Differential inequalities which imply starlikeness

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CRM-2835

March 2002

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[†]R. Fournier acknowledges support of FCAR (Quebec) and NSERC (Canada).

[‡]The research of the authors was done while visiting Universität Wuerzburg (Germany). They wish to thank St. Ruscheweyh for his hospitality and insight.

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Abstract

We study some differential inequalities in the unit disc which imply starlikeness: for example if $f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n$ is analytic in $\mathbb{D} = \{z \mid |z| < 1\}$ and

$$\big|zf''(z) - \alpha\big(f'(z) - 1\big)\big| \leq 1 - \alpha, \quad z \in \mathbb{D}$$

for some $\alpha \in [0, 1)$, then f is one-to-one on $\mathbb{D}$ and $f(\mathbb{D})$ is a starlike domain with respect to the origin.

Keywords and Phrases. differential inequalities, starlike univalent functions

2000 Mathematics Subject Classification. 30C45

Introduction and statement of the results

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions in the unit disc $\mathbb{D} = \{z \mid |z| < 1\}$ of the complex plane $\mathbb{C}$; here we think of $\mathcal{H}(\mathbb{D})$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. We denote by $\mathcal{B}$ the convex and compact subset of $\mathcal{H}(\mathbb{D})$ containing the functions w with $|w(z)| \le 1$ in $\mathbb{D}$ and

$$\mathcal{B}_0 = \{w \in \mathcal{B} \mid w(0) = 0\}.$$

Further, let

$$\mathcal{A} = \{f \in \mathcal{H}(\mathbb{D}) \mid f(0) = f'(0) - 1 = 0\}$$

and St the set of functions $f \in \mathcal{A}$ which are starlike univalent in $\mathbb{D}$, i.e., f is one-to-one in $\mathbb{D}$ and $f(\mathbb{D})$ is a starlike domain with respect to the origin of the complex plane.

Our goal in this paper is to study certain differential operators which map $\mathcal{B}_0$ into St. The study of such operators has been a constant theme of geometric function theory; many sources and references are given in [3]. Our main results are

Theorem 1. *Let $\alpha \in \mathbb{C}$ and $\mathrm{Re}(\alpha) < 2$. Let also*

$$\rho_1(\alpha) := \sup\big\{\rho > 0 | \{f \in \mathcal{A} | |f'(z) - \alpha f(z)/z + \alpha - 1| \le \rho, z \in \mathbb{D}\} \subset \mathrm{St}\big\}.$$

Then

$$\rho_1(\alpha) = \inf_{T \in \mathbb{R}} \left\{ \frac{|2-\alpha||1+iT|}{|2-\alpha| + |\alpha + iT|} \right\}.$$

A simple optimization over T yields the following

Corollary 1. *Let $\alpha \in \mathbb{R}$ and $\alpha < 2$. Then*

$$\rho_1(\alpha) = \begin{cases} \frac{2-\alpha}{2(1-\alpha)} & \textit{if} \quad \alpha \le \frac{1-\sqrt{3}}{2}, \\ \frac{\sqrt{1-\alpha^2}(2-\alpha)}{\sqrt{5-4\alpha}} & \textit{if} \quad \frac{1-\sqrt{3}}{2} \le \alpha \le \frac{1}{2} \\ 1 - \alpha/2 & \textit{if} \quad \frac{1}{2} \le \alpha \le 2. \end{cases}$$

Theorem 1 greatly improves results which appeared in [2] [4], and [5].

We shall also study some second order differential operators and obtain, amongst other results,

Theorem 2. *Let $\alpha \in \mathbb{R}$ with $0 \le \alpha < 1$. Let also*

$$\rho_2(\alpha) = \sup\Big\{\rho > 0 \Big| \big\{f \in \mathcal{A} \big| \big|zf''(z) - \alpha\big(f'(z) - 1\big)\big| \le \rho, \quad z \in \mathbb{D}\big\} \subset \mathrm{St}\Big\}.$$

Then $\rho_2(\alpha) = 1 - \alpha$.

This theorem improves some inequalities which appeared in [6].

Most of the material needed for our proofs is well-known and can be stated in terms of the Hadamard product

$$f * g(z) := \sum_{n=0}^{\infty} a_n(f) a_n(g) z^n, \quad z \in \mathbb{D}$$

of functions $f(z) := \sum_{n=0}^{\infty} a_n(f) z^n$ and $g(z) := \sum_{n=0}^{\infty} a_n(g) z^n$ in $\mathcal{H}(\mathbb{D})$. We shall use over and over the following facts which are stated in [8]:

A function $f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n$ in $\mathcal{A}$ belongs to St if and only if

$$f(z)/z * \sum_{n=1}^{\infty} (n + iT)/(1 + iT) z^{n-1} \ne 0, z \in \mathbb{D}, T \in \mathbb{R}. \tag{1}$$

Let $\mathcal{P} := \{F \in \mathcal{H}(\mathbb{D}) \mid F(0) = 1 \text{ and } \mathrm{Re}\, F(z) > 1/2, \, z \in \mathbb{D}\}$. Then

$$w * F \in \mathcal{B} \text{ if } w \in \mathcal{B} \text{ and } F \in \mathcal{P}. \tag{2}$$

We shall also use the following result [2, 7] in order to establish the sharpness of some of our statements:

Lemma 1. *There exists a sequence $\{W_k\} \subset \mathcal{B}_0$ of functions analytic in the closed unit disc $\overline{\mathbb{D}}$ such that, given $\theta \in \mathbb{R}$, $W_k(1) = 1$ and $\lim_{k \to \infty} W_k(z) = e^{i\theta} z$, uniformly on compact subsets of $\overline{\mathbb{D}} \setminus \{1\}$.*

1 Proof of Theorem 1

Let $\alpha \in \mathbb{C}$. We wish to determine positive numbers $\rho = \rho(\alpha)$ such that any differential equation

$$f'(z) - \alpha f(z)/z + \alpha - 1 = \rho w(z), \quad z \in \mathbb{D} \tag{3}$$

with driving term $w \in \mathcal{B}_0$ admits a solution $f \in \mathrm{St}$. The example $f_N(z) := z + dz^{N+1}$, $N \geq 1$, $|d| > 1/(N+1)$ shows already that some restriction on α shall be necessary since

$$f_N'(z) - (N+1)f_N(z)/z + N = 0, \quad z \in \mathbb{D}$$

and f_N is not even locally univalent in $\mathbb{D}$. We shall therefore assume that α is not an integer ≥ 2.

Let us write $w(z) = \sum_{n=1}^{\infty} a_n(w) z^n$. By solving (3) we obtain $f(z) = z + \rho \sum_{n=1}^{\infty} a_n(w)/(n+1-\alpha) z^n$ and finally any such f belongs to St if and only if $\rho \leq \rho_1(\alpha)$ where

$$\rho_1(\alpha)^{-1} := \sup \left| \sum_{n=1}^{\infty} \frac{a_n(w)}{n+1-\alpha} \frac{n+1+iT}{1+iT} z^n \right|. \tag{4}$$

Here we have used (1) and the sup in (4) is taken over all $T \in \mathbb{R}$, $z \in \mathbb{D}$ and $w \in \mathcal{B}_0$.

We now consider a functional I over $\mathcal{B}_0$ defined as

$$I(w) = \sum_{n=1}^{\infty} \frac{a_n(w)}{n+1-\alpha}.$$

This functional is well-defined because for any $w \in \mathcal{B}_0$,

$$\left| \sum_{n=1}^{\infty} \frac{a_n(w)}{n+1-\alpha} \right|^2 \leq \left(\sum_{n=1}^{\infty} |a_n(w)|^2 \right) \left(\sum_{n=1}^{\infty} \frac{1}{|n+1-\alpha|^2} \right) \leq \sum_{n=1}^{\infty} \frac{1}{|n+1-\alpha|^2} < \infty.$$

The functional I is also continuous over $\mathcal{B}_0$: we choose an integer k greater than the integer part of $\mathrm{Re}(\alpha) + 1$ and represent I as

$$I(w) = \sum_{j=1}^{k-1} \frac{a_j(w)}{j+1-\alpha} + \int_0^1 \frac{w(t) - \sum_{j=1}^{k-1} a_j(w) t^j}{t^\alpha} \, dt.$$

The continuity shall follow as an application of the dominated convergence theorem.

By compacity of $\mathcal{B}_0$, there exists a function B_α in $\mathcal{B}_0$ such that $|I(B_\alpha)| = \sup_{w \in \mathcal{B}_0} |I(w)|$ and according to a result of Cochrane and MacGregor [1] we may assume that B_α is a finite Blaschke product and in particular $|B_\alpha(1)| = 1$. For any fixed $z \in \mathbb{D}$ and $w \in \mathcal{B}_0$,

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \frac{a_n(w)}{n+1-\alpha} \frac{n+1+iT}{1+iT} z^n \right| &\leq \frac{1}{|1+iT|} \left| \sum_{n=1}^{\infty} a_n(w) z^n \right| + \left| \frac{\alpha+iT}{1+iT} \right| \left| \sum_{n=1}^{\infty} \frac{a_n(w) z^n}{n+1-\alpha} \right| \\ &\leq \frac{1}{|1+iT|} + \left| \frac{\alpha+iT}{1+iT} \right| |I(B_\alpha)|. \end{aligned}$$

It then follows from (4) that

$$\rho_1(\alpha)^{-1} \leq \sup_{T \in \mathbb{R}} \frac{1}{|1+iT|} + \left| \frac{\alpha+iT}{1+iT} \right| |I(B_\alpha)| := M(\alpha). \tag{5}$$

We shall now prove that equality indeed holds in (5). Let $M(\alpha) = 1/|1+iT_\alpha| + |\alpha+iT_\alpha|/|1+iT_\alpha||I(B_\alpha)|$ for some $T_\alpha \in \mathbb{R} \cup \{\infty\}$ and $\rho \in \mathbb{R}$ with

$$\left| \frac{1}{1+iT_\alpha} B_\alpha(1) + \frac{\alpha+iT_\alpha}{1+iT_\alpha} \sum_{n=1}^{\infty} \frac{a_n(W_\alpha)}{n+1-\alpha} e^{i\rho} \right| = \frac{1}{|1+iT_\alpha|} + \left| \frac{\alpha+iT_\alpha}{1+iT_\alpha} \right| |I(B_\alpha)|.$$

There exists, by Lemma 1, a sequence $\{W_k\} \subset \mathcal{B}_0$ of functions such that $W_k(1) = 1$ and $\lim_{k\to\infty} W_k(z) = e^{i\rho}z$ in $\mathcal{B}_0$. Then $\{w_k\} \subset \mathcal{B}_0$ if $w_k(z) := B_\alpha(z)W_k(z)/z$ and

$$\begin{aligned}\lim_{k\to\infty}\left|\sum_{n=1}^{\infty}\frac{a_n(w_k)}{n+1-\alpha}\,\frac{n+1+iT_\alpha}{1+iT_\alpha}\right| &= \lim_{k\to\infty}\left|\frac{B_\alpha(1)W_k(1)}{1+iT_\alpha}+\frac{1}{1+iT_\alpha}I(w_k)\right| \\ &= \left|\frac{B_\alpha(1)}{1+iT_\alpha}+\frac{1}{1+iT_\alpha}I(B_\alpha)e^{i\rho}\right| \\ &= \frac{1}{|1+iT_\alpha|}+\left|\frac{\alpha+iT_\alpha}{1+iT_\alpha}\right||I(B_\alpha)|.\end{aligned}$$

This now proves that $\rho_1(\alpha) = M(\alpha)^{-1}$ if $\alpha \notin \{2, 3, \ldots\}$.

In the case where $\mathrm{Re}(\alpha) < 2$, it is possible to identify explicitly an extremal Blaschke product B_α. Given $w \in \mathcal{B}_0$ we consider a function

$$J_w(z) := \sum_{n=1}^{\infty}\frac{a_n(w)}{n+(1-\alpha)}z^n = \frac{1}{2-\alpha}\left(\frac{w(z)}{z} * \sum_{n=1}^{\infty}\frac{1+(1-\alpha)}{n+(1-\alpha)}z^{n-1}\right).$$

It is known [9] that $p(z) := \sum_{n=1}^{\infty}\frac{1+(1-\alpha)}{n+(1-\alpha)}z^{n-1} \in \mathcal{P}$ if and only if $\mathrm{Re}(\alpha) < 2$. According to (2), if this last condition holds we get

$$|J_w(z)| \le \frac{1}{|2-\alpha|} = |J_{w_0}(1)|, \quad w \in \mathcal{B}_0, z \in \mathbb{D}$$

where w_0 denotes the identity function. This clearly means that in that case we may assume that B_α is the identity function and the conclusion of Theorem 1 follows.

We wish to make a last statement about the sharpness of our Theorem. It is also well-known [9] that if $\mathrm{Re}(\alpha) > 2$, the function p defined above does not belong to $\mathcal{P}$ and there exists $\widetilde{w}_0 \in \mathcal{B}_0$ with

$$\sup_{z\in\mathbb{D}}\left|\frac{\widetilde{w}_0(z)}{z} * p(z)\right| > 1,$$

i.e.,

$$\left|\sum_{n=1}^{\infty}\frac{a_n(\widetilde{w}_0)}{n+(1-\alpha)}Z^n\right| > \frac{1}{|2-\alpha|}, \qquad \text{for some } Z \in \mathbb{D}.$$

In other words, the function $W(z) := \widetilde{w}_0(Zz)$ shall satisfy $|I(W)| > 1/|2-\alpha|$ and our Theorem 1 cannot hold if $\mathrm{Re}(\alpha) > 2$.

2 Proof of Theorem 2

We shall split our proof into a sequence of lemmas.

Lemma 2.

$$\sup\left\{d|\{f \in \mathcal{A}||zf''(z)| \le d, z \in \mathbb{D}\} \subset \mathrm{St}\right\} = 1.$$

Proof. We write for $f \in \mathcal{A}$, $f''(z) = dw(z)$ where $w(z) := \sum_{n=0}^{\infty} a_n(w)z^n \in \mathcal{B}$. Then $f(z) = z + d\sum_{n=1}^{\infty}\frac{a_{n-2}(w)}{n(n-1)}z^n$ and by (2) the best d available with the desired properties satisfies

$$d_0^{-1} = \sup\left|\sum_{n=2}^{\infty}\frac{a_{n-2}(w)}{n(n-1)}\,\frac{n+iT}{1+iT}z^n\right| \tag{6}$$

where the sup is taken over all $T \in \mathbb{R}$, $z \in \mathbb{D}$ and $w \in \mathcal{B}$. For $F(z) := \sum_{n=2}^{\infty}\frac{a_{n-2}(w)}{n(n-1)}\frac{n+iT}{1+iT}z^n$, we have

$$F'(z) = \frac{zw(z)}{1+iT} + \int_0^z w(\zeta)\,d\zeta$$

and

$$|F(z)| \le |z|\int_0^1 |F'(tz)|\,dt \le 2|z|^2\int_0^1 t\,dt = |z|^2, \quad z \in \mathbb{D}$$

and by (6), $d_0^{-1} \le 1$. For the constant function $w(z) \equiv e^{i\theta}$ and $T = 0$, we obtain $f(z) = z + (d/2)z^2$, which means that $d_0 = 1$. □

Our next lemma follows from an idea due to St. Ruscheweyh.

Lemma 3. *Let* $G(z) := \sum_{n=1}^{\infty} a_n(G)z^n \in H(\mathbb{D})$ *with* $\sum_{n=1}^{\infty} a_1(G)/a_n(G)z^{n-1} \in \mathcal{P}$*. Then,*

$$\sup\left\{d|\{f \in \mathcal{A}||zf''(z) * G(z)| \leq d, z \in \mathbb{D}\} \subset \mathrm{St}\right\} = |a_1(G)|.$$

Proof. By the hypothesis and (2), for $f \in \mathcal{A}$,

$$\begin{aligned}|a_1(G)|\sup_{z\in\mathbb{D}}|zf''(z)| &= \sup_{z\in\mathbb{D}}\left|\frac{\left(zf''(z) * G(z)\right)}{z} * \sum_{n=1}^{\infty}\frac{a_1(G)}{a_n(G)}z^{n-1}\right| \\ &\leq \sup_{z\in\mathbb{D}}|zf''(z) * G(z)|.\end{aligned}$$

It then follows from Lemma 2 that the condition

$$|zf''(z) * G(z)| \leq |a_1(G)|, \quad z \in \mathbb{D}$$

is sufficient for a function f in $\mathcal{A}$ to be starlike univalent. Let $\varepsilon > 0$: the function $f_\epsilon(z) := z + \left(|a_1(G)\mid +\epsilon\right)/2a_1(G)z^2$ is not a member of St (it is not even locally univalent in $\mathbb{D}$!) and satisfies

$$|zf_\epsilon''(z) * G(z)| \leq |a_1(G)| + \epsilon, \quad z \in \mathbb{D}.$$

This completes the proof of Lemma 3. ∎

Lemma 4. *Let* $\alpha \in \mathbb{C}$*, not a positive integer. Then*

$$F_\alpha(z) := \sum_{n=1}^{\infty}\frac{n(1-\alpha)}{n-\alpha}z^{n-1} \in \mathcal{P} \text{ iff} . \quad 0 \leq \alpha < 1.$$

Proof. The sufficiency of the condition is obvious since

$$F_\alpha(z) = (1-\alpha)\frac{1}{1-z} + \alpha\sum_{n=1}^{\infty}\frac{1-\alpha}{n-\alpha}z^{n-1}, \quad z \in \mathbb{D},$$

the class $\mathcal{P}$ is convex and contains each member of the convex combination above. For the necessity, we may assume that $\mathrm{Re}(\alpha) < 1$ because if $F_\alpha \in \mathcal{P}$ then it follows that [8]

$$F_\alpha(z) * \frac{-\ln(1-z)}{z} = \sum_{n=1}^{\infty}\frac{1-\alpha}{n-\alpha}z^{n-1} \in \mathcal{P}$$

and the last statement is known to be valid in and only if $\mathrm{Re}(\alpha) < 1$. We may then write, for any $w \in \mathcal{B}$,

$$w * F_\alpha(z) = (1-\alpha)\left(w(z) + \frac{\alpha}{1-\alpha}\sum_{n=1}^{\infty}\frac{1-\alpha}{n-\alpha}a_{n-1}(w)z^{n-1}\right). \tag{7}$$

It follows that

$$\sup_{z\in\mathbb{D}, w\in\mathcal{B}}|w * F_\alpha(z)| \leq |1-\alpha| + |\alpha|. \tag{8}$$

We shall now prove that equality holds in (8): this will mean that

$$F_\alpha \in \mathcal{P} \iff |1-\alpha| + |\alpha| \leq 1 \iff \alpha \in [0,1).$$

We consider the functions $\{W_k\}$ given by Lemma 1. According to (7), if $z = 1$,

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{W_k(z)}{z} * F_\alpha(z)\right| &= \lim_{k\to\infty}|1-\alpha|\left|W_k(1) + \alpha\int_0^1 t^{-1-\alpha}W_k(t)\,dt\right| \\ &= |1-\alpha|\left|1 + \frac{\alpha}{1-\alpha}e^{i\theta}\right| \\ &= |1-\alpha| + |\alpha|\end{aligned}$$

if θ is properly chosen. ∎

We may now give a short proof of Theorem 2: assume that, for some $f \in \mathcal{A}$ and $\alpha \in [0,1)$,

$$\big|zf''(z) - \alpha\big(f'(z) - 1\big)\big| \le 1 - \alpha, \quad z \in \mathbb{D}. \tag{9}$$

This means that

$$\left| zf''(z) * \sum_{n=1}^{\infty} \frac{n-\alpha}{n(1-\alpha)} z^n \right| \le 1, \quad z \in \mathbb{D}$$

and according to Lemma 4

$$|zf''(z)| = \left| \left(zf''(z) * \sum_{n=1}^{\infty} \frac{n-\alpha}{n(1-\alpha)} z^n \right) * \sum_{n=1}^{\infty} \frac{n(1-\alpha)}{n-\alpha} z^n \right| \le 1, \quad z \in \mathbb{D}.$$

By Lemma 2 we have $f \in \mathrm{St}$. The fact that the constant $1-\alpha$ to the right of (9) is the best possible in order to insure membership of f in St follows of course from Lemma 3.

We wish to comment on another aspect of the sharpness of this Theorem: an essential element of our proof is the fact that under the condition (9) we have

$$|zf''(z)| \le 1, \quad z \in \mathbb{D}. \tag{10}$$

We point out that the conclusion (10) shall not folllow from the modified hypothesis

$$\big|zf''(z) - \alpha\big(f'(z) - 1\big)\big| \le |1 - \alpha|, \quad z \in \mathbb{D}, \tag{11}$$

for any $\alpha \in \mathbb{C} \setminus [0,1]$: in the latter case, the function F_α does not belong to $\mathcal{P}$ and there exists $w_\alpha \in \mathcal{B}$ such that

$$\sup_{z \in \mathbb{D}} \left| w_\alpha(z) * \sum_{n=1}^{\infty} \frac{n(1-\alpha)}{n-\alpha} z^{n-1} \right| > 1.$$

We now define $f_\alpha \in \mathcal{A}$ by $zf''_\alpha(z) := zw_\alpha(z) * \sum_{n=1}^{\infty} n(1-\alpha)/(n-\alpha) z^n$. Clearly f_α satisfies the hypothesis (11) but does not satisfy (10). There exists however an insidious possibility that $f_\alpha \in \mathrm{St}$ but to decide this seems beyond the scope of this paper.

Many variations on the theme of Theorem 2 can be established. We mention (without proof)

Theorem 3. *: Let $\alpha \in [0,2)$. Then*

$$\left\{ f \in \mathcal{A} \mid \big|zf''(z) - \alpha\big(f'(z) - f(z)/z\big)\big| \le 1 - \frac{\alpha}{2}, \quad z \in \mathbb{D} \right\} \subset \mathrm{St}\,.$$

Theorem 4. *Let $\alpha \in [0,2)$. Then*

$$\left\{ f \in \mathcal{A} \mid \big|zf''(z) - \alpha\big(f(z)/z - 1\big)\big| \le 1 - \frac{\alpha}{2}, z \in \mathbb{D} \right\} \subset \mathrm{St}\,.$$

All the constants involved in the formulation of the above theorems are best possible.

3 Conclusion

We would like to mention a closely related problem whose solution seems unattainable with our method: what are the complex numbers β such that

$$S(\beta) := \{ f \in \mathcal{A} \mid |\beta z f''(z) + f'(z) - 1| < 1, z \in \mathbb{D} \} \subset \mathrm{St}? \tag{12}$$

It is known [2], [4] that $S(\sqrt{5}/2 - 1) \subset \mathrm{St}$ and $S(0) \not\subset \mathrm{St}$.

We first remark that the functions $f \in S(\beta)$ are univalent if $|\beta + 1| \ge 1$. Any such f admits a representation

$$f(z) = z + \sum_{n=1}^{\infty} \frac{a_n(w)}{(n+1)(\beta n + 1)} z^{n+1}, \quad \text{where } w(z) := \sum_{n=1}^{\infty} a_n(w) z^n \in \mathcal{B}_0$$

and because

$$|f'(z) - 1| = \frac{1}{|1+\beta|} \left| \sum_{n=1}^{\infty} \frac{1 + 1/\beta}{n + 1/\beta} z^n * w(z) \right|$$

we see by an already quoted result in [7] that

$$|f'(z)-1| \le 1, \quad z \in \mathbb{D}, \quad f \in S(\beta) \tag{13}$$

if $|1+\beta| \ge 1$ (i.e., $\mathrm{Re}(1/\beta) > -1/2$). The propertie (13) is well-known to imply univalency. If, on the other hand $|\beta+1| < 1$, then $f_\beta(z) := z + 1/2(1+\beta)\, z^2 \in S(\beta)$ and f_β is not locally univalent in $\mathbb{D}$.

We also remark that $S(\beta) \subset \mathrm{St}$ if $|\beta+1| \ge \sqrt{5}/2$. This can be seen in a number of ways; we shall use a variant of Lemma 3. For $f \in S(\beta)$ we have

$$|\beta z f''(z) + f'(z) - 1| = \left| \big(f'(z)-1\big) * (1+\beta) \sum_{n=1}^{\infty} \frac{n+1/\beta}{1+1/\beta} z^n \right| \le 1, \quad z \in \mathbb{D}$$

and if $|\beta+1| \ge \sqrt{5}/2$ we get,

$$|1+\beta||f'(z)-1| = \left| \big(\beta z f''(z) + f'(z) - 1\big) * \sum_{n=1}^{\infty} \frac{1+1/\beta}{n+1/\beta} z^n \right| \le 1, \quad z \in \mathbb{D}$$

i.e.,

$$|f'(z)-1| \le \frac{1}{|1+\beta|} \le \frac{2}{\sqrt{5}}, \quad z \in \mathbb{D},$$

and this last condition is known to implie $f \in St$ [2, 4].

We are unable to use Lemma 1 to prove a sharpness argument concerning (12). Let us remark however that the exterior of the disc $\{\beta \mid |\beta+1| < \sqrt{5}/2\}$ is not the sharp region for (12) to hold: it is a simple consequence of Theorem 2 that $S(\beta) \subset \mathrm{St}$ if $\beta \le -2$.

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