Weyl groups of current algebras and Jacobi groups[*]

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CRM-2845

June 2002

[*]Work supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Fonds FCAR du Québec.

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Abstract

We construct an infinite dimensional Lie algebra of currents on elliptic curves and we show that there exists a suitable Abelian subalgebra (Cartan subalgebra) whose “Weyl” group is isomorphic to the one of the generalized Jacobi groups associated to the complex simple Lie algebras.

1 Introduction

The purpose of this paper is to connect the notion of Jacobi groups with that of Weyl groups for infinite dimensional Lie algebras; the result was only announced in [Be02] (see also [Be99]). Jacobi groups arise in the study of period mappings of simple elliptic singularities e.g. the $\widetilde{E}_6$ one [Lo76, Lo80, Sa74, Sa85, Sa90, Sa98]

$$F(x,y,z) = x^3 + y^3 + z^3 + \lambda xyz \ .$$

They arise in precisely the same way as the simply laced Coxeter groups of $A-D-E$ type arise from the period mapping in the Milnor fibers of the unfolding of *simple* hypersurface singularities [Ar]. Historically it was an unexpected finding that these discrete groups coincided with the totally independent definition in terms of the classification of simple Lie algebras.
In the case of the group of type $\widetilde{E}_6$ hinted at above and its generalization to arbitrary Dynkin diagram [Wi92] (that baptized them "Jacobi groups" following [EZ]) it seems that a group-theoretical interpretation along similar lines to those of Weyl groups is still missing in the literature. In particular there is so far no Lie algebra which gives rise to Jacobi groups in the same way as finite reflection groups arise in the study of simple Lie algebras: the present paper is aimed at filling this gap.

We should also mention that the invariant theory of Jacobi groups leads to the study of Jacobi forms, which constitute a bigraded polynomial algebra over the graded ring of modular forms [Wi92, Be00]. Such invariants have applications in the context of Frobenius manifolds and Chern–Simons–Witten theory over elliptic curves [Be99, Be00].

The paper is structured as follows: in Section 2 we recall the definition of Jacobi groups given abstractly in [Wi92, Lo80] and define their representation on a suitable cone Ω.
In Section 3 we will construct an infinite–dimensional Lie algebra $\mathfrak{G}$ which possesses a "maximal" Abelian subalgebra $\mathfrak{H}$. We will describe the automorphism group $Aut(\mathfrak{G})$ and show that the subgroup $\mathfrak{W} \subset Aut(\mathfrak{G})$ leaving $\mathfrak{H}$ invariant (modulo its centralizer) is isomorphic to the Jacobi group defined independently in Sect. 2. Furthermore, since Jacobi groups are extensions of Complex Coxeter Crystallographic groups [BS86], we provide at the same time a connection of the latter to Weyl groups of infinite dimensional Lie algebras.

2 Jacobi Groups

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of rank n: this is classified in terms of its *Dynkin Diagram* and labeled accordingly by the names $A_n,\ n \geq 1$, $B_n,\ n \geq 2$, $C_n,\ n \geq 3$, $D_n,\ n \geq 4$, $E_6,\ E_7,\ E_8,\ G_2,\ F_4$, where the subscript refers to the rank (see e.g. [Bo]).
Let us fix a Cartan subalgebra $\mathfrak{h} \hookrightarrow \mathfrak{g}$ and denote by W its Weyl group, i.e. the stabilizer of $\mathfrak{h}$ modulo its centralizer. Let $Q \hookrightarrow \mathfrak{h}^\vee$ the root lattice; we will consider it also as an Abelian group. Let $\langle , \rangle$ be the Killing form on $\mathfrak{h}$ (and -by abuse of notation- also on $\mathfrak{h}^\vee$) normalized to 2 for short roots, (this implies that $\forall \lambda \in Q,\ \langle \lambda, \lambda \rangle \in 2\mathbb{Z}$); we will think of Q as embedded directly in $\mathfrak{h}$ by means of the isomorphism induced by the Killing form and hence we will refer to Q from now on as the coroot lattice, $Q \hookrightarrow \mathfrak{h}$. We construct the Heisenberg group H_Q, obtained by central extending the Abelian group $Q \times Q$ by $\mathbb{Z}$ with cocycle

$$C((\lambda,\mu),(\lambda',\mu')) = \langle \mu, \lambda' \rangle - \langle \mu', \lambda \rangle \ . \tag{2-1}$$

The product rule for the set $Q \times Q \times \mathbb{Z}$ is given by

$$\begin{aligned} &\forall (\lambda,\mu,k),\ (\lambda',\mu',k') \in Q \times Q \times \mathbb{Z} \\ &(\lambda,\mu,k)\cdot(\lambda',\mu',k') := (\lambda+\lambda', \mu+\mu', k+k'+\langle \mu,\lambda' \rangle) \ . \end{aligned} \tag{2-2}$$

We remark here and for future convenience that this is not the only group multiplication rule that corresponds to the cocycle (2-1), but that different choices give rise to isomorphic groups.
Since the Killing form $\langle\ ,\ \rangle$ is Weyl invariant and the Weyl group W acts on Q we can construct the semidirect product

$$\mathcal{W} := W \ltimes H_Q \ .$$

As usual the semidirect product is specified by the following multiplication rule on the set $W \times Q \times Q \times Z$

$$\begin{aligned} &\mathcal{W} := W \ltimes H_Q, \qquad s.t.\ \ \forall \mathrm{w},\mathrm{w}' \in W,\ \ \mathrm{t} = (\lambda,\mu,k), \mathrm{t}' = (\lambda',\mu',k') \in H_Q \\ &(\mathrm{w},\mathrm{t})\cdot(\mathrm{w}',\mathrm{t}') := (\mathrm{w}\mathrm{w}', \mathrm{w}\cdot\lambda'+\lambda, \mathrm{w}\cdot\mu'+\mu, k+k'+\langle \mu,\lambda' \rangle) \ . \end{aligned}$$

Note that we have the exact sequence

$$1 \mapsto \mathbb{Z} \mapsto \mathcal{W} \mapsto W \ltimes (Q \times Q) \mapsto 1 \ ,$$

and the group $W \rtimes (Q \times Q)$ is known as Complex Coxeter Crystallographic group [BS86].
The group $\mathcal{W}$ itself is well known and its invariants are theta functions in the sense of [KP84].
We now can give the definition of Jacobi group [EZ, Lo80, Wi92]

Definition 2.1 *The* **Jacobi group** $\mathbb{J}(\mathfrak{g})$ *is the semidirect product* $SL(2,\mathbb{Z}) \rtimes \mathcal{W}$. *The action of* $SL(2,\mathbb{Z})$ *on the group* $W \rtimes H_Q$ *is defined by*

$$\begin{aligned} & Ad_\gamma(\mathrm{w}) := \mathrm{w} \\ & Ad_\gamma(\mathrm{t}) := \left(a\lambda - b\mu, -c\lambda + d\mu, k + \frac{ac}{2}\langle \lambda, \lambda \rangle - bc\langle \lambda, \mu \rangle + \frac{bd}{2}\langle \mu, \mu \rangle \right) \ , \end{aligned} \tag{2-3}$$

for $\mathrm{w} \in W$, $\mathrm{t} = (\lambda, \mu, k) \in H_Q$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \in SL(2,\mathbb{Z})$. *The multiplication rule is defined by (* $\forall (\gamma, \mathrm{w}, \mathrm{t}),\ (\gamma', \mathrm{w}', \mathrm{t}') \in SL(2,\mathbb{Z}) \times W \times H_Q$*)*

$$(\gamma, \mathrm{w}, \mathrm{t}) \cdot (\gamma', \mathrm{w}', \mathrm{t}') := (\gamma \cdot \gamma', \mathrm{w} \cdot \mathrm{w}, \mathrm{t} \cdot Ad_\gamma(\mathrm{w}\mathrm{t}')) \ .$$

2.1 Faithful representation of $\mathbb{J}(\mathfrak{g})$

Let us consider the cone $\Omega := \mathbb{C} \oplus \mathfrak{h} \oplus \mathcal{H} \ni (u, \mathrm{x}, \tau)$, where $\mathfrak{h}$ is the complex Cartan subalgebra of $\mathfrak{g}$ and $\mathcal{H}$ denotes the Poincaré upper half plane.
In the literature it is often called the *Tits cone* [Lo80] and it is the union of all the images under $\mathcal{W}$ of the closure of a fundamental chamber $\mathcal{C}$; therefore it is an invariant cone for the action of the group $\mathcal{W}$ (and of $\mathbb{J}$ as well).

Let us consider $\tau \in \mathcal{H}$; we have an embedding of $Q \times Q$ in $\mathfrak{h}$ as the complex crystallographic lattice $Q + \tau Q$ and we can define the action as in the following

Proposition 2.1 The Jacobi group $\mathbb{J}$ is represented on Ω by definition of the action of $w \in W$, $\mathrm{t} = (\lambda, \mu, k) \in H_Q$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ given by

$$\begin{array}{cl} \text{Weyl} & \mathrm{w}(u, \mathrm{x}, \tau) := (u, \mathrm{w}\mathrm{x}, \tau) \\ H_Q & \mathrm{t}(u, \mathrm{x}, \tau) := \left(u + k - \langle \mathrm{x}, \mu \rangle - \frac{\tau}{2}\langle \mu, \mu \rangle, \mathrm{x} + \lambda + \tau\mu, \tau \right) \\ SL(2,\mathbb{Z}) & \gamma(u, \mathrm{x}, \tau) := \left(u + \frac{c\langle \mathrm{x}, \mathrm{x} \rangle}{2(c\tau + d)}, \frac{\mathrm{x}}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) \ . \end{array}$$

The proof is straightforward although rather long and it is left to the reader.

Remark 2.1 The action of the Jacobi group is non linear, but it could be made such by realizing it as a discrete subgroup of the conformal group: in fact its action is a conformal action for the metric $-2\mathrm{d}u\mathrm{d}\tau + \langle \mathrm{d}\mathrm{x}, \mathrm{d}\mathrm{x} \rangle$.

3 Jacobi group as Weyl group of a current algebra

In this section we realize Jacobi groups as the group of automorphisms preserving the Cartan subalgebra of a suitable Lie algebra.
The algebra which we are going to define is an extension of a variation of the well–known current algebra considered in [EF94] (but see also [PrSe]).

Let us consider the elliptic curve $E_\tau \simeq \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ and the Lie algebra of smooth $\mathfrak{g}$–valued functions on E_τ

$$T(\mathfrak{g}) := \mathcal{C}^\infty(E_\tau, \mathfrak{g}) \ : \qquad [\xi, \eta](z, \overline{z}) := [\xi(z, \overline{z}), \eta(z, \overline{z})] \ , \quad \forall \xi, \eta \in T(\mathfrak{g}) \ .$$

Following essentially [EF94] we define a central extension of $T(\mathfrak{g})$ by the central element c with cocycle

$$\Omega(\xi_1, \xi_2) := \int_{E_\tau} \left\langle \frac{\partial \xi_1}{\partial z}, \xi_2 \right\rangle \mathrm{d}z \wedge \mathrm{d}\overline{z} \ . \tag{3-4}$$

This cocycle is closely related to the one in [EF94] except for the fact that –in their language– we are extending by the space of anti–holomorphic differentials. We will denote the new Lie algebra by

$$\widehat{T(\mathfrak{g})} = T(\mathfrak{g}) + \mathbb{C}\mathrm{c} \ , \quad [\xi + v\mathrm{c}, \eta + v\mathrm{c}] := [\xi, \eta] + \Omega(\xi, \eta)\mathrm{c} \ . \tag{3-5}$$

Note that $\widehat{T(\mathfrak{g})}$ depends on the conformal structure of the elliptic curve (i.e., on τ) in the definition of the cocycle Ω. We will denote the algebra with $\widehat{T_\tau(\mathfrak{g})}$ when emphasizing this dependence.

Remark 3.1 Although it is not relevant for the following, we should remark that this construction can be carried over to any Riemann surface Σ of genus g. As it is explained in [[EF94] Thm 2.2] any such central extension is an homotopy retract of the *universal central extension* [PrSe]: this is an extension of $\mathrm{T}(\mathfrak{g})$ by means of the infinite–dimensional space $\mathfrak{a} := \Omega^1(\Sigma)/d\Omega^0(\Sigma)$ of complex–valued one–forms on Σ modulo exact forms, defined by the $\mathfrak{a}$–valued cocycle

$$u(\xi,\eta) := \langle \xi, \mathrm{d}\eta\rangle \bmod \mathrm{d}\Omega^0(\Sigma)\ , \qquad \forall \xi,\eta \in \mathrm{T}(\mathfrak{g})\ .$$

In [EF94] it is also shown that such central–extended Lie algebra integrates to an infinite dimensional Lie group which is the extension of the gauge group of Σ by its Jacobian variety. However the extra construction we are interested in is naturally present only for genus $g = 1$ curves.

Finally we add to $\widehat{T(\mathfrak{g})}$ a suitable (outer) derivation. The resulting algebra $\mathfrak{G}$ will *not* be the Lie algebra of any Lie group but has the property that it supports in a natural way (an extension of) the coadjoint representation of the gauge group $\widehat{\mathrm{T}(G)}$ of local G-gauge transformations. Nevertheless any automorphism of $\widehat{\mathrm{T}(\mathfrak{g})}$ extends to an automorphism of $\mathfrak{G}$; we will realize explicitly this action and see that the subgroup of automorphisms preserving a Cartan subalgebra is isomorphic to the Jacobi group $\mathbb{J}(\mathfrak{g})$ defined in the previous section.

We recall that the gauge group $\mathrm{T}(G)$ acts on the space of connections

$$\mathcal{A} := \left\{ D = w\frac{\partial}{\partial\overline{z}} + \xi \right\} \simeq \mathbb{C}\delta \oplus \mathrm{T}(\mathfrak{g})\ , \quad \delta := \frac{\partial}{\partial\overline{z}}$$

by

$$Ad_g D = w\partial_{\overline{z}} + g\xi g^{-1} - w\partial_{\overline{z}} g g^{-1}\ , \quad g = g(z,\overline{z}) \in \mathcal{C}^\infty(E_\tau, G)\ .$$

Remark 3.2 In [EF94] the vector space $\mathcal{A}$ is naturally realized as the dual of an algebra $\widehat{\mathrm{T}(\mathfrak{g})}^\vee$ which is constructed as ours but using the anti–holomorphic derivative in the definition of the cocycle (3-4); in that case the gauge action is realized as the coadjoint action on the central–extended Lie algebra by means of the following invariant pairing

$$\begin{aligned}
&\mathcal{B}(D,\widehat{\eta}) := wy + \int_\Sigma \mathrm{d}z\wedge \mathrm{d}\overline{z}\langle \xi,\eta\rangle\ ,\\
&D := w\frac{\partial}{\partial\overline{z}} + \xi\ \in \mathcal{A}\ ,\ \ \widehat{\eta} = \eta + y\mathrm{c}^\vee\ \in \widehat{T(\mathfrak{g})}^\vee\\
&[\eta_1 + y_1\mathrm{c}^\vee, \eta_2 + y_2\mathrm{c}^\vee] = [\eta_1,\eta_2] + \int_{E_\tau} \mathrm{d}z\wedge\mathrm{d}\overline{z}\left\langle \frac{\partial\eta_1}{\partial\overline{z}}, \eta_2\right\rangle \mathrm{c}^\vee\ .
\end{aligned}$$

In the case under exam, however, the above pairing is not invariant due to our choice of the cocycle.

Consider now the vector space

$$\mathfrak{G} := \mathbb{C}\delta \oplus \widehat{\mathrm{T}(\mathfrak{g})} = \mathbb{C}\delta \oplus T(\mathfrak{g}) \oplus \mathbb{C}\mathrm{c} = \mathcal{A} \oplus \mathbb{C}\mathrm{c} \tag{3-6}$$

We introduce coordinates on it of the form $(w,\xi,v) \in \mathbb{C}\delta \oplus \mathrm{T}(\mathfrak{g}) \oplus \mathbb{C}\mathrm{c}$ and we naturally identify $\mathfrak{G}$ with $\mathcal{A} + \mathbb{C}\mathrm{c}$. We endow $\mathfrak{G}$ with the structure of Lie algebra by defining the commutation relations (which extend the previous ones) [see [PrSe] for the similar case of loop algebras]

$$\begin{aligned}
\Big[D_1 + v_1\mathrm{c}, D_2 + v_2\mathrm{c}\Big] &:= [\xi_1,\xi_2] + w_1\frac{\partial}{\partial\overline{z}}\xi_2 - w_2\frac{\partial}{\partial\overline{z}}\xi_1 + \Omega(\xi_1,\xi_2)\mathrm{c} =\\
&= [D_1,D_2] + \Omega(\xi_1,\xi_2)\mathrm{c}\ .
\end{aligned}$$

We leave to the reader the straightforward check that the Jacobi identity is satisfied.
The adjoint action can be computed by straightforward calculations adapting those for ordinary Kac–Moody's algebras.

Proposition 3.1 [Compare with Prop. 4.9.4 in [PrSe]] If $g := \exp(\xi)$ belongs to the identity component of the gauge group $\mathrm{T}(G)$, then the adjoint action of g on the extended algebra $\mathfrak{G}$ is given by

$$\begin{aligned}
Ad_g(D + v\mathrm{c}) = w\frac{\partial}{\partial\overline{z}} + Ad_g\eta - w\frac{\partial g}{\partial\overline{z}} g^{-1} +&\\
+\left(v + \int_{E_\tau}\mathrm{d}z\wedge\mathrm{d}\overline{z}\left\langle g^{-1}\frac{\partial g}{\partial z}, \eta\right\rangle - \frac{w}{2}\int_{E_\tau}\mathrm{d}z\wedge\mathrm{d}\overline{z}\left\langle g^{-1}\frac{\partial g}{\partial z}, g^{-1}\frac{\partial g}{\partial\overline{z}}\right\rangle\right)\mathrm{c}\ .&
\end{aligned}$$

Remark 3.3 The extended algebra $\mathfrak{G}$ fails to be integrated to a Lie group because of the presence of the derivation $\partial_{\overline{z}}$ (as for Kac–Moody algebras, [PrSe])

The algebra $\mathfrak{G}$ contains an obvious Cartan subalgebra that can be conveniently parametrized as follows

$$\mathfrak{H} := \mathbb{C}\delta \oplus \mathfrak{h} \oplus \mathbb{C}\mathrm{c} = \left\{ D + v\mathrm{c} = w\frac{\partial}{\partial \overline{z}} + \frac{2i\pi w}{\tau - \overline{\tau}}\mathrm{x} + v\mathrm{c} \right\} ,$$

where $\mathfrak{h}$ is the Abelian subalgebra of constant maps with values in a fixed Cartan subalgebra of $\mathfrak{g}$. In this definition we consider w as coordinate along the derivation $\delta = \frac{\partial}{\partial \overline{z}}$ and v as coordinate along the central element c. The normalization $\frac{2i\pi w}{\tau - \overline{\tau}}$ in front of x is for later convenience.
We now turn to the subgroup of $Aut(\mathfrak{G})$ which leaves $\mathfrak{H}$ invariant (modulo the centralizer).
First of all we prove

Lemma 3.1 *Any automorphism of $\mathfrak{G}$ is the lift of an automorphism of $\widehat{\mathrm{T}(\mathfrak{g})}$.*

Proof.
We must check that any automorphism $F \in Aut\,(\mathfrak{G})$ leaves invariant $\widehat{\mathrm{T}(\mathfrak{g})} \hookrightarrow \mathfrak{G}$. Indeed, if p_1 is the projection onto the Abelian subalgebra spanned by the derivation δ we have, for any two elements $a, b \in \mathfrak{G}$

$$0 = p_1([Fa, Fb]) = p_1 \circ F[a, b] \ .$$

But $\widehat{\mathrm{T}(\mathfrak{g})} = [\mathfrak{G}, \mathfrak{G}]$, hence $p_1 \circ F \equiv 0$; this implies that $F\widehat{\mathrm{T}(\mathfrak{g})} \subseteq \widehat{\mathrm{T}(\mathfrak{g})}$.
Therefore the automorphisms of $\mathfrak{G}$ are a subgroup of the automorphisms of $\widehat{\mathrm{T}(\mathfrak{g})}$; on the other hand any automorphism of $\widehat{\mathrm{T}(\mathfrak{g})}$ can be lifted to $\mathfrak{G}$ by acting trivially on δ. ■

Therefore we pass to considering all automorphisms of $\mathfrak{G}$ which stabilize $\mathfrak{H}$; we know that any automorphism of $\mathfrak{G}$ must be an extension of an automorphism of the algebra $\widehat{\mathrm{T}(\mathfrak{g})}$. We use the following

Proposition 3.2 [[EF94] Proposition 1.2] (i) Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two Riemann surfaces with complex structure: the Lie algebras $\widehat{\mathrm{T}_1(\mathfrak{g})}, \widehat{\mathrm{T}_2(\mathfrak{g})}$ are isomorphic iff $\mathcal{C}_1$ and $\mathcal{C}_2$ are conformally equivalent (i.e. biholomorphically isomorphic).
(ii) Any automorphism f of $\mathrm{T}(\mathfrak{g})$ can be uniquely represented as a composition $f = h \circ \phi_*$ where h is a conjugation by an element of $\mathrm{T}\,(Aut(\mathfrak{g}))$ and ϕ_* is the direct image map induced by a conformal diffeomorphism $\phi : \mathcal{C}_1 \to \mathcal{C}_2$.

In view of Prop. 3.2 and then of Lemma 3.1 any element of $Aut(\mathfrak{G})$ is the composition of an inner automorphism and an automorphism of the underlying complex curve.
To understand the action of the group of automorphisms of the complex curve on $\mathfrak{G}$ it is convenient to introduce a fibration over the Teichmüller space of elliptic curves: hence the base space will be the Poincaré upper half plane $\mathcal{H} := \{\tau,\ \Im(\tau) > 0\}$ while the fibers will be $\mathfrak{G} = \mathfrak{G}_\tau$ which depend on the complex structure because $\widehat{T_\tau(\mathfrak{g})}$ does

$$\mathcal{G} := \begin{array}{c} \mathfrak{G}_\tau \\ \downarrow \\ \mathcal{H} \end{array} .$$

This fibration is acted upon by the group $SL(2, \mathbb{Z})$, namely the group of complex isomorphisms of elliptic curves in the way to be described below. We introduce coordinates on $\mathcal{G}$ $(D + v\mathrm{c}, \tau)$ by adding the modular parameter of the elliptic curve τ. As far as the action of the gauge group $\widehat{\mathrm{T}(G)}$ on $\mathfrak{G}$ is concerned we can regard τ as a parameter.

Now we are to analyze the automorphisms fixing our Cartan subalgebra, namely the Weyl group of $\mathfrak{G}$. The obvious ones are the element of the Weyl group of $\mathfrak{h}$ considered as constant elements of the gauge group $\widehat{\mathrm{T}(G)}$. The action of such an element w is given by

$$\mathrm{w}\left(w\frac{\partial}{\partial \overline{z}} + \frac{2i\pi w}{\tau - \overline{\tau}}\mathrm{x} + v\mathrm{c}, \tau \right) = \left(w\frac{\partial}{\partial \overline{z}} + \frac{2i\pi w}{\tau - \overline{\tau}}\mathrm{w} \cdot \mathrm{x} + v\mathrm{c}, \tau \right) .$$

Along with these we should also consider the action of the outer automorphisms of $\mathfrak{g}$, namely those corresponding to the symmetries of the Dynkin diagram.
Another class of automorphisms are those deriving by non constant gauge transformations. We have

Lemma 3.2 *Any non–constant gauge transformation $F(z, \overline{z}) \in \mathrm{T}(G)$ leaving invariant $\mathfrak{H}$ is of the form*

$$F(z, \overline{z}) := \mathrm{w} \circ \exp\left[-2i\pi\frac{\rho\overline{z} - \overline{\rho}z}{\tau - \overline{\tau}} \right] , \tag{3-7}$$

with w *an element of the Weyl group of* $\mathfrak{h}$ *and* $\rho = \lambda + \tau\mu \in Q \oplus \tau Q \hookrightarrow \mathfrak{h}$.

Remark 3.4 The embedding of the root lattice Q in $\mathfrak{h}$ via our bilinear form $\langle,\rangle$, coincides with the usual *coroot* lattice in $\mathfrak{h}$: this is due to the unconventional normalization of the Killing form $\langle\alpha,\alpha\rangle = 2$ for short roots. This has the effect that $\exp(2i\pi\lambda)$ equals the identity of the group G iff λ belongs to our *root* lattice. Notice furthermore that with this normalization

$$\langle\lambda,\mu\rangle \in 2\mathbb{Z} \quad , \ \forall\ \lambda,\mu \in Q \ .$$

This fact is essential in what follows.

Proof. Any such transformation $F(z,\overline{z})$ must satisfy $Ad_F H + F^{-1}\partial_{\overline{z}} F = K(H) = \text{const} \in \mathfrak{h}$ for any H belonging to $\mathfrak{h}$. The operator $K(H)$ is clearly affine w.r.t. H and its linear part must be an inner automorphism of G preserving $\mathfrak{h}$ because $K(H) \in \mathfrak{h}$, $\forall H \in \mathfrak{h}$. Therefore there exists some w in the stabilizer of $\mathfrak{h}$ in $Aut(\mathfrak{g})$ and a $L \in \mathfrak{h}$ such that

$$K(H) = \mathrm{w}\cdot H \cdot \mathrm{w}^{-1} + L \ .$$

Clearly w is defined up to the centralizer of $\mathfrak{h}$ in $Aut(\mathfrak{g})$ and hence is an element of the Weyl group. Then it follows that F is of the form $\mathrm{w}F_0(z,\overline{z})$ with F_0 an $\exp(\mathfrak{h})$–valued function. Plugging into the equation we have

$$F_0^{-1}\partial_{\overline{z}}F_0 \equiv L \in \mathfrak{h} \ .$$

The conditions of periodicity force $L = -2i\pi(\lambda+\tau\mu)$ with λ and μ belonging to the (co)root lattice and F_0 to be of the form above. ■

Using Prop. 3.1 the action of an element of the form (3-7) with $\mathrm{w} = \mathrm{id}_{\mathfrak{h}}$ and $\rho = \lambda + \tau\mu$ on $\mathfrak{H}$ is computed to be

$$\begin{aligned} &Ad_F\left(w\frac{\partial}{\partial\overline{z}} + \frac{2i\pi w}{\tau-\overline{\tau}}\mathrm{x} + v\mathrm{c}\right) = \\ &= w\frac{\partial}{\partial\overline{z}} + \frac{2i\pi w}{\tau-\overline{\tau}}(\mathrm{x}+\rho) + \left(v - \frac{(2i\pi)^2 w}{\tau-\overline{\tau}}\left(\langle\mathrm{x},\overline{\rho}\rangle + \frac{1}{2}\langle\rho,\overline{\rho}\rangle\right)\right)\mathrm{c} \ . \end{aligned} \qquad (3\text{-}8)$$

Remark 3.5 We note that the group element

$$F(z,\overline{z}) = \exp\left[-2i\pi\frac{\rho\overline{z}-\overline{\rho}z}{\tau-\overline{\tau}}\right] = \exp[\xi(z,\overline{z})]$$

used in the computation of Eq. (3-8) does not belong to the identity component of $T(G)$ so that -rigorously speaking- Prop. 3.1 cannot be applied directly. However it is very simple to compute directly the *Ad* action by exponentiating the *ad* action, given that $\xi(z,\overline{z})$ belongs to $\mathfrak{h}$. Indeed, even if $\xi(z,\overline{z})$ in Eq. (3-8) is not doubly periodic, its *ad* action is well defined since $\forall \mathrm{x} \in \mathfrak{h}$

$$\begin{aligned} &\left[\xi(z,\overline{z}), w\partial_{\overline{z}} + \frac{2i\pi w}{\tau-\overline{\tau}}\mathrm{x} + v\mathrm{c}\right] = \frac{2i\pi w}{\tau-\overline{\tau}}\rho + \frac{(2i\pi)^2 w}{(\tau-\overline{\tau})^2}\int_{E_\tau}\mathrm{d}z\wedge\mathrm{d}\overline{z}\,\langle\overline{\rho},\mathrm{x}\rangle\,\mathrm{c} = \\ &= \frac{2i\pi w}{\tau-\overline{\tau}}\rho - \frac{(2i\pi)^2 w}{\tau-\overline{\tau}}\langle\overline{\rho},\mathrm{x}\rangle\mathrm{c} \\ &\left[\xi(z,\overline{z}), \left[\xi(z,\overline{z}), w\partial_{\overline{z}} + \frac{2i\pi w}{\tau-\overline{\tau}}\mathrm{x} + v\mathrm{c}\right]\right] = \frac{(2i\pi)^2 w}{(\tau-\overline{\tau})^2}\int_{E_\tau}\mathrm{d}z\wedge\mathrm{d}\overline{z}\,\langle\overline{\rho},\rho\rangle\,\mathrm{c} = \\ &= -\frac{(2i\pi)^2 w}{(\tau-\overline{\tau})}\langle\overline{\rho},\rho\rangle\mathrm{c} \\ &ad_\xi{}^n\left(w\partial_{\overline{z}} + \frac{2i\pi w}{\tau-\overline{\tau}}\mathrm{x} + v\mathrm{c}\right) = 0 \ , \quad n > 2 \\ &\exp(ad_\xi)\left(w\partial_{\overline{z}} + \frac{2i\pi w}{\tau-\overline{\tau}}\mathrm{x} + v\mathrm{c}\right) = \text{RHS of Eq. 3-8} \ . \end{aligned}$$

(see [PrSe] for similar considerations in the case of Kac-Moody algebras and loop groups).

Finally we compute the action of the change of complex structure of the torus to an isomorphic one; this amounts to studying the action of the transformations $\tau \mapsto \tau' = \frac{a\tau+b}{c\tau+d}$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$.

We focus on the action of the two generators $\tau \to \tau+1$ and $\tau \to -\frac{1}{\tau}$. It is easy to see that their action is

$$\begin{aligned} \tau' = \tau+1 : \qquad &\left(w\frac{\partial}{\partial\overline{z}} + \frac{2i\pi w}{\tau-\overline{\tau}}\mathrm{x} + v\mathrm{c}, \tau\right) \mapsto \left(w\frac{\partial}{\partial\overline{z}} + \frac{2i\pi w}{\tau'-\overline{\tau}'}\mathrm{x} + v\mathrm{c}, \tau'\right) \\ \tau' = -\frac{1}{\tau} : \qquad &\left(w\frac{\partial}{\partial\overline{z}} + \frac{2i\pi w}{\tau-\overline{\tau}}\mathrm{x} + v\mathrm{c}, \tau\right) \mapsto \left(w\frac{\partial}{\partial\overline{z}'} + \frac{2i\pi w}{\tau'-\overline{\tau}'}\frac{\mathrm{x}}{\tau} + v\mathrm{c}, \tau'\right) \ , \end{aligned}$$

where, in the second line z' is the holomorphic coordinate on the complex torus $\mathbb{C}/\mathbb{Z}+\tau'\mathbb{Z}$ of parameter $\tau'=-\frac{1}{\tau}$. We summarize these formulas in the following

Proposition 3.3 The action of the group of automorphisms of $\mathfrak{G}$ preserving the Cartan subalgebra $\mathfrak{H}=\Big\{(D+v\mathrm{c})=\Big(w\frac{\partial}{\partial \bar z}+\frac{2i\pi w}{\tau-\overline{\tau}}\mathrm{x}+$ $\mathbb{C}\delta\oplus\mathfrak{h}\oplus\mathbb{C}\mathrm{c}$ is generated by the following transformations

$$(w,\mathrm{x},v,\tau)\mapsto(w,\mathrm{w}\cdot\mathrm{x},v,\tau) \tag{3-9a}$$

$$(w,\mathrm{x},v,\tau)\mapsto\left(w,\mathrm{x}-\lambda-\tau\mu,v-\frac{(2i\pi)^2w}{\tau-\overline{\tau}}\left(\langle\mathrm{x},\lambda+\overline{\tau}\mu\rangle+\frac{1}{2}\langle\lambda+\tau\mu,\lambda+\overline{\tau}\mu\rangle\right),\tau\right) \tag{3-9b}$$

$$(w,\mathrm{x},v,\tau)\mapsto\left(w,\frac{\mathrm{x}}{c\tau+d},v,\frac{a\tau+b}{c\tau+d}\right)\ . \tag{3-9c}$$

The abstract group underlying this action is exactly the Jacobi group $\mathbb{J}(\mathfrak{g})$: the action is clearly different from the one considered in Section 2.1 but can be compared by means of a (nonlinear) change of coordinates. Indeed we introduce the coordinate

$$u:=-\frac{v}{(2i\pi)^2w}-\frac{\langle\mathrm{x},\mathrm{x}\rangle}{2(\tau-\overline{\tau})}\ .$$

We can check the action of the transformations (3-9) for the "coordinates" (w,x,u,τ); since the level w is invariant we disregard it completely in what follows.
The action of the Weyl group clearly gives

$$(\mathrm{x},u,\tau)\mapsto(\mathrm{w}\cdot\mathrm{x},u,\tau)\ ,\ \ \mathrm{w}\in W\ .$$

Let us check the action of the complex crystallographic lattice $Q+\tau Q$ on the new coordinate u. From Eq. (3-9b) we get

$$\begin{aligned}u'&=-\frac{v'}{(2i\pi)^2w}-\frac{\langle\mathrm{x}',\mathrm{x}'\rangle}{2(\tau-\overline{\tau})}= &(3\text{-}10)\\ &=-\frac{v}{(2i\pi)^2w}+\frac{1}{\tau-\overline{\tau}}\left(\langle\mathrm{x},\overline{\rho}\rangle+\frac{1}{2}\langle\rho,\overline{\rho}\rangle\right)-\frac{\langle\mathrm{x}+\rho,\mathrm{x}+\rho\rangle}{2(\tau-\overline{\tau})}=\\ &=-\frac{v}{(2i\pi)^2w}+\frac{1}{\tau-\overline{\tau}}\left(\langle\mathrm{x},\overline{\rho}\rangle+\frac{1}{2}\langle\rho,\overline{\rho}\rangle\right)-\frac{\langle\mathrm{x}+\rho,\mathrm{x}+\rho\rangle}{2(\tau-\overline{\tau})}=\\ &=-\frac{v}{(2i\pi)^2w}-\frac{\langle\mathrm{x},\mathrm{x}\rangle}{2(\tau-\overline{\tau})}-\frac{1}{\tau-\overline{\tau}}\left(\langle\mathrm{x},\overline{\rho}-\rho\rangle+\frac{1}{2}\langle\rho,\overline{\rho}-\rho\rangle\right)=\\ &=u-\langle\mathrm{x},\mu\rangle-\frac{\tau}{2}\langle\mu,\mu\rangle-\frac{1}{2}\langle\lambda,\mu\rangle &(3\text{-}11)\end{aligned}$$

Finally let us check the action of the modular group: the translation $\tau\mapsto\tau+1$ leaves u invariant and the inversion $\tau'=-\frac{1}{\tau}$ (corresponding to $\begin{pmatrix}0&-1\\1&0\end{pmatrix}\in SL(2,\mathbb{Z})$) gives

$$\begin{aligned}u'&=\frac{-v'}{(2i\pi)^2w}-\frac{\langle\mathrm{x}',\mathrm{x}'\rangle}{2(\tau'-\overline{\tau}')}=\frac{-v}{(2i\pi)^2w}-\frac{\overline{\tau}\langle\mathrm{x},\mathrm{x}\rangle}{2\tau(\tau-\overline{\tau})}=\\ &=\frac{-v}{(2i\pi)^2w}-\frac{\langle\mathrm{x},\mathrm{x}\rangle}{2(\tau-\overline{\tau})}+\frac{\overline{\tau}\langle\mathrm{x},\mathrm{x}\rangle}{2\tau}=u+\frac{\langle\mathrm{x},\mathrm{x}\rangle}{2\tau}\end{aligned}$$

Summarizing, the action of the "Weyl" group of $\mathfrak{H}$ on the parameters (x,u,τ) reads

$$\text{Weyl:}\quad \mathrm{w}(\mathrm{x},u,\tau)=(\mathrm{w}\mathrm{x},u,\tau) \tag{3-12a}$$

$$H_Q:\quad \mathrm{t}(\mathrm{x},u,\tau)=\left(\mathrm{x}+\lambda+\tau\mu,u+k-\langle\mathrm{x},\mu\rangle-\frac{\tau}{2}\langle\mu,\mu\rangle-\frac{1}{2}\langle\lambda,\mu\rangle,\tau\right) \tag{3-12b}$$

$$SL(2,\mathbb{Z}):\quad \begin{pmatrix}a&b\\c&d\end{pmatrix}(\mathrm{x},u,\tau)=\left(\frac{\mathrm{x}}{c\tau+d},u+\frac{c\langle\mathrm{x},\mathrm{x}\rangle}{2(c\tau+d)},\frac{a\tau+b}{c\tau+d}\right) \tag{3-12c}$$

We should now compare the action in Eqs. (3-12) with the representation of the Jacobi group given in Prop. 2.1: the only difference is the action of the Heisenberg group. However, it is not difficult to realize that the group underlying the action described in Eq. (3-12b) is the central extension of $Q\times Q$ by $\mathbb{Z}$ given by the multiplication rule

$$(\lambda,\mu,k)\cdot(\lambda',\mu',k')=\left(\lambda+\lambda',\mu+\mu',k+k'+\frac{1}{2}\left(\langle\mu,\lambda'\rangle-\langle\lambda,\mu'\rangle\right)\right) \tag{3-13}$$

Such central extension corresponds to the same cocycle as the one defined by the multiplication rule in Eq. (2-2) and hence it is isomorphic; in fact the two multiplication rules (3-13) and (2-2) are intertwined by the group-isomorphism

$$(\lambda, \mu, k) \mapsto (\lambda, \mu, k + \frac{1}{2}\langle \lambda, \mu \rangle) \ .$$

To conclude we can formulate the main theorem motivating this paper

Theorem 3.1 The Jacobi group $J(\mathfrak{g})$ is isomorphic to the Weyl group of $\mathfrak{G}$, namely the group of automorphisms of $\mathfrak{G}$ which preserve the Cartan subalgebra $\mathfrak{H}$ modulo its centralizer.

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