Bilinear semi-classical moment functionals and their integral representation*

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CRM-2842

May 2002

*Work supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Fonds FCAR du Québec.

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Abstract

We introduce the notion of bilinear moment functional and study their general properties. The analogue of Favard's theorem for moment functionals is proven. The notion of semi-classical bilinear functionals is introduced as a generalization of the corresponding notion for moment functionals and motivated by the applications to multi-matrix random models. Integral representations of such functionals are derived and shown to be linearly independent.

Keywords. moment functionals; biorthogonal polynomials; semiclassical functionals

1 Introduction

The notion of moment functional is most commonly encountered as a generalization of the context of Orthogonal Polynomials (OP) [1]. These are generally defined as a graded polynomial orthonormal basis in $L^2(\mathbb{R}, \mathrm{d}\mu)$ where $\mathrm{d}\mu$ is a given positive measure for which all *moments*

$$\mu_i := \int_{\mathbb{R}} \mathrm{d}\mu(x)\, x^i \ , \tag{1-1}$$

exist finite. The moment functional associated to such a measure is then the element $\mathcal{L}$ in the dual space of polynomials, $\mathbb{C}[x]^\vee$ defined by

$$\mathcal{L}(p(x)) := \int_{\mathbb{R}} \mathrm{d}\mu\, p(x) \ , \tag{1-2}$$

and it is uniquely characterized by its moments. The positivity of the measure implies that we can always find orthogonal polynomials which are real, so that the orthogonality relation reads

$$\mathcal{L}(p_m(x)p_n(x)) = h_n\delta_{nm} \ . \tag{1-3}$$

$$p_n(x) = x^n + \mathcal{O}(x^{n-1}) \in \mathbb{R}[x] \ , \quad h_n \in \mathbb{R}_+^\times. \tag{1-4}$$

Generalizing this picture one is led to consider *complex* funtionals [2], i.e. whose moments are not necessarily real. The associated OPs are then defined by the same relations (1-3) where now the polynomials belong to the ring $\mathbb{C}[x]$ and h_n are nonzero complex numbers.

One of the main applications of OPs is in the context of random matrices [3, 4] where they allow to write explicit expressions for the correlation functions of eigenvalues and of the partition function of these models.

Recently [5, 6, 7, 8] growing attention is devoted to the 2-matrix models (or the multi matrix models) in which the probability space is the space of couples (or n-tuples) of matrices. Also such models can be "solved" along lines similar to the one matrix models by finding certain bi-orthogonal polynomials (BOP). The probability measure is given by

$$\mathrm{d}\mu(M_1, M_2) = \frac{1}{\mathcal{Z}_n} \mathrm{e}^{\mathrm{Tr}(M_1 M_2)}\, \mathrm{d}\mu_1(M_1)\, \mathrm{d}\mu_2(M_2) \tag{1-5}$$

where M_i are $N \times N$ Hermitian matrices (usually) and the positive measures $\mathrm{d}\mu_i$ are $U(N)$ invariant. The relevant BOPs are then a pair of graded polynomial bases $\{p_n(x)\}, \ \{s_n(y)\}$ "dual" to each other in the sense that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}\mu_1(x)\mathrm{d}\mu_2(y)\, p_n(x)s_m(y)\mathrm{e}^{xy} = h_n\delta_{nm} \ , \tag{1-6}$$

$$p_n \in \mathbb{R}[x], \quad s_n \in \mathbb{R}[y] \ , \quad h_n \in \mathbb{R}^\times. \tag{1-7}$$

The integral in Eq. (1-6) defines a particular kind of *bi*-moment functional, that is an element of the dual to the tensor of two spaces of polynomials $\mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y]$

$$\mathcal{L}\big(p(x)\big|s(y)\big) := \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}\mu_1(x)\mathrm{d}\mu_2(y)\, p(x)s(y)\mathrm{e}^{xy} \ , \tag{1-8}$$

provided all its *bi-moments* μ_{ij} are finite

$$\mu_{ij} := \mathcal{L}(x^i|y^j) \in \mathbb{R} \ . \tag{1-9}$$

Generalizing this picture we now consider complex bi-moment functionals which are uniquely characterized by their (complex) bi-moments $\mu_{ij} \in \mathbb{C}$.

The notion of semiclassical moment functional for a functional of the form (1-2) requires that the measure $\mathrm{d}\mu(x)$ has a density $W(x)$ whose logaritmic derivative is a rational function of x and the support is a finite union of intervals. This condition can be translated into a distributional equation for the moment functional itself and then generalized to the complex case [9, 10, 11].

Motivated by the applications to 2-matrix models, we are interested in the corresponding notion of semiclassical bi-moment functionals (which we will define properly later on) and in studying their properties: we will produce (complex path) integral representations for them, generalizing the framework of [12, 13, 14] to this situation.

We quickly recall that [9, 10, 11] a moment functional $\mathcal{L}$ is called semi-classical if there exist two (minimal) fixed polynomials $A(x)$ and $B(x)$ with the properties that

$$\mathcal{L}\left(-B(x)p'(x) + A(x)p(x)\right) = 0, \ \forall p(x) \in \mathbb{C}[x] \ . \tag{1-10}$$

The integral representation was obtained in [12, 13, 14]: we can quickly reprove here their result (without details) in a different way which was not used there and which is in the line of approach of this paper. Consequence of the definition is that the (possibly formal) generating power series

$$F(z) := \sum_{k=0}^{\infty} \mu_k \frac{z^k}{k!} \, ('' ='' \, \mathcal{L}(\mathrm{e}^{xz})) \ , \quad \mu_k := \mathcal{L}(x^k) \ , \tag{1-11}$$

satisfies the n-th order ODE

$$\left[zB\left(\frac{d}{dz}\right) - A\left(\frac{d}{dz}\right)\right] F(z) = 0 \ . \tag{1-12}$$

The order n is the highest of the degrees of $A(x), B(x)$ and it is referred to –in this context– as the *class*. A distinction occurs according to the cases $\deg(A) < \deg B$ (Case A in [13]) or $\deg(A) \geq \deg(B)$ (Case B). By looking at the recursion relation satisfied by the moments μ_k one realizes that there are precisely n linearly independent solutions if in Case B or $n-1$ in Case A[3] and hence the functionals are in one–to–one correspondence with the solutions of Eq. (1-12) which are analytic at $z = 0$.
It is precisely the result of [15] that the fundamental system of solutions of Eq. (1-12) are expressible as Laplace integral transform of the weight density

$$W(x) := \exp\left(\int \mathrm{d}x \frac{A(x) + B'(x)}{B(x)}\right) \ , \tag{1-13}$$

(which may have also branch-points) over n distinct suitably chosen contours Γ_j;

$$F_j(z) := \int_{\Gamma_j} \mathrm{d}x W(x) \mathrm{e}^{xz} \ . \tag{1-14}$$

In Case A one should actually reject one solution among them, i.e. the one with a singularity at the origin, or better consider only the linear combinations which are analytic at $z = 0$.

In the present paper the bi-moment functionals we consider will rather correspond to generating functions in two variables satisfying an over-determined (but compatible) system of PDEs, and the fundamental solutions will be representable as suitably chosen double Laplace integrals. The paper is organized as follows:
in Section 2 we introduce the basic objects and definitions, recalling how to explicitly construct the BOPs from the matrix of bimoments. We also prove that the BOPs uniquely determine the bi-moment functional: this is the analog in this setting of Favard's Theorem which allows to reconstruct a moment functional from any sequence of polynomials which satisfy a three–term recurrence relation.
In Section 3 we introduce the definition of semiclassical functionals and then prove that (under certain general assumpions) they are representable as integrals of suitable 2-forms over Cartesian products of complex paths. The starting point is the fact already mentioned that the generating function of bi-moments now depends on two variables z, w and satisfies an over-determined system of PDEs. We will prove the compatibility of this system (in the class of cases specified in the text) and then we will solve it. The solutions that we obtain (in the cases we consider) are entire functions of both variables z, w so that one could derive bounds on the growth of the bi-moments (the coefficients of the Taylor series centered at $z = 0 = w$).
It should also be remarked that all semiclassical linear moment functionals can be recovered as a special case of bilinear ones (see Remark 3.1): this correspond to the fact that one-matrix models can be recovered from two-matrix models in which one of the measures is Gaussian.

2 Definitions and first properties

By bi-moment functional we mean a functional $\mathcal{L}$ on the tensor product of two copies of the space of polynomials

$$\mathcal{L} : \mathbb{C}[x] \otimes \mathbb{C}[y] \to \mathbb{C} \ . \tag{2-1}$$

Although the two polynomial spaces are just copies of the same space, we use two different indeterminates x and y in order to distinguish them.
Such a functional is uniquely determined by its bi-moments

$$\mu_{ij} := \mathcal{L}(x^i | y^j). \tag{2-2}$$

It makes sense to look for bi-orthogonal polynomials. We recall their definition and some standard facts [16, 4]

[3] In Case A and if $A(x) \not\equiv 0$ there is a linear constraint on the initial conditions for the recurrence relation, which decreases the dimension of solution space by one. If $A(x) \equiv 0$ then the solutions of the functional equation can be found easily.

Definition 2.1 *Two sequences of polynomials* $\{\pi_n(x)\}_{n\in\mathbb{N}}$ *and* $\{\sigma_n(y)\}_{n\in\mathbb{N}}$ *of exact degree* n *are said to be biorthogonal with respect to the bi-moment functional* $\mathcal{L}$ *if*

$$\mathcal{L}(\pi_n|\sigma_m) = \delta_{nm} \ . \tag{2-3}$$

If such two sequences exist then we denote by $\{p_n(x)\}_{n\in\mathbb{N}}$ and $\{s_n(y)\}_{n\in\mathbb{N}}$ the corresponding sequences of monic polynomials, which then satisfy

$$\mathcal{L}(p_n|s_m) = h_n\delta_{nm} \ , \qquad h_n \neq 0 \ , \forall n \in \mathbb{N}. \tag{2-4}$$

It is an adaptation of the classical result for orthogonal polynomials to write a formula for the monic sequences

Proposition 2.1 The biorthogonal polynomials exist if and only if

$$\Delta_n \neq 0, \ n \in \mathbb{N}, \qquad \Delta_n := \det \begin{pmatrix} \mu_{0,0} & \mu_{0,1} & \cdots & \mu_{0,n-1} \\ \mu_{1,0} & \mu_{1,1} & \cdots & \mu_{1,n-1} \\ \vdots & \cdots & \cdots & \vdots \\ \mu_{n-1,0} & \mu_{n-1,1} & \cdots & \mu_{n-1,n-1} \end{pmatrix} \ , \tag{2-5}$$

Under this hypothesis the monic sequences $\{p_n\}_{n\in\mathbb{N}}$ and $\{s_n\}_{n\in\mathbb{N}}$ are given by the formulas

$$p_n(x) := \frac{1}{\Delta_n} \det \begin{pmatrix} \mu_{0,0} & \cdots & \mu_{0,n-1} & 1 \\ \mu_{1,0} & \cdots & \mu_{1,n-1} & x \\ \vdots & \cdots & \cdots & \vdots \\ \mu_{n,0} & \cdots & \mu_{n,n-1} & x^n \end{pmatrix} \ ; \tag{2-6}$$

$$s_n(y) := \frac{1}{\Delta_n} \det \begin{pmatrix} \mu_{0,0} & \cdots & \mu_{0,n-1} & \mu_{0,n} \\ \mu_{1,0} & \cdots & \mu_{1,n-1} & \mu_{1,n} \\ \vdots & \cdots & \cdots & \vdots \\ 1 & \cdots & y^{n-1} & y^n \end{pmatrix} \ . \tag{2-7}$$

The proof of this simple proposition is essentially the same as for the orthogonal polynomials and it is left to the reader (see [4, 16]).
With formula (2-7) we can also compute

$$\mathcal{L}(p_n|s_m) = \frac{\Delta_{n+1}}{\Delta_n}\delta_{nm} \ . \tag{2-8}$$

The relation with the normalized polynomials is

$$\pi_n(x) = c_n p_n(x) \ ; \ \ \sigma_n(y) := \tilde{c}_n s_n(y) \ , \tag{2-9}$$

where the complex constants c_n and $\tilde{c}_n$ are such that $c_n\tilde{c}_n = \frac{\Delta_{n+1}}{\Delta_n}$.
If biorthogonal polynomials exist they in general do not satisfy a three terms recurrence relation as for the ordinary orthogonal polynomials: they rather satisfy recurrence relations which generally are not of finite bands

$$x\pi_n(x) = \gamma_n\pi_{n+1}(x) + \sum_{j=0}^{n} a_j(n)\pi_{n-j}(x) \tag{2-10}$$

$$y\sigma_n(y) = \tilde{\gamma}_n\sigma_{n+1}(y)\sum_{j=0}^{n} b_j(n)\sigma_{n-j}(y) \ . \tag{2-11}$$

In the case of orthogonal polynomials the three terms recurrence relation is sufficient for reconstructing the moment functional (Favard's Theorem [2]). A natural question is whether the recurrence relations (2-10, 2-11) are also sufficient for the existence of a moment bifunctional for which the two sequences are bi-orthogonal polynomials. Note that the specification of the numbers $\gamma_n, \alpha_i(n)$, $i \leq n$ and $\tilde{\gamma}_n, \beta_i(n)$, $i \leq n$ determines uniquely the two sequences of polynomials (with the understanding that $\pi_{-n} \equiv 0 \equiv \sigma_{-n}$) in Eqs. (2-10,2-11) provided that $\gamma_n \neq 0 \neq \tilde{\gamma}_n$, $\forall n \in \mathbb{N}$. The following theorem answers positively to the existence of the moment bifunctional

Theorem 2.1 [Favard-like Theorem for biorthogonal polynomials] If the constants $\gamma_n, \tilde{\gamma}_n$ do not vanish for all $n \in \mathbb{N}$ then there exists a unique moment bifunctional $\mathcal{L}$ for which the two sequences of polynomials π_n, σ_n as in Eq. (2-10, 2-11) are biorthogonal.

Proof. As for the ordinary Favard's theorem we proceed to the construction of the bi-moments $\mu_{ij} = \mathcal{L}(x^i|y^j)$ by induction. We introduce the associated monic polynomials by defining

$$p_n(x) := \frac{1}{\pi_0}\pi_n(x)\prod_{k=0}^{n-1}\gamma_k \ , \qquad p_0(x) \equiv 1, \tag{2-12}$$

$$s_n(y) := \frac{1}{\sigma_0}\sigma_n(y)\prod_{k=0}^{n-1}\tilde{\gamma}_k \ , \qquad s_0(y) \equiv 1 \ . \tag{2-13}$$

The corresponding recurrence relations have the same form as in Eq. (2-10, 2-11) except that now the constants $\gamma_n, \tilde{\gamma}_n$ are replaced by 1.
The first moment μ_{00} is fixed by the requirement

$$1 = \mathcal{L}(\pi_0|\sigma_0) = \mu_{00}\pi_0\sigma_0 \ , \tag{2-14}$$

since the polynomials $\pi_0, \ \sigma_0$ are just nonzero constants.
Suppose now that the moments μ_{ij} have already been defined for $i, j < N$. We need then to define the moments μ_{Nj} for $j = 0, \dots N-1$, and μ_{iN} for $i = 0, \dots, N-1$ and μ_{NN}. By imposing the orthogonality

$$0 = \mathcal{L}(p_N|s_0) = \mu_{N0} + \dots \ , \tag{2-15}$$

we define μ_{N0}, where the dots represent an expression which contains only moments already defined (i.e. μ_{i0}, $i < N$). We define by induction on j the moments μ_{Nj}, the first having been defined above. We have, for $j < N-1$

$$0 = \mathcal{L}(p_N|s_{j+1}) = \mu_{N,j+1} + \dots \ , \tag{2-16}$$

where again the dots is an expression involving only previously defined moments. This defines $\mu_{N,j+1}$. We can repeat the arguments for the moments μ_{iN}, $i < N$ by reversing the role of the p_i's and s_j's.
Finally the moment μ_{NN} is defined by

$$\det\begin{pmatrix} \mu_{00} & \cdots & \mu_{0N} \\ \vdots & & \vdots \\ \mu_{N0} & \cdots & \mu_{NN} \end{pmatrix} = \frac{1}{\pi_0\sigma_0}\prod_{k=0}^{N-1}\gamma_k\tilde{\gamma}_k \ , \tag{2-17}$$

where the only unknown is precisely μ_{NN} and its coefficient in the LHS does not vanish since the corresponding minor is just

$$\det\begin{pmatrix} \mu_{00} & \cdots & \mu_{0N-1} \\ \vdots & & \vdots \\ \mu_{N-10} & \cdots & \mu_{N-1N-1} \end{pmatrix} = \frac{1}{\pi_0\sigma_0}\prod_{k=0}^{N-2}\gamma_k\tilde{\gamma}_k \neq 0 \ . \tag{2-18}$$

This completes the definition of the moment bifunctional $\mathcal{L}$. Q.E.D.

We now turn our attention to some specific class of bilinear functionals $\mathcal{L}$. We do not require for the analysis to come that the biorthogonal polynomials exist, although for applications to multimatrix models this is essential. In those applications the determinants Δ_n are proportional to the partition functions for the corresponding multi-matrix integrals (up to a multiplicative factor of $n!$) and are also interpretable as tau functions of KP and 2-Toda hierarchies [17, 18]

3 Bilinear semiclassical functionals

The notion of semiclassical for ordinary moment functionals and the applications to random matrices suggest the following

Definition 3.1 *We say that a bilinear functional $\mathcal{L} : \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y] \to \mathbb{C}$ is* semiclassical *if there exist four polynomials $A_1(x), B_1(x)$ and $A_2(y), B_2(y)$ of degrees $a_1+1, b_1+1, a_2+1, b_2+1$ respectively, such that the following distributional equations are fulfilled*

$$\left\{ \begin{array}{l} (D_x \circ B_1(x) + A_1(x)) \otimes 1\mathcal{L} = B_1(x) \otimes y\mathcal{L} \\ 1 \otimes (D_y \circ B_2(y) + A_2(y))\,\mathcal{L} = x \otimes B_2(y)\,\mathcal{L} \ . \end{array} \right. \tag{3-1}$$

Explicitly these equations mean that, for any polynomials $p(x), s(y)$

$$\mathcal{L}\Big(- B_1(x)p'(x) + A_1(x)p(x)\Big|s(y)\Big) = \mathcal{L}\Big(B_1(x)p(x)\Big|ys(y)\Big) \tag{3-2}$$

$$\mathcal{L}\Big(p(x)\Big| - B_2(y)s'(y) + A_2(y)s(y)\Big) = \mathcal{L}\Big(xp(x)\Big|B_2(y)s(y)\Big) \tag{3-3}$$

Remark 3.1 We mentioned that any semi-classical moment functional is –in a certain sense– a special case of bilinear semi-classical functional. We want to clarify this relation here. Let us consider a semiclassical bifunctional in which $A_2(y) = ay$ and $B_2(y) = 1$. The defining relations become

$$\mathcal{L}(-B_1p' + A_1p|s) = \mathcal{L}(B_1p|ys)\ ,\quad \mathcal{L}(p| - s' + ays) = \mathcal{L}(xp|s). \tag{3-4}$$

In particular for $s(y) = 1$ the second in Eq. (3-4) reads

$$\mathcal{L}(p|y) = \frac{1}{a}\mathcal{L}(xp|1)\ . \tag{3-5}$$

The claim that the reader can check directly is that the moment functional $\mathcal{L}_r(\cdot) := \mathcal{L}(\cdot|1)$ is a semiclassical functional in the sense explained in the introduction with $A(x) = A_1(x) - \frac{x}{a}B_1(x)$ and $B(x) = B_1(x)$. It will be clear later on that this "reduction" corresponds to a partial integration of a Gaussian weight.

In analogy with the orthogonal polynomials case we also define the class

Definition 3.2 *For a semi-classical bi-functional $\mathcal{L}$ we define its* bi-class *as the pair of integers*

$$(s_1, s_2) = (\max(a_1, b_1) + 1, \max(a_2, b_2) + 1)\ . \tag{3-6}$$

Note that from the definition some recurrence relations follow for the moments μ_{ij}. In order to spell them out we introduce the following notations for the coefficients of the polynomials A_i, B_i

$$A_1(x) = \sum_{j=0}^{a_1+1} \alpha_1(j)x^j\ ;\quad B_1(x) := \sum_{j=0}^{b_1+1} \beta_1(j)x^j \tag{3-7}$$

$$A_2(y) = \sum_{j=0}^{a_2+1} \alpha_2(j)y^j\ ;\quad B_2(y) := \sum_{j=0}^{b_2+1} \beta_2(j)y^j\ . \tag{3-8}$$

Then the aforementioned recurrence relations are given by

Proposition 3.1 The moments μ_{ij} of the classical bi-functional $\mathcal{L}$ are subject to the relations

$$\sum_{j=0}^{b_1+1} \beta_1(j)\mu_{n+j,m+1} = -n\sum_{j=0}^{b_1+1} \beta_1(j)\mu_{n-1+j,m} + \sum_{j=0}^{a_1+1} \alpha_1(j)\mu_{n+j,m} \tag{3-9}$$

$$\sum_{j=0}^{b_2+1} \beta_2(j)\mu_{n+1,m+j} = -m\sum_{j=0}^{b_2+1} \beta_2(j)\mu_{n,m-1+j} + \sum_{j=0}^{a_2+1} \alpha_2(j)\mu_{n,m+j}\ . \tag{3-10}$$

Proof.
From the definition of semi-classicity by setting $p(x) = x^n$ and $s(y) = y^m$ in the two relations (3-2, 3-3). Q.E.D.

The two recurrence relations give an overdetermined system for the moments: it is not guaranteed a priori that solutions exist and if they do, how many. There are now four different cases, according to $\deg(B_i) \lesseqgtr \deg(A_i)$; we address in the present paper the case $\deg(A_i) > \deg(B_i)$, $i = 1, 2$ (most relevant in the applications to random matrix models) which is the analog of Case B in [13] and we could call "Case BB". The other cases have less interesting applications in matrix models because they correspond to potentials (in a sense which will be clear below) which are bounded at infinity. They are certainly interesting from the point of view of Eqs. (3-9, 3-10); for example it is a simple exercise to check that if $\deg(B_1) = \deg(B_2) = 1$ and $\deg(A_1) = \deg(A_2) = 0$ then in general no nontrivial solutions exist for Eqs (3-9, 3-10).

For the rest of this paper we will make the following
Assumptions ($\mathcal{A}$)

$$\deg(B_i) + 1 \le \deg(A_i)\ ,\qquad i = 1, 2. \tag{3-11}$$

Moreover in the case $\deg(B_1) + 1 = \deg(A_1)$ and $\deg(B_2) + 1 = \deg(A_2)$ we impose

$$\det\begin{pmatrix} \alpha_1(a_1+1) & \beta_1(b_1+1) \\ \beta_2(b_2+1) & \alpha_2(a_2+1)\end{pmatrix} \neq 0 \ \text{ when } a_1 = b_1 + 1,\ a_2 = b_2 + 1\ . \tag{3-12}$$

Under this assumption we can prove

Proposition 3.2 The solutions to Eqs. (3-9, 3-10) form a vector space of dimension $M := s_1 \cdot s_2 = (a_1+1)\cdot(a_2+1)$.

Proof. The fact that the space of solutions is a vector space is obvious from the linearity of the defining equations. We need to prove the assertion regarding the dimension.
We define the (possibly formal) generating function of moments

$$F(z,w) := \sum_{j,k=0}^{\infty} \frac{z^j w^k}{j!k!}\mu_{jk} = \mathcal{L}\Big(\mathrm{e}^{xz}\Big|\mathrm{e}^{yw}\Big)\ . \tag{3-13}$$

From the recursion relation for the moments or (equivalently) from the definition of semi-classicity, it follows that such function satisfies the system of PDEs

$$\left\{\begin{array}{l} \Big[(\partial_z + w)B_2(\partial_w) - A_2(\partial_w)\Big]F(z,w) = 0 \\ \Big[(\partial_w + z)B_1(\partial_z) - A_1(\partial_z)\Big]F(z,w) = 0 \end{array}\right. \tag{3-14}$$

Conversely, any solution of this system which is analytic at $z = 0 = w$ provides a semi-classical bi-moment functional associated with the data A_i, B_i. We now count the solutions of this system. It will be clear later on that all the solutions are analytic at $z = 0 = w$ (in fact entire) so that any solution does define a moment functional.

The system (3-14) is a higher order overdetermined system of PDEs for the single function (or formal power series) $F(z,w)$ and the compatibility is readily seen since

$$\Big[(\partial_z + w)B_2(\partial_w) - A_2(\partial_w), (\partial_w + z)B_1(\partial_z) - A_1(\partial_z)\Big] = \tag{3-15}$$

$$= \Big[(\partial_z + w)B_2(\partial_w), (\partial_w + z)B_1(\partial_z)\Big] = \tag{3-16}$$

$$= \Big[(\partial_z + w), (\partial_w + z)\Big]B_2(\partial_w)B_1(\partial_z) = (1-1)B_2(\partial_w)B_1(\partial_z) = 0. \tag{3-17}$$

Now we express the system as a first order system of PDE's on the suitable jet extension. Let us introduce the notation

$$F_{\mu,\nu}(z,w) := {\partial_z}^{\mu}{\partial_w}^{\nu}F(z,w). \tag{3-18}$$

The proof now proceeds according to the three different cases:

Case BB1: $\deg(A_i) \geq \deg(B_i) + 1$, $i = 1, 2$;

Case BB2: $\deg(A_1) = \deg(B_1) + 1$ but $\deg(A_2) > \deg(B_2) + 1$ (or vice-versa);

Case BB3: $\deg(A_1) = \deg(B_1) + 1$, $\deg(A_2) = \deg(B_2) + 1$.

For convenience we set the leading coefficients of the two polynomials A_i to unity as this does not affect the dimension of the solution space of the system but make the formulas to come shorter to write.
In Case BB1 ($a_i \geq b_i + 2$) we can write the two first order systems for the systems

$$\left\{\begin{array}{ll} \partial_w F_{\mu,\nu} = F_{\mu,\nu+1} & \mu = 0,\cdots,a_1,\ \nu = 0,\cdots a_2 - 1 \\ \partial_w F_{\mu,a_2} = \displaystyle\sum_{k=0}^{b_2+1}\beta_2(k)(wF_{\mu,k} + F_{\mu+1,k}) - \sum_{k=0}^{a_2}\alpha_2(k)F_{\mu,k} & \mu = 0..a_1 - 1 \\ \partial_w F_{a_1,a_2} = \displaystyle\sum_{k=0}^{b_2+1}\beta_2(k)\left[wF_{a_1,k} + \left(\sum_{j=0}^{b_1+1}\beta_1(j)\Big(zF_{j,k} + F_{j,k+1}\Big) - \sum_{j=0}^{a_1}\alpha_1(j)F_{j,k}\right)\right] - \sum_{k=0}^{a_2}\alpha_2(k)F_{a_1,k} & \end{array}\right. \tag{3-19}$$

$$\left\{\begin{array}{ll} \partial_z F_{\mu,\nu} = F_{\mu+1,\nu} & \mu = 0,\cdots,a_1 - 1,\ \nu = 0,\cdots a_2 \\ \partial_z F_{a_1,\nu} = \displaystyle\sum_{j=0}^{b_1+1}\beta_1(j)(zF_{j,\nu} + F_{j,\nu+1}) - \sum_{j=0}^{a_1}\alpha_1(j)F_{j,\nu} & \nu = 0..a_2 - 1 \\ \partial_z F_{a_1,a_2} = \displaystyle\sum_{j=0}^{b_1+1}\beta_1(j)\left[zF_{j,a_2} + \left(\sum_{k=0}^{b_2+1}\beta_2(k)\Big(wF_{j,k} + F_{j+1,k}\Big) - \sum_{k=0}^{a_2}\alpha_2(k)F_{j,k}\right)\right] - \sum_{j=0}^{a_1}\alpha_1(j)F_{j,a_2} & \end{array}\right. \tag{3-20}$$

Note that the two systems are consistent for the unknowns $F_{\mu,\nu}$, $\mu = 0,...,a_1$, $\nu = 0,...,a_2$ if we have $b_i + 2 \leq a_i$, $i = 1, 2$.

In Case BB2 with $a_1 = b_1 + 1$ the second system is not anymore consistent because the RHS of the third equation in system (3-20) contains F_{a_1+1,a_2}. It must be replaced by

$$\left\{ \begin{array}{ll} \partial_z F_{\mu,\nu} = F_{\mu+1,\nu} & \mu = 0, \cdots, a_1 - 1, \ \nu = 0, \cdots a_2 \\ \partial_z F_{a_1,\nu} = \sum_{j=0}^{a_1} \Big(\beta_1(j)(zF_{j,\nu} + F_{j,\nu+1}) - \alpha_1(j) F_{j,\nu} \Big) & \nu = 0..a_2 - 1 \\ \partial_z F_{a_1,a_2} = \sum_{j=0}^{a_1} \beta_1(j) \left[zF_{j,a_2} + \left(\sum_{k=0}^{b_2+1} \beta_2(k) w F_{j,k} - \sum_{k=0}^{a_2} \alpha_2(k) F_{j,k} \right) \right] - \sum_{j=0}^{a_1} \alpha_1(j) F_{j,a_2} + & \\ + \sum_{j=0}^{a_1-1} \sum_{k=0}^{b_2+1} \beta_2(k)\beta_1(j) F_{j+1,k} + \beta_1(a_1) \sum_{k=0}^{b_2+1} \beta_2(k) \left(\sum_{j=0}^{a_1} \Big(\beta_1(j)(zF_{j,k} + F_{j,k+1}) - \alpha_1(j) F_{j,k} \Big) \right) & \end{array} \right. \tag{3-21}$$

Finally in the Case BB3 ($a_1 = b_1 + 1$ and $a_2 = b_2 + 1$) we have the two systems

$$\left\{ \begin{array}{ll} \partial_z F_{\mu,\nu} = F_{\mu+1,\nu} & \mu = 0, \cdots, a_1 - 1, \ \nu = 0, \cdots a_2 \\ \partial_z F_{a_1,\nu} = \sum_{j=0}^{a_1} \Big(\beta_1(j)(zF_{j,\nu} + F_{j,\nu+1}) - \alpha_1(j) F_{j,\nu} \Big) & \nu = 0..a_2 - 1 \\ (1 - \beta_1(a_1)\beta_2(a_2)) \partial_z F_{a_1,a_2} = \sum_{j=0}^{a_1} \beta_1(j) \left[zF_{j,a_2} + \sum_{k=0}^{a_2} \Big(w\beta_2(k) - \alpha_2(k) \Big) F_{j,k} \right] + & \\ - \sum_{j=0}^{a_1} \alpha_1(j) F_{j,a_2} + \sum_{j=0}^{a_1-1} \sum_{k=0}^{a_2} \beta_1(j)\beta_2(k) \partial_z F_{j,k} & \end{array} \right. \tag{3-22}$$

and a similar system for the ∂_w derivative. Note that in the third equation the derivatives $\partial_z F_{j,k}$ are defined by the first and second equation.
Since now $(1 - \beta_1(a_1)\beta_2(a_2)) \neq 0$ as per the **Assumption** (which is $(\alpha_1(a_1 + 1)\alpha_2(a_2 + 2) - \beta_1(a_1)\beta_2(a_2)) \neq 0$ if we do not assume that the polynomials A_1, A_2 are monic) then the system is still well defined; on the other hand, if $(1 - \beta_1(a_1)\beta_2(a_2)) = 0$ then the last equation becomes a *constraint*[4].

It is a lengthy but straightforward check that the two systems are indeed compatible in each of the three cases. Since the size of the system is $M = (a_1 + 1) \cdot (a_2 + 1) = s_1 s_2$ then there are precisely M linearly independend solutions. Q.E.D.

Remark 3.2 In principle we would not have to check the compatibility because we will construct later $M = s_1 s_2$ solutions to the system, which therefore will be proven to be compatible *a posteriori*: the point of Prop. 3.2 is principally that the dimension of the solution space certainly does not exceed M because that is the dimension of a closed system in the jet space.

The Proposition implies that the recurrence relations (3-9, 3-10) determine uniquely the functional $\mathcal{L}$ in terms of the moment μ_{ij} with $i = 0, \ldots, a_1, j = 0, \ldots, a_2$. We need to produce $M = s_1 s_2$ linearly independent semiclassical functionals associated to the same data (A_1, B_1, A_2, B_2) by means of integral representations.
Equivalently we can produce integral representation for the M linearly independent solutions of the overdetermined system of PDE's (3-14). It is precisely in this form that we will solve the problem, showing contextually that the generating functions are indeed entire functions of w, z. The starting point is to assume that such an integral representation exists: so suppose that

$$F(z, w) = \int_{\Gamma_{(x)}} \int_{\Gamma_{(y)}} \mathrm{d}x \wedge \mathrm{d}y W(x, y) \mathrm{e}^{xz+yw} \ , \tag{3-23}$$

is a double Laplace integral representation for a solution of (3-14)[5].
Plugging such representation in the two equations in (3-14) and assuming that the contours are so chosen as to allow integration by parts without boundary terms, we obtain two first order equations for the bi-weight $W(x, y)$

$$\Big(B_1(x)\partial_x + A_1(x) + B_1'(x) \Big) W(x, y) = y\, B_1(x) W(x, y) \tag{3-24}$$

$$\Big(B_2(y)\partial_y + A_2(y) + B_2'(y) \Big) W(x, y) = x\, B_2(y) W(x, y) \ . \tag{3-25}$$

We make the **Assumption** $(\mathcal{B})$ that each pair (A_i, B_i) are relatively prime or at most share a factor $(x - c)$ (or $(y - s)$). The reason is similar to the case of ordinary semiclassical functionals. We will return on this genericity assumption later on.

[4]We are not going to examine this case in this paper because it is more natural to study in the context of semiclassical functionals of type AB or AA, i.e. when $\deg(A_i) \leq \deg(B_i)$

[5]In principle one could integrate the two-form $W(x, y)\mathrm{e}^{xz+yw}\mathrm{d}x \wedge \mathrm{d}y$ over any 2-cycle, but here we do not need such generality

The two differential equations (3-24,3-25) form an overdetermined system for the biweight $W(x,y)$ which is compatible and can be solved to give the only solution (up to a multiplicative nonzero constant)

$$W(x,y) = W_1(x)W_2(y)\mathrm{e}^{xy} = \exp\left(-V_1(x) - V_2(x) + xy\right) \ , \tag{3-26}$$

$$\frac{W_1'(x)}{W_1(x)} = \frac{A_1(x)+B_1'(x)}{B_1(x)}, \qquad \frac{W_2'(y)}{W_2(y)} = \frac{A_2(y)+B_2'(y)}{B_2(y)} \ , \tag{3-27}$$

$$V_1(x) := \int \mathrm{d}x \frac{A_1(x)+B_1'(x)}{B_1(x)} \tag{3-28}$$

$$V_2(y) := \int \mathrm{d}y \frac{A_2(y)+B_2'(y)}{B_2(y)} \ . \tag{3-29}$$

We call the two functions $V_1(x)$, $V_2(y)$ the *potentials* (borrowing the name from the statistical mechanic and random matrix context).
Note that if there are nonzero residues at the poles of $\frac{A_i+B_i'}{B_i}$ then the corresponding potential have logarithmic singularities or poles. The general form of the biweight is

$$W_1(x) := \prod_{j=1}^{p_1}(x-X_j)^{\lambda_j} \exp\left[V_1^+(x) + \frac{M_1(x)}{\prod_{j=1}^{p_1}(x-X_j)^{g_j}}\right] \ , \tag{3-30}$$

$$\deg(M_1) \le \sum_{j=1}^{p_1} g_j \ , \ M_1(X_j) \neq 0$$

$$W_2(y) := \prod_{k=1}^{p_2}(y-Y_j)^{\rho_k} \exp\left[V_2^+(y) + \frac{M_2(y)}{\prod_{k=1}^{p_2}(y-Y_k)^{h_k}}\right] \ , \tag{3-31}$$

$$\deg(M_2) \le \sum_{k=1}^{p_2} h_k \ , \ M_2(Y_k) \neq 0 \ .$$

In this formulas and in the rest of the paper X_j denote the zeroes of $B_1(x)$, g_j+1 the corresponding multiplicities and $-\lambda_j$ are the residues at X_j of the differential $\mathrm{d}V_1(x)$; similarly, Y_k denote the zeroes of $B_2(y)$, h_k+1 the corresponding multiplicities and $-\rho_k$ the residues at Y_k of the differential $\mathrm{d}V_2(y)$.

The bi-class of the corresponding semiclassical bifunctional is then the total degree of the divisor of poles of the derivatives of the two potentials on the Riemann spheres whose affine coordinates are x and y

$$s_1 = d_1 + \sum_{j=1}^{p_1}(g_j+1) \ , \quad s_2 = d_2 + \sum_{j=1}^{p_2}(h_j+1) \ . \tag{3-32}$$

We will also use the notations $X_0 = \infty \in \mathbb{P}^1_x$, $Y_0 = \infty \in \mathbb{P}^1_y$.

3.1 The functionals

We will define two sets of paths in the two punctured Riemann spheres $\mathbb{P}^1_x$ and $\mathbb{P}^1_y$. We focus on the first sphere, the paths in the second being defined in analogous way.
More precisely we define s_1 "homologically" independent paths in $\mathbb{P}^1_x \setminus C_x$ and s_2 paths in $\mathbb{P}^2_y \setminus C_y$ where C_x and C_y are suitable union of cuts and points: for example the set C_x is the union of all poles and essential singularities of $W_1(x)$ and cuts extending from the branchpoints to infinity.
The reference to the homology is not in the ordinary sense: here we are considering in fact the *relative* homology of the cut-punctured sphere with prescribed sectors around the punctures.
We first define some sectors $S_k^{(j)}$, $j = 1,\dots p_1$, $k = 0,\dots g_j - 1$. around the points X_j for which $g_j > 0$ (the multiple zeroes of $B_1(x)$) in such a way that

$$\Re\left(V_1(x)\right) \underset{\substack{x \to X_j, \\ x \in S_k^{(j)}}}{\longrightarrow} +\infty \ . \tag{3-33}$$

The number of sectors for each pole is the degree of that pole in the exponential part of $W_1(x)$, that is d_1+1 for the pole at infinity and g_j for the j-th pole. Explicitly

$$S_k^{(0)} := \left\{ x :\in \mathbb{C}; \ \frac{2k\pi - \frac{\pi}{2} + \epsilon}{d_1+1} < \arg(x) + \frac{\arg(v_{d_1+1})}{d_1+1} < \frac{2k\pi + \frac{\pi}{2} - \epsilon}{d_1+1} \right\} \ , \quad k = 0\dots d_1 \ ; \tag{3-34}$$

$$S_k^{(j)} := \left\{ x :\in \mathbb{C};\ \ \frac{2k\pi - \frac{\pi}{2} + \epsilon}{g_j} < \arg(x - X_j) + \frac{\arg(M_1(X_j))}{g_j} < \frac{2k\pi + \frac{\pi}{2} - \epsilon}{g_j} \right\} , \qquad (3\text{-}35)$$
$$k = 0, \ldots, g_j - 1, \quad j = 1, \ldots, p_1 \ .$$

These sectors are defined precisely in such a way that approaching any of the essential singularities (i.e. an X_j such that $g_j > 0$) the function $W_1(x)$ tends to zero faster than any power.

Definition of the contours

The definition of the contours follows directly [15], but we have to repeat it in both Riemann spheres. For the sake of completeness we recall the way they are defined.

1. For any X_j for which there is *no essential singularity* (i.e. $g_j = 0$), then we have two subcases
   - (a) Corresponding to the X_j's which are branch points or a pole ($\lambda_j \in \mathbb{C} \setminus \mathbb{N}$), we take a loop starting at infinity in some fixed sector $S^{(0)}_{k_L}$ encircling the singularity and going back to infinity in the same sector.
   - (b) For the X_j's which are regular points ($\lambda_j \in \mathbb{N}$) we take a line joining X_j to infinity and approaching ∞ in the same sector $S^{(0)}_{k_L}$ as before.
2. For any X_j for which there is an essential singularity (i.e. for which $g_j > 0$) we define g_j contours starting from X_j in the sector $S^{(j)}_0$ and returning to X_j in the next (counterclockwise) sector. Finally we join the singularity X_j to ∞ by a path approaching ∞ within the sector $S^{(0)}_{k_L}$ chosen at point 1(a).
3. For $X_0 := \infty$ we take d_1 contours starting at X_0 in tha sector $S^{(0)}_k$ and returning at X_0 in the sector $S^{(0)}_{k+1}$.[6].

For later convenience we also fix a sector $\mathcal{S}_L$ of width $\beta < \pi - \epsilon$ which contains the sector $S^{(0)}_{k_L}$ used above. The picture below gives an example of the typical situation, where the light grey sector represents $\mathcal{S}_L$. We will make use also of the sector $\mathcal{E}$ which is a sector within the dual sector[7] of $\mathcal{S}_L$ (in dark shade of grey in the picture): it is not difficult to realize that we can always arrange contours in such a way that $\mathcal{E}$ is a small sector below the real positive axis (if the leading coefficient of V_1^+ is real and positive, otherwise the whole picture should be rotated appropriately).
We shall also require that all contours do not intersect except possibly at some X_j and that each closed loop should either encircle only one singularity or have one of the X_j on its support.

The result of this procedure produces precisely s_1 contours. By virtue of Cauchy's theorem the choice is largely arbitrary.
An important feature for what follows is that *when a contour Γ_j is closed (on the sphere $\mathbb{P}^1_x$), then $W_1(x)$ has a singularity and/or is unbounded in the region inside Γ_j*. We will call this property the **Property ($\wp$)**.

We then define the fundamental functionals by

$$\mathcal{L}_{ij}(x^n|y^m) := \int_{\Gamma_i^{(x)} \times \Gamma_j^{(y)}} \mathrm{d}x \wedge \mathrm{d}y \, W_1(x) W_2(y) \mathrm{e}^{xy} x^n y^m \ , \qquad (3\text{-}36)$$
$$i = 1, \ldots, s_1, \ j = 1, \ldots s_2 \ , n, m \in \mathbb{N}.$$

We point out that such contours are chosen so that the corresponding functionals are defined on any monomials $x^j y^k$ and such that integration by parts does not give any boundary contribution. Each such functional is a semi-classical functional associated to the data A_1, B_1, A_2, B_2 and their number is precisely the expected number $s_1 s_2$ for the solutions of Eqs. (3-14) for the generating functions. The problem now is to show that they are linearly independent.

Remark 3.3 A special care should be directed at the case $d_1 = d_2 = 1$, i.e. when $a_1 = b_1 + 1$ and $a_2 = b_2 + 1$. Indeed in this circumstance the two polynomials $V_1^+(x) = \frac{\delta}{2}x^2 + ..$ and $V_2^+(y) = \frac{\sigma}{2}y^2 + ..$ are just quadratic. The biweight $W(x, y)$ has then the form

$$W(x, y) = \exp\left(-\frac{\delta}{2}x^2 - \frac{\sigma}{2}y^2 + xy + \ldots\right)[\ldots] \ . \qquad (3\text{-}37)$$

The condition on the determinant (3-12) is precisely the nondegeneracy of the quadratic form $-\frac{\delta}{2}x^2 - \frac{\sigma}{2}y^2 + xy$. However, if $|\delta||\sigma| \leq 1$ then the integrals as we have defined are always divergent when two contours which stretch to infinity are involved. This simply means that we cannot choose the surface of integration in factorized form $\Gamma^{(x)} \times \Gamma^{(y)}$ but need to resort to a surface which is not factorized.
Alternatively we can analytically continue from the region of δ, σ for which the integrals are convergent.

[6]Note that in our assumptions on the degrees of A_i, B_i the degrees of the essential singularity at infinity satisfy $d_1 \geq 1 \leq d_2$

[7] We recall that for a given sector $\mathcal{S}$ centered around a ray $\arg(z) = \alpha_0$ with width $A < \pi$, the **dual sector** $\mathcal{S}^\vee$ is the sector centered around the ray $\arg(z) = \alpha_0 + \pi$ and with width $\pi - A$.

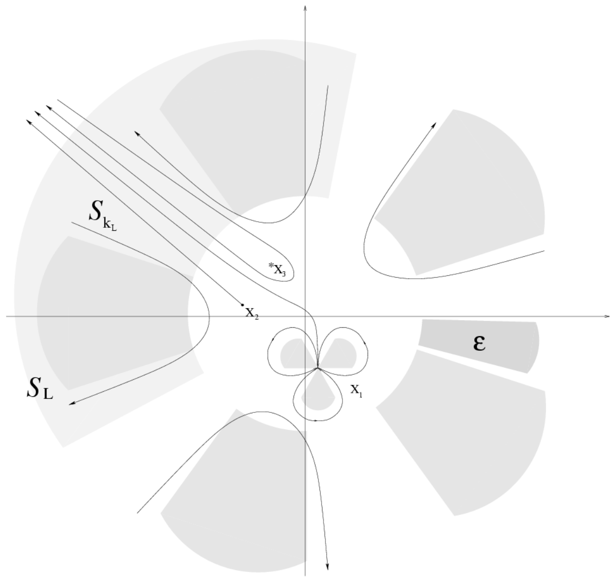

Figure 1: The set of contours in the x Riemann sphere $\mathbb{P}^1_x$. Here we have three zeroes of $B(x)$, X_1, X_2, X_3, and the singularity at infinity X_0 of order $d_1+1=5$. The zero X_1 has multiplicity $g_j+1=4$ and the corresponding essential singularity behaves like $\exp{(x-X_1)^{-3}}$, the zero X_2 is a regular point for $W_1(x)$, namely $\lambda_2 \in \mathbb{N}$ and finally the zero X_3 is either a branch point of W_1, in which case the cut extends to infinity "inside" the contour (in the picture), or a pole ($\lambda_3 \notin \mathbb{N}$).

Some important remarks are in order. Consider the generating functions associated to these contours

$$F_{ij}(z,w) := \int_{\Gamma_i^{(x)}\times\Gamma_j^{(y)}} \mathrm{d}x\wedge\mathrm{d}y\, W_1(x)W_2(y)\mathrm{e}^{xy}\mathrm{e}^{xz+yw} \ . \tag{3-38}$$

They are entire functions of z, w and hence are indeed generating functions of the bi-moment functionals $\mathcal{L}_{ij}(\cdot|\cdot)$. Indeed our assumptions on the degrees guarantees that V_i^+ have degree at least 2, which is sufficient to guarantee analyticity w.r.t. z, w in the whole complex plane.

Remark 3.4 If the index i corresponds to a bounded contour $\Gamma_i^{(x)}$ then $F_{ij}(z,w)$ is a function of exponential type in z (similarly for w if $\Gamma_j^{(y)}$ is bounded).

Remark 3.5 If the index i corresponds to one of the contours $\Gamma_i^{(x)}$ defined at point 1(a) or 1(b) above, then $F_{ij}(z,w)$ is of exponential type only for z in an appropriate sector which contains the sector $\mathcal{E}$ dual to the sector $\mathcal{S}_L$.

Before entering into the details of the proof of linear independence let us return to the Assumption ($\mathcal{B}$) about the pairs (A_i, B_i). Suppose that -say- A_1 and B_1 have a common factor $(x-c)^K$, $K\geq 1$ and that they have no other common factor. That is let us suppose that

$$A_1(x) = (x-c)^l\tilde{A}_1(x)\ ,\quad B_1(x) = (x-c)^r\tilde{B}_1(x)\ ,\qquad l>0<r,\ K:=\min(l,r)\ , \tag{3-39}$$

with $\tilde{A}_1(c)\neq 0\neq \tilde{B}_1(c)$. Then formula (3-27) would give

$$V_1'(x) = -\frac{W_1'(x)}{W_1(x)} = \frac{(x-c)^l\tilde{A}_1 + r(x-c)^{r-1}\tilde{B}_1 + (x-c)^r\tilde{B}_1'}{(x-c)^r\tilde{B}_1}\ , \tag{3-40}$$

so that the divisor of poles od $\mathrm{d}V_1(x)$ has degree *less* than s_1. Now we have two possible cases:
(i) if $l\geq r-1$ then we can recast Eq. (3-40) in the form

$$-\frac{W_1'(x)}{W_1(x)} = \frac{(x-c)^{l-r+1}\tilde{A}_1 + (r-1)\tilde{B}_1 + \left((x-c)\tilde{B}_1\right)'}{(x-c)\tilde{B}_1}\ . \tag{3-41}$$

which is equivalent to a problem in which the polynomials A_1, B_1 are substituted by $\underline{A_1} := (x-c)^{l-r+1}\tilde{A}_1 + (K-1)\tilde{B}_1$ and $\underline{B_1} := (x-c)\tilde{B}_1$ respectively, which now satisfy the assumption (**F**). In particular the definition of the contours

provides the correct number of distinct contours for the new pair $(\underline{A_1}, \underline{B_1})$, that is $s_1 - r + 1$ distinct contours (in the x plane). We need to recover $(K-1)s_2$ solutions if $l > r-1$ or $ls_2 = Ks_2$ if $l = r-1$.
(ii) If $l \le r-2$ then we can recast Eq. (3-40) in the form

$$-\frac{W_1'(x)}{W_1(x)} = \frac{\tilde{A}_1 + l(x-c)^{r-1-l}\tilde{B}_1 + \left((x-c)^{r-l}\tilde{B}_1\right)'}{(x-c)^{r-l}\tilde{B}_1} \ . \tag{3-42}$$

now equivalent to a problem in which the polynomials A_1, B_1 are substituted by $\underline{A_1} := \tilde{A}_1 + K(x-c)^{r-l-1}\tilde{B}_1$ and $\underline{B_1} := (x-c)^{r-l}\tilde{B}_1$ respectively, which do not have the factor $(x-c)$ in common and hence satisfy the assumption (**F**). The definition of the contours provides the correct number of distinct contours for the new pair $(\underline{A_1}, \underline{B_1})$, and we need to recover Ks_2 solutions.
The next proposition shows how to recover the missing solutions.

Proposition 3.3 If

$$A_1(x) = (x-c)^K \tilde{A}_1(x) \ , \quad B_1(x) = (x-c)^K \tilde{B}_1(x) \ , \quad K \ge 1 \ , \tag{3-43}$$

and $\tilde{A}_1(x)$, $\tilde{B}_1(x)$ do not vanish both at c then Eqs. (3-14) have also the solutions

$$F_k^{(j)}(z,w) = \mathrm{e}^{cz} \int_{\Gamma_k^{(y)}} \mathrm{d}y (y+z)^j \mathrm{e}^{y(w+c)} W_2(y) \ , \quad j = 0, ..., K-1. \tag{3-44}$$

Proof.
The fact that the functions (3-44) solve our system can be checked directly.
Indeed the first eq. in (3-14) is satisfied because the differential operator reads

$$(\partial_w + z)B_1(\partial_z) - A_1(\partial_z) = \left[(\partial_w + z)\tilde{B}_1(\partial_z) - \tilde{A}_1(\partial_z)\right](\partial_z - c)^K \ , \tag{3-45}$$

and the proposed solutions are linear combination of functions of the form $z^r \mathrm{e}^{cz} f_r(w)$, $r < K$ which are all in the kernel of $(\partial_z - c)^K$. The second equation in (3-14) now reads

$$\begin{aligned}
&[(\partial_z + w)B_2(\partial_w) - A_2(\partial_w)]\, \mathrm{e}^{cz} \int_{\Gamma_k^{(y)}} \mathrm{d}y (y+z)^j \mathrm{e}^{y(w+c)} W_2(y) = \\
&= c\, \mathrm{e}^{cz} \int_{\Gamma_k^{(y)}} \mathrm{d}y\, B_2(y)(y+z)^j \mathrm{e}^{y(w+c)} W_2(y) + \mathrm{e}^{cz} \int_{\Gamma_k^{(y)}} \mathrm{d}y \left(B_2(y)(\partial_z + w) - A_2(y) \right) (y+z)^j \mathrm{e}^{y(w+c)} W_2(y) = \\
&= \mathrm{e}^{cz} \int_{\Gamma_k^{(y)}} \mathrm{d}y \left(B_2(y)(c + \partial_z) - A_2(y) \right) (y+z)^j \mathrm{e}^{y(w+c)} W_2(y) + \mathrm{e}^{cz} \int_{\Gamma_k^{(y)}} \mathrm{d}y B_2(y) W_2(y)(y+z)^j \mathrm{e}^{yc} \partial_y (\mathrm{e}^{yw}) = \\
&= \mathrm{e}^{cz} \int_{\Gamma_k^{(y)}} \mathrm{d}y \left(B_2(y)(\partial_z + c) - A_2(y) \right) (y+z)^j \mathrm{e}^{y(w+c)} W_2(y) + \mathrm{e}^{cz} \int_{\Gamma_k^{(y)}} \mathrm{d}y B_2(y) W_2(y)(y+z)^j \mathrm{e}^{cy} \partial_y (\mathrm{e}^{yw}) = \\
&= \mathrm{e}^{cz} \int_{\Gamma_k^{(y)}} \mathrm{d}y\, W_2(y) \mathrm{e}^{y(w+c)} \left[B_2(y)\, [\partial_z - \partial_y] \right] (y+z)^j + \\
&\qquad + \mathrm{e}^{cz} \int_{\Gamma_k^{(y)}} \mathrm{d}y \left(W_2'(y) B_2(y) - (A_2(y) + B_1'(y)) W_2(y) \right) (y+z)^j \mathrm{e}^{y(w+c)} = 0.
\end{aligned}$$

In Case **(ii)** (or in Case **(i)** but with $l = r-1$) these solutions are precisely the Ks_2 missing solutions.
In Case **(i)** with $l \ge r$ only $l - 1 = K - 1$ among the solutions (3-44) are linearly independent from those defined in terms of the contour integrals. To see this we write the weight

$$-\frac{W_1'(x)}{W_1(x)} = \frac{r}{x-c} + \frac{\tilde{A}_1 + \tilde{B}_1'}{\tilde{B}_1} \ . \tag{3-46}$$

Since $\tilde{B}_1(c) \neq 0$ then $W_1(x)$ has a pole of order r at $x = c$ and can be written as

$$W_1(x) = (x-c)^{-r} w_1(x) \ , \tag{3-47}$$

with $w_1(x)$ analytic at $x = c$ and $w_1(c) \neq 0$. The contour which comes from infinity, encircles c and goes back to infinity can be retracted to a circle around the pole, so that the corresponding solutions given by the integral representation would be

$$\begin{aligned}
&\int_{\Gamma_y^{(k)}} \oint_{|x-c|=\epsilon} \mathrm{d}x \wedge \mathrm{d}y\, (x-c)^{-r} w_1(x) \mathrm{e}^{x(z+y)+wy} W_2(y) = \\
&= 2i\pi (r-1)! \int_{\Gamma_y^{(k)}} \mathrm{d}y \partial_x^{r-1} \left(w_1(x) \mathrm{e}^{x(z+y)} \right)\Big|_{x=c} W_2(y) \ .
\end{aligned}$$

Such a solution is clearly an appropriate linear combination of the $F_k^{(j)}$s $j = 0, \dots r-1 \le K-1$ with the nonzero coefficient $w_1(c)$ in front of $F_k^{(r-1)}$. Q.E.D

Remark 3.6 The function in Eq. (3-44) with $j = 0$ corresponds to a moment functional $\mathcal{L} = \delta_c \otimes \mathcal{Y}$, where $\mathcal{Y}$ is any semi-classical moment functional associated to $A_2(y), B_2(y)$ and δ_c is the delta functional supported at $x = c$ on the space of polynomials $\mathbb{C}[x]$. The other solutions in Eq. (3-44) with $j > 0$ are also supported at c but are not factorized and have the form

$$\mathcal{L} = \sum_{k=0}^{j} \delta_c^{(k)} \otimes \mathcal{Y}_k \ , \tag{3-48}$$

for suitable moment functionals $\mathcal{Y}_k$.

If there are other roots common to A_i, B_i we can repeat the procedure until we have a reduce problem which satisfies the Assumption $(\mathcal{B})$.

Therefore from this point on we will assume that the data (A_1, B_1, A_2, B_2) satisfy the Assumption $(\mathcal{B})$.

Theorem 3.1 The functionals $\mathcal{L}_{ij}$ or –equivalently– the generating functions

$$F_{ij}(z,w) := \int_{\Gamma_i^{(x)} \times \Gamma_j^{(y)}} \mathrm{d}x \wedge \mathrm{d}y \, W_1(x) W_2(y) \mathrm{e}^{xy} \mathrm{e}^{xz+yw} \tag{3-49}$$

are linearly independent

The proof is an adaptation of [15] with a small improvement (and a correction). We prepare a few lemmas.

Lemma 3.1 *[Theorem of Mergelyan ([19], p. 367)] If E is a closed bounded set not separating the plane and if $F(z)$ is continuous on E and analytic at the interior points of E, then $F(z)$ can be uniformly approximated on E by polynomials.*

The next Theorem is a rephrasing of the content of [15] for the proof of which we refer ibidem.

Theorem 3.2 [Miller-Shapiro Theorem] If Γ is a closed simple Jordan curve and $F(z)$ is an analytic function (possibly with singularities and/or multivalued) in the points inside Γ such that the equation

$$\oint_\Gamma F(z) p(z) \mathrm{d}z = 0 \tag{3-50}$$

holds for any polynomial $p(z) \in (z - z_0)\mathbb{C}[z]$ (for some fixed $z_0 \in \Gamma$), then $F(z)$ has no singularities inside Γ and it is bounded in the interior region of and on Γ.

Suppose now by contradiction that there exist constants C_{ij} not all of which zero such that

$$\sum_{i=1}^{s_1} \sum_{j=1}^{s_2} C_{ij} \int_{\Gamma_i^{(x)} \times \Gamma_j^{(y)}} \mathrm{d}x \wedge \mathrm{d}y \, W_1(x) W_2(y) \mathrm{e}^{xy} \mathrm{e}^{xz+yw} \equiv 0. \tag{3-51}$$

Reduction of the problem
We claim that if Eq. (3-51) holds then we also have

$$0 \equiv \sum_{i=1}^{s_1} \sum_{j=1}^{s_2} C_{ij} \int_{\Gamma_i^{(x)} \times \Gamma_j^{(y)}} \mathrm{d}x \wedge \mathrm{d}y \, W_1(x) W_2(y) \mathrm{e}^{xz+yw} = \sum_{i=1}^{s_1} \sum_{j=1}^{s_2} C_{ij} \, \Xi_i(z) \Psi_j(w) \ , \tag{3-52}$$

where we have defined

$$\Xi_i(z) := \int_{\Gamma_i^{(x)}} \mathrm{d}x \, W_1(x) \mathrm{e}^{xz} \tag{3-53}$$

$$\Psi_j(w) := \int_{\Gamma_j^{(y)}} \mathrm{d}y \, W_2(y) \mathrm{e}^{yw} \ . \tag{3-54}$$

Indeed consider the auxiliary function of the new variable ρ

$$A(\rho; z, w) := \sum_{i=1}^{s_1} \sum_{j=1}^{s_2} C_{ij} \int_{\Gamma_i^{(x)} \times \Gamma_j^{(y)}} \mathrm{d}x \wedge \mathrm{d}y \, W_1(x) W_2(y) \mathrm{e}^{\rho xy + zx + wy} \ . \tag{3-55}$$

Here z, w play the role of parameters. This function is entire in ρ (because by our assumptions $\deg(V_i^+) \geq 2$ and hence for all contours going to infinity the integrand goes to zero at least as $\exp(-|x|^2 - |y|^2)$), and by applying $(\partial_z \partial_w)^K$ to Eq. (3-51) we have

$$0 \equiv (\partial_z \partial_w)^K A(1; z, w) = \left(\frac{d}{d\rho}\right)^K A(\rho; z, w)\bigg|_{\rho=1}, \quad \forall K \in \mathbb{N} . \tag{3-56}$$

Therefore we also have $A(0; z, w) \equiv 0$, $\forall z, w \in \mathbb{C}$, which is Eq. (3-52).
This shows that proving that the functions F_{ij} are linearly independent is equivalent to proving that the two sets of functions $\{\Xi_i(z)\}_{i=1\dots s_1}$ and $\{\Psi_j(w)\}_{j=1\dots s_2}$ are (separately) linearly independent.
Both the Ξ_is and the Ψ_js are now solutions of the decoupled ODEs of the same type (i.e. with linear coefficients)

$$\left[zB_1\left(\frac{d}{dz}\right) - A_1\left(\frac{d}{dz}\right)\right]\Xi_i(z) = 0 \tag{3-57}$$

$$\left[wB_2\left(\frac{d}{dw}\right) - A_2\left(\frac{d}{dw}\right)\right]\Psi_j(w) = 0 . \tag{3-58}$$

Equivalently we may say that Ξ_is and Ψ_js are generating functions for the moments of semiclassical functionals associated to (A_1, B_1) and (A_2, B_2) respectively. Their linear independence was proven in [15]. Unfortunately this latter paper has a small flaw that makes one step of the proof impossible when $\deg(A_i) > \deg(B_i) + 2$ (while it is correct if $\deg(A_i) \leq \deg(B_i) + 2$) [20].
On the other side the linear independence of certain integral representation for semi-classical moment functionals was obtained in [13]; however their definitions for the contours forces them to a procedure of regularization in certain cases which is elegantly bypassed by the definition of the contours in [15]. We prefer to fix the proof of [15] since then we will not need any regularization.

3.2 Linear independence of the Ξ_is

In this section we prove the linear independence of the functions Ξ_i. This will also prove the linear independence of the functions Ψ_j since they are precisely of the same form. We assume that the polynomial $V_1^+(x)$ appearing in Eq. (3-30) has the form

$$V_1^+(x) = \frac{1}{d+1}x^{d+1} + \sum_{j=0}^{d} v_j x^j \quad (d := d_1 \geq 1). \tag{3-59}$$

This does not affect the generality of the problem inasmuch as it amounts to a rescaling of the variable x. To prove their linear independence we can reduce further the problem to the case where $V_1^+(x) = \frac{1}{d+1}x^{d+1}$. Indeed, suppose that there exist constants A_j such that

$$\mathcal{W}(z; v_0, ..., v_d) := \sum_{j=1}^{s_1} A_j \int_{\Gamma_j} \mathrm{d}x\, W_1(x) \mathrm{e}^{xz} \equiv 0 , \tag{3-60}$$

where we have emphasized the dependence on the subleading coefficients of V_1^+ as given in Eqs. (3-59, 3-30). Considering it as a function of the variables $v_0, ..., v_d$ then Eq. (3-60) implies that

$$\frac{\partial^{|\alpha|}}{\partial \underline{\tilde{v}}^\alpha}\mathcal{W}(z; \underline{\tilde{v}})\bigg|_{\tilde{v}_i = v_i} = 0 , \quad \forall\, \alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d , \ \forall z \in \mathbb{C}. \tag{3-61}$$

Since $\mathcal{W}(z; \tilde{v}_0, ..., \tilde{v}_d)$ is clearly entire in the variables $\tilde{v}_i$, Eq. (3-61) implies that actually it does not depend on them. In other words if the Ξ_is are linearly dependent with constants A_i then also the Ξ_is where we "switch off" the coefficients v_i of the potential are linearly dependent with the *same* constants A_i.
Therefore it also does not affect the generality of the problem of showing linear independence to assume the specific form for V_1^+

$$V_1^+(x) = \frac{1}{d+1}x^{d+1} . \tag{3-62}$$

We now analyze the asymptotic behavior, and we need the following definition (here given for a V_1^+ more general than the one above).

Definition 3.3 *The steepest descent contours (SDCs) for integrals of the form*

$$I_\Gamma(z) := \int_\Gamma \mathrm{d}x\, \mathrm{e}^{-V_1^+(x)+xz} H(x) \ , \tag{3-63}$$

with $H(x)$ of polynomial growth at $x = \infty$, are the contours γ_k uniquely defined, as $z \to \infty$ within the sector $\mathcal{E} = \left\{\arg(z) \in \left(-\frac{\pi}{2(d+1)}, 0\right)\right\}$, by

$$\gamma_k := \left\{ x \in \mathbb{C};\ \Im(V_1^+(x) - xz) = \Im\left(V_1^+(x_k(z)) - zx_k(z)\right) \ , \Re(V_1^+(x)) \underset{\substack{x\to\infty \\ x\in\gamma_k}}{\longrightarrow} +\infty \ . \right\} \ , \tag{3-64}$$

where $x_k(z)$ are the d_1 branches of the solution to

$$V_1^{+\prime}(x) = z \ , \tag{3-65}$$

which behave as $z^{\frac{1}{d_1}}$ as $z \to \infty$ in the sector, for the different determinations of the roots of z.
Their homology class is constant as $x \to \infty$ within the sector.

With reference to Figure 1, the sector $\mathcal{E}$ is the narrow dark-shaded dual sector of $\mathcal{S}_L$ (light-shaded).

Proposition 3.4 Let $\mathcal{E}$ be the sector $\arg(z) \in \left(-\frac{\pi}{2(d+1)}, 0\right)$ at $z = \infty$. Then the Laplace-Fourier transforms over the SDCs γ_k

$$F_k(z) := \int_{\gamma_k} \mathrm{d}x\, W_1(x) \mathrm{e}^{zx} \ , k = 1, \dots d \tag{3-66}$$

have the following asymptotic leading behavior in the sector $\mathcal{E}$

$$F_k(z) = K\sqrt{\frac{2\pi}{d}} z^{\frac{2A+1-d}{2d}} \omega^{k(A-\frac{1}{2})} \exp\left[\frac{d}{d+1} z^{\frac{d+1}{d}} \omega^k\right] \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \ , \tag{3-67}$$

$$A := \sum_{j=1}^p \lambda_j \ , \ \omega := \mathrm{e}^{\frac{2i\pi}{d}} \ , \tag{3-68}$$

where $K \neq 0$ is a constant found in the proof.

Proof.
The proof of this asymptotic is an application of the saddle point method. Writing $z = |z|\mathrm{e}^{i\theta}$ with the change $x = |z|^{1/d}\xi$ we can rewrite the integrals

$$\int_\Gamma \mathrm{e}^{-\frac{1}{d+1}x^{d+1}+xz} \prod_{j=1}^p (x - X_j)^{\lambda_j} \mathrm{e}^{T(x)} \mathrm{d}x = \tag{3-69}$$

$$= |z|^{\frac{1}{d}} |z|^{\frac{A}{d}} \int_\Gamma \exp\left[-|z|^{\frac{d+1}{d}} \left(\frac{1}{d+1}\xi^{d+1} - \xi \mathrm{e}^{i\theta}\right)\right] \xi^A \prod_{j=1}^p \left(1 - \frac{X_j}{\xi |z|^{\frac{1}{d}}}\right)^{\lambda_j} \mathrm{e}^{T(|z|^{1/d}\xi)} \mathrm{d}\xi \ , \tag{3-70}$$

$$T(x) := \exp\left[\frac{M_1(x)}{\prod_{j=1}^p (x - X_j)^{g_j}}\right] \to K \neq 0 \ , |x| \to \infty. \tag{3-71}$$

Let us set $\lambda := |z|^{\frac{d+1}{d}}$ and change integration variable

$$s = S(\xi) := \frac{1}{d+1}\xi^{d+1} - \xi \mathrm{e}^{i\theta} \ . \tag{3-72}$$

Note that the rescaling of variable leaves the contour Γ in the same "homology" class, so that we can take the contour as fixed in the ξ-plane. The saddle points for this exponential are the roots of

$$0 = S'(\xi) = \xi^d - \mathrm{e}^{i\theta} \ , \tag{3-73}$$

that is the d roots of $\mathrm{e}^{i\theta}$. The corresponding critical values are

$$s_{cr}^{(k)}(\theta) := -\frac{d}{d+1}\omega^k \mathrm{e}^{i\theta\frac{d+1}{d}} \ , \ \ \omega := \mathrm{e}^{2i\pi/d}, \ \ k = 0, \dots, d-1. \tag{3-74}$$

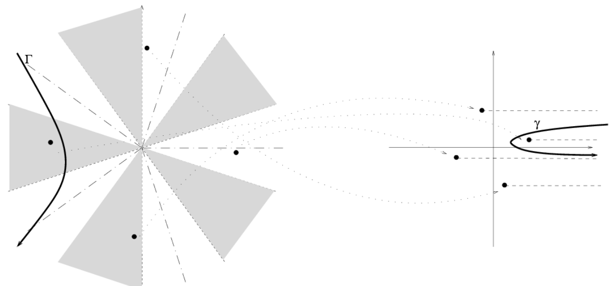

Figure 2: The Steepest Descent contours for $d = 4$. The left depicts the ξ-plane, the right the s-plane.

The map $s = S(\xi)$ is a $d+1$-fold covering of the s plane by the ξ-plane with square-root-type branching points at the $s_{cr}^{(k)}(\theta)$. Moreover each of the $d+1$ sectors (around $\xi = \infty$) for which $\Re(\xi^{d+1}) > 0$ is mapped to the single sector

$$\mathcal{S} := \{s \in \mathbb{C}, \ \ -\frac{\pi}{2} + \epsilon < \arg(s) < \frac{\pi}{2} - \epsilon\} \ . \tag{3-75}$$

The inverse map $\xi = \xi(s)$ is univalued if we perform the cuts on the s plane starting at each $s_{cr}^{(j)}(\theta)$ and going to $\Re(s) = +\infty$ parallel to the real axis. Such cuts are distinct for generic values of θ. We obtain a simply connected domain in the s plane (see picture). By their definiton the SDCs γ_j corresponds to (the two rims of) the horizontal cuts in the s-plane that go from the critical points $s_{cr}^{(j)}(\theta)$ to $\Re(s) = +\infty$.
The cuts are distinct if $\Im\left(\mathrm{e}^{i\frac{d+1}{d}\theta + 2ik\frac{\pi}{d}}\right) \neq \Im\left(\mathrm{e}^{i\frac{d+1}{d}\theta+2ij\frac{\pi}{d}}\right)$, for $j \neq k$, that is away from the Stokes' lines at infinity

$$l_k = \left\{\arg(z) = \frac{\pi k}{d+1}, \ \ k \in \frac{1}{2}\mathbb{Z}\right\} . \tag{3-76}$$

Therefore if z approaches infinity along a ray distinct from the Stokes' lines and within the same sector between them, the asymptotic expansion does not change.

Asymptotic evaluation of the steepest descent integrals

The integrals corresponding to the steepest descent path γ_k become

$$|z|^{\frac{A+1}{d}} \int_{\gamma_k} \mathrm{e}^{-\lambda s} \xi(s)^A g(s,|z|) \frac{d\xi}{ds} \mathrm{d}s \ , \tag{3-77}$$

$$g(s,|z|) := \prod_{j=1}^{p} \left(1 - \frac{X_j}{\xi(s)|z|^{\frac{1}{d}}}\right)^{\lambda_j} \mathrm{e}^{T(|z|^{1/d}\xi(s))}, \ \ \lim_{|z|\to\infty} g(s,|z|) = K \neq 0. \tag{3-78}$$

where $\lambda := |z|^{\frac{d+1}{d}}$. The Jacobian of the change of variable has square-root types singularity at the critical point $s_{cr}^{(k)}$ since the singularities (in the sense of singularity theory) of $S(\xi)$ are simple and nondegenerate.

Then the above integral becomes, upon developing the Jacobian in Puiseux series,

$$|z|^{\frac{A+1}{d}} \int_{\gamma_k} \mathrm{e}^{-\lambda s} g(s,|z|) \xi(s)^A \frac{d\xi}{ds}(s) = \tag{3-79}$$

$$= |z|^{\frac{A+1}{d}} \mathrm{e}^{-\lambda s_{cr}} \int_{\gamma_k} \mathrm{d}s \, \mathrm{e}^{-\lambda(s-s_{cr})} \xi(s)^A g(s,|z|) \frac{1}{\sqrt{2\frac{d^2 s}{d\xi^2}(s_{cr})(s-s_{cr})}} (1+\ldots) = \tag{3-80}$$

$$\simeq K|z|^{\frac{A+1}{d}} \mathrm{e}^{i\frac{A}{d}\theta} \omega^{kA} \mathrm{e}^{-\lambda s_{cr}} \left(2d\mathrm{e}^{\frac{d-1}{d}\theta}\omega^k\right)^{-\frac{1}{2}} 2\int_{\mathbb{R}_+} \mathrm{e}^{-\lambda t} \frac{\mathrm{d}t}{\sqrt{t}} = \tag{3-81}$$

$$= K|z|^{\frac{A+1}{d}} \omega^{kA} \mathrm{e}^{i\frac{A}{d}\theta} \mathrm{e}^{-\lambda s_{cr}} \left(2d\mathrm{e}^{\frac{d-1}{d}\theta}\omega^k\right)^{-\frac{1}{2}} 2\sqrt{\pi}\lambda^{-\frac{1}{2}} = \tag{3-82}$$

$$= K\sqrt{\frac{2\pi}{d}} z^{\frac{2A+1-d}{2d}} \omega^{k(A-\frac{1}{2})} \exp\left[\frac{d}{d+1} z^{\frac{d+1}{d}} \omega^k\right] \ .Q.E.D. \tag{3-83}$$

In particular Proposition 3.4 shows that the SDC integrals F_k are linearly independent because their asymptotics clearly is.

Since the SDCs γ_k and the contours Γ_k span the same homology, we can always assume that the Ξ_i corresponding to the closed loops attached to ∞ are integrals over the SDC γ_k Suppose now that there exist constants A_i such that

$$\sum_{j=1}^{s_1} A_i \Xi_i(z) \equiv 0 \ . \tag{3-84}$$

We split the sum into two parts; the first one contains all contour integrals corresponding to the bounded paths, the paths joining the finite zeroes X_is to infinity, and loops attached to $X_0 = \infty$ approaching ∞ within the sector $\mathcal{S}_L$. We denote the subset of the corresponding indices by I_L. Now it is a simple check which we leave to the reader that all these integrals are of exponential type in the sector $\mathcal{E}$ dual to $\mathcal{S}_L$[8].
The second subset of indices I_R corresponds to the remaining contour integrals over paths which come from and return to ∞ outside the sector $\mathcal{S}_L$; a careful counting gives $|I_R| = [d/2]$. The sum in (3-84) can be accordingly separated in

$$\sum_{i\in I_L} A_i \Xi_i(z) = -\sum_{i\in I_R} A_i \Xi_i(z) \ . \tag{3-85}$$

We want to conclude that the two sides of Eq. (3-85) must vanish separately. Indeed we have remarked above that the LHS in (3-85) is of exponential type in the sector $\mathcal{E}$.
On the other hand we now prove that the RHS *cannot* be of exponential type unless each of $A_i,\ i\in I_R$ vanishes. From Prop. 3.4 we deduce that among the SDC integrals there are precisely $[d/2]$ that have a dominant exponential behavior of the type $\exp\left(\frac{d}{d+1} z^{\frac{d+1}{d}} \omega^k\right)$ with $\Re(z^{\frac{d+1}{d}}\omega^k) > 0$ in the sector $\mathcal{E}$, whic is *not* of exponential type; since the SDC's can be obtained by suitable linear combinations with integer coefficients of the chosen contours then the $[d/2]$ functions $\Xi_i,\ i\in I_R$ must span the same space as the dominant $[d/2]$ linearly independent SDC's in the sector $\mathcal{E}$, *modulo* the span of $\Xi_i,\ i\in I_L$. In formulae

$$\mathbb{Z}\{F_k:\ F_k \text{ dominant in } \mathcal{E}\} \ \simeq\ \mathbb{Z}\{\Xi_i,\ \forall i\} \mathrm{mod}\, \mathbb{Z}\{\Xi_i,\ i\in I_L\} \ =\ \mathbb{Z}\{\Xi_i,\ i\in I_R\} \ . \tag{3-86}$$

Since no nontrivial linear combination of the $[d/2]$ dominant SDC integrals F_k's in $\mathcal{E}$ can be of exponential type, the only possibility for the RHS of Eq. (3-85) to be of exponential type in the sector $\mathcal{E}$ is that

$$A_i = 0,\ \forall i\in I_R \ .$$

Let us now focus on the terms in the LHS of Eq. (3-85). We must now prove that also $A_i = 0,\ i\in I_L$. We can now follow [15] without hurdles. We sketch the main steps below for the sake of completeness.
We need to prove that

$$Q(z) := \sum_{i\in I_L} A_i \int_{\Gamma_i} \mathrm{d}x\, W_1(x)\, \mathrm{e}^{xz} \equiv 0 \ \Leftrightarrow\ A_i = 0\ \forall i\in I_L. \tag{3-87}$$

Let a be a point within the sector $\mathcal{E}$ and far enough from the origin so as to leave all contours $\Gamma_i,\ i\in I_L$ to the left[9]. Let us choose a contour $\mathcal{C}$ starting at z and going to infinity in the sector to $\mathcal{E}$. Then we integrate $Q(\zeta)\mathrm{e}^{-a\zeta}$ along $\mathcal{C}$. Since $\mathrm{e}^{\zeta(x-a)}W_1(x)$ is jointly absolutely integrable with respect to the arc-length on each of the $\Gamma_i,\ i\in I_L$ and $\mathcal{C}$, we may interchange the order of integration to obtain

$$\sum_{i\in I_L} A_i \int_{\Gamma_i} \frac{1}{x-a} \mathrm{e}^{z(x-a)} W_1(x) \equiv 0. \tag{3-88}$$

Repeating this $r-1$ times and then setting $z=0$ at the end, we obtain

$$\sum_i A_i \int_{\Gamma_i} (x-a)^{-r} W_1(x)\mathrm{d}x \equiv 0,\ \forall r\in\mathbb{N}. \tag{3-89}$$

Let us define

$$\tilde{v}(x) := W_1(x)(x-a)^2 \tag{3-90}$$

so that Eq. (3-89) is turned into

$$\sum_i A_i \int_{\Gamma_i} (x-a)^{-r} \tilde{v}(x) \frac{\mathrm{d}x}{(x-a)^2} \equiv 0,\ \forall r\in\mathbb{N}. \tag{3-91}$$

[8] Saying that a function is of exponential type in a given sector means that there exist constants K and C such that the function is bounded by $|z|^K \mathrm{e}^{C|z|}$ in that sector.

[9] More precisely in the half plane to the left of the perpendicular to the bi-secant of the sector $\mathcal{E}$

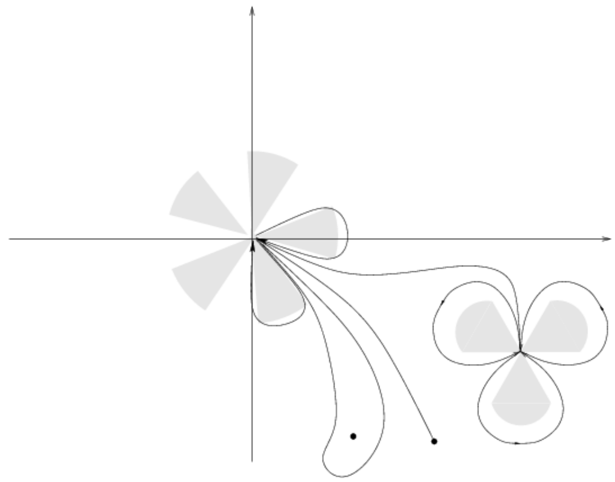

Figure 3: The contours $\gamma_i,\ i \in I_L$ in the ω plane.

Let us perform the change of variable $\omega = \frac{1}{x-a}$ (a homographic transformation). We denote by γ_i the images of the contours Γ_i and by $f(\omega)$ the function $\tilde{v}(x(\omega))$.
Eq. (3-89) (or equivalently Eq. (3-91)) now becomes

$$\sum_{i\in I_L} A_i \int_{\gamma_i} \mathrm{d}\omega f(\omega) P(\omega) = 0\ , \quad \forall P \in \mathbb{C}[\omega]\ . \tag{3-92}$$

Note that in the variable ω all contours are in the finite region of the ω-plane and the contours look like the ones in Figure 3 (the missing loops attached to $0 = \omega(X_0) = \omega(\infty)$ were the contours indexed by I_R).

We denote by E the closed and bounded set in the ω plane constituted by all contours γ_i, $i \in I_L$ and the interiors of the closed loops. This set E satisfies the requirements of Lemma (3.1). Moreover the contours γ_i have all the Property ($\wp$) with respect to $f(\omega)$.

We now start proving that the A_is vanish.
First consider a contour γ_i without interior points (i.e. those segments which join two different X_is). Let $\omega(t)$ be a parametric representation where $t \in [0, L]$ is the arc length parameter parameter so that $\omega'(t)$ is continuous and nonvanishing on $[0, L]$. Therefore it follows that the function

$$\chi_i(\omega) := \begin{cases} \frac{\overline{f(\omega)}}{\omega'(t)}, & \omega \in \gamma_i \\ 0, & \omega \in E \setminus \gamma_i \end{cases} \tag{3-93}$$

is continuous on E and analytic in the interior points of E. Hence there exists a sequence of polynomials $P_n(\omega)$ converging uniformly to $\chi_i(\omega)$ on E (by Lemma 3.1). Plugging into Eq. (3-92) and passing to the limit we obtain

$$A_i \int_0^L \mathrm{d}t\, |f(\omega(t))|^2 = 0\ , \tag{3-94}$$

which implies that A_i vanishes.

Let us now consider a closed loop, say γ_l. Let $T(\omega)$ be any polynomial vanishing at $\omega_0 \in \gamma_l$ where ω_0 is the image of the (unique) zero of $B_1(x)$ on the contour Γ_l. Then we define

$$\Phi_l(\omega) := \begin{cases} T(\omega), & \omega \in \gamma_l \text{ and its interior} \\ 0, & \omega \in E \setminus \{\gamma_l \text{ and its interior}\} \end{cases} \tag{3-95}$$

Again, $\phi_l(\omega)$ satisfies the requirement of Lemma (3.1) and hence can be approximated uniformly by a sequence of polynomials. Passing the limit under the integral we then obtain

$$A_l \int_{\gamma_l} \mathrm{d}\omega f(\omega) T(\omega) = 0\ , \quad \forall T \in (\omega - \omega_0)\mathbb{C}[\omega]\ . \tag{3-96}$$

We then use Theorem 3.2 to conclude that f should be bounded inside γ_l. But this is a contradiction because $f(\omega)$ has the Property ($\wp$) w.r.t. γ_l since $\tilde{v}(x) = W_1(x)(x-a)^2$ had the same Property w.r.t. the closed contour Γ_l. This is a contradiction unless the A_l vanishes.

Therefore we have proven that all the A_i must vanish, i.e. the $\Xi_i(z)$ are linearly independent.

Repeating for the $\Psi_j(w)$ we conclude the proof of Theorem 3.1.

4 Conclusion

We make a few remarks on the cases we have not considered, i.e. when $\deg(A_i) \le \deg(B_i)$ for one or both $i = 1, 2$. Indeed (up to some care in the definition of the contours for reasons of convergence) one can easily define *some* solutions of Eqs. (3-14) in the form of double Laplace–Fourier integrals and also prove their linear independence. More complicated is to produce the analog of Prop. 3.2, that is to have an a-priori knowledge of the dimension of the space of solutions to Eqs. (3-14): the result (which we do not prove here) is that there are $M = s_1 s_2 + 1$ solutions. The moment recurrences (3-9,3-10) say then that the bifunctionals are actually $M - 1$ in Case AB or $M - 2$ in Case AA. That is one has to give a criterion to select amongst the solutions to Eq. (3-14) the ones which are analytic at $w = 0 = z$. We will return on this point in a future publication.

Suffices here to say that a similar problem occurs for the semi-classical moment functionals $\mathcal{L} : \mathbb{C}[x] \to \mathbb{C}$. As we have illustrated in the introduction the generating function satisfies Eq. 1-12, but in general not all solutions are analytic at $z = 0$ and hence do not define any moment functional. This can be understood by looking at the recurrence relations satisfied by the moments:

$$n \sum_{j=0}^{d} \beta(j) \mu_{n+j-1} = \sum_{j=0}^{k} \alpha(j) \mu_{n+j} \ , \tag{4-1}$$

where $d = \deg(B) > \deg(A) + 1 = k + 1$. In this case the resulting d-terms recurrence relation has actually only $d - 1$ solutions because, for $n = 0$ the above equation gives a *constraint* on the initial conditions[10]

$$0 = \sum_{j=0}^{k} \alpha(j) \mu_j \ . \tag{4-2}$$

This should be regarded as the requirement that the solution of Eq. (1-12) be analytic at $z = 0$.

Now, in the bilinear case we have the additional problem that the recurrence relations for the bi-moments are overdetermined and hence the corresponding constraint on the initial conditions must be shown to be compatible as well. We postpone the more detailed discussion of this problem to a future publication.

Acknowledgments

The author wishes to thank Prof. B. Eynard and Prof. J. Harnad for stimulating discussion, and Prof. H. S. Shapiro for helpful hints in amending the proof of linear independence.

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[10] When $\deg(A) + 1 = \deg(B) = d$ then *generically* there are $d - 1$ solutions, except in some cases when $\exists n$ *s.t.* $\alpha(d-1) = n\beta(d)$. See [14] for more details.

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