Brockett's condition for stabilization in the state constrained case

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Abstract

A variant of Brockett's necessary condition for feedback stabilization is derived, in the state constrained case.

Key words: Nonlinear control system, feedback stabilization, state constraint, Brockett's condition, proximal smoothness, proximal normal cone.

Mathematical Subject Classification: 93D15, 93D20

1 Introduction

Consider a control system

$$\dot{x}(t) = f(x(t), u(t)), \quad t \geq 0, \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz. We shall say that a locally Lipschitz feedback law $k : \mathbb{R}^n \to \mathbb{R}^m$ stabilizes (1) provided that 0 is both Lyapunov stable and an attractor with respect to the ordinary differential equation

$$\dot{x}(t) = g(x(t)), \quad t \geq 0, \tag{2}$$

where we have denoted

$$g(x) := f(x, k(x)). \tag{3}$$

It is a classical fact that in the linear case, $f(x,u) = Ax + Bu$, a necessary and sufficient condition for stabilization (by a linear feedback law $k(x) = Cx$) is that rank $[B, AB, \ldots, A^{n-1}B] = n$, which in turn is equivalent to complete controllability (in the open loop sense). The situation is quite different in the nonlinear case. The control system with dynamics $f(x,u)$ specified next, is easily seen to completely controllable:

$$f(x_1, x_2, x_3, u_1, u_2) = (u_1 \cos x_3, u_1 \sin x_3, u_2) \tag{4}$$

However, the system cannot be stabilized to 0 by a locally Lipschitz feedback, because of the following result, which we refer to as *Brockett's theorem*. (We denote by B_δ the open ball of radius δ, centered at 0.)

Theorem 1.1. *Let $f(\cdot,\cdot)$ be locally Lipschitz, and suppose that (1) is stabilizable by a locally Lipschitz feedback law $k(\cdot)$. Then the maping $g : \mathbb{R}^n \to \mathbb{R}$ defined in (**??**) is open at the origin; that is*

$$\forall\, \delta > 0,\ \exists\, \gamma = \gamma(\delta) \quad \ni \quad B_\gamma \subset g(B_\delta). \tag{5}$$

To see that the above example does not satisfy Brockett's necessary condition (**??**), note that points $(0, a, b)$ with $a \neq 0$ are not contained in the image of f for x_3 near 0.

Actually, Brockett's original result in [3] pertains to smooth $f(\cdot,\cdot)$ and $k(\cdot)$, and is not localized as above, but Theorem **??** is well known; see Sontag [15]. (It is also explained in that reference that the above example is equivalent, under a transformation, to one provided by Brockett.) Zabczyk [18] derived certain variants of the result under relaxations of the definition of stabilization, and Ryan [14] proved that Brockett's necessary condition for stabilizability persists when a certain class of multivalued feedback laws is considered. The general problem of constructing *discontinuous* feedback laws $k(x)$ which achieve stabilization of asymptotically controllable systems has been stimulated by Brockett's theorem as well as by early results of Sontag and Sussman [17]. References on the stabilizing feedback design problem include Hermes [10], Clarke, Ledyaev, Sontag and Subottin [5], Ancona and Bressan [1], Rifford [12], [13], Clarke, Ledyaev, Rifford and Stern [4], Sontag [16], and in problems with state constraints, Clarke and Stern [**?**]. On the other hand, Coron [8] (see also Coron and Rosier [9]) showed that in the case of controllable systems affine in the control, there exist continuous, but time-dependent feedbacks $k(t,x)$ which achieve stabilization.

Our purpose in the present work is to provide a variant of Theorem **??**, when trajectories of (1) are required to satisfy a state constraint $x(t) \in S$ for all $t \geq 0$. We will restrict our attention to a class of constraint sets S which includes the compact convex ones.

2 Necessary condition for state constrained stabilization

Throughout, $S \subset \mathbb{R}^n$ will denote a compact set containing 0, and it will be assumed that S is *proximally smooth*. This means that there exists $r_S > 0$ such that $d_S(\cdot)$, the euclidean distance function to S, is C^1 on the tube

$$U(r_S) := \{x \in \mathbb{R}^n : 0 < d_S(x) < r_S\}.$$

Our reference regarding this property is Clarke, Stern and Wolenski [7], where alternate characterizations were derived (in an infinite dimensional setting). Of particular use to us is the fact that S is proximally smooth if and only if for every $x \in S + B_{r_S}$, there exists a unique closest point $p(x) \in S$. It was shown it [7] that when this is the case, the function $p(\cdot)$ is Lipschitz on the tube $U(r)$ for every $r \in (0, r_S)$; it readily follows that then $p(\cdot)$ is Lipschitz on $S + B_{r_S}$. Note that convexity implies proximal smoothness, and that in the convex case, r_S can be taken to be arbitrarily large.

Definition 2.1. The locally Lipschitz feedback law $k : \mathbb{R}^n \to \mathbb{R}^m$ is said to *S-stabilize* the control system (1) provided that solutions $x(\cdot)$ of (**??**) satisfy the following two properties:

(a) *S-constrained Lyapunov stability of* 0*:* Given $r_1 > 0$, there exists $r_2 > 0$ such that

$$x(0) \in S \cap B_{r_2} \Longrightarrow x(t) \in S \cap B_{r_1} \quad \forall\, t \geq 0.$$

(b) *S-constrained attractivity:* One has

$$x(0) \in S \Longrightarrow x(t) \in S \quad \forall\, t \geq 0, \quad \lim_{t\to\infty} x(t) = 0.$$

Property (b) above implies in particular that S is (forward time) invariant with respect to the dynamics $\dot{x}(t) = g(x(t))$. Note also that since $f(\cdot,\cdot)$ and $k(\cdot)$ are locally Lipschitz, so is $g(\cdot)$, and therefore one is assured of uniqueness of the solution $x(\cdot)$ for any initial point $x(0) \in S$.

Our main result is the following:

Theorem 2.2. *Let S be a compact, proximally smooth set containing* 0*. Let $f(\cdot,\cdot)$ be locally Lipschitz, and assume that a locally Lipschitz feedback law $k(\cdot)$ achieves S-stabilization of the control system (1). Define $h : S + B_{r_S} \to \mathbb{R}^n$ by*

$$h(x) := f(p(x), k(p(x))) + p(x) - x.$$

Then $h(\cdot)$ is open at 0*; that is*

$$\forall\, \delta > 0,\ \exists\, \gamma = \gamma(\delta) \quad \ni \quad B_\gamma \subset h(B_\delta). \tag{6}$$

Remark 2.3.

(a) Note that $h(\cdot)$ is Lipschitz on $S + r_S B$ and that $h \equiv g$ on S, because $p(x) = x$ if $x \in S$.

(b) Also, observe that if 0 is an interior point of S, then the necessary condition (**??**) reduces to that of Brockett, (**??**), because then $h(x) = g(x)$ for all x near 0.

(c) The statement and proof of Theorem **??** remain unchanged if the original dynamics $f(x,u)$ are specified only for $x \in S$, which is realistic in certain state constrained models.

We require three lemmas—the hypotheses of the theorem remain in effect throughout. Solutions of the ordinary differential equation

$$\dot{y}(t) = h(y(t)), \quad t \geq 0, \quad y(0) = \alpha, \tag{7}$$

will be denoted by $y(t;\alpha)$. Note that if $\alpha \in S$, then $y(t;\alpha)$ is in fact a solution of (**??**).

Lemma 2.4. *Given $R > 0$, there exists $T_1(R) > 0$ such that*

$$\alpha \in S \Longrightarrow y(t;\alpha) \in S \cap B_R \quad \forall\, t \geq T_1(R). \tag{8}$$

Proof: By the S-stabilization property, there exists $R' > 0$ such that solutions of (**??**) emanating from initial points in $S \cap B_{R'}$ remain in $S \cap B_R$ for all $t \geq 0$. Furthermore, for each $\alpha \in S$, there exists $\tau_\alpha > 0$ such that $y(\tau_\alpha;\alpha) \in S \cap B_{R'}$, and continuity of solutions in the initial state implies that there exists $r_\alpha > 0$ such that

$$\alpha' \in S \cap \{\alpha + B_{r_\alpha}\} \Longrightarrow y(\tau_\alpha;\alpha') \in S \cap B_{R'}.$$

The family of sets

$$\{S \cap \{\alpha + B_{r_\alpha} : \alpha \in S\}$$

is an open (relative to S) cover of the compact set S and as such, possesses a finite subcover

$$\bigcup_{i=1}^{k} S \cap \{\alpha_i + B_{r_{\alpha_i}}\}.$$

Then

$$T_1(R) := \max\{\tau_{\alpha_i} : 1 \leq i \leq k\}$$

has the required property. $\square$

Lemma 2.5. *Let $\alpha \in \mathbb{R}^n$ be such that $\|\alpha\| < r_S$. Then the solution $y(t;\alpha)$ of (3) is defined on $[0,\infty)$ and satisfies*

$$d_S(y(t;\alpha)) \le e^{-t}\|\alpha\| \quad \forall\, t \ge 0. \tag{9}$$

Proof: For any $\alpha \in S + r_S B$, one has

$$\langle \alpha - p(\alpha), g(p(\alpha))\rangle \le 0, \tag{10}$$

due to the fact that S is invariant with respect to the dynamics (**??**). (Here, upon noting that $\alpha - p(\alpha)$ is a *proximal normal direction* to S at $p(\alpha)$, we have applied Theorem 4.2.10 in Clarke, Ledyaev, Stern and Wolenski [6].) Then

$$\langle \alpha - p(\alpha), h(\alpha)\rangle \le -\|\alpha - p(\alpha)\|^2 = -d_S^2(\alpha). \tag{11}$$

Exercise 4.2.2(b) in [6] (in conjunction with Corollary 4.3.7) on the "proximal aiming" method then yields the differential inequality

$$\frac{d}{dt} d_S(y(t;\alpha)) \le -d_S(y(t;\alpha)) \quad a.e., \quad t \ge 0 \tag{12}$$

on any interval such that $y(t;\alpha) \notin S$. Also, by invariance, if the trajectory $y(t;\alpha)$ enters S, then it remains in S thereafter. Then (**??**) follows readily. Clearly the solution $y(t;\alpha)$ is defined on $[0,\infty)$ since (**??**) precludes finite time blow-up. $\square$

Lemma 2.6. *Let $R > 0$ be such that $3R < r_S$. Then there exists $T_2(R) > 0$ such that*

$$\|\alpha\| < 3R \Longrightarrow \|y(t;\alpha)\| < 2R \quad \forall\, t > T_2(R). \tag{13}$$

Proof: Let $\|\alpha\| < 3R$. In view of the previous lemma, one has $d_S(y(T;\alpha)) \le e^{-T}\|\alpha\|$ for any $T > 0$. Denote by K a Lipschitz constant for $h(\cdot)$ on $S + r_S B$. For $T > 0$ (to be determined), we want to compare the trajectory of (3) emanating from $\alpha' := y(T;\alpha)$ with the one emanating from $\alpha'' := p(y(T;\alpha))$. A standard estimate based on Gronwall's inequality yields

$$\begin{aligned} \|y(t;\alpha'') - y(t;\alpha'))\| &\le e^{Kt}\|\alpha'' - \alpha'\| \\ &< 3Re^{Kt}e^{-T} \end{aligned}$$

for all $t \ge 0$. It follows that for $T > \ln(3)$, one has

$$0 < t < \frac{T - \ln(3)}{K} \Longrightarrow \|y(t;\alpha'') - y(t;\alpha'))\| < R.$$

In view of Lemma 2.4, we deduce that large enough T,

$$T_1(R) < t < \frac{T - \ln(3)}{K} \Longrightarrow \|y(t;\alpha')\| < 2R.$$

Then for the trajectory emanating from the original start point α, one has

$$t \in \left(T + T_1(R), T + \frac{T - \ln(3)}{K}\right) \Longrightarrow \|y(t;\alpha)\| < 2R. \tag{14}$$

Let $\gamma > 0$ be given, and choose T so large that

$$\frac{T - \ln(3)}{K} > T_1(R) + \gamma;$$

for example, we can take $T = KT_1(R) + \ln(3) + 1 + K\gamma$. Then (**??**) implies that for any start point α such that $\|\alpha\| < 3R$, one has $\|y(t;\alpha)\| < 2R$ for all t in the interval

$$((K+1)T_1(R) + \ln(3) + 1 + K\gamma, (K+1)T_1(R) + \ln(3) + 1 + (K+1)\gamma) \tag{15}$$

We claim that $T_2(R) = (K+1)T_1(R) + \ln(3) +$ has the required property. To see this, note that for any $t > T_2(R)$, one can choose γ (depending on t) so that t is contained in the interval (**??**). $\square$

We are now in position to complete the proof of the theorem. Like the proof of Theorem **??** provided in [15] and the method in Ryan [14], the argument rests upon properties of the topological (or Brouwer) degree of a mapping; see also Krasnoselskii and Zabreiko [11]. A useful reference on the Brouwer degree is Berger and Berger [2].

Proof of Theorem **??**: Let R be as in Lemma 2.6. Consider the function $H : [0,1] \times \mathrm{cl}(B_{2R}) \to \mathbb{R}^n$ given by

$$H(t,\alpha) := \begin{cases} h(\alpha) & \text{if } t = 0 \\ \frac{1}{tT_2(R)}[y(tT_2(R);\alpha) - \alpha] & \text{if } 0 < t \leq 1 \end{cases}$$

In view of Lemma 2.6, $\mathrm{bdry}(B_{2R})$ cannot contain rest points or periodic points of (3). It follows that

$$H(t,\alpha) \neq 0 \quad \forall\, (t,\alpha) \in [0,1] \times \mathrm{bdry}(B_{2R}). \tag{16}$$

We claim that $H(\cdot,\cdot)$ is continuous on $[0,1]\times\mathrm{cl}(B_{2R})$. Continuity on $(0,1]\times\mathrm{cl}(B_{2R})$ follows readily from continuity of solutions of (3) as a function of initial data. It remains to verify that continuity holds at points of the form $(0,\alpha)$, where $\|\alpha\| \leq 2R$. For such a point and given any $\varepsilon > 0$, it suffices to show that there exists $\delta > 0$ such that

$$0 \leq t \leq \delta, \quad \|\alpha' - \alpha\| \leq \delta, \quad \|\alpha'\| \leq 2R \Longrightarrow \|H(t,\alpha') - H(0,\alpha)\| < \varepsilon. \tag{17}$$

Denote by M a norm bound for $h(\cdot)$ on the set $S + r_S B$, and let K be a Lipschitz constant as in the preceding lemma. Then, since for $t > 0$ one has

$$\frac{1}{tT_2(R)}[y(tT_2(R);\alpha') - \alpha'] - h(\alpha') = \frac{1}{tT_2(R)}\int_0^{tT_2(R)} [h(y(s;\alpha')) - h(\alpha')]ds,$$

it follows that

$$\begin{aligned} \|H(t,\alpha') - H(0,\alpha)\| &\leq \|H(t,\alpha') - H(0,\alpha')\| + H(0,\alpha') - H(0,\alpha)\| \\ &\leq KMtT_2(R) + K\|\alpha' - \alpha\|, \end{aligned}$$

which shows that the required δ exists. Bearing (**??** in mind, it follows that $H(\cdot,\cdot)$ is a homotopy between the functions $h(\cdot)$ and $w(\cdot)$, where

$$w(\alpha) := \frac{1}{T_2(R)}[y(T_2(R);\alpha) - \alpha].$$

Now consider the mapping $\tilde{H} : [0,1] \times \mathrm{cl}(B_{2R}) \to \mathbb{R}^n$ given by

$$\tilde{H}(t,\alpha) := (1-t)w(\alpha) - \frac{t}{T_2(R)}\alpha.$$

It is easy to see that this mapping is continuous, and what is more,

$$\tilde{H}(t,\alpha) \neq 0 \quad \forall\, (t,\alpha) \in [0,1] \times \mathrm{bdry}(B_{2R}). \tag{18}$$

To verify (7), first note that $\tilde{H}(0,\alpha) = w(\alpha) \neq 0$ for $\alpha \in \mathrm{bdry}(B_{2R})$, since periodicity is precluded. In addition, $\tilde{H}(1,\alpha)$ is obviously nonzero. Now consider $t \in (0,1)$ and $\alpha \in \mathrm{bdry}(B_{2R})$. If $\tilde{H}(t,\alpha) = 0$, then one would have

$$\|y(T_2(R);\alpha)\| = \frac{\|\alpha\|}{1-t} > 2R,$$

which is in violation of Lemma 2.6.

Hence $\tilde{H}(\cdot,\cdot)$ provides a homotopy between $w(\cdot)$ and the function $v(x) = -\frac{1}{T_2(R)}x$, and therefore $h : \mathrm{cl}(B_{2R}) \to \mathbb{R}^n$ and $v : \mathrm{cl}(B_{2R}) \to \mathbb{R}^n$ are homotopically equivalent as well. It follows that the topological degree of these mappings, with respect to points near 0, have the same value. Since $v(\cdot)$ is an odd mapping, it has nonzero degree with respect to such points. Then the equation $h(x) = p$ is solvable by $x \in B_{2R}$ for any p sufficiently near 0; therefore $h(B_{2R})$ contains an open neighborhhod of 0. Since R can be taken as small as desired, the proof of the theorem is completed. $\square$.

Let us denote the *proximal normal cone* to S at $x \in S$ by $N_S^P(x)$. We refer the reader to [6] for background on this construct of nonsmooth analysis, and make note of the fact that when S is closed and convex, this cone is simply the classical normal cone of convex analysis. The following consequence of Theorem **??** follows immediately from the definition of the proxmal normal cone. It provides a slightly more crude, but more convenient, necessary condition for S-constrained stabilization.

Corollary 2.7. *Let the hypotheses of Theorem* **??** *be in effect. Then for every $\delta > 0$, the set*

$$f(S \cap B_\delta(\tilde{x}), \mathbb{R}^n) - N_S^P(S \cap B_\delta) \tag{19}$$

contains an open neighborhood of 0.

Remark 2.8. It is worth noting that only the stated localized form of the corollary is interesting, since one always has $N_S^P(S) = \mathbb{R}^n$. This follows from the fact that for *every* compact set S, $\mathrm{cl}[N_S^P(S)] = \mathbb{R}^n$ (by Exercise 1.11.4 in [6]), and the property that $N_S^P(\cdot)$ is a closed multifunction on S when S is proximally smooth.

In the following example, simple cases (without control) serve to illustrate the corollary.

Example 2.9. Consider the planar system given by

$$\dot{x}(t) = \begin{pmatrix} -x_1(t) \\ 0 \end{pmatrix}.$$

(a) Consider the compact convex constraint set

$$S = \{(x_1, x_2) : \|x_1\| \leq 1, x_2 = 0\}.$$

It is clear that in the absence of a state constraint, the system is not stabilizable to the origin (or, since there is no control, not stable), and it is equally clear that it fails Brockett's necessary condition in Theorem **??**. On the other hand, the system is obviously S-stabilizable (or just "S-stable" here). That the necessary condition of Corollary **??** holds is easily verified, upon noting that

$$N_S^P(0,0) = \{(\alpha, \beta) : \alpha = 0\}.$$

(b) Now consider

$$S = \{(x_1, x_2) : 0 \leq x_1| \leq 1, x_2 = 0\}.$$

The same comments as in part (a) apply, because now one has available the larger cone

$$N_S^P(0,0) = \{(\alpha, \beta) : \alpha \leq 0\}.$$

In both cases (a) and (b), the necessary condition (**??**) of Theorem **??** is easy to check as well.

An example related to the one provided in the introduction is given next.

Example 2.10. Consider the control system given by (2), and let

$$S = \{(x_1, x_2, x_3) : |x_1| \leq 1, 0 \leq x_2 \leq 1, |x_3| \leq \pi\}.$$

Then $0 \in \mathrm{bdry}(S)$, and for $\delta > 0$ sufficiently small, one has

$$N_S^P(S \cap B_\delta) = \{(0, x_2, 0) : x_2 \leq 0\}. \tag{20}$$

It is readily noted that every point in S can be steered (by open loop control) to the origin via an S-constrained trajectory; in fact, S is completely controllable via S-constrained trajectories. But there there does not exist a locally Lipschitz S-stabilizing feedback, because the necessary condition of Corollary **??** fails to hold. To see this, note (again) that points $(0, a, b)$ with $a \neq 0$ are not contained in the image of f for x_3 near 0. Hence, in view of (**??**), the set in (**??**) cannot contain points $(0, b, c)$ with $b < 0$, for such x_3.

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