

Weak Transversality and Partially Invariant Solutions

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Abstract

New exact solutions are obtained for several nonlinear physical equations, namely the Navier–Stokes and Euler systems, an isentropic compressible fluid system and a vector nonlinear Schrödinger equation. The solution methods make use of the symmetry group of the system in situations when the standard Lie method of symmetry reduction is not applicable.

Résumé

De nouvelles solutions exactes sont présentées pour plusieurs systèmes d'équations de la physique, telles les équations de Navier–Stokes, d'Euler, d'un fluide isentropique compressible et l'équation de Schrödinger non linéaire vectorielle. La méthode de résolution utilise le groupe de symétrie du système dans les situations où la méthode de Lie de réduction par symétrie n'est pas applicable.

1 Introduction

Lie group theory provides very general and efficient methods for obtaining exact analytic solutions of systems of partial differential equations, specially nonlinear ones [1]–[8]. The different methods have in common that they provide a reduction of the original system. This reduction usually means the reduction of the number of independent variables occurring, possibly a reduction of the number of dependent ones too.

The "standard", or "classical" reduction method goes back to the original work of Sophus Lie and is explained in many modern texts [1]–[8]. Essentially, it amounts to requiring that a solution of the equation should be invariant under some subgroup $G_0 \subseteq G$, where G is the symmetry group of the considered system of equations. The subgroup G_0 must satisfy certain criteria, in order to provide such group invariant solutions (see below).

The purpose of this article is to further develop, compare and apply alternative reduction methods. They have in common the fact that they provide solutions not obtainable by Lie's classical method. We shall survey the "tool kit" available for obtaining particular solutions of systems of partial differential equations, and further refine some of the tools. In the process we shall obtain new solutions of some physically important equations such as the Navier–Stokes equations, the Euler equations, the equations describing an isentropic compressible fluid model and the vector nonlinear Schrödinger equation.

We shall consider a system of m partial differential equations of order n , involving p independent variables (x_1, x_2, \dots, x_p) and q dependent variables (u_1, u_2, \dots, u_q)

$$\Delta_\nu \left(x, u^{(n)} \right) = 0, \quad \nu = 1, \dots, m, \quad (1)$$

where $u^{(n)}$ denotes all partial derivatives of u_α , up to order n .

S. Lie's classical method of symmetry reduction consists of several steps.

1. Find the local Lie group G of local point transformations taking solutions into solutions. Realize its Lie algebra L in terms of vector fields and identify it as an abstract Lie algebra. The vector fields will have the form

$$v_a = \sum_{i=1}^p \xi_a^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \varphi_a^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \quad a = 1, \dots, r = \dim G. \quad (2)$$

2. Classify the subalgebras $L_i \subset L$ into conjugacy classes under the action of the largest group \tilde{G} leaving the system of equations invariant (we have $G \subseteq \tilde{G}$). Choose a representative of each class.

3. For each representative subalgebra L_i find the invariants $I_\gamma(x_1, \dots, x_p, u_1, \dots, u_q)$ of the action of the group $G_i = \langle \exp L_i \rangle$ in the space $M \sim X \times U$ of independent and dependent variables. Let us assume that $p + q - s$ functionally independent invariants I_γ exist, at least locally (s is the dimension of the generic orbits of G_i).

4. Divide (if possible) the invariants I_γ into two sets: $\{F_1, \dots, F_q\}$ and $\{\xi_1, \dots, \xi_k\}$, $k = p - s$ in such a manner that the Jacobian relating F_μ and u_α ($\alpha, \mu = 1, \dots, q$) has maximal rank

$$J = \frac{\partial(F_1, \dots, F_q)}{\partial(u_1, \dots, u_q)}, \quad \text{rank } J = q. \quad (3)$$

Then consider F_μ to be functions of the other invariants $\{\xi_i\}$ which serve as the new independent variables and use condition (3) to express the dependent variables u_α in terms of the invariants.

5. Substitute the obtained expressions for u_α into the original system and obtain the reduced system, involving only invariants. The reduced system will involve only $k = p - s$, $s \geq 1$ independent variables.

6. Solve the reduced system. If the variables ξ^i depend on x_1, \dots, x_p only, this will yield explicit solutions of system (1). Otherwise, if ξ_i depend also on the original dependent variables u_α , we obtain implicit solutions.

7. Apply a general symmetry group transformation to these solutions.

This procedure provides a family of particular solutions that can be used to satisfy particular boundary or initial conditions. The classification of subgroups G_i can be viewed as a classification of conditions that can be imposed on the obtained solutions.

Steps 3, 4 and 5 of Lie's method can be reformulated as follows. Take the vector fields $\{v_a\}$ forming a basis of the considered Lie subalgebra $\mathfrak{g}_i \subset \mathfrak{g}$ and set their characteristics equal to zero

$$Q_a^\alpha = \varphi_a^\alpha(x, u) - \sum_{i=1}^p \xi_a^i(x, u) u_{x_i}^\alpha = 0, \quad \alpha = 1, \dots, q, \quad a = 1, \dots, r_i = \dim \mathfrak{g}_i. \quad (4)$$

Solve the systems (1) and (4) simultaneously.

Several alternative reduction procedures have been proposed, going beyond Lie’s classical method and providing further solutions. They all have in common that they add some system of equations to the original system (1) and that the extended system is solved simultaneously. These additional equations replace the characteristic system (4).

In its generality, this was proposed as the method of ”differential constraints” [9], and independently as the method of ”side conditions” [10, 11]. Different methods differ by the choice of this system of side conditions.

Basically two different approaches exist in the literature. The first makes further use of the symmetry group G of system (1), the second approach goes beyond this group of point transformations, or even gives up group theory altogether. Let us briefly discuss some of these methods.

1. ”Group invariant solutions without transversality”. This method was proposed by Anderson et al. [12] quite recently and deals with the situation when the rank condition (3) is not satisfied. It was shown [12] that under certain conditions on the subgroup $G_i \subset G$ one can still obtain G_i invariant solutions. We recall that a group invariant solution, with or without transversality, is transformed into itself by the subgroup $G_i \subset G$.

2. The method of partially invariant solutions. A solution $u(x)$ is ”partially invariant” [3, 13] under a subgroup $G_i \subset G$ of the invariance group, if G_i , when acting on $u(x)$ sweeps out a manifold of a dimension that is larger than that of the graph of the solution, but less than the dimension of the entire space M . A group G_i may provide both invariant and partially invariant solutions (see below). However, if the rank condition (3) for the Jacobian is not satisfied, at least on a set of solutions, then G_i will not provide invariant solutions, but may provide partially invariant ones. If the rank of the Jacobian J is q' , with $q' < q$, then we can express $u_1, \dots, u_{q'}$ in terms of invariants and let $u_{q'+1}, \dots, u_q$ depend on all of the original variables x_1, \dots, x_p . We then substitute the dependent variables back into the original system and obtain a ”partial reduction”. Solving this system, we obtain the partially invariant solutions. Irreducible partially invariant solutions are partially invariant solutions that cannot be obtained by Lie’s method using the subgroup G_i or any other subgroup G'_i of the symmetry group of the considered equations. Such solutions were constructed for certain nonlinear Klein–Gordon and Schrodinger equations, in [14] and [15], and for some equations of hydrodynamics in Ref. [16, 17, 18].

The theory of partially invariant solutions was further developed by Ondich [19, 20] who formulated irreducibility criteria for certain classes of equations. For other applications, see Ref. [21, 22, 23].

In terms of the equations (4), the method described above boils down to taking only $q' < q$ of the equations (4). As we shall show, it is possible to obtain further partially invariant solutions by different methods.

Among methods that go beyond the use of the symmetry group G we mention the following.

3. The Clarkson–Kruskal direct reduction method [24, 25] does not make explicit use of group theory. It is postulated that the dependent variables should be expressed in terms of new dependent variables that depend on fewer independent ones. The corresponding Ansatz is substituted into the original equation, which must then be solved. It has been shown that the direct method is intimately related to the method of conditional symmetries [26, 27, 28], to the ”nonclassical method” proposed by Bluman and Cole [29, 30], and to potential symmetries [31]. This method can also be interpreted in terms of side conditions [10, 11]. The differential constraints added to system (1) in this case have the form of first order quasilinear partial differential equations of the form (4). However the coefficients φ_a^α and ξ_α^i are not related to a Lie point symmetry of eq. (1).

4. The group foliation method. The method goes back to S. Lie, is described by Ovsianikov [3] and has recently been applied to obtain solutions of self dual Einstein equations [32, 33]. In terms of differential constraints this method amounts to embedding the system (1) into a larger system, consisting of all equations up to some definite order, invariant under the same Lie point symmetry group as (1) (and involving the same variables).

5. The method of ”partial Lie–point symmetries”, proposed by Cicogna and Gaeta [34]. This is a modification of the method of conditional symmetries. The method is in some cases easier to use and may provide solutions in cases when the equations of the conditional symmetry method prove to be untractable.

6. For integrable equations [35] generalized symmetries can be used to generate side conditions. These will be higher order equations, rather than first order ones.

7. The method of nonlocal symmetries. This consists in extending the space of dependent variables by adding some auxiliary variables, which can be potentials or pseudopotentials associated to the system of equations under analysis [36, 8, 37, 38]. A nonlocal symmetry will be then a symmetry of the original system augmented with the equations defining the new nonlocal variables.

This article is organized as follows. In Section 2 we introduce the concept of ”weak transversality” allowing us to simplify the method of ”group invariant solutions without transversality” [12] and to relate the rank of the matrix of invariants (3) to that of the coefficients of vector fields. The method of weak transversality is applied in Section 3 to obtain new invariant solutions of the Navier–Stokes equations and of the isentropic compressible fluid model. In Section 4 we establish a relation between partially invariant solutions and the transversality condition. This is then applied to obtain new irreducible partially invariant solutions of the vector nonlinear Schrödinger equation, of the Euler equations, of the Navier–Stokes equations and of the isentropic compressible fluid model in $(3+1)$ dimensions.

The concept of the irreducibility of partially invariant solutions is discussed. Some conclusions are presented in the final Section 5.

2 Group invariant solutions, strong and weak transversality

S. Lie's classical method of symmetry reduction was outlined in the Introduction. The first two steps are entirely algorithmic and we shall assume that they have already been performed. Thus, we are given a system of equations (1) and have found its Lie point symmetry group G , the Lie algebra of which is the symmetry algebra L . The symmetry algebra has dimension r and has a basis realized by vector fields of the form (2). Each vector field has the property that its n -th prolongation annihilates the system (1) on its solution set

$$pr \ v \Delta_\nu |_{\Delta_\mu=0} = 0, \quad \nu, \mu = 1, \dots, m. \quad (5)$$

The functions $\xi_a^i(x, u)$ and $\varphi_a^\alpha(x, u)$ are thus explicitly known.

Let us now consider a subgroup $G_0 \subset G$ and its Lie algebra \mathfrak{g}_0 . A solution $u = f(x)$ of the system (1) is G_0 invariant if its graph $\Gamma_f \sim \{x, f(x)\}$ is a G_0 invariant set:

$$g \cdot \Gamma_f = \Gamma_f, \quad g \in G_0. \quad (6)$$

The vector field (2) can be written in evolutionary form [1] as

$$v_{E,a} = \sum_{\alpha=1}^q \left(\varphi_a^\alpha(x, u) - \sum_{i=1}^p \xi_a^i(x, u) u_{x_i} \right) \partial_{u_\alpha}, \quad a = 1, \dots, r. \quad (7)$$

A G_0 -invariant solution will satisfy the $q \times r_0$ characteristic equations (4) associated with the basis elements of the Lie algebra L_0 ($\dim L_0 = r_0$).

The following matrices play an essential role in symmetry reduction using the symmetry group of the considered system of equations.

1. The matrices Ξ_1 and Ξ_2 of the coefficients of the vector fields v_a spanning the algebra L , or its subalgebra L_0 , and defined as follows:

$$\Xi_1 = \{ \xi_a^i(x, u) \}, \quad \Xi_1 \in \mathbb{R}^{r \times p}$$

$$\Xi_2 = \{ \xi_a^i(x, u), \varphi_a^\alpha(x, u) \}, \quad \Xi_2 \in \mathbb{R}^{r \times (p+q)} \quad (8)$$

where $a = 1, \dots, r$ labels the rows, i and α labels the columns.

2. The matrix of characteristics of the vector fields (2) (or (7)) spanning the considered algebra L (or its subalgebra L_0)

$$Q_a^\alpha = \{ v_a^E u_\alpha \} \quad a = 1, \dots, r; \quad \alpha = 1, \dots, q. \quad (9)$$

3. The Jacobian matrix J of the transformation relating the dependent variables u_α and the invariants of the action of G_0 on the space $M \sim X \times U$ of independent and dependent variables.

Let us now consider a specific subalgebra $L_0 \subset L$ and use it to obtain group invariant solutions via symmetry reduction. If the group G_0 acts regularly and transversally on $M \sim X \times U$ then

$$\text{rank} \{ \xi_a^i(x, u) \} = \text{rank} \{ \xi_a^i(x, u), \varphi_a^\alpha(x, u) \}. \quad (10)$$

This rank is equal to the dimension of the generic orbits of G_0 on M . If the transversality condition (10) is satisfied, at least locally, for all $\{x, u\} \in M$, then Lie's classical reduction method is directly applicable. Indeed, if (10) is satisfied then the rank of the matrix J of eq. (3) is maximal, $\text{rank} J = q$ (for a proof, see e.g. Ref. [1], Chapter 3.5). It follows that all dependent variables can be expressed in terms of invariants and a reduction is immediate (to a system with q dependent variables and $p - s$ independent ones).

If the action of G_0 on M is fiber preserving (i.e. the new independent variables only depend on the old independent ones), Lie's method provides explicit solutions. This happens because the new invariant independent variables z_i can be chosen to depend only on the original independent variables

$$z_i = z_i(x_1, \dots, x_p) \quad i = 1, \dots, p - s. \quad (11)$$

More generally, if we have $z_i = z_i(x, u)$, we obtain implicit solutions.

We shall call the rank condition (10) "strong transversality". Quite recently [12] a method was proposed for obtaining group invariant solutions when equation (10) is not satisfied. The method of Ref. [12] can actually be simplified by introducing the concept of "weak transversality".

Definition 1. The local transversality condition will be said to be satisfied in the weak sense if it holds only on a subset $\widetilde{M} \subset M$, rather than on the entire space M :

$$\text{rank} \{ \xi_a^i(x, u) \} |_{\widetilde{M}} = \text{rank} \{ \xi_a^i(x, u), \varphi_a^\alpha(x, u) \} |_{\widetilde{M}} \quad (12)$$

In other words, even if the transversality condition is not in general satisfied, there may exist a class S of functions $u = f(x)$ such that for them the condition (12) holds.

The "weak transversality" method is quite simple, when applicable. It consists of several steps.

1. Determine the conditions on the functions $u = f(x)$ under which eq. (12) is satisfied. Solve these conditions to obtain the general form of these functions.

2. Substitute the obtained expressions into the matrix of characteristics (9) and require that the condition $\text{rank} Q = 0$ be satisfied. This further constrains the functions $f(x)$.

3. Substitute the obtained expressions into the system (1). By construction, the solutions, if they exist, will be G_0 -invariant.

This method can only be applied if the matrix elements in the matrix Ξ_2 depend explicitly on the variables u_α . This poses strong restrictions on the considered algebra \mathfrak{g}_0 . We shall give some examples of the method in Section 3.

3 Examples of invariant solutions obtained by the weak transversality method.

3.1 The Navier–Stokes equations

The Navier–Stokes equations in (3+1) dimensions describing the flow of an incompressible viscous fluid are:

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \nabla p - \nu \nabla^2 \vec{u} = 0, \quad (13)$$

$$\nabla \cdot \vec{u} = 0, \quad (14)$$

where $\vec{u} = (u_1(x, y, z, t), u_2(x, y, z, t), u_3(x, y, z, t))$ is the velocity field, $p = p(x, y, z, t)$ the fluid pressure and ν the viscosity coefficient.

The symmetry properties of these equations have been intensively investigated by many authors from different points of view (see, for instance, [39] and references therein). It is well known [40] that eqs. (13)–(14) are invariant under the flow generated by the following vector fields:

$$B_1 = \alpha \partial_x + \dot{\alpha} \partial_{u_1} - \ddot{\alpha} x \partial_p, \quad (15)$$

$$B_2 = \beta \partial_y + \dot{\beta} \partial_{u_2} - \ddot{\beta} y \partial_p, \quad (16)$$

$$B_3 = \gamma \partial_z + \dot{\gamma} \partial_{u_3} - \ddot{\gamma} z \partial_p, \quad (17)$$

$$T = \partial_t, \quad (18)$$

$$Q = \partial_p, \quad (19)$$

$$D = x \partial_x + y \partial_y + z \partial_z + 2t \partial_t - u_1 \partial_{u_1} - u_2 \partial_{u_2} - u_3 \partial_{u_3} - 2p \partial_p, \quad (20)$$

$$L_1 = z \partial_y - y \partial_z + u_3 \partial_{u_2} - u_2 \partial_{u_3}, \quad (21)$$

$$L_2 = x \partial_z - z \partial_x + u_1 \partial_{u_3} - u_3 \partial_{u_1}, \quad (22)$$

$$L_3 = y \partial_x - x \partial_y + u_2 \partial_{u_1} - u_1 \partial_{u_2}, \quad (23)$$

where α, β, γ and δ are arbitrary functions of time. The operators B_j generate symmetry transformations that can be interpreted as boosts to frames moving with arbitrary velocities $\vec{v}(t) = \lambda \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}$, where λ is a constant. Space translations and Galilei boosts are obtained if α, β and γ are linear in t . The operators T and Q express the invariance of the eqs. (13)–(14) under translations of time and pressure, D generates scaling transformations, and L_1, L_2 and L_3 are the generators of the group of the rotations of the Euclidean space.

Example 1. Let us consider the subalgebra generated by L_1, L_2 and L_3 . Here we apply the ideas discussed in Section 2 to determine rotationally invariant solutions for the Navier–Stokes equations.

The matrices of the coefficients $\Xi_1 = (\xi_a^i(x, u))$ and $\Xi_2 = (\xi_a^i(x, u), \phi_a^\alpha(x, u))$ are represented by

$$\Xi_1 = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}, \quad \Xi_2 = \begin{pmatrix} 0 & z & -y & 0 & u_3 & -u_2 \\ -z & 0 & x & -u_3 & 0 & u_1 \\ y & -x & 0 & u_2 & -u_1 & 0 \end{pmatrix} \quad (24)$$

We observe that the matrix Ξ_1 has rank 2, whereas the matrix Ξ_2 has rank 3. In this situation, the transversality condition is violated in the strong sense. In other words, it is not true that for *every* function $\vec{u} = (u_1, u_2, u_3)$ the ranks of the matrices Ξ_1 and Ξ_2 coincide. In general, the system of characteristics is not compatible, the Jacobian matrix J will not have maximal rank, and it is *not* possible to use the classical symmetry reduction approach. To overcome these difficulties, let us force the matrix Ξ_2 to be of rank 2. This requirement is equivalent to a system of algebraic equations for \vec{u} , obtained imposing that the determinants of all 3×3 matrices constructed using the rows and the columns of Ξ_2 be equal to zero. Once this algebraic system is solved, we get the class S of functions $\vec{u} = (u_1, u_2, u_3)$ on which transversality is weakly restored. The class S in this case is characterized by the conditions:

$$u_1 = f(x, y, z, t) x, \quad u_2 = f(x, y, z, t) y, \quad u_3 = f(x, y, z, t) z, \quad p = p(x, y, z, t) \quad (25)$$

The second step consists in solving the characteristic system $Q_k^\alpha(x, u^{(1)}) = 0$ for the class S of eqs. (25). This forces the functions f and p to have the form:

$$f = f(r, t), \quad p = p(r, t), \quad (26)$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Relations (26) represent the most general form for the function \vec{u} and p to be rotationally invariant in the Euclidean space. Substituting these expressions for \vec{u} and p into the eqs. (13)–(14), we find the solution

$$\vec{u} = \frac{a(t)}{r^3} \vec{x}, \quad p = \frac{\dot{a}}{r} - \frac{a^2}{2r^4} + b. \quad (27)$$

The same vector fields L_1, L_2 and L_3 provide also a subalgebra of the symmetry algebra of the Euler equations. Anderson et al.[12] obtained a class of rotationally invariant solutions of the Euler equations by means of their technique of reduction diagrams. In Ref. [16] partially invariant solutions related to the subalgebra $\{L_1, L_2, L_3\}$ for the equations describing a nonstationary and isentropic flow for an ideal and compressible fluid in $(3 + 1)$ dimensions have been constructed using the transformation (25). A similar situation is also observed in magnetohydrodynamics [17]. Our solutions of the Navier–Stokes equations coincide with the solutions of the Euler equations found in Ref. [12]. Physically that means that the solutions (27) describe a laminar flow for which viscosity plays no role. This phenomenon occurs because the components of the vector \vec{u} in eq. (27) are all harmonic functions, i.e. they satisfy the Laplace equation, in addition to the Navier–Stokes equations.

Example 2. Now, Let us analyze the subalgebra defined by the operators

$$\begin{aligned} D &= x\partial_x + y\partial_y + z\partial_z + 2t\partial_t - u_1\partial_{u_1} - u_2\partial_{u_2} - u_3\partial_{u_3} - 2p\partial_p, \\ L_3 &= y\partial_x - x\partial_y + u_2\partial_{u_1} - u_1\partial_{u_2}, \\ X &= t^k \partial_x + k t^{k-1} \partial_{u_1} - k(k-1) t^{k-2} x \partial_p, \\ Y &= t^k \partial_y + k t^{k-1} \partial_{u_2} - k(k-1) t^{k-2} y \partial_p, \end{aligned} \quad (28)$$

which is a subalgebra of the Galilei–similitude algebra for a given $k \in \mathbb{R}$. It is immediate to check that the local transversality is violated for this subalgebra in the strong sense. The matrix Ξ_2 is now represented by

$$\begin{pmatrix} x & y & z & 2t & -u_1 & -u_2 & -u_3 & -2p \\ y & -x & 0 & 0 & u_2 & -u_1 & 0 & 0 \\ t^k & 0 & 0 & 0 & k t^{k-1} & 0 & 0 & -k(k-1) t^{k-2} x \\ 0 & t^k & 0 & 0 & 0 & k t^{k-1} & 0 & -k(k-1) t^{k-2} y \end{pmatrix}$$

If we impose that the matrix Ξ_2 should have rank 3 (that is weak transversality), we get

$$\begin{aligned} u_1 &= k \frac{x}{t}, & u_2 &= k \frac{y}{t} \\ u_3 &= u_3(x, y, z, t), & p &= p(x, y, z, t). \end{aligned}$$

As a second step, let us solve the characteristic system $Q_a^\alpha(x, u^{(1)}) = 0$, which consists of 16 linear differential equations of first order in the derivatives of the velocity components u_j and the pressure p . The most general function living in the space of the dependent variables and invariant under the flow associated to the generators (28) is

$$u_1 = k \frac{x}{t}, \quad u_2 = k \frac{y}{t}, \quad u_3 = \frac{\alpha\left(\frac{t}{z^2}\right)}{z^2} \quad (29)$$

$$p = -\frac{k(k-1)(x^2 + y^2)}{2t^2} + \frac{\beta\left(\frac{t}{z^2}\right)}{z^2}, \quad (30)$$

with α and β arbitrary functions of $t z^{-2}$. Substituting into the Navier–Stokes equations (13)–(14), we obtain the following three parameter set of solutions:

$$u_1 = k \frac{x}{t}, \quad u_2 = k \frac{y}{t}, \quad u_3 = \frac{c_1}{\sqrt{t}} - 2 \frac{kz}{t} \quad (31)$$

$$p = \frac{1}{2t^2} \left\{ c_1 \sqrt{t} z + k \left(x^2 + y^2 + 4c_1 \sqrt{t} z - 2z^2 \right) - k^2 \left(x^2 + y^2 + 4z^2 \right) + 2c_2 t \right\}, \quad (32)$$

with $c_1, c_2, k \in \mathbb{R}$. This solution is invariant under the group generated by the algebra (28) and satisfies weak (but not strong) transversality.

3.2 The isentropic compressible fluid model

The equations describing the non–stationary isentropic flow of a compressible ideal fluid are [41]

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + k a \nabla a = 0 \quad (33)$$

$$a_t + \vec{u} \cdot \nabla a + k^{-1} a \nabla \cdot \vec{u} = 0, \quad (34)$$

where $\vec{u} = u_1(x, y, z, t), u_2(x, y, z, t), u_3(x, y, z, t)$ is the velocity field, $a = a(x, y, z, t)$ is the velocity of sound, related to the pressure p and the density ρ by the formula $a = \left(\frac{\gamma p}{\rho}\right)^{1/2}$, γ is the adiabatic exponent and $k = 2/(\gamma - 1)$. The symmetry group G of eqs. (33)–(34) was derived in ref. [18]. For $k \neq 3$, G is generated by the following vector fields:

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad P_2 = \partial_y, \quad P_3 = \partial_z \quad (35)$$

$$K_1 = t\partial_x + \partial_{u_1}, \quad K_2 = t\partial_y + \partial_{u_2}, \quad K_3 = t\partial_z + \partial_{u_3} \quad (36)$$

$$L_1 = z\partial_y - y\partial_z + u_3\partial_{u_2} - u_2\partial_{u_3}, \quad (37)$$

$$L_2 = x\partial_z - z\partial_x + u_1\partial_{u_3} - u_3\partial_{u_1}, \quad (38)$$

$$L_3 = y\partial_x - x\partial_y + u_2\partial_{u_1} - u_1\partial_{u_2}, \quad (39)$$

$$F = x\partial_x + y\partial_y + z\partial_z + t\partial_t, \quad G = -t\partial_t + u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3} + a\partial_a. \quad (40)$$

We mention that for $k = 3$ the symmetry algebra contains an additional element generating projective transformations. The operators P_i, K_i and L_i are the infinitesimal generators of space translations, Galilei boosts and rotations, respectively. The operators F and G generate scaling transformations.

Example 3. Let us consider the subalgebra $\{L_3, F + G, K_1, K_2\}$. The matrix Ξ_2 is given by

$$\begin{pmatrix} y & -x & 0 & 0 & u_2 & -u_1 & 0 & 0 \\ x & y & z & 0 & u_1 & u_2 & u_3 & a \\ t & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad (41)$$

and the transversality is again violated in the strong sense, because $\text{rank } \Xi_1 = 3$ and $\text{rank } \Xi_2 = 4$. If we force the matrix Ξ_2 to be of rank 3, then we get the following constraints

$$u_1 = \frac{x}{t}, \quad u_2 = \frac{y}{t}, \quad u_3 = u_3(x, y, z, t), \quad a = a(x, y, z, t). \quad (42)$$

From the characteristic system $Q_a^\alpha(x, u^{(1)}) = 0$ we deduce

$$u_3 = z W(t), \quad a = z A(t), \quad (43)$$

where W and A are arbitrary functions of time. Now, substituting relations (42)–(43) into the system (33)–(34) we obtain the relation

$$A(t) = \sqrt{-\frac{1}{k}(W^2 + W')} \quad (44)$$

and a second order ODE for W

$$W'' + 2\left(2 + \frac{1}{k}\right)WW' + 2\left(1 + \frac{1}{k}\right)W^3 + \frac{4}{kt}(W' + W^2) = 0 \quad (45)$$

In general, eq. (45) does not have the Painlevé property. For special values of the parameter k , namely $k = -1$ and $k = -2$, it does. In these cases it can be reduced to a canonical form (see Ref. [42], p.334) via a linear transformation of the type

$$W = \alpha(t) U(z(t)) + \beta(t). \quad (46)$$

1. For $k = -1$ we have

$$W'' = -2WW' + p(t)(W' + W^2), \quad (47)$$

where $p(t) = 4/t$. This equation can be integrated and its solution, which is regular, is

$$W = \frac{c_1 t^2 \left(I_{-\frac{5}{6}}\left(\frac{c_1 t^3}{3}\right) + c_2 I_{\frac{5}{6}}\left(\frac{c_1 t^3}{3}\right) \right)}{I_{\frac{1}{6}}\left(\frac{c_1 t^3}{3}\right) + c_2 I_{-\frac{1}{6}}\left(\frac{c_1 t^3}{3}\right)}, \quad (48)$$

where c_1 and c_2 are constants and $I_n(x)$ is the modified Bessel function of the first kind. Correspondingly we find

$$A = c_1 t^2 \quad (49)$$

in eq. (44).

2. For $k = -2$ we have

$$W'' = -3WW' - W^3 + q(t)(W' + W^2), \quad (50)$$

where $q(t) = 2/t$. In this case, we can integrate and the general solution is:

$$W = \frac{4t^3 + c_1}{t^4 + c_1 t + c_2}. \quad (51)$$

The solution of eq. (44) is

$$A = 2\sqrt{3} \sqrt{\frac{t^2}{t^4 + c_1 t + c_2}}, \quad (52)$$

where c_1 , and c_2 are constants. The solutions for $k = -1$ and $k = -2$ represent nonscattering waves.

4 Partially invariant solutions of systems of differential equations and the transversality condition

A useful tool applicable to the study of systems of differential equations, and intimately related to the standard Lie approach, is the theory of partially invariant solutions. The relevant notion in this context is the defect δ of a k -dimensional manifold M with respect to a Lie group G . When the group acts on a p -dimensional submanifold $\Gamma \subseteq M$, it sweeps out an orbit $G(\Gamma)$. The manifold Γ will be identified with the graph Γ_f of a function $u = f(x)$, so its dimension will coincide with the number of independent variables (also denoted p). As we already said, the case $G(\Gamma_f) = \Gamma_f$ corresponds to the G -invariance of the manifold. Otherwise, $G(\Gamma_f)$ will be a more generic subset of M . There is no guarantee that this subset will be a submanifold. However, if the intersection between an orbit O of G and Γ_f has a dimension which is constant in a neighbourhood N of a point of Γ_f , then there exists a neighbourhood \tilde{G} of the identity of G such that the subset $\tilde{G}(N \cap \Gamma_f)$ is a submanifold [19]. In the subsequent considerations, $G(\Gamma_f)$ will be considered as a submanifold.

Let G be a group, acting regularly with s -dimensional orbits. We call the number

$$\delta = \dim G(\Gamma_f) - \dim \Gamma_f \quad (53)$$

the defect δ of the function f with respect to G . The usual G -invariant functions correspond to the case $\delta = 0$. A function will be said to be *generic* if $\delta = m_0 = \min\{s, k - p\}$. The more interesting situation is when $0 < \delta < m_0$, which is the case we will dealing with. In this case, the function f will be said to be partially invariant [3].

Let us consider the system (1) of partial differential equations, whose symmetry group G acts on the $p + q$ -dimensional space $M = X \times U$. Let \mathfrak{g} be a subalgebra of the symmetry algebra of Δ , and Q the characteristic matrix associated to the set of its generators. Then $u = f(x)$ is a partially invariant solution of Δ with defect δ with respect to \mathfrak{g} if and only if [3, 19, 43]

$$\text{rank} \left(Q \left(x, u^{(1)} \right) \right) = \delta. \quad (54)$$

The condition (54) provides a system of differential equations involving the dependent variables $u = (u_1, \dots, u_q)$. In order to determine partially invariant solutions, we can extend the original system Δ by adding the set of differential constraints given by the condition (54). We must then solve the extended system consisting of eq. (1) and (54). The set of equations given by the prescription (54) is less constraining than the set required to obtain G -invariance, as in formulas (4).

In this section we will study the role of the local transversality condition (and in particular of the notion of weak transversality) in the theory of partially invariant solutions and propose a strategy to find them.

Let us start by noticing that a partially invariant solution of a system of differential equations can be naturally related to the violation of the transversality condition. Indeed, let $\Delta(x, u^{(n)}) = 0$ be a system of differential equations defined over $M \subset X \times U$ and $u_0 = u_0(x_0)$ be a solution of Δ . Let G be an r -dimensional subgroup of the symmetry group of Δ , acting regularly on M , whose generators are given by (2). If the condition

$$\text{rank} \left(\xi_a^i(x_0, u_0) \right) < \text{rank} \left(\xi_a^i(x_0, u_0), \phi_a^\alpha(x_0, u_0) \right) \quad (55)$$

is satisfied, then $u_0 = f(x_0)$ is a partially invariant solution of Δ (or possibly a generic one).

Example 4. The vector nonlinear Schrödinger equation

$$i\psi_t + \Delta\psi = (\bar{\psi}\psi)\psi, \quad (56)$$

where $\psi \in \mathbb{C}^N$ and Δ is the Laplace operator in n dimensions, plays an important role in many areas of physics. For instance, in nonlinear optics it describes the interaction of electromagnetic waves propagating with different polarizations in nonlinear media [45]. In hydrodynamics, it furnishes a model for the description of the interactions of N water waves in a deep fluid [46]–[48]. For these and other applications of the vector nonlinear Schrödinger equation, see also Ref. [49].

Let us consider the case of three components ($N = 3$) and two spatial dimensions ($n = 2$). The symmetry algebra has been computed in Ref. [50]. In terms of amplitude and phase, the components of the wave function will be written as $\psi_i = \rho_i e^{i\omega_i}$. In particular, we will discuss the role of the subalgebra

$$\{\partial_x, \partial_y, y\partial_x - x\partial_y + a_1\partial\omega_1 + a_2\partial\omega_2 + a_3\partial\omega_3\}, \quad (57)$$

generated by the two translations in the plane and a rotation combined with a transformation of the phases. The matrix of the coefficients Ξ_2 , given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & -x & a_1 & a_2 & a_3 & 0 & 0 & 0 \end{pmatrix}$$

has rank 3, unless $(a_1, a_2, a_3) = (0, 0, 0)$. Let us consider the case $a_2 = a_3 = 0$, $a_1 \neq 0$. We obtain $\rho_i = \rho_i(t)$, ($i = 1, 2, 3$), $\omega_1 = \omega_1(t)$, $\omega_2 = \omega_2(t)$, but ω_3 is not an invariant, so we keep $\omega_3 = \omega_3(x, y, z, t)$. Substituting into eq. (56), we obtain:

$$\rho_1 = \frac{\gamma_1}{\sqrt{t(t-t_0)}}, \quad \rho_2 = \gamma_2, \quad \rho_3 = \gamma_3, \quad (58)$$

$$\omega_1 = \frac{x^2}{4(t-t_0)} + \frac{y^2}{4t} + \frac{\gamma_1^2}{t_0} \ln \frac{t}{t-t_0} - (\gamma_2^2 + \gamma_3^2) t, \quad (59)$$

$$\omega_2 = \omega_3 = \frac{\gamma_1^2}{t_0} \ln \frac{t}{t-t_0} - (\gamma_2^2 + \gamma_3^2) t. \quad (60)$$

This solution is partially invariant with respect to the subgroup corresponding to the subalgebra (57) with $a_2 = a_3 = 0$ and $a_1 \neq 0$, and is not reducible (unless we choose $t_0 = 0$). For $t_0 = 0$ it is invariant under rotations in the xy-plane.

Example 5. Let us consider again the isentropic compressible model (formulas (33)–(34)). For the subalgebra $\mathfrak{g} = \{K_1, K_2, K_3, P_3\}$ the matrix Ξ_2 is given by

$$\begin{pmatrix} t & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here transversality is violated also in the weak sense. The invariants of the corresponding Lie group are $F = u_1 t - x$, $G = u_2 t - y$, a and t . The matrix J of (3) is not invertible, but we can write

$$u_1 = \frac{F(t) + x}{t}, \quad u_2 = \frac{G(t) + y}{t}, \quad a = A(t), \quad (61)$$

but leave u_3 general, i.e. $u_3 = u_3(x, y, z, t)$. Substituting eqs. (61) into (33), (34) we obtain the solution [16]

$$u_1 = \frac{x}{t}, \quad u_2 = \frac{y}{t}, \quad u_3 = \frac{z + \lambda(\xi_1, \xi_2)}{t + t_0}, \quad a = c \left(\frac{1}{t^2(t+t_0)} \right)^{\frac{1}{k}}, \quad (62)$$

where $\xi_1 = \frac{x}{t}$, $\xi_2 = \frac{y}{t}$, λ is an arbitrary function of ξ_1 and ξ_2 , c and t_0 are constants. The rank of the matrix Q_a^α of the characteristics associated to (62) is equal to one, and therefore this solution is partially invariant with respect to the subalgebra \mathfrak{g} , with $\delta = 1$. Now let us check if it is reducible under any other subalgebra of the full symmetry algebra (35)–(40). To do this, it is useful to study the kernel K of the characteristic matrix Q for the full symmetry algebra (35)–(40) associated to the solution (62) and to determine its generators $\{k_1, \dots, k_l\}$. It is clear that if at least a subspace of K can be generated by constant vectors, then the solution will be reducible with respect to the subalgebra identified by these vectors.

In the case of the solution (62), the kernel is generated by 8 vectors, each having 12 components. It is possible to show that there exists only one constant generator, namely

$$k = (0, 0, 0, t_0, 0, 0, 0, 0, 0, 1, 0, 0).$$

This implies that the solution (62) is reducible with respect to the one-dimensional subalgebra $\{K_3 + t_0 P_3\}$.

The concept of "irreducibility" of a partially invariant solutions needs further clarification. Once a solution $u = f(x)$, partially invariant under G_i is found, it is of course possible to verify whether it is invariant under some other subgroup $G'_i \subset G$. Let this be the case, let G'_i satisfy the strong transversality condition, and have generic orbits of dimension r'_i . The standard Lie method using the subgroup G'_i would then reduce system (1) to a system with $q - r'_i$ variables. In general, specially for $q - r'_i > 1$, we may not be able to solve this system and the methods of partial invariance for the original subgroup G_i may be more tractable. If the invariance subgroup G'_i does not satisfy the weak transversality condition, it may not help us at all.

We observe that the violation of the transversality condition is *not* a necessary condition for the existence of partially invariant solutions of a system $\Delta(x, u^{(n)}) = 0$. In fact, there could be solutions of Δ which are not solutions of the characteristic system, even if it is compatible as an algebraic system.

Example 6. As a matter of fact, a counterexample was provided by Ondich [19], namely the two variable Laplace equation, expressed in the following form:

$$v = u_x, \quad v_y = w_x, \quad w = u_y, \quad v_x = -w_y. \quad (63)$$

This system is clearly invariant with respect to the translations in the plane, generated by the vector fields $\{\partial_x, \partial_y\}$. The characteristic matrix Q has the form

$$\begin{pmatrix} -u_x & -v_x & -w_x \\ -u_y & -v_y & -w_y \end{pmatrix}. \quad (64)$$

Invariant solutions are obtained if and only if the rank of this matrix is equal to zero. This implies that u, v and w are constants:

$$u = k, \quad v = \lambda, \quad w = \mu, \quad k, \lambda, \mu \in \mathbb{R}.$$

Let us impose that the rank of Q is equal to 1. This means that the two rows in eq. (64) are proportional. Then solving the corresponding equations and replacing the result into eqs. (63), we get the following solution:

$$(u, v, w) = (ax + by + c, a, b), \quad a, b \in \mathbb{R}. \quad (65)$$

By construction, this solution is partially invariant with respect to the group of translations in the plane, with defect $\delta = 1$. Nevertheless, the transversality condition is satisfied. More generally, the transversality condition is always satisfied if we have $\text{rank}(\xi_a^i(x, u)) = \dim L$.

We mention that the previous solution (65) can be also obtained starting from the symmetry subalgebra $\{\partial_x, \partial_y, \partial_u\}$ for which both the weak and strong transversality conditions are violated. In this case, the invariants are $I_1 = v = a$, $I_2 = w = b$, where a and b are arbitrary constants. The function u remains arbitrary. Putting these constraints into eqs. (63), we recover immediately the solution (65). However, it should be noticed that the analysis of the rank of the characteristic matrix furnishes a complete characterization of the invariant and partially invariant solutions of a system of differential equations, and provides a more general procedure than the use of the invariants. For instance, this second approach does not allow to recognize the solution (65) as a partially invariant one with respect to the subalgebra $\{\partial_x, \partial_y\}$.

Example 7. To show that a symmetry group satisfying the strong transversality can produce both invariant and partially invariant solutions, let us consider the Euler equations for an incompressible nonviscous fluid in (3+1) dimensions:

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \nabla p = 0, \quad (66)$$

$$\nabla \cdot \vec{u} = 0, \quad (67)$$

The symmetry group of the Euler equations is well-known [1, 5]. It coincides with the symmetry group of the Navier–Stokes equations, except that it contains an additional dilation. Thus, D of eq. (20) is replaced by

$$D_1 = x\partial_x + y\partial_y + z\partial_z + t\partial_t, \quad (68)$$

$$D_2 = t\partial_t - u_1\partial_{u_1} - u_2\partial_{u_2} - u_3\partial_{u_3} - 2p\partial_p. \quad (69)$$

We consider here the subgroup of Galilei transformations. Its Lie algebra is given by

$$K_1 = t\partial_x + \partial_{u_1}, \quad K_2 = t\partial_y + \partial_{u_2}, \quad K_3 = t\partial_z + \partial_{u_3}. \quad (70)$$

Here the transversality holds in the strong sense and indeed an invariant solution will have the form

$$u_1 = \frac{x}{t} + F_1(t), \quad u_2 = \frac{y}{t} + F_2(t), \quad u_3 = \frac{z}{t} + F_3(t), \quad p = P(t). \quad (71)$$

Let us now look for partially invariant solutions of the system (66)–(67) with respect to the same subgroup and impose that the defect be $\delta = 2$. Writing down the characteristic system associated to (70) and imposing that $\text{rank}Q = 2$, we get the following constraints on u_1 and u_2

$$u_1 = \frac{x}{t} - \mu\lambda\frac{z}{t} + \mu\lambda u_3 + h_1(t) \quad (72)$$

$$u_2 = \mu u_3 + \frac{y}{t} - \mu\frac{z}{t} + h_2(t) \quad (73)$$

but $u_3 = u_3(x, y, z, t)$ and $p = p(x, y, z, t)$ remain arbitrary.

Substituting the relations (72)–(73) into the Euler equations, and choosing for simplicity $h_1 = h_2 = 0$, we obtain the solution:

$$u_1 = \frac{1}{t [\mu^2 (1 + \lambda^2) + 1]} \{x [\mu^2 (1 - 2\lambda^2) + 1] - 3\lambda\mu (\mu y + z)\} + \lambda\mu t^2 F\left(\frac{\lambda y - x}{t}, \frac{\lambda\mu z - x}{t}\right), \quad (74)$$

$$u_2 = \frac{1}{t [\mu^2 (1 + \lambda^2) + 1]} \{y [\mu^2 (\lambda^2 - 2) + 1] - 3\mu (\lambda\mu x + z)\} + \mu t^2 F\left(\frac{\lambda y - x}{t}, \frac{\lambda\mu z - x}{t}\right), \quad (75)$$

$$u_3 = \frac{1}{t [\mu^2 (1 + \lambda^2) + 1]} \{z [\mu^2 (1 + \lambda^2) - 2] - 3(\lambda x + y)\} + t^2 F\left(\frac{\lambda y - x}{t}, \frac{\lambda\mu z - x}{t}\right), \quad (76)$$

$$p = -\frac{3\mu^2}{t^2 (\lambda^2 \mu^2 + \mu^2 + 1)} \left(\lambda x + y + \frac{z}{\mu}\right)^2 + p(t), \quad (77)$$

where λ, μ are constants, F is an arbitrary function of its arguments and p is an arbitrary function of t . We have checked explicitly that, if F is kept arbitrary, this solution is not invariant under any subgroup of the symmetry group. Thus, it represents an irreducible partially invariant solution of the Euler equations of defect $\delta = 2$, with respect to a subgroup satisfying the strong transversality condition. If

$$F = \left(\frac{\lambda y - x}{t}\right)^{\frac{3a+2b}{a}} \Phi\left(\frac{\lambda y - x}{\lambda\mu z - x}\right), \quad a, b \in \mathbb{R},$$

where Φ is an arbitrary function of its argument, then the solution (74)–(77) is invariant under the subgroup generated by $aD_1 + bD_2$, where D_1 and D_2 are defined in eqs. (68)–(69).

However, this subgroup provides a reduced system with three independent variables that would be very difficult to solve.

Particularly interesting is the case when partially invariant solutions can be found that satisfy weak but not strong transversality. Indeed, imposing weak transversality basically means that a class of functions $u = f(x)$ is chosen in such a way that the characteristic system is algebraically compatible. This condition is of course not sufficient to guarantee the invariance of these functions under the action of the considered group G . Indeed, if we compute the rank of the matrix Q_a^α on this class of functions, in general it will be not equal to zero. Then, once weak transversality is satisfied, we can choose either to have group invariant solutions, using the method outlined in Section 3, or to use the class of functions $u = f(x)$ to get partially invariant solutions.

In the next two examples, we will see how this approach can be used to obtain in a simple and straightforward way new classes of solutions of hydrodynamic systems.

Example 8. In Example 2 we studied the algebra (28) which is a subalgebra of the symmetry algebra of both the Euler and the Navier–Stokes equations. We showed that the requirement of weak transversality implies

$$u_1 = k\frac{x}{t}, \quad u_2 = k\frac{y}{t}, \quad (78)$$

$$u_3 = u_3(x, y, z, t), \quad p = p(x, y, z, t). \quad (79)$$

These formulas define a class of functions which is partially invariant with defect $\delta = 2$ with respect to the subalgebra (28). At this stage, we can choose either to have group invariant solutions or partially invariant ones. Indeed, in Example 2 we forced the class of functions (78)–(79) to be a solution of the characteristic system, and then substituting the obtained expressions (29)–(30) into the system (13)–(14) we constructed the group invariant solutions (31)–(32).

Another possibility is to substitute formulas (78)–(79) directly into the Euler equations (66)–(67) or the Navier–Stokes equations (13)–(14) without requiring further invariance properties. For the case of the Euler equations we get the constraints

$$t^2 p_x + k(k-1)x = 0, \quad (80)$$

$$t^2 p_y + k(k-1)y = 0, \quad (81)$$

$$u_{3z} + \frac{2k}{t} = 0, \quad (82)$$

$$u_{3t} + u_3 u_{3z} + \frac{k}{t} (x u_{3x} + y u_{3y}) + p_z = 0. \quad (83)$$

We solve this system and obtain the following partially invariant solution of the Euler equations:

$$u_1 = k \frac{x}{t}, \quad u_2 = k \frac{y}{t}, \quad (84)$$

$$u_3 = -\frac{2kz}{t} + x^2 F\left(tx^{-\frac{1}{k}}, \frac{y}{x}\right) \quad (85)$$

$$p = -\frac{k(k-1)(x^2+y^2)}{2t^2} - \frac{k(2k+1)z^2}{t^2} + f(t), \quad (86)$$

where $\xi = tx^{-\frac{1}{k}}, \eta = \frac{y}{x}$ and $f(t)$ and $F(\xi, \eta)$ are arbitrary functions of their arguments. This solution is irreducible for a generic function F . If F satisfies the equation

$$\left[\left(c_1 \frac{k-1}{k} + c_2 \right) \xi - c_3 \frac{\xi \eta}{k} - \frac{c_4}{k} \xi^{k+1} + c_5 \xi^k \right] F_\xi +$$

$$[-c_3(\eta^2+1) - c_4 \xi^k \eta + c_5 \xi^k] F_\eta + (2c_1 + c_2 + 2c_3 \eta) F - c_6(4k+1)\xi^{2k} = 0,$$

then it is invariant under the subgroup generated by

$$X = c_1 D_1 + c_2 D_2 + c_3 L_3 + c_4 B_1 + c_5 B_2 + c_3 B_3,$$

where c_1, \dots, c_6 are real constants, D_1 and D_2 are the dilations given by eqs. (68)–(69), L_3 is the generator (23), and the functions appearing in the boosts (15)–(17) are now monomials in t , namely

$$\alpha = t^k, \quad \beta = t^k, \quad \gamma = t^{2k+1}.$$

The same procedure can be applied to the Navier–Stokes equations. Repeating the previous steps, we obtain the following solution:

$$u_1 = k \frac{x}{t}, \quad u_2 = k \frac{y}{t}, \quad (87)$$

$$u_3 = -\frac{2kz}{t} + \alpha(x, y, t) \quad (88)$$

$$p = -\frac{k(k-1)(x^2+y^2)}{2t^2} - \frac{k(2k+1)z^2}{t^2} + f(t), \quad (89)$$

where $\alpha(x, y, t)$ satisfies the following equation

$$\alpha_t + \frac{k}{t}(x\alpha_x + y\alpha_y - 2\alpha) - \nu(\alpha_{xx} + \alpha_{yy}) = 0. \quad (90)$$

We thus obtain a large class of partially invariant solutions of the Navier–Stokes equations parametrized by the solutions of the *linear* partial differential equation (90).

Example 9. Partially invariant solutions with weak transversality can be also found for the case of the compressible fluid model (33)–(34). Let us again consider the subalgebra $\{L_3, F+G, K_1, K_2\}$ and the corresponding weak transversality condition (42). Substituting into equations (33)–(34) we obtain a coupled system of quasilinear first order PDE's, namely:

$$a_x = 0, \quad a_y = 0,$$

$$u_{3t} + u_3 u_{3z} + \frac{x}{t} u_{3x} + \frac{y}{t} u_{3y} + k a a_z = 0$$

$$a_t + u_3 a_z + \frac{a}{k} \left(\frac{2}{t} + u_{3z} \right) = 0. \quad (91)$$

In particular, if we assume $a_z = 0$, we reobtain the solution (62). However, the system (91) allows much more general solutions.

5 Conclusions

The main conclusion of this article is that one can do considerably more with the symmetry group G of a system of partial differential equations than apply the standard method for finding group invariant solutions.

Indeed, let us assume that the largest group G (of local Lie point transformations) leaving the system (1) invariant has been found and its subgroups classified. For each subgroup G_0 , or its Lie algebra L_0 we should proceed as follows.

1. Obtain invariant solutions. First check whether the transversality condition (10) is satisfied (in the strong sense). If it is, we apply Lie's classical method. This is always possible since transversality assures that the rank condition (3) is satisfied. If the (strong) transversality condition is not satisfied, we may still be able to obtain solutions invariant under G_0 [12], by imposing "weak transversality" on solutions, as described in Section 2 above. This is illustrated in Section 3 by Examples 1,2 and 3.

2. Obtain partially invariant solutions. These can be obtained by (at least) three complementary methods. If the transversality condition (10) is not satisfied and weak transversality cannot be imposed, then the characteristic system (4) is not consistent and the rank condition (3) for the invariants is not satisfied. We then choose a subset of $q' < q$ invariants such that we can express q' dependent variables in terms of invariants. The remaining $q - q'$ variables u_α are considered as functions of all the original variables x_1, \dots, x_p . Examples 4 and 5 of Section 4 are of this type, as are those of Ref. [14], [15].

If the transversality condition is satisfied for G_0 we may still be able to obtain partially invariant solutions, in addition to the invariant ones. Instead of imposing that the matrix of characteristics (4) have rank zero (i.e. that all equations (4) be satisfied) we require that on solutions we have $rank Q = \delta$, with $\delta = 1, 2, \dots$, as the case may be. This rank condition must be solved explicitly for u_α and the result substituted into eq. (1) (see Examples 6,7). The third possibility is to impose weak transversality (if possible), but still to impose $rank Q = \delta \geq 1$ (see Examples 8 and 9).

3. Go beyond invariant and partially invariant solutions, either by the method of group foliation [3, 32, 33], or by other methods not using the symmetry group G [24]–[31].

Missing at this stage are clear criteria that tell us which approach will be fruitful. Furthermore, the same solution may be obtained by different methods and it is not clear which of these method will lead to the least amount of calculations. For instance, solutions partially invariant under some subgroup $G_1 \subset G$ have been called "reducible" [14]–[20], [16]–[18] if they are actually invariant under some other subgroup $G_2 \subset G$. However, it may be more difficult to use G_2 , than G_1 , specially if the dimension of G_2 is small with respect to the number of independent variables.

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