2}\cosh\beta[e^{i(\varphi_+-\theta_-)}a^\dagger J_- + e^{-i(\varphi_+-\theta_-)}aJ_+] \\ &+|x|^{1/2}\sinh\beta[e^{i(\varphi_--\theta_-)}a^\dagger J_+ + e^{-i(\varphi_--\theta_-)}aJ_-] \\ &+|x|\sinh\beta\cosh\beta[e^{i(\varphi_+-\varphi_-)}J_-^2 + e^{-i(\varphi_+-\varphi_-)}J_+^2]\},\end{aligned} \tag{4.39}$$

if $x<0$. If $x>0$, we get a similar expression except that we must make the change $\sinh\beta \leftrightarrow \cosh\beta$.

In the spin-1/2 representation, we have

$$J_-^2 = J_+^2 = 0, \quad J_+J_- = \frac{I}{2} + J_3 \quad \text{and} \quad J_-J_+ = \frac{I}{2} - J_3, \tag{4.40}$$

hence (4.39) becomes

$$\begin{aligned}\mathcal{H}_0 \;=\;& w\bigg\{\left(a^\dagger a + \frac{I}{2}\right) - xJ_3 + |x|^{1/2}\cosh\beta[e^{i(\varphi_+-\theta_-)}a^\dagger J_- + e^{-i(\varphi_+-\theta_-)}aJ_+] \\ &+|x|^{1/2}\sinh\beta[e^{i(\varphi_--\theta_-)}a^\dagger J_+ + e^{-i(\varphi_--\theta_-)}aJ_-] + \big(|x|\cosh(2\beta)-1\big)\frac{I}{2}\bigg\}\end{aligned} \tag{4.41}$$

and a similar expression when $x>0$, making the literal change $\sinh\beta \leftrightarrow \cosh\beta$. If we take $x=-w_0/w$, $\varphi_+ = \theta_-$ and the limit $\beta \mapsto 0$, then $\mathcal{H}_0 \equiv$ (4.41) becomes

$$\mathcal{H}_0 = w\left(a^\dagger a + \frac{1}{2}\right) + w_0J_3 + \sqrt{ww_0}(a^\dagger J_- + aJ_+) + \frac{w-w_0}{2}I, \tag{4.42}$$

which is the Jaynes–Cummings Hamiltonian [16] up to a constant term and for a coupling constant given by $\kappa = \sqrt{ww_0}$. Let us recall that this Hamiltonian describes the interaction of a cavity mode (with frequency w) with a two level-system (w_0 being the atomic frequency). When $x=-1$, i.e., for $w=w_0$, (4.42) becomes the strong-coupling limit of the Jaynes–Cummings Hamiltonian.

In the case $\beta=0$, the new operator $\mathcal{A}_0 \equiv$ (4.16) reduces now to

$$\mathcal{A}_0(x) = \begin{cases} e^{i\theta_-}a + |x|^{1/2}e^{i\varphi_+}J_-, & \text{if } x<0 \\ e^{i\theta_-}a + |x|^{1/2}e^{i\varphi_-}J_+, & \text{if } x>0. \end{cases} \tag{4.43}$$

As we have here $b=0$, according to the expressions (3.18) and (3.19), the orthonormalized eigenstates of $\mathcal{A}_0$ are given by

$$\begin{aligned}|Z(x)\rangle_m^j =& \big(\widetilde{C}_m^j(x)\big)^{1/2} D(ze^{-i\theta_-}) \\ &\times \sum_{k=0}^{j-m}(-1)^k\binom{j-m}{k}\frac{(2j-k)!}{(2j)!}(e^{-i\theta_-}a^\dagger)^{j-m-k}\left(J_\mp\frac{e^{-i\varphi_\mp}}{\sqrt{|x|}}\right)^k\bigg|0;j,\frac{x}{|x|}j\bigg\rangle,\end{aligned} \tag{4.44}$$

where the $-$ sign refers to $x>0$ and the sign $+$ to $x<0$ and

$$\widetilde{C}_m^j(x) = (j-m)!\sum_{k=0}^{j-m}\binom{j-m}{k}\frac{(2j-k)!}{(2j)!}\left(\frac{1}{|x|}\right)^k. \tag{4.45}$$

Since in this case, we have

$${}_m^j\big\langle Z(x)\big|J_3\big|Z(x)\big\rangle_m^j = \frac{x}{|x|}\left[j + |x|\frac{\partial}{\partial|x|}\ln\big(\widetilde{C}_m^j(x)\big)\right], \tag{4.46}$$

the dispersion of $\mathcal{H}_0 = w\mathcal{A}_0^\dagger\mathcal{A}_0$ in the states (4.44) is given by

$$\big((\Delta\mathcal{H}_0)^2\big)_m^j = w^2|z|^2\left[1 + 2|x|j + 2|x|^2\frac{\partial}{\partial|x|}\ln\big(\widetilde{C}_m^j(x)\big)\right]. \tag{4.47}$$

When $m=j$, we have $\widetilde{C}_m^j(x)=1$, so that we get

$$\big((\Delta\mathcal{H}_0)^2\big)_j^j = w^2|z|^2\big(1+2|x|j\big). \tag{4.48}$$

For example, when $j = 1/2$, the dispersion corresponding to $m = 1/2$ is given by

$$\big((\Delta\mathcal{H}_0)^2\big)_+ = w^2|z|^2\big(1 + |x|\big) \tag{4.49}$$

and one obtains the same result as in the preceding case when we take the limit $\beta \mapsto 0$. On the other hand, for $m = -1/2$, we get

$$\big((\Delta\mathcal{H}_0)^2\big)_- = w^2|z|^2\left[1 + |x|\frac{(|x|-1)}{(|x|+1)}\right] \tag{4.50}$$

and it is always smaller than $\big((\Delta\mathcal{H}_0)^2\big)_+$. In this last case, we see that if $|x| > 1$, the dispersion is bigger than $w^2|z|^2$ while if $|x| < 1$ it is smaller than $w^2|z|^2$ and if $|x| = 1$ it is equal to $w^2|z|^2$. Furthermore, the dispersion reaches its minimum $0.83w^2|z|^2$ when $|x| = (\sqrt{2}-1)$. Fig. 8 shows the behaviour of dispersions $\big((\Delta\mathcal{H}_0)^2\big)_\pm$ as function of $|x|$.

Let us finally mention that the Hamiltonian $\mathcal{H}_0$ in this case and for $j = 1/2$ corresponds to (4.41) when $\beta = 0$. A special case is again the Jaynes–Cummings Hamiltonian (4.42) so we get eigenstates of $\mathcal{A}_0 \equiv$ (4.43) such that the dispersion of this Hamiltonian is minimized and lower than $w^2|z|^2$.

4.3 Generalized $\boldsymbol{h}(1) \oplus \mathrm{su}(2)$ non-canonical commutation relation

In the case where we have

$$[\mathcal{A}, \mathcal{A}^\dagger] = I + \gamma J_+ + \bar{\gamma} J_-, \quad \gamma \in \mathbb{C},\ \gamma \neq 0, \tag{4.51}$$

according to (4.3), the necessary conditions on the original parameters are

$$|\alpha_-|^2 - |\alpha|_+{}^2 = 1, \quad |\beta_-| = \beta_+, \quad \beta_3\bar{\beta}_+ - \bar{\beta}_3\beta_- = \gamma = \rho e^{i\nu}, \tag{4.52}$$

where $\rho \in \mathbb{R}_+$. A suitable choice of the parameters is

$$\alpha_- = \cosh\alpha e^{i\theta_-}, \quad \alpha_+ = \sinh\alpha e^{i\theta_+}, \quad \beta_\pm = \beta e^{i\varphi_\pm}, \quad \beta_3 = re^{i\varphi_3}, \quad \beta \neq 0,\ r \neq 0, \tag{4.53}$$

such that

$$r\beta[e^{i(\varphi_3-\varphi_+)} - e^{-i(\varphi_3-\varphi_+)}] = \rho e^{i\nu}. \tag{4.54}$$

Equation (4.54) implies that

$$\rho = 2r\beta\left|\sin\left(\varphi_3 - \frac{(\varphi_+ + \varphi_-)}{2}\right)\right| \tag{4.55}$$

and the following conditions on the phases: $\varphi_3 \neq (\varphi_+ + \varphi_-)/2$, $\varphi_3 \neq (\varphi_+ + \varphi_-)/2 + \pi$ and$\varphi_+ - \varphi_- = \pi - 2\nu, \nu \in [0, 3\pi/2]$ or $\varphi_+ - \varphi_- = 3\pi - 2\nu$, $\nu \in [\pi/2, 2\pi]$. Thus, the operator $\mathcal{A}$ compatible with all the previous conditions is

$$\mathcal{A} = \cosh\alpha e^{i\theta_-}a + \sinh\alpha e^{i\theta_+}a^\dagger + \alpha_3 I + e^{i(\varphi_- - \nu)}\left[\beta(e^{i\nu}J_+ - e^{-i\nu}J_-) + \frac{\rho}{2\beta|\cos\theta|}e^{i\theta}J_3\right], \tag{4.56}$$

where

$$\theta = \varphi_3 - (\varphi_- - \nu), \quad -\frac{\pi}{2} < \theta < 3\frac{\pi}{2}. \tag{4.57}$$

The new operator $\mathcal{A}_0$ defined in (4.16) is then given by

$$\mathcal{A}_0 = e^{i\theta_-}a + e^{i(\varphi_- - \nu)}\left[-\beta(e^{i\nu}J_+ - e^{-i\nu}J_-) + \frac{\rho}{2\beta|\cos\theta|}e^{i\theta}J_3\right]. \tag{4.58}$$

The parameter $b \equiv$ (3.16) is now $b = i\sqrt{16\beta^2\cos^2(\theta) - \rho^2 e^{2i\theta}}e^{i(\varphi_- - \nu)}/(2\beta|\cos\theta|)$, i.e. $b = 0$ if and only if $\beta = \sqrt{\rho}/2$ and $\theta = \pi$.

Here we can proceed as before, that is, when $b = 0$, find, by means of the equation (3.20) the eigenstates of $\mathcal{A}_0$ and, when $b \neq 0$, find the solutions by means of the equation (3.17) and then calculate the dispersions of $\mathcal{H}_0$.

But, we will follow another treatment which teachs us about the similarities between the canonical and the non-canonical cases. Indeed, seen in another perspective, the commutation relation (4.51) can be expressed in the form

$$[\mathcal{A}_0, \mathcal{A}_0^\dagger] = I + 2\rho\mathbb{J}_3, \tag{4.59}$$

where we have set

$$\mathbb{J}_3 = \frac{(e^{i\nu}J_+ + e^{-i\nu}J_-)}{2}. \tag{4.60}$$

Thus, when $b=0$, $\mathcal{A}_0$ becomes

$$\mathcal{A}_0 = e^{i\theta_-}a + \sqrt{\rho}e^{i(\varphi_- - \nu)}\mathbb{J}_+, \tag{4.61}$$

with

$$\mathbb{J}_\pm = \pm\frac{(e^{i\nu}J_+ - e^{-i\nu}J_-)}{2} - J_3. \tag{4.62}$$

The operators $\mathbb{J}_3$, $\mathbb{J}_\pm$ satisfy the su(2) algebra and let us denote by $|J,M\rangle$ the eigenstates of both $\mathbb{J}^2$ and $\mathbb{J}_3$. We have again:

$$\mathbb{J}_3|J,M\rangle = M|J,M\rangle, \quad \mathbb{J}_\pm|J,M\rangle = \sqrt{(J\mp M)(J\pm M+1)}|J,M\rangle. \tag{4.63}$$

Now, it is clear that the resolution of the problem to find the eigenstates of $\mathcal{A}_0$ is similar to the canonical case. Indeed, the normalized eigenstates of $\mathcal{A}_0$ are given by

$$\begin{aligned}|Z(\rho)\rangle_M^J = &\left(\widetilde{C}_M^J(\rho)\right)^{1/2} D(ze^{-i\theta_-})\\ &\times \sum_{k=0}^{J-M}(-1)^k\binom{J-M}{k}\frac{(2J-k)!}{(2J)!}\left(e^{-i\theta_-}a^\dagger\right)^{J-M-k}\left(\frac{\mathbb{J}_- e^{-i(\varphi_- - \nu)}}{\sqrt{\rho}}\right)^k|0;J,J\rangle,\end{aligned} \tag{4.64}$$

where $\widetilde{C}_M^J(\rho)$ is given as in (4.45).

As before, the dispersion of $\mathcal{H}_0$ in the states (4.64) is given by

$$\left((\Delta\mathcal{H}_0)^2\right)_M^J = w^2|z|^2\left[1 + 2J\rho + 2\rho^2\frac{\partial}{\partial\rho}\ln\left(\widetilde{C}_M^J(\rho)\right)\right]. \tag{4.65}$$

For example, when $J=1/2$, we have

$$\left((\Delta\mathcal{H}_0)^2\right)_+ = w^2|z|^2(1+\rho), \quad \left((\Delta\mathcal{H}_0)^2\right)_- = w^2|z|^2\left[1+\rho\frac{(\rho-1)}{(\rho+1)}\right]. \tag{4.66}$$

Evidently, the behaviour of these dispersions as functions of ρ is identical to that described in the last paragraph of the previous section.

In the general case where $b\neq 0$, $\mathcal{A}_0$ can be expressed in the form

$$\mathcal{A}_0 = e^{i\theta_-}a + e^{i(\varphi_- - \nu)}\left\{\left[\frac{4\beta^2|\cos\theta| - \rho e^{i\theta}}{4\beta|\cos\theta|}\right]\mathbb{J}_+ - \left[\frac{4\beta^2|\cos\theta| + \rho e^{i\theta}}{4\beta|\cos\theta|}\right]\mathbb{J}_-\right\}. \tag{4.67}$$

From (3.17), we see that the eigenstates of $\mathcal{A}_0$ are

$$|Z\rangle_M^J = (C_m^j)^{-1/2}D(ze^{-i\theta_-})T_{\text{eff}}|0;J,M\rangle, \tag{4.68}$$

where

$$T_{\text{eff}} = e^{\Phi_-\mathbb{J}_+}e^{\Phi_+\mathbb{J}_-}, \tag{4.69}$$

with

$$\Phi_- = i\frac{[4\beta^2|\cos\theta| - \rho e^{i\theta}]}{R^{1/2}e^{i\tilde{\varphi}/2}}, \quad \Phi_+ = i\frac{[4\beta^2|\cos\theta| + \rho e^{i\theta}]}{2R^{1/2}e^{i\tilde{\varphi}}/2}. \tag{4.70}$$

The dispersion of $\mathcal{H}_0$ in these states is

$$\left((\Delta\mathcal{H}_0)^2\right)_M^J = w^2|z|^2\left[1 + 2\rho_M^J\langle Z|\mathbb{J}_3|Z\rangle_M^J\right], \tag{4.71}$$

where [17]

$${}_M^J\langle Z|\mathbb{J}_3|Z\rangle_M^J = M\left(\frac{1-|\Phi_-|^2}{1+|\Phi_-|^2}\right) + \frac{(J-M+1)}{2}\frac{P_{J+M-1}^{1,-2M+1}(\Lambda)}{P_{J+M}^{0,-2M}(\Lambda)}\tilde{\Lambda}, \tag{4.72}$$

with

$$\Lambda = 1 + 2\left|\Phi_- + \bar{\Phi}_+\left(1+|\Phi_-|^2\right)\right|^2 \tag{4.73}$$

and

$$\tilde{\Lambda} = 2\left[|\Phi_-|^2(1+\Phi_-\Phi_+ + \bar{\Phi}_-\bar{\Phi}_+) + |\Phi_+|^2\left(|\Phi_-|^4 - 1\right)\right]. \tag{4.74}$$

Thus, in the spin-1/2 representation, we get

$${}_\pm\langle Z|\mathbb{J}_3|Z\rangle_\pm = \frac{1}{2}\left(\frac{|\Phi_-|^2-1}{1+|\Phi_-|^2}\right). \tag{4.75}$$

Finally by direct computation, we find

$$\left((\Delta\mathcal{H}_0)^2\right)_\pm = w^2|z|^2\left[1+\rho\frac{[16\beta^4\cos^2(\theta)+\rho^2-8\rho\beta^2\cos\theta|\cos\theta|]-R}{[16\beta^4\cos^2(\theta)+\rho^2-8\rho\beta^2\cos\theta|\cos\theta|]+R}\right], \tag{4.76}$$

where

$$R=\sqrt{\left[16\beta^4\cos^2(\theta)-\rho^2\cos(2\theta)\right]^2+\rho^4\sin^2(2\theta)}. \tag{4.77}$$

We see that, for fixed value of ρ, Equation (4.76) as a function of β is symmetric around $\theta=\pi$.

Fig. 9 shows the behaviour of the dispersions (4.76) as functions of $\beta>0$ when $\theta=\pi$ and for different values of parameter ρ. Let us notice the similarity between these curves starting from a certain value of β and the curves for the canonical case showed in Fig. 7.

Fig. 10 shows the behaviour of the same functions as functions of $\beta>0$, for different values of θ when $\rho=1$. We observe that when the angle θ is different from π the curves have a continuous derivative with respect to β but, when the angle $\theta=\pi$, the derivative of the curve at the point $\beta=0.5=\sqrt{\rho}/2$ is not continuous.

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Appendix A Algebra eigenstates associated to su(2)

In this appendix we want to solve the eigenvalue equation

$$[\vec{\beta}\cdot\vec{J}]|\psi\rangle=[\beta_1J_1+\beta_2J_2+\beta_3J_3]|\psi\rangle=\Gamma|\psi\rangle,\quad \beta_1,\beta_2,\beta_3\in\mathbb{C}, \tag{A.1}$$

where J_1, J_2 and J_3 are the su(2) generators which have already been given in Section 2.3. The eigenvalue equation (A.1) can also be written as

$$[\beta_-J_++\beta_+J_-+\beta_3J_3]|\psi\rangle=\Gamma|\psi\rangle, \tag{A.2}$$

where J_1 and J_2 have been expressed in terms of the usual operators $J_\pm$ and

$$\beta_\pm=\frac{\beta_1\pm i\beta_2}{2}. \tag{A.3}$$

We see that Eq. (2.45) is just a particular case of equation (A.2). The eigenvalue equation (A.2) has already been solved by Brif [8] by expanding the state $|\psi\rangle$ in the standard coherent-state basis [9], introducing in this way analytic functions and asking for solving a first order differential equation. Here, we consider a different method based on the operator algebra technique.

For j fixed, we can show that (A.2) admits the eigenvalues

$$\Gamma^j_m=mb, \tag{A.4}$$

with $m=-j,\ldots,j$ and $b=\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}=\sqrt{4\beta_+\beta_-+\beta_3^2}$. We then solve

$$[\beta_-J_++\beta_+J_-+\beta_3J_3]|\psi\rangle^j_m=\Gamma^j_m|\psi\rangle^j_m, \tag{A.5}$$

by using

$$|\psi\rangle^j_m=(N^j_m)^{-1/2}T|j,m\rangle, \tag{A.6}$$

where the N^j_m are normalization constants and T is an operator that has to be determined. We take it as

$$T=\exp\left(-\frac{\tilde{\theta}}{2}[e^{-i\tilde{\phi}}J_+-e^{i\tilde{\phi}}J_-]\right),\quad \tilde{\phi},\tilde{\theta}\in\mathbb{C}. \tag{A.7}$$

Inserting (A.6) with (A.7) into (A.5), that leads to

$$[\vec{\beta}\cdot\vec{J}]T|j,m\rangle=mbT|j,m\rangle. \tag{A.8}$$

Using the usual decomposition

$$T = \exp\Big(-e^{-i\tilde{\phi}}\tan\Big(\frac{\tilde{\theta}}{2}\Big)J_+\Big) \quad \exp\Big(\ln\sec^2(\frac{\tilde{\theta}}{2}\Big)J_3\Big) \quad \exp\Big(e^{i\tilde{\phi}}\tan\Big(\frac{\tilde{\theta}}{2}\Big)J_-\Big) \tag{A.9}$$

and the relations

$$e^{\eta J_3}J_\pm e^{-\eta J_3} = e^{\pm\eta}J_\pm, \quad e^{\eta J_\pm}J_3 e^{-\eta J_\pm} = J_3 \mp \eta J_\pm, \quad e^{\eta J_\pm}J_\mp e^{-\eta J_\pm} = J_\mp \pm 2\eta J_3 - \eta^2 J_\pm, \tag{A.10}$$

we can show that, for $\beta_+ \neq 0$, $\beta_- \neq 0$ and $b \neq 0$, we have

$$e^{i\tilde{\phi}} = \sqrt{\frac{\beta_+}{\beta_-}}, \tag{A.11}$$

and

$$\frac{\tilde{\theta}}{2} = \arctan\Big(\sqrt{\frac{b-\beta_3}{b+\beta_3}}\Big). \tag{A.12}$$

Inserting the results (A.11) and (A.12) in (A.9), we obtain

$$T = \exp\Big(-\frac{2\beta_-}{b+\beta_3}J_+\Big)\exp\Big(\ln\Big(\frac{2b}{b+\beta_3}\Big)J_3\Big)\exp\Big(\frac{2\beta_+}{b+\beta_3}J_-\Big). \tag{A.13}$$

The original form (A.7) of the T operator allows us to look easily for the special cases studied in [6, 9] and in the preceding sections while the form (A.13) allows to calculate directly the explicit form of the eigenstates (A.6). Indeed, the first relation (A.10) allows us to pass the exponential term $\exp\big(\ln(2b/(b+\beta_3))J_3\big)$ to the right in (A.13) and this without changing essentially the operator action on the pure states $|j,m\rangle$ because $|j,m\rangle$ is an eigenstate of the operator J_3. Thus, in equation (A.6), we can replace the operator T by the operator

$$T_{\text{eff}} = \Big(\frac{b}{\beta_+}\Big)^{j+m}\sqrt{\frac{(j+m)!(j-m)!}{(2j)!}}\exp\Big(-\frac{2\beta_-}{b+\beta_3}J_+\Big)\exp\Big(\frac{\beta_+}{b}J_-\Big), \tag{A.14}$$

such that

$$|\psi\rangle^j_m = (\widetilde{N}^j_m)^{-1/2}T_{\text{eff}}|j,m\rangle, \tag{A.15}$$

where $\widetilde{N}^j_m$ are new normalization constants. Redefining the summation indices, we get

$$\begin{aligned} |\psi\rangle^j_m \quad &= \quad (\widetilde{N}^j_m)^{-1/2}\sum_{u=-j}^{j}\sqrt{\frac{(j+u)!(j-u)!}{(2j)!}}\Big(\frac{b}{\beta_+}\Big)^{j+u} \\ &\quad \times\frac{(j+m)!}{(j-u)!}\sum_{n=0}^{j+u}(-1)^n\frac{(j-u+n)!}{n!(m-u+n)!(j+u-n)!}\Big(\frac{(1-\beta_3/b)}{2}\Big)^n|j,u\rangle. \end{aligned} \tag{A.16}$$

We also have an expression in terms of the Jacobi polynomials (see [18]):

$$|\psi\rangle^j_m = (\widetilde{N}^j_m)^{-1/2}\sum_{u=-j}^{j}\sqrt{\frac{(j+u)!(j-u)!}{(2j)!}}\Big(\frac{b}{\beta_+}\Big)^{j+u}P^{-u+m,-u-m}_{j+u}\Big(\frac{\beta_3}{b}\Big)|j,u\rangle, \tag{A.17}$$

which is the result obtained by Brif [8].

For the special case where $\beta_+ = 0$, $\beta_3 \neq 0$ so that, in connection with (A.4), we have $b = \beta_3$, we find the operator

$$T_{\text{eff}} = \exp\Big(-\frac{\beta_-}{\beta_3}J_+\Big). \tag{A.18}$$

The eigenstates are

$$|\psi\rangle^j_m = (C^j_m)^{-1/2}\sum_{u=m}^{j}\sqrt{\frac{(j+u)!}{(j-u)!}}\frac{1}{(u-m)!}\Big(-\frac{\beta_-}{\beta_3}\Big)^{u-m}|j,u\rangle, \tag{A.19}$$

and become the standard CS of SU(2) [9] when $m = -j$.

For the special case where $\beta_- = 0$, $\beta_3 \neq 0$, we have similar results. Indeed, the new operator T_{eff} is

$$T_{\text{eff}} = \exp\left(\frac{\beta_+}{\beta_3} J_-\right) \tag{A.20}$$

and the eigenstates write

$$|\psi\rangle^j_m = (C^j_m)^{-1/2} \sum_{u=-j}^{m} \sqrt{\frac{(j-u)!}{(j+u)!}} \frac{1}{(m-u)!} \left(\frac{\beta_+}{\beta_3}\right)^{u-m} |j,u\rangle, \tag{A.21}$$

which become the standard CS of SU(2) [9] when $m = j$.

Now for the case $\beta_+ = 0$ and $\beta_3 = 0$ ($\beta_- = 0$ and $\beta_3 = 0$), the only normalizable solution is $|j,-j\rangle$ $\big(|j,j\rangle\big)$. For $\beta_+ = \beta_- = 0$ and $\beta_3 \neq 0$, the AES are evidently the pure states $|j,m\rangle$.

Finally, the degenerate case $b = 0$ leads to the solution $|\psi\rangle^j_{-j} = (C^j_{-j})^{-1/2} T_{\text{eff}} |j,-j\rangle$ with $T_{\text{eff}} = \exp(-2(\beta_-/\beta_3)J_+)$, that is the standard CS of SU(2).

The mean value of J_3 in the states (A.17) has already been calculated by Brif [8]. We have

$$\langle J_3\rangle^j_m = \frac{jY + m(S_+ - S_-)}{S_+S_-} - \frac{(j+|m|)Yt}{S_+^2 S_-^2}\Omega, \tag{A.22}$$

where

$$S_\pm = 1 + \left|\frac{2\beta_-}{\beta_3 \mp b}\right|^2, \quad t = \left|\frac{b}{\beta_+}\right|^2, \quad Y = S_+S_- - S_+ - S_- \tag{A.23}$$

and

$$\Omega = \frac{P^{(-2j,1)}_{j-|m|-1}(1-(2t/S_+S_-))}{P^{(-2j-1,0)}_{j-|m|}(1-(2t/S_+S_-))}, \quad \text{if } |m| < j; \qquad \Omega = 0, \quad \text{if } |m| = j. \tag{A.24}$$

Appendix B Resolution of a first order system of differential equations

Let us recall that a realization [9] of the Fock space $\mathcal{F}_b = \big\{|n\rangle, n = 0,1,2,\ldots\big\}$ of energy eigenstates of the harmonic oscillator as a space $\mathcal{X}$ of analytic functions $f(\zeta)$ is obtained by expanding this function in the basis of analytic functions $\{\varphi_n(\zeta) = \zeta^n/\sqrt{n!}, n = 0,1,2,\ldots\}$, that is

$$f(\zeta) = \sum_{n=0}^{\infty} c_n \varphi_n(\zeta) = \sum_{n=0}^{\infty} c_n \frac{\zeta^n}{\sqrt{n!}}, \quad \zeta \in \mathbb{C}. \tag{B.1}$$

The scalar product is

$$(f_1, f_2) = \int_C \bar{f}_1(\zeta) f_2(\zeta) e^{-|\zeta|^2} \frac{d\zeta\, d\bar{\zeta}}{2\pi i}, \quad \forall f_1, f_2 \in \chi, \tag{B.2}$$

the integral being extended to the complex plane. The action of the creation $a^\dagger$ and annihilation a operators on the $\mathcal{X}$ space is then given by

$$a^\dagger \equiv \zeta, \quad a \equiv \frac{d}{d\zeta}. \tag{B.3}$$

The eigenvalues equation (2.26) thus becomes a first order differential equation

$$\frac{1}{\sqrt{2}}\left((1+\lambda)\frac{d}{d\zeta} + (1-\lambda)\zeta\right) f(\zeta) = \beta f(\zeta), \tag{B.4}$$

for which normalized solutions are obtained for $\lambda \neq -1$. The general solution of (B.4) is

$$f(\zeta) = f(0) \exp\left(\frac{2\sqrt{2}\beta\zeta - (1-\lambda)\zeta^2}{2(1+\lambda)}\right). \tag{B.5}$$

With respect to the scalar product (B.2), the normalization constant $f(0)$ is computed by imposing

$$\int_C \big|f(\zeta)\big|^2 e^{-|\zeta|^2} \frac{d\zeta\, d\bar{\zeta}}{2\pi i} = 1, \tag{B.6}$$

and we find the normalized solution of (B.4) as

$$f_(\zeta) = \left(1-|\eta_1|^2\right)^{1/4} \exp\left(-\frac{1}{2}\left[\frac{|\eta_2|^2 - \mathrm{Re}\,(\bar{\eta}_1\eta_2^2)}{1-|\eta_1|^2}\right]\right) \exp\left(\eta_2\zeta - \frac{\eta_1}{2}\zeta^2\right), \tag{B.7}$$

with

$$\eta_1 = \frac{(1-\lambda)}{(1+\lambda)} = \delta e^{i\phi} \quad \text{and} \quad \eta_2 = \frac{\sqrt{2}\beta}{(1+\lambda)} = \frac{\beta}{\sqrt{2}}(1+\delta e^{i\phi}). \tag{B.8}$$

This corresponds to the states (2.33) after normalization.

Now we are concerned with the algebra eigenstates satisfying the equation (3.10) in the Fock space $\mathcal{F} \equiv$ (3.4). A realisation of $\mathcal{F}$ can be easily given from the preceding considerations and the expression (3.12) of a state $|\psi\rangle$ for a fixed j. Indeed, we have

$$\psi_m^j(\zeta) = \langle \zeta; j, m|\psi\rangle \tag{B.9}$$

and the eigenvalue equation (3.10) then becomes a system of first order differential equations

$$\left(\alpha_- \frac{d}{d\zeta} + \alpha_+\zeta + \alpha_3\right)\psi_m^j(\zeta) \quad + \quad \left[\beta_- \sqrt{(j-m+1)(j+m)}\psi_{m-1}^j(\zeta) \right.$$
$$\left. +\beta_+\sqrt{(j+m+1)(j-m)}\psi_{m+1}^j(\zeta) + \beta_3 m \psi_m^j(\zeta)\right] = \beta\psi_m^j(\zeta), \tag{B.10}$$

where j is fixed but m takes the values $-j,\ldots,j$. Let us now solve this system by first introducing the differential operator

$$L = \alpha_- \frac{d}{d\zeta} + \alpha_+\zeta + \alpha_3 - \beta \tag{B.11}$$

and, second, defining the vector

$$\Psi = \begin{pmatrix} \psi_{-j}^j \\ \psi_{-j+1}^j \\ \vdots \\ \psi_{j-1}^j \\ \psi_j^j \end{pmatrix}. \tag{B.12}$$

The system (B.10) thus becomes a matrix differential system

$$L\Psi = -A\Psi, \tag{B.13}$$

with A a $(2j+1)\times(2j+1)$ matrix given by

$$A = \begin{pmatrix} -j\beta_3 & \sqrt{2j}\beta_+ & 0 & 0 & \ldots & 0 \\ \sqrt{2j}\beta_- & (-j+1)\beta_3 & \sqrt{(2j-1)2}\beta_+ & 0 & \ldots & 0 \\ 0 & \sqrt{(2j-1)2}\beta_- & (-j+2)\beta_3 & \sqrt{(2j-2)3}\beta_+ & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ldots & \vdots \\ 0 & 0 & \sqrt{(2j-2)3}\beta_- & (j-2)\beta_3 & \sqrt{(2j-1)2}\beta_+ & 0 \\ 0 & 0 & 0 & \sqrt{(2j-1)2}\beta_- & (j-1)\beta_3 & \sqrt{2j}\beta_+ \\ 0 & 0 & 0 & 0 & \sqrt{2j}\beta_- & j\beta_3 \end{pmatrix}. \tag{B.14}$$

If we can find a nonsingular matrix S that diagonalizes A on the form $D = S^{-1}AS$ where

$$D = \mathrm{diag}(\lambda_{-\mathrm{j}}^{\mathrm{j}}, \lambda_{-\mathrm{j}+1}^{\mathrm{j}}, \ldots, \lambda_{\mathrm{j}}^{\mathrm{j}}), \tag{B.15}$$

the system (B.13) will reduce to

$$L\tilde{\Psi} = -D\tilde{\Psi}, \quad \tilde{\Psi} = S^{-1}\Psi. \tag{B.16}$$

Thus, for $\alpha_- \neq 0$, the direct integration of (B.16) will lead to

$$\tilde{\psi}_m^j = \tilde{\psi}_m^j(0)\exp\left(\frac{\beta - \alpha_3 - \lambda_m^j}{\alpha_-}\zeta - \frac{\alpha_+}{2\alpha_-}\zeta^2\right) \tag{B.17}$$

and the general solution Ψ will be obtained as

$$\begin{pmatrix} \psi^j_{-j} \\ \psi^j_{-j+1} \\ \vdots \\ \psi^j_{j-1} \\ \psi^j_j \end{pmatrix} = S \begin{pmatrix} \tilde{\psi}^j_{-j} \\ \tilde{\psi}^j_{-j+1} \\ \vdots \\ \tilde{\psi}^j_{j-1} \\ \tilde{\psi}^j_j \end{pmatrix} = \sum_{m=-j}^{j} \tilde{\psi}^j_m(0) \exp\left(\frac{\beta - \alpha_3 - \lambda^j_m}{\alpha_-}\zeta - \frac{\alpha_+}{2\alpha_-}\zeta^2\right) \begin{pmatrix} S_{-j,m} \\ S_{-j+1,m} \\ \vdots \\ S_{j-1,m} \\ S_{j,m} \end{pmatrix}, \tag{B.18}$$

where S is assumed to be on the form:

$$S = \begin{pmatrix} S_{-j,-j} & S_{-j,-j+1} & \dots & S_{-j,j-1} & S_{-j,j} \\ S_{-j+1,-j} & S_{-j+1,-j+1} & \dots & S_{-j+1,j-1} & S_{-j+1,j} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ S_{j-1,-j} & S_{j-1,-j+1} & \dots & S_{j-1,j-1} & S_{j-1,j} \\ S_{j,-j} & S_{j,-j+1} & \dots & S_{j,j-1} & S_{j,j} \end{pmatrix}. \tag{B.19}$$

Computing the eigenvalues of A, we find that we have to distinguish two cases, i.e. the one with $b = \sqrt{4\beta_+\beta_- + \beta_3^2} \neq 0$ and the one with $b = 0$. For the first case $b \neq 0$, all eigenvalues are different and given by

$$\lambda^j_m = mb, \quad m = -j, \dots, j. \tag{B.20}$$

The system is diagonalizable and the general solution is given by (B.18) with

$$S_{u,m} = \sqrt{\frac{(j+u)!(j-u)!}{(2j)!}} \left(\frac{b}{\beta_+}\right)^{j+u} P^{-u+m,-u-m}_{j+u}\left(\frac{\beta_3}{b}\right), \quad u = -j, \dots, j, \tag{B.21}$$

when $\beta_- \neq 0$, $\beta_+ \neq 0$ and $\beta_3 \neq 0$,

$$S_{u,m} = \sqrt{\frac{(j-u)!}{(j+u)!}} \frac{1}{(m-u)!} \left(\frac{\beta_+}{\beta_3}\right)^{u-m}, \quad -j \leq u \leq m, \qquad S_{u,m} = 0, \quad m < u \leq j, \tag{B.22}$$

when $\beta_- = 0$, $\beta_+ \neq 0$ and $\beta_3 \neq 0$ and

$$S_{u,m} = \sqrt{\frac{(j+u)!}{(j-u)!}} \frac{1}{(u-m)!} \left(-\frac{\beta_-}{\beta_3}\right)^{u-m}, \quad m \leq u \leq j, \qquad S_{u,m} = 0, \quad -j \leq u < m, \tag{B.23}$$

when $\beta_- \neq 0$, $\beta_+ = 0$ and $\beta_3 \neq 0$.

In the Fock space representation, the solutions (B.18) with (B.21), (B.22) and (B.23) correspond, apart from an superfluous change of notation, exactly to the states (3.17) with T_{eff} given by (A.14), (A.18) and (A.20) respectively.

For the second case $b = 0$, the matrix A can not be diagonalized. We could use the Jordan form or start from the differential equation system again and include this condition. Taking the second way, we can express the $\psi^j_m(\zeta)$ components on the form

$$\psi^j_m(\zeta) = \exp\left[-\frac{\alpha_+}{2\alpha_-}\zeta^2 + \frac{(\beta - \alpha_3 - m\beta_3)}{\alpha_-}\zeta\right] \tilde{\psi}^j_m(\zeta), \tag{B.24}$$

and insert these in equation (B.10). We get to the following system:

$$\begin{aligned} \alpha_- \frac{d}{d\zeta}\tilde{\psi}^j_m(\zeta) + \beta_- \sqrt{(j-m+1)(j+m)} e^{\beta_3\zeta/\alpha_-} \tilde{\psi}^j_{m-1}(\zeta) \\ + \beta_+ \sqrt{(j+m+1)(j-m)} e^{-\beta_3\zeta/\alpha_-} \tilde{\psi}^j_{m+1}(\zeta) = 0, \end{aligned} \tag{B.25}$$

when $m = -j, \dots, j$. By handling these equations suitably we can, for example, obtain an ordinary differential equation of the $2j+1$ order for $\tilde{\psi}^j_{-j}(\zeta)$, namely:

$$\left[\prod_{-j}^{j}\left(\frac{d}{d\zeta} - \mu^j_m\right)\right] \tilde{\psi}^j_{-j}(\zeta) = 0, \tag{B.26}$$

where

$$\mu_m^j = -j\frac{\beta_3}{\alpha_-} + m\frac{b}{\alpha_-}. \tag{B.27}$$

When $b = 0$, we have $2j+1$ equal roots. This means that the solutions for $\tilde{\psi}_{-j}^j(\zeta)$ take the form:

$$\tilde{\psi}_{-j}^j(\zeta) = \exp\left(\frac{-j\beta_3\zeta}{\alpha_-}\right)\sum_{q=0}^{2j} A_q\zeta^q. \tag{B.28}$$

Then, we can insert (B.28) in (B.25) and thus obtain, in an iterative way, all solutions $\tilde{\psi}_m^j(\zeta)$ and thereafter, using (B.24), all solutions $\psi_m^j(\zeta)$.

For example, in the case $\beta_+ = \beta_3 = 0$ and $\beta_- \neq 0$, we have

$$\tilde{\psi}_{-j}^j(\zeta) = \psi_{-j}^j(0), \tag{B.29}$$

i.e. a constant and, consequently, by integrating one by one the equations of the system (B.25), we obtain

$$\tilde{\psi}_m^j(\zeta) = \sum_{k=0}^{j+m}\left(-\frac{\beta_-}{\alpha_-}\right)^k\frac{\zeta^k}{k!}\sqrt{\frac{(j+m)!(j-m+k)!}{(j-m)!(j+m-k)!}}\psi_{m-k}^j(0), \tag{B.30}$$

when $m = -j, \ldots, j$. The general solution (B.12) is then given by

$$\Psi = \exp\left[-\frac{\alpha_+}{2\alpha_-}\zeta^2 + \frac{(\beta-\alpha_3)}{\alpha_-}\zeta\right] \times \sum_{m=-j}^{j}\psi_m^j(0)\left[\sum_{k=0}^{j-m}\frac{(-1)^k}{k!}\sqrt{\frac{(j-m)!(j+m+k)!}{(j-m-k)!}}\left(\frac{\beta_-}{\alpha_-}\right)^k\zeta^k\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}\right], \tag{B.31}$$

where, in each sum, the 1 in the vector column is placed in the $(j+m+k+1)$ row. We thus obtain the $(2j+1)$ independent solutions of the system of differential equations.

In the Fock space representation, we can show that the independent solutions given by equation (B.31) correspond, apart from a superfluous change of notation, to the states (3.18). In the case $\beta_- = \beta_3 = 0$ with $\beta_+ \neq 0$, following a similar procedure, one finds the expression (3.19).

Finally, when $\beta_+,\ \beta_-,\ \beta_3 \neq 0$, by inserting (B.28) in (B.25) and ordering the independent solutions with respect to the arbitrary constants A_q, one finds:

$$\Psi(\zeta) = \exp\left[-\frac{\alpha_+}{2\alpha_-}\zeta^2 + \frac{(\beta-\alpha_3)}{\alpha_-}\zeta\right] \times \sum_{q=0}^{2j} A_q\left[\sum_{k=0}^{q}(-1)^k\binom{q}{k}\frac{(2j-k)!}{(2j)!}\zeta^{q-k}\left(\frac{\alpha_-}{\beta_+}\right)^k\left[\frac{d^k}{d\vartheta^k}\sum_{r=0}^{2j}\sqrt{\frac{(2j)!}{(2j-r)!r!}}\vartheta^r\right]\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}\right], \tag{B.32}$$

where, $\vartheta = \beta_3/2\beta_+ = -2\beta_-/\beta_3$ and in each sum, the 1 in the vector column is placed in the $r+1$ row. In the Fock space representation, these solutions, with a slight change of notation, correspond to Eq. (3.20).

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Figure 1: Graphs of the dispersions $(\Delta x)^2$, $(\Delta p)^2$ and the Δ factor as functions of δ for $\phi = \pi/6$.

Figure 2: Graphs of the dispersions $(\Delta x)^2$, $(\Delta p)^2$ and the Δ factor as functions of ϕ for $\delta = 0.5$.

Figure 3: Graphs of the dispersions $\left((\Delta J_1)^2\right)_\pm$, $\left((\Delta J_2)^2\right)_\pm$ and the $\Delta_\pm$ factor as functions of δ for $\phi = \pi/6$ and $j = 1/2$.

Figure 4: Graphs of the dispersions $\left((\Delta J_1)^2\right)_\pm$, $\left((\Delta J_2)^2\right)_\pm$ and the $\Delta_\pm$ factor as functions of ϕ for $\delta = 0.5$ and $j = 1/2$.

Figure 5: Graphs of the dispersions $(\Delta X)^2$, $(\Delta P)^2$ and the factor Δ as functions of $x \equiv \delta$ for $\phi = \pi/6$, $|\tau| = |\mu| = 1$, $j = 1/2$.

Figure 6: Graphs of the dispersions $(\Delta X)^2$, $(\Delta P)^2$ and the factor Δ as functions of $x \equiv \phi$, $\delta = 0.5$, $|\tau| = |\mu| = 1$, $j = 1/2$.

Figure 7: Graphs of the dispersions $\left((\Delta\mathcal{H})^2\right)_{\pm} \equiv (4.37)$ as functions of $\beta > 0$ for $|x| = 0, 1, 2, 4$.

Figure 8: Graphs of the dispersions $\left((\Delta\mathcal{H}_0)^2\right)_\pm$ as given by (4.49) and (4.50) as functions of $|x|$.

Figure 9: Graphs of the dispersions $\left((\Delta\mathcal{H}_0)^2\right)_{\pm} \equiv$ (4.76) as functions of $\beta > 0$, $\theta = \pi$ and $\rho = 1, 2, 4$.

Figure 10: Graphs of the dispersions $\left((\Delta\mathcal{H}_0)^2\right)_\pm \equiv$(4.76) as functions of $\beta > 0$ for $\rho = 1$, $\theta = 5\pi/8, 3\pi/4, 7\pi/8$ and π.