# State Constrained Feedback Stabilization

F. H. Clarke[*][†] R. J. Stern[‡][§]

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[*]Institut Desargues (Bât 101), Université Claude Bernard Lyon I, 69622 Villeurbanne, France

[†]Professeur, Institut universitaire de France

[‡]Department of Mathematics and Statistics, Concordia University, Montreal, Quebec H4B 1R6, Canada

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**Abstract**

A standard finite dimensional nonlinear control system is considered, along with a state constraint set $S$ and a target set $\Sigma$. It is proven that open loop $S$-constrained controllability to $\Sigma$ implies closed loop $S$-constrained controllability to the closed $\delta$-neighborhood of $\Sigma$, for any specified $\delta > 0$. When the target set $\Sigma$ satisfies a small time $S$-constrained controllability condition, conclusions on closed loop $S$-constrained stabilizability ensue. The (necessarily discontinuous) feedback laws in question are implemented in the sample-and-hold sense and possess a robustness property with respect to state measurement errors. The feedback constructions involve the quadratic infimal convolution of a control Lyapunov function with respect to a certain modification of the original dynamics. The modified dynamics in effect provide for constraint removal, while the convolution operation provides a useful semiconcavity property.

**Key words:** Asymptotic controllability, state constraint, semiconcave control Lyapunov function, constraint removal, feedback, robustness.

**Mathematical Subject Classification:** 93D15, 93D20

# 1 Introduction

We shall consider a control system of the form

$$\dot{x}(t) = f(x(t), u(t)) \quad a.e., \quad u(t) \in U. \tag{1}$$

The state trajectory $x(\cdot)$ evolves in $\mathbb{R}^n$ and control functions $u(\cdot)$ are Lebesgue measurable functions $u : \mathbb{R} \to U$, where $U \subset \mathbb{R}^m$ is a compact control constraint set. We shall assume throughout that the above dynamics satisfy the following standard hypotheses:

(F1) The function $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuous and is locally Lipschitz in the state variable $x$, uniformly for $u \in U$; that is, for each bounded set $\Gamma \subset \mathbb{R}^n$, there exists $K_\Gamma > 0$ such that

$$\|f(x, u) - f(y, u)\| \leq K_\Gamma \|x - y\|,$$

whenever $(x, u)$ and $(y, u)$ are in $\Gamma \times U$.

(F2) The function $f$ possesses linear growth; that is, there exist positive numbers $c_1, c_2$ such that

$$\|f(x, u)\| \leq c_1 \|x\| + c_2 \quad \forall\, (x, u) \in \mathbb{R}^n \times U.$$

(F3) The velocity set

$$f(x, U) := \{f(x, u) : u \in U\}$$

is convex for every $x \in \mathbb{R}^n$.

Under (F1)-(F2), for every initial phase $(\tau, \alpha) \in \mathbb{R} \times \mathbb{R}^n$ and every control function $u(\cdot)$, there exists a unique trajectory $x(t) = x(t; \tau, \alpha, u(\cdot))$ defined for $t \geq \tau$ and satisfying $x(\tau) = \alpha$.

**Remark 1.1.** Actually, for our purposes, (F2) could be replaced by the somewhat weaker hypothesis that $f(\Gamma, U)$ be bounded for any bounded set $\Gamma \subseteq \mathbb{R}^n$. (See also §5.3 below with regard to this issue.) Assumption (F3) will be needed below in order to have available a required sequential compactness property of trajectories. On the other hand, in the absence of (F3), the results of this article could be framed in the context of relaxed controls.

A general problem of considerable theoretical as well as applied interest, and one which has received much attention in recent years, is whether *open loop asymptotic controllability* of the origin implies *closed loop stabilization*. We need not give precise definitions of these properties here, but roughly speaking, open loop asymptotic controllability means that for every initial state in $\mathbb{R}^n$, there exists a control function so that the resulting trajectory of (1) is driven asymptotically to the origin, and that this property holds in a certain uniform and Lyapunov stable manner. Closed loop stabilizability of the origin involves the existence of a feedback law $k : \mathbb{R}^n \to U$ such that all solutions of the ordinary differential equation

$$\dot{x}(t) = f(x(t), k(x(t))) \tag{2}$$

asymptotically approach the origin, again, in a uniform and Lyapunov stable manner.

A minimal condition for the existence of classical solutions to the ordinary differential equation (2) is that the feedback law $k(\cdot)$ be continuous on $\mathbb{R}^n \backslash \{0\}$. However, as was shown by Sontag and Sussman [34], even when $m = n = 1$, such a feedback law $k(\cdot)$ need not exist. A further negative result in this regard was provided by Brockett [4], who derived a topological condition on the dynamics which is necessary for the existence of a stabilizing feedback law which is continuous on $\mathbb{R}^n$, and exhibited an example violating this condition, in spite of its global open loop controllability to the origin. In addition, Ryan [30] showed that Brockett's necessary condition persists even when Filippov solutions are considered. The upshot is that in addressing the above problem, due to the fact that continuity of feedback laws cannot be expected, it is advantageous to work with an alternative solution concept for (2), rather than the classical or Filippov ones. On the other hand, if nonautonomous feedbacks of the form $k(t, x)$ are allowed (and they are not, in our problem, which calls for a purely positional feedback law $k(x)$), then continuity is not precluded; see Coron [15] and Coron and Rosier [16].

Clarke, Ledyaev, Sontag and Subbotin [8] obtained an affirmative answer to the above problem in terms of the following "sample-and-hold" solution concept for (2), where $k(\cdot)$ is in general discontinuous. Let an initial state $\alpha \in \mathbb{R}^n$ be specified. Then given a partition

$$\pi = \{t_0, t_1, t_2, \ldots\} \tag{3}$$

of $[0,\infty)$ (where $t_0 = 0$), the associated *$\pi$-trajectory* $x(\cdot)$ on $[0,\infty)$ with $x(0) = x(t_0) = \alpha$ is the curve satisfying interval-by-interval dynamics as follows: Set $x_0 = \alpha$. Then on the interval $[t_0, t_1]$, $x$ is the classical solution of the differential equation

$$\dot{x}(t) = f(x(t), k(x_0)), \quad x(t_0) = x_0, \quad t \in (t_0, t_1). \tag{4}$$

We then set $x_1 := x(t_1)$, and restart the system on the next interval as follows:

$$\dot{x}(t) = f(x(t), k(x_1)), \quad x(t_1) = x_1, \quad t \in (t_1, t_2). \tag{5}$$

The process is continued in this manner through each interval. Note that $x$ is the unique solution on $[0,\infty)$ of the differential equation $\dot{x}(t) = f(x(t), u(t))$ satisfying $x(\tau) = \alpha$, with a certain piecewise constant control function $u$ determined by the control feedback $k(x)$. The sample-and-hold solution procedure is sometimes referred to as "closed loop system sampling", and is the same as the "step-by-step" solution concept employed by Krasovskiĭ and Subbotin [23] in differential game theory. We refer the reader to the introduction in Clarke, Ledyaev, Rifford and Stern [7] for more detail on the history of the problem, nonsmooth Lyapunov functions, Filippov solutions, and related topics. Other references relevant to the present work are Sontag [33]), Clarke, Ledyaev, Stern and Wolenski [9], [11], and Rifford [25], [26], and [27], Hermes [19], [20], Kokotovic and Sussman [22], Bacciotti [2], Ancona and Bressan [1], Teel and Praly [35], and Kellett and Teel [21].

In the present article, we shall address a variant of the problem discussed above, in which a state constraint is imposed; to the best of our knowledge, this is the first such endeavor. Specifically, for a given constraint set $S \subset \mathbb{R}^n$ and target set $\Sigma$ such that $S \cap \Sigma \neq \emptyset$, we introduce the following definitions:

**Definition 1.2.** *Open loop $S$-controllability* to $\Sigma$ holds provided that for any initial state $\alpha \in S$, there exists a control function $u(\cdot)$ and a time $t(\alpha) \geq 0$ such that

$$x(t) = x(t; 0, \alpha, u(\cdot)) \in S \quad \forall\, t \in [0, t(\alpha)], \tag{6}$$

and

$$x(t(\alpha)) \in \Sigma. \tag{7}$$

Note that the controllability property in the preceding definition is not an asymptotic one. On the other hand, if open loop *asymptotic* $S$-controllability holds for a given target, then obviously open loop $S$-controllability to the closed $\gamma$-neighborhood of $\Sigma$ holds for every $\gamma > 0$.

**Definition 1.3.** *Closed loop $S$-controllability* to $\Sigma$ holds provided that there exists a feedback law $k : \mathbb{R}^n \to U$ along with reals $T_1 > 0$ and $\beta > 0$, such that the following holds: If

$$\operatorname{diam}(\pi) := \max\{t_{i+1} - t_i \,:\, i = 0, 1, \ldots\} \leq \beta,$$

then for every $\alpha \in S$, there exists $t_1(\alpha) \in [0, T_1]$ such that the $\pi$-trajectory associated with the ordinary differential equation

$$\dot{x}(t) = f(x(t), k(x(t))$$

and initial condition $x(0) = \alpha$, satisfies

$$x(t) \in S \quad \forall\, t \in [0, t_1(\alpha)], \tag{8}$$

and

$$x(t_1(\alpha)) \in \Sigma. \tag{9}$$

Our first main result (Theorem 4.1) asserts that when certain geometric conditions are imposed upon $S$, then open loop $S$-controllability to $\Sigma$ implies closed loop $S$-controllability to the closed $\delta$-neighborhood of $\Sigma$, for any specified $\delta > 0$. No geometric assumptions are imposed upon the target set $\Sigma$, beyond nonemptiness of $S \cap \Sigma$.

**Definition 1.4.** *closed loop $S$-stabilizability to* $\Sigma$ holds provided that closed loop $S$ controllability to $\Sigma$ holds, with (9) in Definition 1.3 fortified to

$$x(t) \in S \cap \Sigma \quad \forall\, t \geq t_1(\alpha). \tag{10}$$

In the second main result (Theorem 4.7), we impose a small time $S$-constrained controllability hypothesis, and prove that in the presence of that hypothesis, open loop $S$-controllability to $\Sigma$ implies closed loop $S$-stabilizability to the closed $\delta$-neighborhood of $S$, for any specified $\delta > 0$. Our feedback constructions in these results involve the quadratic infimal convolution of a control Lyapunov function with respect to a certain modification of the original dynamics. The modified dynamics in effect provide for constraint removal, while the convolution operation provides a useful semiconcavity property.

The layout of this article is as follows. In the next section, we will present preliminaries from nonsmooth analysis. Then in §3, certain required geometric results pertaining to the constraint removal method in Clarke, Rifford and Stern [12] are recalled. The main results are provided in §4, while §5 contains concluding comments, inlcuding a robustness property with respect to state measurement error of the feedback laws constructed in §4.

# 2 Nonsmooth analysis background

Our general reference on nonsmooth analysis employed in this article is [11]. Other useful references are [9], Clarke [5], [6], Loewen [24] and Vinter [36].

## 2.1 Notation and definitions

The Euclidean norm is denoted $\|\cdot\|$, and $\langle\, ,\, \rangle$ is the usual inner product. The open unit ball in $\mathbb{R}^n$ is denoted $B$. For a set $Z \subset \mathbb{R}^n$, we denote by $\mathrm{co}(Z)$, $\mathrm{cl}(Z)$, $\mathrm{bdry}(Z)$, and $\mathrm{int}(Z)$ the convex hull, closure, boundary, and interior of $Z$, respectively. We denote the closure of the complement of $Z$ by $\hat{Z} := \mathrm{cl}\{\mathbb{R}^n \backslash Z\}$. Given $\delta > 0$, we denote $Z^\delta := Z + \delta\overline{B}$. For a closed set $Z$, the distance of a point $u$ to $Z$ is denoted

$$d_Z(u) := \min\{\|u - x\| : x \in Z\}.$$

Let $g : \mathbb{R}^n \to (-\infty, \infty]$ be an extended real valued function which is lower semicontinuous; that is, for each $x \in \mathbb{R}^n$, $g(x) \le \liminf_{y\to x} g(y)$. A vector $\zeta \in \mathbb{R}^n$ is said to be a *proximal subgradient* (or P-subgradient) of $g$ at a point $x$ such that $g(x) < \infty$ provided that there exists $\sigma > 0$ such that

$$g(y) - g(x) + \sigma\|y - x\|^2 \ge \langle \zeta, y - x\rangle \tag{11}$$

for all $y$ near $x$; this is known as the *proximal subgradient inequality*. The set of all such vectors $\zeta$ is called the P-subdifferential of $g$ at $x$, denoted $\partial_P g(x)$. One can show that $\partial_P g(x) \neq \emptyset$ for a dense subset of $\mathrm{dom}(g)$, the set of points where $g$ is finite. The limiting, or L-subdifferential is defined via limits as

$$\partial_L g(x) := \{\lim \zeta_i : \zeta_i \in \partial_P g(x_i), x_i \to x, g(x_i) \to g(x)\},$$

and for $g$ locally L:ipschitz, the C-subdifferential is defined as $\partial_C g(x) := \mathrm{co}\{\partial_L g(x)\}$. For such $g$, one has the containments

$$\partial_P g(x) \subset \partial_L g(x) \subset \partial_C g(x) \quad \forall\, x \in \mathbb{R}^n. \tag{12}$$

## 2.2 Semiconcavity

**Definition 2.1.** Let $U \subset \mathbb{R}^n$ be open. Then a continuous function $\varphi : U \to \mathbb{R}$ is *semiconcave* on $U$ provided that there exists $c \ge 0$ such that for any $x \in U$

$$\varphi(x + h) + \varphi(x - h) - 2\varphi(x) \le c\|h\|^2$$

whenever $\|h\|$ is sufficiently small (depending on $x$).

Semiconcavity is an important regularity property in the theory of nonlinear partial differential equations, and as first demonstrated by Rifford [26], [27], in Lyapunov theory as well. The following proposition summarizes some useful equivalences involving semiconcavity,It is essentially known, following as it does from facts in Bardi and Capuzzo-Dolcetta [3] and [11].

**Proposition 2.2.** *For $U \subset \mathbb{R}^n$ open and $\varphi : U \to \mathbb{R}$ locally Lipschitz, the following three properties are equivalent:*

*(i) $\varphi$ is semiconcave on $U$.*

*(ii)* *There exists* $c \geq 0$ *such that*

$$g(y) := \varphi(y) - c\|y\|^2$$

*is locally concave on* $U$*; that is, for every* $x \in U$*, there exists* $r_x > 0$ *such that* $x + r_x B \subset U$ *and* $g(\cdot)$ *is concave on* $x + r_x B$*.*

*(iii)* *There exists* $c \geq 0$ *such that given* $x \in U$*, there exists* $r_x > 0$ *for which*

$$-\varphi(y) + \varphi(x) + c\|y - x\|^2 \geq \langle \zeta, y - x \rangle \quad \forall\, \zeta \in \partial_P(-\varphi)(x), \quad \forall\, y \in x + r_x B. \tag{13}$$

*Furthermore, if* $\varphi$ *is semiconcave on* $U$*, then at every* $x \in U$ *one has*

$$\partial_P(-\varphi)(x) = \partial_L(-\varphi)(x) = \partial_C(-\varphi)(x) = -\partial_C \varphi(x). \tag{14}$$

Of particular use to us below will be the following.

**Corollary 2.3.** *Suppose that* $U \subset \mathbb{R}^n$ *is open and that* $\varphi$ *is semiconcave on* $U$*. Then for any open convex subset* $U'$ *of* $U$ *and any* $x, y \in U'$*, one has*

$$-\varphi(y) + \varphi(x) + c\|y - x\|^2 \geq \langle \zeta, y - x \rangle \quad \forall\, \zeta \in \partial_P(-\varphi)(x) \tag{15}$$

*and*

$$\varphi(y) - \varphi(x) \leq \langle \zeta, y - x \rangle + c\|y - x\|^2 \quad \forall\, \zeta \in \partial_L \varphi(x), \tag{16}$$

*where* $c$ *is as in (13).*

# 3 State constrained tracking and constraint removal

Our methods will utilize recent results in Clarke, Rifford and Stern [12] (see also Clarke and Stern [13]), which dealt with the construction of feedback control laws for a general class of state constrained optimal control problems, via a constraint removal method. In that work, extra hypotheses were imposed upon $S$, so as to have available certain geometric properties of *inner approximations* of $S$, given by

$$S_r := \{x \in \mathbb{R}^n : d_{\hat{S}}(x) \geq r\}.$$

for $r \geq 0$; note that $S_0 = S$. Inner approximations were studied in earlier work by Clarke, Ledyaev and Stern [10], as well as in [12]. Several important properties verified in [12] regarding inner approximations will also be required in the present article, and will be summarized in this section.

The augmented geometric hypotheses on $S$ are now posited. We denote the *lower Hamiltonian* $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$h(x, p) := \min_{u \in U} \langle f(x, u), p \rangle. \tag{17}$$

(S1) $S$ is a compact subset of $\mathbb{R}^n$ which is *wedged* at each $x \in S$; that is, $N_S^C(x)$, the *Clarke normal cone* to $S$ at $x$, is pointed. This means that $N_S^C(x) \cap \{-N_S^C(x)\} = \{0\}$, or equivalently, $\mathrm{int}[T_S^C(x)] \neq \emptyset$ for each $x \in S$, where $T_S^C(x)$ denotes the *Clarke tangent cone* to $S$ at $x$. (We again refer the reader to [11] for the defintions and properties of these geometric constructs.)

(S2) The following "strict inwardness" condition holds:

$$h(x, \zeta) < 0 \quad \forall\, 0 \neq \zeta \in N_S^C(x), \quad \forall\, x \in \mathrm{bdry}(S). \tag{18}$$

- Hypotheses (S1)-(S2) will be assumed to hold in all that follows.

**Remark 3.1.**

*(a)* Any convex body (i.e. a convex set with nonempty interior) is wedged, but convexity is not required; for example, the closed complement of a convex body is necessarily wedged. Wedgedness of $S$ at $x \in \mathrm{bdry}(S)$ is also referred to in the literature as *epi-Lipschitzness* at $x$, since the property amounts to $S$ being locally linearly homeomorphic to the epigraph of a Lipschitz function; see Rockafellar [28] and Clarke [5].

*(b)* The set $S$ is said to be *weakly invariant* provided that for any initial state $\alpha \in S$, there exists a control function $u(\cdot)$ such that

$$x(t) = x(t; 0, \alpha, u(\cdot)) \in S \quad \forall\, t \geq 0.$$

This is equivalent to the proximal condition

$$h(x, \zeta) \leq 0 \quad \forall\, \zeta \in N_S^P(x), \quad \forall\, x \in S; \tag{19}$$

see [11]. Hence conditions (S1)-(S2) are sufficient for weak invariance.

### 3.1 State constrained trajectory tracking

Required properties of inner approximations are summarized in the following lemma. Part (a) asserts that for small positive $r$, inner approximations $S_r$ of $S$ inherit weak invariance from $S$ itself, while parts (b) and (c) provide trajectory tracking properties relative to inner approximations, in a uniform manner with respect to $r$. Among references involving state constrained tracking are the seminal results of Soner [31]; see also Forcellini and Rampazzo [17] and Frankowska and Rampazzo [18]. Part (a) of the lemma is included in Corollary 3.4 of [12], while parts (b) and (c) are provided by the proofs of Theorem 3.10 and Proposition 3.13 of [12], respectively.)

**Lemma 3.2.** *There exists a constant $r_0 > 0$ satisfying the following three properties:*

*(a)* *For every $r \in [0, r_0]$, the set $S_r$ is nonempty and weakly invariant.*

*(b)* *Given $T > 0$, there exists a constant $M(T) > 0$ such that for every $r \in [0, r_0]$, the following holds: Let $\alpha_0$ and $\alpha_1$ be initial states in $S_r$, and let $u_0(\cdot)$ be a control function producing a trajectory which satisfies*

$$x(t; 0, \alpha_0, u_0(\cdot)) \in S_r \quad \forall\, t \in [0, T]. \tag{20}$$

*Then there exists a control function $u_1(\cdot)$ which produces a trajectory which satisfies*

$$\|x(t; 0, \alpha_1, u_1(\cdot)) - x(t; 0, \alpha_0, u_0(\cdot))\| \leq M(T)\|\alpha_1 - \alpha_0\| \quad \forall\, t \in [0, T] \tag{21}$$

*and*

$$x(t; 0, \alpha_1, u_1(\cdot)) \in S_r \quad \forall\, t \in [0, T]. \tag{22}$$

*(c)* *Given $T > 0$, there exists a constant $W(T) > 0$ such that for any initial state $\alpha \in \mathrm{int}(S)$, if $r \in [0, r_0]$ is such that $\alpha \in S_r$ and $u(\cdot)$ is a control function such that*

$$x(t; 0, \alpha, u(\cdot)) \in S \quad \forall\, t \in [0, T], \tag{23}$$

*then there exists a control function $\bar{u}(\cdot)$ such that*

$$\|x(t; 0, \alpha, \bar{u}(\cdot)) - x(t; 0, \alpha, u(\cdot))\| \leq rW(T) \quad \forall\, t \in [0, T], \tag{24}$$

*and*

$$x(t; 0, \alpha, \bar{u}(\cdot)) \in S_r \quad \forall\, t \in [0, T]. \tag{25}$$

### 3.2 Modified dynamics and constraint removal

It will be convenient to denote

$$F(x) := f(x, U) = \{f(x, u) : u \in U\}.$$

Let us recall that in view of Filippov's lemma, an absolutely conhtinuous arc $x(\cdot)$ is a trajectory of the differential inclusion

$$\dot{x}(t) \in F(x(t)) \quad \text{a.e.} \tag{26}$$

on a given time interval, if and only if for some control function $u(\cdot)$, $x(\cdot)$ is a trajectory of the original control system (1).

We will require the following lemma, which summarizes certain technical facts from [12].

**Lemma 3.3.** *If $r_0 > 0$ is sufficiently small, then for each $r \in [0, r_0]$, there exists a multifunction $F_r$ with the following properties.*

*(a) $F_r(x)$ is a compact convex subset of $\mathbb{R}^n$ for every $x \in \mathbb{R}^n$. Also,*

$$F_r(x) = F(x) \quad \forall\, x \in S_r, \tag{27}$$

*and*

$$F_r(x) \subset F(x) \quad \forall\, x \in S. \tag{28}$$

*(b) There exists $K > 0$ (independent of $r$) such that*

$$F_r(x) \subset F_r(y) + K\|y - x\|B \quad \forall\, x, y \in \mathbb{R}^n;$$

*that is, $F_r(\cdot)$ is globally Lipschitz of rank $K$.*

*(c) For every initial phase $(\tau, \alpha) \in \mathbb{R} \times \mathbb{R}^n$, there exists a trajectory $x(\cdot)$ satisfying the differential inclusion*

$$\dot{x}(t) \in F_r(x(t)) \quad \text{a.e.} \tag{29}$$

*on $[\tau, \infty)$, such that $x(\tau) = \alpha$.*

*(d) The set $S$ is strongly invariant with respect to $F_r$; that is, for every initial state $\alpha \in S$, every trajectory $x(\cdot)$ of the differential inclusion (29) with $x(0) = \alpha$ satisfies $x(t) \in S$ for all $t \geq 0$.*

*(e) There exists $\varepsilon_0 > 0$ such that for any $\alpha \in S + \varepsilon_0 B$, there exists a trajectory $x(\cdot)$ of (29) with $x(0) = \alpha$, such that $x(1) \in S$.*

*(f) There exists $C > 0$ (independent of $r$) such that the following holds: For any $\alpha \in S \backslash S_r$, there exists a trajectory of $x(\cdot)$ of (29) such that $x(0) = \alpha$ and*

$$t(\alpha) := \sup\{t : d_{\hat{S}}(x(t)) \leq r\} \leq Cr. \tag{30}$$

*(g) There exists $T_2(r) > 0$ such that if $\alpha \in S$ and $u_\alpha \in U$ are such that $f(\alpha, u_\alpha) \in F_r(\alpha)$, then the (unique) solution $x_\alpha(\cdot)$ of the differential equation*

$$\dot{x}(t) = f(x(t), u_\alpha), \tag{31}$$

*with $x(0) = \alpha$, satisfies $x_\alpha(t) \in S$ for all $t \in [0, T_2(r)]$.*

According to (a) and (b), the multifunction $F_r$ in the statement is globally Lipschitz and agrees with $F$ on the inner approximation $S_r$, and is contained in $F(x)$ for other points $x \in S$. Part (c) of the lemma follows from the standard existence theory for differential inclusions; observe that the usual "linear growth" condition is implied by the *global* Lipschitz nature of $F_r$. Note that (d) provides for what we refer to as "constraint removal", in the sense that for any initial state $x(0) \in S$, the state constraint $x(t) \in S\ \forall\, t \geq 0$ is *implicit* for the differential inclusion (29). Full details of the construction of $F_r$ satisfying (a)-(d) and (f)-(g) are provided in [12]. We mention that part (e) follows independently from that construction and a result on set attainability in [11].

# 4 Main results

Our first main result is the following.

**Theorem 4.1.** *For any $\delta > 0$, open loop $S$-controllability to $\Sigma$ implies closed loop $S$-controllability to $\Sigma^\delta$.*

We shall require the following lemma.

**Lemma 4.2.** *Assume that open loop $S$-controllability to $\Sigma$ holds. Then for any $\gamma > 0$, there exists $T_3 = T_3(\gamma) > 0$ such that the following hold:*

*(a) For any initial state $\alpha \in S$, there exists a control function $u(\cdot)$ such that for some $\bar{t} = \bar{t}(\alpha, \gamma) \in [0, T_3]$ one has*

$$x(t) = x(t; 0, \alpha, u(\cdot)) \in S \quad \forall\, t \in [0, \bar{t}\,] \tag{32}$$

*and*

$$x(\bar{t}) \in \Sigma^\gamma. \tag{33}$$

*(b) There exists $r(\gamma) \in (0, r_0]$ such that if $0 \le r \le r(\gamma)$, then the following holds: For any initial state $\alpha \in S$ there exist $\tilde{t} = \tilde{t}(\alpha, \gamma) \in [0, T_3 + Cr_0]$ and a trajectory $x(\cdot)$ of the differential inclusion (29) satisfying $x(0) = \alpha$, such that*

$$x(t) \in S \quad \forall\, t \in [0, \tilde{t}\,] \tag{34}$$

*and*

$$x(\tilde{t}) \in \Sigma^{2\gamma}. \tag{35}$$

*Proof:* In order to prove part (a), note that open loop $S$-controllability to $\Sigma$ implies that for given $\alpha \in S$, there exists $t(\alpha) \ge 0$ such that for some trajectory $x(\cdot)$ of the control system (1) with $x(0) = \alpha$, one has $x(t) \in S$ for all $t \in [0, t(\alpha)]$ and $x(t(\alpha)) \in \Sigma$. By the $S$-constrained tracking property (b) of Lemma 3.2, there exists $Q(\alpha) > 0$ such that the following holds: For each

$$\alpha_1 \in N(\alpha) := \{\alpha + Q(\alpha)B\} \cap S,$$

there exists a trajectory $x_1(\cdot)$ of (1) with $x_1(0) = \alpha_1$ such that $x_1(t) \in S$ for all $t \in [0, t(\alpha)]$ and $x_1(t(\alpha)) \in \Sigma^{\gamma}$. In particular, using the notation of Lemma 3.2 (b), we can take

$$Q(\alpha) = \frac{\gamma}{M(t(\alpha))}.$$

The family of sets $N(\alpha)$ forms a relatively open cover of $S$, and since $S$ is compact, we have a finite subcover $\{N(\alpha_i)\}_{i=1}^{k}$. It is readily noted that

$$T_3 = \max\{t(\alpha_i) : 1 \le i \le k\}$$

has the required properties.

As for part (b) of the assertion, consider any initial state $\alpha \in S$, and note that by part (f) of Lemma 3.3, for $r \in [0, r_0]$, there exists a trajectory $x_1(\cdot)$ of differential inclusion (29) emanating from $\alpha$ such that $\alpha_1 := x_1(t_1) \in S_r$ for some $t_1 \in [0, Cr_0]$. Furthermore, by part (d) of that lemma (strong invariance), $x_1(t) \in S$ on the interval $[0, t_1]$. By part (a) of the present lemma, there exists a trajectory $x_2(\cdot)$ of the control system (1) such that $x_2(0) = \alpha_1$ and such that for some $t_2 \in [0, T_3]$ one has $x_2(t) \in S$ for all $t \in [0, t_2]$, and $x_2(t_2) \in \Sigma^{\gamma}$. According to tracking property (c) of Lemma 3.2, if $r \in [0, r(\gamma)]$, where

$$r(\gamma) := \min\left\{r_0, \frac{\gamma}{W(T_3)}\right\},$$

then there also exists a trajectory $x_3(\cdot)$ of the control system (1) such that $x_3(0) = \alpha_1$, $x_3(t) \in S_r$ for all $t \in [0, t_2]$, and $\|x_3(t_2) - x_2(t_2)\| \le \gamma$, implying $x_3(t_2) \in \Sigma^{2\gamma}$. Now note that in view of (27), on the interval $[0, t_2]$, the trajectory $x_3(\cdot)$ is also a trajectory of the differential inclusion (29). The required trajectory $x(\cdot)$ of the assertion is the concatenation of $x_1(\cdot)$ and $x_3(\cdot)$. $\square$.

Assume that the hypotheses and notations of the preceding lemma are still in effect. For a given $r \in [0, r(\gamma)]$, we introduce the following modification of the multifunction $F_r$:

$$F_{r,\gamma}(x) := \begin{cases} \operatorname{co}\{F_r(x) \cup \overline{B}\} & \text{if } \|x\| \in \Sigma^{2\gamma} \\ \operatorname{co}\{F_r(x)\} \cup \frac{[3\gamma - d_\Sigma(x)]}{\gamma}\overline{B} & \text{if } x \in \Sigma^{2\gamma}, x \notin \Sigma^{3\gamma} \\ F_r(x) & \text{if } x \notin \Sigma^{3\gamma} \end{cases} \tag{36}$$

By part (b) of the preceding lemma and Lemma 3.3(e), any $\alpha \in S + \varepsilon_0 \overline{B}$ is the start point of some trajectory of the differential inclusion (29) which reaches the target $\Sigma^{2\gamma}$ at a time not exceeding $T_3 + Cr_0 + 1$. Hence the same is true for the differential inclusion

$$\dot{x}(t) \in F_{r,\gamma}(x(t)) \quad \text{a.e.} \tag{37}$$

This is due to the fact that one has the obvious containment

$$F_r(x) \subset F_{r,\gamma}(x) \quad \forall\, x \in \mathbb{R}^n. \tag{38}$$

Furthermore, since the values of the multifunction $F_{r,\gamma}$ are compact convex subsets of $\mathbb{R}^n$ and since this multifunction is globally Lipschitz, it follows from the standard theory of differential inclusions that the set of trajectories of (37) on any compact time interval, emanating from a given start point, is nonempty and sequentially compact in the uniform topology. Hence the minimum time $\tau_{r,\gamma}(\alpha)$ to the target $\Sigma^{2\gamma}$ from any start point $\alpha \in S + \varepsilon_0 \overline{B}$ is attained; here

$$\tau_{r,\gamma}(\alpha) := \min\{\tilde{t} \ge 0 : x(\tilde{t}) \in \Sigma^{2\gamma}, \dot{x}(t) \in F_{r,\gamma}(x(t)) \text{ a.e.}, \text{x}(0) = \alpha\}. \tag{39}$$

Note that in this minimum time problem, there is no state constraint imposed.

We go on to define, for $r, \gamma$ as above, an extended real valued function $V_{r,\gamma} : \mathbb{R}^n \to (-\infty, \infty]$ as

$$V_{r,\gamma}(\alpha) := \begin{cases} \tau_{r,\gamma}(\alpha) & \text{if } \alpha \in S + \varepsilon_0 \overline{B} \\ \infty & \text{if } \alpha \in \mathbb{R}^n \backslash \{S + \varepsilon_0 \overline{B}\} \end{cases} \tag{40}$$

Important properties of the $V_{r,\gamma}$ are provided by the following lemma, which follows directly from the more general result, Theorem 3 in [27].

**Lemma 4.3.** *Let $\gamma > 0$, assume that the origin is open loop $S$-controllable to $\Sigma$, and that $0 < r \leq r(\gamma)$. Then the following properties hold:*

*(a) $V_{r,\gamma}$ is Lipschitz on $S + \varepsilon_0 \overline{B}$, where $\varepsilon_0$ is as in Lemma 3.3(e).*

*(b) One has*

$$\min_{v \in F_{r,\gamma}(x)} \langle v, \zeta \rangle \leq -1 \quad \forall \zeta \in \partial_P V_{r,\gamma}(x), \quad \forall x \in \{S + \varepsilon_0 \overline{B}\} \backslash \{\Sigma^{2\gamma}\}. \tag{41}$$

**Remark 4.4.** Actually, (41) holds in equality form, but it is the stated inequality, which encapsulates *weak decrease* of the "control Lyapunov function" $V_{r,\gamma}$, that is of interest to us; see [11] for a discussion of this property.

For $\lambda > 0$, the *quadratic infimal convolution* of (the lower semicontinuous extended real valued function) $V_{r,\gamma}$ is the function $V_{r,\gamma}^{\lambda} : \mathbb{R}^n \to \mathbb{R}$ given by

$$V_{r,\gamma}^{\lambda} := \inf_{y \in \mathbb{R}^n} \{V_{r,\gamma}(y) + \lambda \|y - x\|^2\}. \tag{42}$$

This function, which is clearly majorized by $V_{r,\gamma}$, is locally Lipschitz on $\mathbb{R}^n$. Furthermore, if $x \in \mathbb{R}^n$ is such that $\partial_P V_{r,\gamma}^{\lambda}(x) \neq \emptyset$, then there exists a point $\bar{y} \in \mathbb{R}^n$ such that the infimum in (42) is uniquely attained at $\bar{y}$ and

$$\partial_P V_{r,\gamma}^{\lambda}(x) \subset \partial_P V_{r,\gamma}(\bar{y}). \tag{43}$$

It can also be shown that in fact, $\partial_P V_{r,\gamma}^{\lambda}(x)$ reduces to a singleton, the Fréchet derivative of $V_{r,\gamma}^{\lambda}$ at $x$, a fact we will not require. These properties of quadratic infimal convolutions are all verified in the reference [11]. We shall in addition require the following lemma concerning the function $V_{r,\gamma}^{\lambda}$; the hypotheses of Lemma 4.3 are assumed to still be in effect.

**Lemma 4.5.** *There exists $\lambda(\gamma) > 0$ such that if $\lambda > \lambda(\gamma)$, then $V_{r,\gamma}^{\lambda}$ is semiconcave on the set $S + \frac{\varepsilon_0}{2} B$, and*

$$\min_{v \in F_{r,\gamma}(x)} \langle v, \zeta \rangle \leq -\frac{1}{2} \quad \forall \zeta \in \partial_P V_{r,\gamma}^{\lambda}(x), \quad \forall x \in \{S + \frac{\varepsilon_0}{2} B\} \backslash \{\Sigma^{3\gamma}\}. \tag{44}$$

*Proof:* One has

$$0 \leq V_{r,\gamma}^{\lambda}(x) \leq V_{r,\gamma}(x) \leq T_3 + C r_0 + 1 \quad \forall x \in S + \varepsilon_0 \overline{B}, \quad \forall \lambda > 0.$$

Hence for any $x \in S + \varepsilon_0 \overline{B}$ and any given $\lambda > 0$, there exist points $y \in \mathbb{R}^n$ such that

$$V_{r,\gamma}(y) + \lambda \|y - x\|^2 \leq T_3 + C r_0 + 2. \tag{45}$$

Then

$$\|y - x\| \leq \sqrt{\frac{T_3 + C r_0 + 2}{\lambda}} =: w(\gamma, \lambda), \tag{46}$$

and therefore

$$V_{r,\gamma}^{\lambda}(x) = \min_{y \in x + w(\gamma,\lambda)\overline{B}} \{V_{r,\gamma}(y) + \lambda \|y - x\|^2\}, \tag{47}$$

or equivalently,

$$-V_{r,\gamma}^{\lambda}(x) = \max_{y \in x + w(\gamma,\lambda)\overline{B}} \{-V_{r,\gamma}(y) - \lambda \|y - x\|^2\}. \tag{48}$$

Now consider $x \in S + \frac{\varepsilon_0}{2} B$, and note that the condition

$$\lambda > \hat{\lambda}(\gamma) := 4 \left( \frac{T_3 + C r_0 + 2}{(\varepsilon_0)^2} \right) \tag{49}$$

implies

$$x + w(\gamma,\lambda)\overline{B} \subset S + \varepsilon_0 B. \tag{50}$$

By Lemma 4.3(a), $V_{r,\gamma}$ is Lipschitz on the ball $x + w(\gamma,\lambda)\overline{B}$, and it follows from (48) that the function $-V^{\lambda}_{r,\gamma}$ is $\lambda$-*lower* $C^2$ on that ball, in the terminology of Rockafellar, who studied this class of functions in [29]. Furthermore, as was shown in Clarke, Stern and Wolenski [14], this property implies that one has the uniform proximal subgradient inequality given by

$$-V^{\lambda}_{r,\gamma}(y) + V^{\lambda}_{r,\gamma}(x) + \lambda\|y-x\|^2 \geq \langle \zeta, y - s\rangle \;\forall\,\zeta \in \partial_P(-V^{\lambda}_{r,\gamma})(x),\;\forall\, y \in x + w(\gamma,\lambda)B. \tag{51}$$

According to Proposition 2.2(iii) (with $c=\lambda$), since (51) holds for every $x \in S + \frac{\varepsilon_0}{2}B$, it follows that the function $V^{\lambda}_{r,\gamma}$ is semiconcave on that set, for every $\lambda > \hat{\lambda}(\gamma)$.

Now let $x \in \{S + \frac{\varepsilon_0}{2}B\}\backslash\{\Sigma^{3\gamma}\}$, and assume that

$$\lambda > \tilde{\lambda}(\gamma) := \max\left\{\hat{\lambda}(\gamma)\;,\;\frac{T_3 + Cr_0 + 2}{\gamma^2}\right\} \tag{52}$$

Then

$$x + w(\gamma,\lambda)\overline{B} \subset \{S + \varepsilon_0 B\}\backslash\{\Sigma^{2\gamma}\}, \tag{53}$$

and therefore by (41), for any $y \in x + w(\gamma,\lambda)\overline{B}$ one has

$$\min_{v\in F_{r,\lambda}(y)} \langle v,\zeta\rangle \leq -1 \quad \forall\,\zeta \in \partial_P V_{r,\lambda}(y). \tag{54}$$

Suppose that $\zeta \in \partial_P V^{\lambda}_{r,\lambda}(x)$. Then by property (43) (in the present context), one has that $\zeta \in \partial_P V_{r,\lambda}(\bar{y})$ for some $\bar{y} \in x + w(\gamma,\lambda)B$. In view of (54), there exists $\bar{v} \in F_{r,\lambda}(\bar{y})$ such that $\langle v,\zeta\rangle \leq -1$. Let us denote by $K_{r,\lambda}$ a (global) Lipschitz constant for the multifunction $F_{r,\lambda}$. Then there exists $\hat{v} \in F_{r,\lambda}(x)$ such that $\|\bar{v} - \hat{v}\| \leq K_{r,\lambda} w(\gamma,\lambda)$. Now denote by $K'_{r,\lambda}$ a Lipschitz constant for $V_{r,\lambda}$ on $S + \varepsilon_0\overline{B}$. then $\|\zeta\| \leq K'_{r,\lambda}$, by a standard fact regarding norm bounds on proximal subgradients of Lipschitz functions. We then obtain

$$\langle \hat{v},\zeta\rangle \leq \langle \bar{v},\zeta\rangle + K_{r,\lambda}K'_{r,\lambda} w(\gamma,\lambda). \tag{55}$$

It follows that $\langle \hat{v},\zeta\rangle \leq -\frac{1}{2}$ provided that

$$\lambda > \bar{\lambda}(\gamma) := \frac{T_3 + Cr_0 + 2}{4(K_{r,\lambda}K'_{r,\lambda})^2}.$$

Upon setting

$$\lambda(\gamma) := \max\{\tilde{\lambda}(\gamma)\;,\;\bar{\lambda}(\gamma)\},$$

(44) holds and the proof is completed. □.

*Proof of Theorem 4.1:* As in the preceding lemma, let us fix $0 < r \leq r(\gamma)$ and $\lambda > \lambda(\gamma)$, where $\gamma > 0$ has been chosen a priori so that $4\gamma < \delta$.

- *For ease of notation, from this point on we will denote* $V^{\lambda}_{r,\gamma} = V$.

Since

$$F_{r,\gamma}(x) = F_r(x) \quad \forall\, x \in \mathbb{R}^n\backslash\{\Sigma^{3\gamma}\}, \tag{56}$$

the inequality (44) can be written as

$$\min_{v\in F_r(x)} \langle v,\zeta\rangle \leq -\frac{1}{2} \quad \forall\,\zeta \in \partial_P V(x), \quad \forall\, x \in \{S + \frac{\varepsilon_0}{2}B\}\backslash\{\Sigma^{3\gamma}\}. \tag{57}$$

This proximal Hamilton-Jacobi inequality is in turn readily seen to be equivalent to the limiting version

$$\min_{v\in F_r(x)} \langle v,\zeta\rangle \leq -\frac{1}{2} \quad \forall\,\zeta \in \partial_L V(x), \quad \forall\, x \in \{S + \frac{\varepsilon_0}{2}B\}\backslash\{\Sigma^{3\gamma}\}. \tag{58}$$

A feedback law $k : \mathbb{R}\times\mathbb{R}^n \to U$ is now defined as follows.

- Let $x \in \mathbb{R}^n$.

– If $x \in S \backslash \{\Sigma^{4\gamma)}\}$, arbitrarily choose $\zeta \in \partial_L V(x)$, and then set $k(x) = u \in U$ such that $f(x,u) \in F_r(x)$ and

$$\min_{v \in F_r(x)} \langle v, \zeta \rangle = \langle f(x,u), \zeta \rangle.$$

– Otherwise take $k(x)$ to be any element of $U$.

**Remark 4.6.** It is the $L$-subdifferential of $V$ that features in the definition of the feedback, and not the $P$-subdifferential. The advantage of this choice is that $\partial_L V(x) \neq \emptyset$ for every $x$, whereas the possible emptiness of $\partial_P V(x)$ would be problematic in our ensuing construction. Also observe that in the construction of a $\pi$-trajectory associated with the control feedback $k(x)$, the choice of $\zeta \in \partial_L V(x_i)$ does not need to be "remembered" at the next node $x_{i+1}$, so in the case of an "on-line" procedure, it suffices to calculate an *arbitrary* $L$-subgradient when a given state is reached.

Given an initial state

$$\alpha \in S \backslash \{\Sigma^{\delta}\} \subset S \backslash \{\Sigma^{4\gamma}\},$$

and a partition

$$\pi = \{t_0 = 0, t_1, t_2, \ldots\}$$

of $[0,\infty)$, let us consider the $\pi$-trajectory $x(\cdot)$ associated with the ordinary differential equation $\dot{x}(t) = f(x(t), k(x(t))$, where the initial node is $x(t_0) = x(0) = x_0 = \alpha$. Our goal is to produce positive numbers $\beta$ and $T_1$ such that if $\operatorname{diam}(\pi) \leq \beta$, then for every such $\alpha$, there exists $t_1(\alpha) \in [0, T_1]$ such that

$$x(t) \in S \quad \forall\, t \in [0, t_1(\alpha)], \tag{59}$$

and

$$x(t_1(\alpha)) \in \Sigma^{4\gamma} \subset \Sigma^{\delta}. \tag{60}$$

Of course, if $\alpha \in S \cap \{\Sigma^{4\gamma}\}$, then we can take $t_1(\alpha) = 0$.

We shall assume that

$$\operatorname{diam}(\pi) \leq T_2(r), \tag{61}$$

where $T_2(r)$ is as in Lemma 3.3(g). Then the $\pi$-trajectory satisfies

$$x_i \in S \backslash \{\Sigma^{4\gamma)}\} \Longrightarrow x(t) \in S \quad \forall\, t \in [t_i, t_{i+1}], \tag{62}$$

since $F_{r,\gamma}(x_i) = F_r(x_i)$.

By Lemma 4.5, $V$ is semiconcave on $S + \frac{\varepsilon_0}{2} B$, and therefore Corollary 2.3 yields

$$V(y) - V(x) \leq \langle \zeta, y - x \rangle + \lambda \|y - x\|^2 \quad \forall\, \zeta \in \partial_L V(x) \quad \forall\, x, y \in U', \tag{63}$$

for every open convex set $U' \subset S + \frac{\varepsilon_0}{2} B$. Let

$$M := \max\{\|f(x,u)\| : x \in S + \varepsilon_0 \overline{B}\ ,\ u \in U\}. \tag{64}$$

In order to be able to apply (63) to the evolving $\pi$-trajectory, we will assume (further to (61)) that

$$\operatorname{diam}(\pi) \leq \frac{\min\{\gamma, \frac{\varepsilon_0}{2}\}}{M} =: \rho. \tag{65}$$

This together with (62) yields the implication

$$x_i \in S \backslash \{\Sigma^{4\gamma}\} \Longrightarrow x(t) \in x_i + \rho B \subset S \backslash \{\Sigma^{3\gamma}\} \quad \forall\, t \in [t_i, t_{i+1}]. \tag{66}$$

Let us further consider the evolution of the $\pi$-trajectory. Pick $\zeta_0 \in \partial_L V(x_0)$. Then by (63) and (66), for $t \in [t_0, t_1]$ one has

$$\begin{aligned} V(x(t)) - V(x_0) &\leq \langle \zeta_0, x(t) - x_0 \rangle + \lambda \|x(t) - x_0\|^2 \\ &= \langle \zeta_0, \int_{t_0}^{t_1} f(x(s), k(x_0) ds \rangle + \lambda \|x(t) - x_0\|^2. \end{aligned}$$

For ease of notation, let us abbreviate $K = K_{r,\lambda}$, which we recall denotes a Lipschitz constant for $V = V_{r,\gamma}$ on $S + \varepsilon_0\overline{B}$. We also denote by $\hat{K} = K_\Gamma$ a Lipschitz constant for $f$ as in (F1), with $\Gamma = S + \varepsilon_0\overline{B}$. Then, bearing (58), (66) in mind, one has

$$V(x_1) - V(x_0) \leq -\frac{1}{2}(t - t_0) + (KM\hat{K} + \lambda M)(t_1 - t_0)^2. \tag{67}$$

Now pick $\zeta_1 \in \partial_L V(x_1)$. Upon repeating the above steps on the interval $[t_1, t_2]$ and combining this with (67), we obtain

$$V(x_2) - V(x_0) \leq -\frac{1}{2}(t_2 - t_0) + (KM\hat{K} + \lambda M)[(t_1 - t_0)^2 + (t_2 - t_1)^2, \tag{68}$$

which readily yields

$$V(x_2) - V(x_0) \leq \left[-\frac{1}{2} + (KM\hat{K} + \lambda M)\,\mathrm{diam}(\pi)\right](t_2 - t_0). \tag{69}$$

Continuing this process, we arrive at

$$V(x_i) - V(x_0) \leq \left[-\frac{1}{2} + (KM\hat{K} + \lambda M)\,\mathrm{diam}(\pi)\right](t_i - t_0). \tag{70}$$

Let us assume that

$$\mathrm{diam}(\pi) \leq \frac{1}{4(KM\hat{K} + \lambda M)} =: \beta_1 \tag{71}$$

Then due to (70),

$$V(x_i) - V(x_0) \leq -\frac{1}{4}(t_i - t_0). \tag{72}$$

It is readily noted that $V = V^\lambda_{r,\gamma} \equiv 0$ on $\Sigma^{2\gamma}$. Then since $V$ is continuous on $S + \varepsilon_0\overline{B}$, it follows from (72) that $x(\cdot)$ must enter $\Sigma^{4\gamma}$ at a time not exceeding

$$T_1 := 4\max\{V(x) : x \in S + \varepsilon_0\overline{B}\} \leq 4(T_3 + Cr_0 + 1). \tag{73}$$

Upon taking

$$\beta := \min\{T_2(r), \rho, \beta_1\}, \tag{74}$$

the proof of the theorem is completed. □

Theorem 4.7, below, provides a strengthening of the conclusion of Theorem 4.1 from $S$-constrained controllability to $S$-constrained stabilizability, when an $S$-constrained small time controllability hypothesis is posited. The need for strengthened hypotheses is well understood and illustrated by the following example. Consider

$$S = \{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2, \}, \quad \Sigma = \{(2, 0)\},$$

where the dynamics are given by the bilinear system (a perturbed harmonic oscillator)

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x(t) + u(t)x(t), \quad U = [-1, 1].$$

It is easy to check that all the hypotheses of Theorem 4.1 hold, including open loop $S$-controllability, but closed loop $S$-stabilization does not hold.

**Theorem 4.7.** *Let open loop $S$-controllability to $\Sigma$ hold, and let $\delta > 0$ be given. Further assume that there exists $\varepsilon_1 > 0$ such that the $S$-constrained minimum time function $\tau : S \to \mathbb{R}$ to the target $\Sigma$,*

$$\tau(\alpha) := \min\{\tilde{t} \geq 0 : x(\tilde{t}) \in \Sigma,\ \dot{x}(t) \in F(x(t)) \text{ a.e., } \mathrm{x(t) \in S\ \forall\, t \in [0, \tilde{t}],\ x(0) = \alpha}\}, \tag{75}$$

*is continuous on $S \cap \{\Sigma^{\varepsilon_1}\}$. Then closed loop $S$-stabilizability to $\Sigma^\delta$ holds for every $\delta > 0$.*

We remark that in view of sequential compactness of trajectories of the differential inclusion (26) on any compact time interval, the minimum in (75) is indeed attained, for any $\alpha \in S$.

The proof of Theorem 4.7 is actually a continuation of the proof of Theorem 4.1. There we showed that if $\mathrm{diam}(\pi) \leq \beta$, then for any startpoint $\alpha \in S$, the $\pi$ trajectory associated with our feedback $k(\cdot)$ enters $\Sigma^{4\gamma}$ not later than time $T_1$, and is contained in $S$ until its entry into $\Sigma^{4\gamma}$.

In the present proof, we will specialize (but not violate the definition of) the feedback $k(\cdot)$ from the proof of Theorem 4.1, as follows:

- Let $x \in \mathbb{R}^n$.
  - If $x \in S \backslash \{\Sigma^{4\gamma}\}$, arbitrarily choose $\zeta \in \partial_L V(x)$, and then set $k(x) = u \in U$ such that $f(x,u) \in F_r(x)$ and
  $$\min_{v \in F_r(x)} \langle v, \zeta \rangle = \langle f(x,u), \zeta \rangle.$$
  - If $x \in S \cap \{\Sigma^{4\gamma}\}$, choose $k(x) = u \in U$ such that $f(x,u) \in F_r(x)$.
  - Otherwise (i.e. when $x \notin S$), take $k(x)$ to be any element of $U$.

We shall require the following consequence of $S$-constrained trajectory tracking.

**Lemma 4.8.** *There exists $\hat{r}(\gamma) > 0$ such that if $0 < r < \hat{r}(\gamma)$, the following holds: For any $\alpha \in S \cap \{\Sigma^{\frac{\varepsilon_1}{2}}\}$, there exists a trajectory of the differential inclusion (29) ($\dot{x}(t) \in F_r(x(t))$) such that $x(0) = \alpha$ and $x(\tau(\alpha)) \in \Sigma^{2\gamma}$.*

*Proof:* Let $\alpha \in S \cap \{\Sigma^{\frac{\varepsilon_1}{2}}\}$. In view of Lemma 3.3(f), there exists a trajectory of (29) (necessarily $S$-constrained by part (d) of that lemma) such that $x(0) = \alpha$ and $x(t') \in S_r$ for some time $t' = t'(\alpha) \in [0, Cr]$. We now denote $\alpha' = x(t')$. If $r$ is small enough, then $\alpha' \in S \cap \{\Sigma^{\varepsilon_1}\}$, for any $\alpha$ as above.

Let $u(\cdot)$ be a control function such that

$$x(t; 0, \alpha', u(\cdot)) \in S \quad \forall t \in [0, \tau(\alpha')]$$

and

$$x(\tau(\alpha'); 0, \alpha', u(\cdot)) \in \Sigma.$$

that is, $u(\cdot)$ is optimal in the $S$-constrained minimum time problem with target $\Sigma$ and startpoint $\alpha'$ ; recall that we are presently assuming open loop $S$-controllability to $\Sigma$. In view of the tracking result given by Lemma 3.2(c), there exists a trajectory $\bar{x}(\cdot)$ of (29) such that $\bar{x}(0) = \alpha'$, $\bar{x}(t) \in S_r$ for all $t \in [0, \tau(\alpha')]$, and

$$\bar{x}(\tau(\alpha')) \in \Sigma + rW(\tilde{T})\overline{B}, \tag{76}$$

where $\tilde{T}$ is an upper bound on (the continuous function) $\tau(\cdot)$ on $S$. It follows that

$$\tilde{x}(t' + \tau(\alpha')) \in \Sigma + rW(\tilde{T})\overline{B}, \tag{77}$$

where $\tilde{x}(\cdot)$ denotes the concatenation of the trajectories $x(\cdot)$ and $\bar{x}(\cdot)$ of differential inclusion (29); it is therefore a trajectory of (29), and as such, remains in $S$. One has

$$\|\tilde{x}(t' + \tau(\alpha')) - \tilde{x}(\tau(\alpha)\| \leq Mt' + M\|\tau(\alpha') - \tau(\alpha)\|, \tag{78}$$

where $M$ is as in (64), although any norm bound on $F(x) = f(x,U)$ over $S$ will do. Now, $0 < t' < Cr$, $\tau(\cdot)$ is continuous on $S \cap \{\Sigma^{\varepsilon_1}\}$ (where both $\alpha$ and $\alpha'$ lie when $r$ is sufficiently small), and $\|\alpha - \alpha'\| \leq MCr$. It then follows from (77) and (78) that $\tilde{x}(\tau(\alpha)) \in \Sigma^{2\gamma}$ if $r$ is sufficiently small, independently of $\alpha \in S$. □

*Completing the proof of Theorem 4.7:* In view of the definitions of the functions $V_{r,\gamma}$, $V = V^{\lambda}_{r,\gamma}$, the preceding lemma, and (38), one has

$$V(\alpha) \leq V_{r,\gamma}(\alpha) \leq \tau(\alpha) \quad \forall \alpha \in S \cap \{\Sigma^{\frac{\varepsilon_1}{2}}\}. \tag{79}$$

In view of the small time controllability hypothesis and (79), $\gamma > 0$ can be taken small enough (say $0 < \gamma \leq \hat{\gamma}$) to ensure that

$$V(\alpha) < \frac{4\gamma}{M} \quad \forall \alpha \in S \cap \{\Sigma^{5\gamma}\}. \tag{80}$$

Let us now reconsider the $\pi$-trajectory $x(\cdot)$ generated by the feedback $k(\cdot)$, emanating from an arbitrary $\alpha \in S$. We choose

$$0 < \gamma \leq \max\{\frac{\delta}{6}, \hat{\gamma}\},$$

$$\lambda > \lambda(\gamma),$$

$$0 < r < \max\{r(\gamma), \hat{r}(\gamma)\}$$

and

$$\operatorname{diam}(\pi) \leq \beta := \min\{T_2(r), \rho, \beta_1\}.$$

We know that $x(\cdot)$ enters $\Sigma^{4\gamma}$ not later than time $T_1$. Denote by $t_{i^*}$ the first node after the $\pi$-trajectory enters $\Sigma^{4\gamma}$. (If $\alpha \in \Sigma^{4\gamma}$, then $i^* = 1$.) Since $M\,\mathrm{diam}(\pi) \leq \gamma$ (by the bound involving $\rho$),

$$x(t) \in \Sigma^{5\gamma} \quad \forall\, t \in [t_{i^*}, t_{i^*+1}]. \tag{81}$$

In view of (80),

$$V(x(t_{i^*+1})) \leq \frac{\gamma}{4M}. \tag{82}$$

Then similarly to the proof of Theorem 4.1, $x(\cdot)$ re-enters $\Sigma^{4\gamma}$ not later than time $t_{i^*+1} + \frac{\gamma}{M}$, and consequently, from time $t_{i^*+1}$ until this re-entry, one has $\|x(t) - x(t_{i^*+1})\| \leq \gamma$. Hence, from time $t_{i^*}$ until re-entry to $\Sigma^{4\gamma}$, the $\pi$-trajectory remains in $\Sigma^{6\gamma} \subset \Sigma^{\delta}$. The above arguments show that for any $\alpha \in S$, after the $\pi$-trajectory enters $\Sigma^{4\gamma} \subset \Sigma^{\delta}$, it thereafter remains in $\Sigma^{\delta}$. During its evolution, the $\pi$-trajectory never leaves $S$, since it is a trajectory of (29); recall Lemma 3.3(d). □.

# 5 Concluding remarks

## 5.1 Robustness

It transpires that the feedback law in Theorem 4.7 possesses a robustness property with respect to state measurement errors which are small in an appropriate sense, and when the partition in the discretization scheme has the additional requirement of being "reasonably uniform", an insight first brought to light in [7].

The perturbed system under study is modeled by

$$\dot{x}(t) = f(x(t), \tilde{k}(x(t) + p(t))), \tag{83}$$

where the function $p(\cdot)$ represents the observational error present in applying the feedback law.

Given a partition $\pi$ of of $[0, \infty]$, the $\pi$-trajectory $x_\pi$ obtained in the model (83) is the curve satisfying the following interval-by-interval dynamics: Upon setting $x_0 = \alpha$, on the interval $[t_0, t_1]$, $x_\pi$ is the classical solution of

$$\dot{x}_\pi(t) = f(x_\pi(t), \tilde{k}(x_0 + p_0)), \quad x_\pi(t_0) = x_0, \quad t \in (t_0, t_1). \tag{84}$$

We then set $x_1 := x_\pi(t_1)$, and restart the process on the next interval:

$$\dot{x}_\pi(t) = f(x_\pi(t), \tilde{k}(x_1 + p_1)), \quad x_\pi(t_1) = x_1, \quad t \in (t_1, t_2), . \tag{85}$$

Here the continuous function $x_\pi(t)$ is the *actual* state of the system at time $t$, and the values $x_i + p_i$ correspond to the inexact measurements used to generate the piecewise constant control function in the scheme.

We have the following robust version of Theorems 4.1 and 4.7. The result allows for erroneous measurements of the state giving values exterior to $S$, while the $\pi$-trajectory that is generated remains in $S$.

**Theorem 5.1.** *Let $\delta > 0$ be given, and assume that open loop $S$-controllability to $\Sigma$ holds. Then there exists a feedback law $\tilde{k} : \mathbb{R}^n \to U$ and positive reals $T_1$ and $\beta$ such that the following hold:*

*(a) For every $b \in (0, \beta)$, there exists $E(b) > 0$ with the property that for any partition $\pi$ of $[0, T]$ having*

$$\frac{b}{2} \leq t_{i+1} - t_i \leq b \quad \forall\, i = 0, 1, \ldots, \tag{86}$$

*the error bounds*

$$\|p_i\| < E(\delta) \quad \forall\, i = 0, 1, \ldots, \tag{87}$$

*imply that for any initial state $\alpha \in S$, the $\pi$-trajectory $x_\pi$ in the model above, with $x_\pi(0) = \alpha$, satisfies*

$$x(t') \in \Sigma \tag{88}$$

*for some $t_1(\alpha) \in [0, T_1]$, and*

$$x_\pi(t) \in S \quad \forall\, t \in [0, t_1(\alpha)]. \tag{89}$$

*(b) If the $S$-constrained small time controllability hypothesis of Theorem 4.7 is posited, then the conclusions of part (a) can be strengthened by replacing (89) with*

$$x_\pi(t) \in S \cap \Sigma \quad \forall\, t \geq t_1(\alpha). \tag{90}$$

For each $x \in \mathbb{R}^n$, choose $s(x) \in \text{proj}_S(x)$. Then the feedback law featuring in the theorem is simply given by

$$\tilde{k}(x) := k(s(x)) \quad \forall\, x \in \mathbb{R}^n, \tag{91}$$

where $k(\cdot)$ is the feedback law from Theorem 4.7

The proof of Theorem 5.1 follows from arguments similar to those employed in §4.2 of [12]. There a finite time optimal control problem was studied, but the technique needed to extend the proofs of Theorems 4.1 and 4.7 to the above robust versions, is provided there. The fact that partitions with sufficiently small diameter are required in Theorems 4.1, 4.7 and 5.1 is to be expected, since this is what is needed in order for the decrease property (as manifested by proximal Hamilton-Jacobi inequalities) to come to bear in a sample-and-hold scheme such as ours. On the other hand, as was pointed out in [12], [7] and Sontag [33] (with the latter two references dealing with robust feedback stabilization via a shell-based approach), the near-uniformity of partitions posited in condition (86) precludes a possible "chattering phenomenon" which could otherwise occur in the presence of state measurement errors.

### 5.2 $S$-restricted dynamics

Suppose that the function $f$ in the dynamics (1) is only defined for state values $x \in S$, where $S$ is the state constraint set in the problem we have studied. In many problems arising in economics and engineering, for example, such a restricted definition is quite reasonable, since the dynamics might not make sense or break down when $x \notin S$. So suppose that $f(x, u)$ is only defined on $S \times U$, while corresponding versions of (F1)-(F3) hold. In this situation, it is possible to now extend $f$ from $S \times U$ to $\mathbb{R}^n \times U$ in a suitable way.

Let $f_i$ denote the $i^{\text{th}}$ component function of $f$, $i = 1, 2, \ldots, n$. For each fixed $u \in U$, define a function $x \to \hat{f}_i(x, u)$ on $\mathbb{R}^n$ as follows.

$$\hat{f}_i(x, u) = \min_{y \in S}\{f_i(y, u) + K\|y - x\|\}.$$

It is not difficult to show that $x \to \hat{f}_i(x, u)$ agrees with $f_i(x, u)$ on $S$, and is globally Lipschitz of rank $K$. We extend $f$ by componentwise by setting $f_i(x, u) = \hat{f}_i(x, u)$ for every $(x, u) \in \mathbb{R}^n \times U$. The resulting function $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ satisfies (F1)-(F3), as required. Note that the velocity sets $f(x, U)$ need not be convex for $x \notin S$, but this poses no difficulty; in particular the tracking results in Lemma 3.2 still hold, as was pointed out in [12].

### 5.3 The case of unbounded $S$

The main results in this article (as well as [12]) have been stated for the case of compact $S$. It is worth noting that if compactness of $S$ is relaxed to mere closedness, corresponding versions can be framed. In particular, in the corresponding versions of Definitions 1.2 and 1.3, the open and closed loop controllability properties to target $\Sigma$ are provided not for any $\alpha \in S$, but for any $\alpha$ in a specified bounded subset of $S$. In order to show that this is valid, the essential task (and a somewhat routine one) is to obtain appropriately localized versions of the tracking properties in Lemma 3.2 as well as Lemma 3.3 on modified dynamics; we omit these details. Note, however, that in carrying this out, (F2) is required, unlike the weakened version of this condition mentioned in Remark 1.1.

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