Geometry of quadratic differential systems in the neighbourhood of the line at infinity

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Abstract

In this article we consider the behavior in the vicinity of infinity of the class Σ of all planar quadratic differential systems. This family depends on twelve parameters but due to the affine group action, the family actually depends on five parameters. We give simple, integer-valued geometric invariants for this group action which classify this family according to the topology of their phase portraits in the vicinity of infinity. For each one of the classes obtained we give necessary and sufficient conditions in terms of algebraic invariants and comitants so as to be able to easily retrieve for any system, in any chart, the geometric as well as the dynamic characteristics of the systems in the neighborhood of infinity. The program was implemented for computer calculations.

Résumé

Dans cet article nous considérons le comportement au voisinage de l'infini de la classe Σ de tous les systèmes différentiels quadratiques dans le plan. Cette famille depend de douze paramètres mais à cause de l'action du groupe affine la famille ne dépend que de cinq paramètres. Nous donnons des invariants simples, à valeurs dans les entiers, pour cette actions de groupe, classifiant cette famille d'après la topology des portraits de phases au voisinage de l'infini des systèmes. Pour chaque classe ainsi obtenue, nous donnons des conditions necessaires et suffisantes en termes d'invariants algébriques et de comitants afin de pouvoir determiner pour tout système, par rapport à toute carte, les caractéristiques au voisinage de l'infini, géométriques et dynamiques de ces systèmes. Ces calculs ont été programmés pour être effectués à l'ordinateur.

1 Introduction

We consider real planar polynomial differential system, i.e. systems of the form

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{S}$$

where P and Q are polynomials in x and y with real coefficients ($P, Q \in \mathbb{R}[x, y]$). In this article, a system of the above form with $\max(\deg(P), \deg(Q)) = 2$ will be called quadratic.

These are the simplest nonlinear differential systems. However, global problems regarding this class are difficult to solve. More than a century has passed since Hilbert gave his list of 23 problems in 1900 and one of the few problems still unsolved, Hilbert 16th problem (its second part), is on planar polynomial systems. This problem asks for the maximum H(n) of the numbers of limit cycles occurring in differential systems with $\max(\deg(P), \deg(Q)) \leq n$. This problem is still unsolved even for quadratic differential systems. These problems are hard because we are interested here in the global behavior of all solutions in the whole plane and even at infinity (for behavior at infinity see [9]) and this for a whole family of systems. The set Σ of quadratic systems depends on 12 parameters, the coefficients of the two polynomials P and Q. But on Σ acts the group $\mathrm{Aff}[2, \mathbb{R}]$ of affine transformations as well as the group of positive time homotheties. So the space actually depends on five parameters. But even five is a large number considering that we expect this class to yield thousands of distinct phase portraits. For this reason people have attempted to study particular classes of quadratic systems and for some classes a complete classification of phase portraits with respect to topological equivalence was obtained (quadratic systems with a center (cf.[23], [19]), quadratic Hamiltonian systems (cf. [1]), quadratic chordal systems ([8]), quadratic systems with a weak focus of third order ([2]), etc.).

The goal in most of these articles was to obtain all topologically distinct phase portraits for the particular class considered. These were obtained by using specific charts and normal forms for the systems in these specific charts and the classifications were obtained with respect to parameters satisfying certain inequalities or equalities. If we need to apply the results for systems given with respect to other charts, we would need to perform a change of coordinates, a noncontsructive procedure. Thus the results are not easily applicable to other specific situations.

Ever since Klein gave his famous Erlangen program, we are used to calling a property geometric, if it is invariant under the action of some group. In this sense, most of the results obtained are not geometrical since they are not independent of the charts considered.

Chart independent classification results were obtained by K.S. Sibirskii and his school (cf. [22], [18], [6]) using the algebraic invariant theory of differential equations developed by Sibirskii (cf. [21]). However most of the articles of the school of Sibirskii were published in Russian, only some appeared in translations which partly explains why this theory is rather unknown in the west. In these articles invariants and comitants are introduced in their multi-index tensorial form, certain rather artificial polynomial combinations of these are chosen and classifications are given in terms of these combinations. In the end these classifications remain insufficiently related to the geometry of the systems.

We need here a number of much simpler invariants, simpler than the configuration space of Markus (cf.[13]), possibly even integer valued invariants which could convey to us in simple terms properties of the global geometry of the systems. We would also need a way of computing at least some of these simple invariants for any system in whatever chart it may be given to us.

In [15], [12] the authors gave topological classifications in terms of the global geometry of the classes of systems considered. These classifications are affine invariant and they are expressed in terms of the geometry of algebraic invariant curves of the systems considered in [15] or in terms of very simple integer-valued invariants in [12] reflecting the properties of the systems.

In spite of their awesome character, polynomial invariants and comitants are a very powerful computational tool applicable to any canonical form and they can be programmed on a computer. There is thus a need to merge the geometric methods above mentioned with the algebraic invariant approach. In this work we propose to do just that for the specific problem of topologically classifying quadratic systems in the neighbourhood of the line at infinity.

In [11] Kooij and Reyn obtained all possible local phase portraits around a single singular point at infinity of an arbitrary quadratic vector field. In [11] they did not consider in this work the possible ways of combining such singularities so as to obtain a topological classification of quadratic systems in a neighborhood of the line at infinity. In [14] I. Nicolaev and N. Vulpe obtained such a classification in terms of algebraic invariants and comitants and in [3] affine invariant classification of quadratic system with respect to the possible distributions of the multiplicities of singularities at infinity is obtained by V.Baltag and N.Vulpe. These classifications use the technical language of algebraic invariant theory developed by the school of Sibirskii ([21],[24],[5], etc). Since the geometric meaning of these invariants or comitants is in most cases not clear, a topological classification in much simpler terms with a clear geometric and dynamic meaning is needed. Examples where classifications are done in terms of the global geometry of the systems are [15] (where the classification is done in terms of the singular algebraic invariant curves of the systems), [20] (where the quadratic systems with a weak focus are topologically classified in the neighborhood of the line at infinity) and [12] (where the topological classification of quadratic systems with a weak focus of third order is done in terms of simple, integer valued affine invariants).

The goal of this work is to combine the geometric approach in [15], [12] and [20] with the algebraic invariant approach in [14] and [3] so as to obtain simplicity and clarity in the resulting geometric classifications as well as applicability to any particularly chosen chart. In this article we introduce the notions and prove the necessary results which permit this. We do this in as self-contained a way as possible.

The article is organized as follows: In §2 we consider the two compactifications of real planar polynomial systems and the foliations with singularities, real and complex, on the real and complex projective planes, associated to these systems.

In §3 we describe the purely geometric objects, i.e. the divisorsattached to the line at infinity which encode the multiplicities at infinity of the systems, and attach to these some integer-valued global affine invariants.

In §4 we consider group actions on quadratic differential systems and define algebraic invariants and comitants with respect to these group actions. We also give, using a linear comitant, canonical forms for these differential systems according to their behavior at infinity.

In §5 we state and prove the classification theorem (Theorem 5.1) of the quadratic differential systems according to their divisors at infinity and for each class we give the necessary and sufficient conditions in terms of algebraic invariants and comitants with respect to the group actions. These conditions allow us to compute for any system and in any chart the types of the multiplicity divisors associated to the system.

In §6 we introduce new classifying tools, among them the index divisor encoding globally the topological indices of the singularities at infinity of any polynomial differential system without a line of singularities at infinity. We also introduce a divisor encoding globally the number of local separatrices bounding a hyperbolic sector of a singular point at infinity.

In §7 we state and prove the topological classification theorem (Theorem 7.1). This classification is expressed on one side in terms of geometrical, affine integer-valued invariants, which convey in simple terms the geometric and dynamic properties of the systems according to their behavior in the vicinity of infinity; on the other hand in terms of algebraic invariants and comitants so as to be able to read for any system and in any chart, its geometric and dynamical properties at infinity once these algebraic invariants and comitants are calculated. These calculations could be done on a computer.

In the Appendix we list the invariants and comitants used in [14] and which are needed for the proofs of the main results. These are also listed for the purpose of comparison with the simpler algebraic invariants and comitants used in this article. Highligthing the geometry of the systems via the integer-valued invariants introduced, helped us to choose better algebraic invariants and comitants than those in [14], closer to the geometry of the systems.

2 The two compactifications of real planar polynomial vector fields

A real planar polynomial system (S) can be compactified on the sphere as follows: Consider the x, y plane as being the plane $Z = 1$ in the space $\mathbb{R}^3$ with coordinates X, Y, Z. The central projection of the vector field $P\partial/\partial x + Q\partial/\partial y$ on the sphere of radius one yields a diffeomorphic vector field on the upper hemisphere and also another vector field on the lower hemisphere. There exists (for a proof cf. [9]) an analytic vector field $\mathcal{V}$ on the whole sphere such that its restriction on the upper hemisphere has the same phase curves as the one induced by the phase curves of (S) via the central projection on the upper hemisphere. In [9] it is shown that the phase curves of $\mathcal{V}$ coincide in each chart with phase curves of planar polynomial vector fields, in particular in the chart corresponding to $Z = 1$, denoting the two coordinate axis x, y corresponding to the OX and OY directions, they coincide with the phase curves of (S). The two planar polynomial vector fields U, V associated to analogue charts for $X = 1$ (with local coordinates (u, z)) and for $Y = 1$ (with local coordinates (v, w)) and changes of coordinates $u = y/x$, $z = 1/x$, or $v = x/y$, $w = 1/y$ are as follows:

$$U \left\{ \begin{aligned} \frac{du}{dt} &= C^*(1, u, z), \\ \frac{dz}{dt} &= zP^*(1, u, z), \end{aligned} \right. \qquad \text{and} \qquad V \left\{ \begin{aligned} \frac{dv}{dt} &= C^*(v, 1, w), \\ \frac{dw}{dt} &= -wQ^*(v, 1, w), \end{aligned} \right.$$

where P^*, Q^* and C^* are defined below.

By the compactification of the planar polynomial vector field associated to (S) we understand the restriction $\mathcal{V}|_{\mathcal{H}'}$ (where by H' we understand the upper hemisphere H completed with the equator) of the analytic vector field $\mathcal{V}$ on the sphere. In this work we are interested in the topological classification of (S) on $\mathbb{R}^2$ completed with its points at infinity i.e. such vector fields $\mathcal{V}|_{\mathcal{H}'}$. Since the vertical projection is a diffeomorphism of H' on the disk $\{(x, y)|x^2 + y^2 \leq 1\}$ we can view the phase portraits of our systems (S) on this disk, called the Poincaré disk.

We shall also use the compactifications (real or complex) associated to the foliations with singularities (real or complex) attached to a real polynomial system (S) (cf. [7] or [17]). These foliations can be described as follows: For a real polynomial system (S) with $n = \max(\deg(P), \deg(Q))$ we associate to the two polynomials $P, Q \in \mathbb{R}[x, y]$ defining (S) the homogeneous polynomials P^*, Q^* in X, Y, Z, of degree n with real coefficients, defined as follows:

$$P^*(X, Y, Z) = Z^n P(X/Z, Y/Z), \quad Q^*(X, Y, Z) = Z^n Q(X/Z, Y/Z).$$

The real (respectively complex) foliations with singularities associated to (S) on the real (respectively, complex) projective plane $\mathrm{P}^2(\mathbb{R})$ (respectively $\mathrm{P}^2(\mathbb{C})$) are then described in homogeneous coordinates by the equations:

$$A^*(X,Y,Z)dX + B^*(X,Y,Z)dY + C^*(X,Y,Z)dZ = 0, \tag{2.1}$$
$$A^*(X,Y,Z)X + B^*(X,Y,Z)Y + C^*(X,Y,Z)Z = 0, \tag{2.2}$$

where $A^* = ZQ^*$, $B^* = -ZP^*$, $C^*(X,Y,Z) = YP^*(X,Y,Z) - XQ^*(X,Y,Z)$. (For these foliations cf. [7]).

Our goal in this work is to give a topological classification, in terms of both geometric and algebraic invariants, of the quadratic systems (S) and their compactification on H' in the neighbourhood of the equator in the closed upper hemisphere H' of the Poincaré sphere. Correspondingly this yields a topological classification of the real foliations, in the neighbourhood of the line at infinity associated to the imbedding of the affine plane:

$$j: \ \mathrm{A}^2(\mathbb{R}) = \mathbb{R}^2 \longrightarrow \mathrm{P}^2(\mathbb{R})$$

where $j(x,y) = [x : y : 1]$. The line at infinity in this case is therefore $Z = 0$.

3 Divisors on the line at infinity encoding globally the multiplicities ofsingularities

In this section we consider real polynomial systems (S) with $n = \max(\deg(P), \deg(Q))$ and their associated foliations with singularities,real or complex, defined in the previous section by the equations (2.1) and (2.2).

Definition 3.1 *We call* ***divisor on the line at infinity*** *for a system (S) a formal expression of the form $D = \sum n(p)p$ where p belongs to the complex line Z=0 of the complex projective plane, $n(p)$ is an integer and only a finite number of the numbers $n(p)$ are not zero. We call* **degree** *of the divisor D the integer* $\deg(D) = \sum n(p)$. *We call* ***support*** *of the divisor D the set* $\mathrm{Supp}(D)$ *of points p such that $n(p) \neq 0$.*

In [20] two divisors on the line at infinity were introduced for systems (S) which were also used in [12].

Definition 3.2 *Assume that the system (S) is such that $P(x,y)$ and $Q(x,y)$ are relatively prime over $\mathbb{C}$. We also assume that the $yP_n - xQ_n$ is not identically zero (i.e. $Z \nmid C^*$), where P_n (respectively Q_n) is the sum of terms of degree n of P (respectively of Q) in case at list one of them is with a non-zero coefficient and zero otherwise.*

The following divisor on the line at infinity is introduced:

$$D_S(P^*, Q^*; Z) = \sum I_p(P^*, Q^*)p$$

where the sum is taken for all points $p = [X : Y : 0]$ on the line $Z = 0$ and $I_p(P^*, Q^*)$ is the intersection number (or multiplicity of intersection) at p of the complex projective curves

$$P^*(X,Y,Z) = 0 \qquad \textit{and} \qquad Q^*(X,Y,Z) = 0.$$

This is a well defined, purely geometric object which encodes the contribution to the multiplicities of the singularities at infinity of the system (S), arising from singularities in the finite plane, i.e.

how many singular points in the finite plane could appear from those singularities at infinity in quadratic perturbations of (S).

We thus have

$$\mathrm{Supp}(D_S(P^*, Q^*; Z)) = \{p \in \{Z = 0\} | P^*(p) = 0 = Q^*(p)\}.$$

Let us list a number of integer-valued invariants which are attached to this divisor.

Notation 3.1

$$N_{\infty,f}(S) = \#\ \mathrm{Supp}(D_S(P^*, Q^*; Z));$$
$$\nu_S = \max\{I_p(P^*, Q^*) \,|\, p \in \mathrm{Supp}(D_S(P^*, Q^*; Z))\};$$
$$\textit{for every } m \leq \nu_S \ \ s(m) = \#\ \{p \in \{Z = 0\} \,|\, I_p(P^*, Q^*) = m\}\ .$$

Note that $N_{\infty,f}$ is the number of distinct infinite singularities of (S) which could produce finite singular points in a quadratic perturbation of (S).

Definition 3.3 *We call type of the divisor $D_S(P^*, Q^*; Z)$ the set*

$$\{(s(m),\, m) \,|\, m \leq \nu_S\}.$$

We also need another divisor on the line at infinity which was used in [20] and [12] and which is defined as follows:

Definition 3.4

$$D_S(C^*, Z) = \sum I_p(C^*, Z)p$$

where the sum is taken for all points $p = [X : Y : 0]$ on the line $Z = 0$ of the complex projective plane.

Clearly for quadratic differential systems $\deg(D_S(C^*, Z)) = 3$.

Definition 3.5 *A point p of the projective plane $\mathrm{P}^2(\mathbb{C})$ is said to be of multiplicity (r, s) for a system (S) if*

$$(r, s) = (I_p(P^*, Q^*), I_p(C^*, Z)).$$

Following [20] we fuse the above two divisors on the line at infinity into just one but with values in the ring Z^2:

Definition 3.6

$$D_S = \sum \begin{pmatrix} I_p(P^*, Q^*) \\ I_p(C^*, Z) \end{pmatrix} p$$

where p belongs to the line $Z = 0$ of the complex projective plane.

The above defined divisor describes the number of singularities which could arise in a perturbation of S from singularities at infinity of (S) in both the finite plane and at infinity, in a smallperturbation of (S).

Remark 3.1 *We observe that the types of $D_S(P^*, Q^*; Z)$ and of $D_S(C^*, Z)$ are affine invariants since both $I_p(P^*, Q^*)$ and $I_p(C^*, Z)$ remain invariant under the action of the affine group on systems (S) ([16], [17]).*

Notation 3.2 *Let us introduce for quadratic systems (*3.1*) the following notations:*

$$\Delta_S = \deg D_S(P^*, Q^*; Z), \quad M_{C^*} = \max\{I_p(C^*, Z) \,|\, p \in \mathrm{Supp}(D_S(C^*, Z))\}.$$

Consider a real quadratic differential system (S). This is a system of the form:

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + \, p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + \, q_2(x,y) \equiv Q(x,y). \end{aligned} \tag{3.1}$$

with $\gcd(P,Q) =$ constant, where p_i (respectively q_i) is the sum of terms in x and y of degree i of P (respectively of Q) in case at least one such term has non-zero coefficient and zero otherwise. Recall that Σ denotes the class of all real quadratic systems.

We want to list all possible divisors D_S for quadratic systems (S) and characterize in terms of invariants and comitants the types of these divisors. This would make possible for any given system and in any chart the computation of the type of its divisor . To do this we need to construct invariants and comitants with respect to group actions, which we do in the next section.

4 Group actions on quadratic systems (3.1) and invariants and comitants with respect to these actions

4.1 Group actions on quadratic systems (3.1)

More explicitly the systems (3.1) can be written in the form:

$$\begin{aligned} \frac{dx}{dt} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ \frac{dy}{dt} &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + 2b_{11}xy + b_{02}y^2, \end{aligned}$$

and let $a = (a_{00}, \ldots, b_{02})$.

On the set Σ of all quadratic differential systems (3.1) acts the group $Aff(2,\mathbb{R})$ of affine transformation on the plane. Indeed $\forall g \in Aff(2,\mathbb{R})$, $g : \ \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ we have:

$$g : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} + B; \qquad g^{-1} : \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} - M^{-1}B,$$

where $M = ||M_{ij}||$ is a 2×2 nonsingular matrix and B is a 2×1 matrix over $\mathbb{R}$. $\forall S \in \Sigma$ we can form its transformed system $\tilde{S} = gS$:

$$\frac{\partial \tilde{x}}{\partial t} = \tilde{P}(\tilde{x}, \tilde{y}), \qquad \frac{\partial \tilde{y}}{\partial t} = \tilde{Q}(\tilde{x}, \tilde{y}), \tag{$\tilde{S}$}$$

where

$$\begin{pmatrix} \tilde{P}(\tilde{x}, \tilde{y}) \\ \tilde{Q}(\tilde{x}, \tilde{y}) \end{pmatrix} = M \begin{pmatrix} (P \circ g^{-1})(\tilde{x}, \tilde{y}) \\ (Q \circ g^{-1})(\tilde{x}, \tilde{y}) \end{pmatrix}.$$

The map

$$\begin{array}{rcl} Aff(2,\mathbb{R})\times\Sigma & \longrightarrow & \Sigma \\ (g, \quad S) & \longrightarrow & \tilde{S}=gS \end{array}$$

verifies the axioms for a left group action. $\forall$ subgroup $G\subseteq Aff(2,\mathbb{R})$ we have an induced action of G on Σ. We can identify the set Σ of system (3.1) with $\mathbb{R}^{12}$ via the map $\Sigma\longrightarrow\mathbb{R}^{12}$ which associates to each system (3.1) the 12-tuple $(a_{00},\dots,b_{02})$ of its coefficients.

The action of $Aff(2,\mathbb{R})$ on Σ yields an action of this group on $\mathbb{R}^{12}$. $\forall g\in Aff(2,\mathbb{R})$ let $r_g:\mathbb{R}^{12}\longrightarrow\mathbb{R}^{12}$ be the map which corresponds to g via this action. We know (cf. [22]) that r_g is linear and that the map $r:\ Aff(2,\mathbb{R}\longrightarrow GL(12,\mathbb{R})$ thus obtained is a group homomorphism. $\forall$ subgroup G of $Aff(2,\mathbb{R})$, r induces a representation of G onto $\mathcal{G}$ of $GL(12,\mathbb{R})$.

4.2 Invariants and comitants associated to the group actions

Definition 4.1 *A polynomial $U(a\,,x,y)\in\mathbb{R}[a,x,y]$ is called a comitant of systems (3.1) with respect to a subgroup G of $Aff(2,\mathbb{R})$, if there exists $\chi\in\mathrm{Z}$ such that for every $(g,\,\boldsymbol{a})\in G\times\mathbb{R}^{12}$ and $\forall\ (x,y)\in\mathbb{R}^2$ the following relation holds:*

$$U(r_g(\boldsymbol{a}),\ g(x,y)\,)\equiv\ (\det\,g)^{-\chi}\,U(\boldsymbol{a},x,y),$$

*where $\det g=\det M$. If the polynomial U does not explicitly depend on x and y then it is called invariant. The number $\chi\in\mathrm{Z}$ is called*the weight *of the comitant $U(a,x,y)$. If $G=GL(2,\mathbb{R})$ (or $G=Aff(2,\mathbb{R})$) then the comitant $U(a,x,y)$ of systems (3.1) is called GL-comitant (respectively,* ***affine*** *comitant).*

Definition 4.2 *A subset $X\subset\mathbb{R}^{12}$ will be called G-invariant, if $\forall g\in G$ we have $r_g(X)\subseteq X$.*

As it can easily be verified, the following polynomials are GL-comitants of system (3.1):

$$\begin{aligned}
&K(a,x,y)=\frac{1}{4}Jacob(p_2,q_2),\ \mu_0(a)=Res_\gamma(p_2,q_2),\ (\gamma=\frac{x}{y}\ or\ \gamma=\frac{y}{x}),\\
&H(a,x,y)=\frac{1}{4}Discrim_\gamma(\alpha p_2(x,y)+\beta q_2(x,y))\Big|_{\{\alpha=y,\,\beta=-x\}},\\
&C_i(a,x,y)=yp_i(x,y)-xq_i(x,y),\ i=0,1,2,\ \eta(a)=Discrim_\gamma(C_2),\\
&M(a,x,y)=\frac{1}{4}Hess(C_2)=\frac{1}{4}\left[\frac{\partial^2C_2}{\partial x^2}\frac{\partial^2C_2}{\partial y^2}-\left(\frac{\partial^2C_2}{\partial x\partial y}\right)^2\right],\\
&K_1(a,x,y)=p_1(x,y)q_2(x,y)-p_2(x,y)q_1(x,y),\ L(a,x,y)=2K-4H-M.
\end{aligned}\tag{4.1}$$

Let $T(2,\mathbb{R})$ be the subgroup of $Aff(2,\mathbb{R})$ formed by translations. Consider the linear representation of $T(2,\mathbb{R})$ into its corresponding subgroup $\mathcal{T}\subset GL(12,\mathbb{R})$, i.e. $\forall\tau\in T(2,\mathbb{R})$, $\tau:\ x=\tilde{x}+\alpha, y=\tilde{y}+\beta$ we consider as above $r_\tau:\ R^{12}\longrightarrow R^{12}$.

Definition 4.3 *A GL-comitant $U(a,x,y)$ of systems (3.1) is called a T-comitant if for every $(\tau,\,\boldsymbol{a})\in T(2,\mathbb{R})\times\mathbb{R}^{12}$ and $\forall\ (\tilde{x},\tilde{y})\in\mathbb{R}^2$ the relation $U(r_\tau\cdot\boldsymbol{a},\ \tilde{x},\ \tilde{y})=U(\boldsymbol{a},\ \tilde{x},\ \tilde{y})$ holds.*

Let

$$U_i(a,x,y)=\sum_{j=0}^{d_i} U_{ij}(a)x^{d_i-j}y^j,\quad i=\overline{1,s}$$

be a set of GL-comitants of systems (3.1) where d_i denotes the degree of the binary form $U_i(a,x,y)$ in x and y with coefficients in $\mathbb{R}[a]$ where $\mathbb{R}[a]=\mathbb{R}[a_{00},\dots,b_{02}]$. We denote by

$$\mathcal{U}=\left\{\, U_{ij}(a)\in\mathbb{R}[a] \mid i=\overline{1,s},\ j=\overline{0,d_i}\,\right\},$$

the set of the coefficients in $\mathbb{R}[a]$ of the GL-comitants $U_i(a,x,y)$, $i=\overline{1,s}$, and by $V(\mathcal{U})$ its associated algebraic set:

$$V(\mathcal{U})=\left\{\, \boldsymbol{a}\in\mathbb{R}^{12} \mid U_{ij}(\boldsymbol{a})=0\ \forall\ U_{ij}(a)\in\mathcal{U}\,\right\}.$$

Definition 4.4 *A GL-comitant $U(a,x,y)$ of systems (3.1) is called a conditional T-comitant (or CT-comitant) modulo $\langle U_1,U_2,...,U_s\rangle$ if the following two conditions are satisfied:*
(i) the algebraic subset $V(\mathcal{U})\subset\mathbb{R}^{12}$ is affine invariant (see Definition 4.2);
(ii) for every $(\tau,\ \boldsymbol{a})\in T(2,\mathbb{R})\times V(\mathcal{U})$ we have $U(r_\tau\cdot\boldsymbol{a},\ \tilde{x},\ \tilde{y})=U(\boldsymbol{a},\ \tilde{x},\ \tilde{y})$ in $\mathbb{R}[\tilde{x},\ \tilde{y}]$.

In other words, a CT-comitant $U(a,x,y)$ is a T-comitant on the algebraic subset $V(\mathcal{U})\subset\mathbb{R}^{12}$.

Remark 4.1 *We give below the geometrical meaning of the comitants $\mu_0(a)$, $K(a,x,y)$ and $H(a,x,y)$:*

$$\gcd(p_2(x,y),q_2(x,y))=\begin{cases}\text{constant} & iff\quad \mu_0(a)\neq 0;\\ bx+cy & iff\quad \mu_0=0,\ K(a,x,y)\neq 0;\\ (bx+cy)(dx+ey) & iff\quad \begin{cases}\mu_0=0, K(a,x,y)=0\\ and\ \ H(a,x,y)\neq 0;\end{cases}\\ (bx+cy)^2 & iff\quad \begin{cases}\mu_0=0, K(a,x,y)=0,\\ and\ \ H(a,x,y)=0;\end{cases}\end{cases}$$

where $bx+cy, dx+ey\in\mathbb{C}[x,y]$ are some linear forms and $be-cd\neq 0$.

Remark 4.2 *The comitants $K_1(a,x,y)$ and $L(a,x,y)$have the following geometrical implications:*

$$\forall\boldsymbol{a}\in V(K_1)\quad we\ have:\ \ if\quad p_1(\boldsymbol{a},x,y)^2+q_1(\boldsymbol{a},x,y)^2\not\equiv 0\quad then\quad \Delta_S\geq 2;$$
$$\forall\boldsymbol{a}\in V(L)\ we\ have:\ \ M_{C^*}\geq 2\quad and$$

$quad\Delta\geq 1$.

Definition 4.5 *The polynomial $U(a,x,y)\in\mathbb{R}[a,x,y]$ has well determined sign on $V\subset\mathbb{R}^{12}$ with respect to x, y if $\forall\boldsymbol{a}\in V$ fixed, the sign of the polynomial function induced by $U(a,x,y)$ on V is constant where this function is not zero.*

Observation 4.1 *We draw the attention to the fact, that if a CT-comitant $U(a,x,y)$ of even weight is a binary form of even degree in the coefficients of (3.1) and has well determined sign on some affine invariant algebraic subset $V(\mathcal{U})$ then this property is conserved by any affine transformation.*

4.3 Canonical forms of planar quadratic systems for the neighbourhood of infinity

According to the well-known geometrical meaning of the polynomials $C_2(a,x,y)$, $\eta(a)$ and $M(a,x,y)$ the following assertion is valid

Lemma 4.1 *For $C_2(a,x,y)\not\equiv 0$ the values of the divisors $D_S(C^*,Z)$ for systems (3.1) are determined by the corresponding conditions indicated in Table 4.1,*

Table 4.1:

M_{C^*}	*Value of* $D_S(C^*, Z)$	*Necessary and sufficient conditions on the comitants*	*Notation for the conditions*
1	$p+q+r$	$\eta > 0$	$(\mathcal{I}_1)$
	$p+q^c+r^c$	$\eta < 0$	$(\mathcal{I}_2)$
2	$2p+q$	$\eta = 0, \quad M \neq 0$	$(\mathcal{I}_3)$
3	$3p$	$M = 0$	$(\mathcal{I}_4)$

where we write $p+q^c+r^c$ *if two of the points, i.e.* q^c, r^c, *are complex but not real and the indicated inequalities determine sets in* $\mathbb{R}^{12}$ *(respectively in* Σ*), for example* $\eta > 0$ *means* $\{\boldsymbol{a} \in \mathbb{R}^{12} \mid \eta(\boldsymbol{a}) > 0\}$.

Let us consider the GL-comitant $C_2(a, x, y) \not\equiv 0$ as a cubic binary form.

It is well known that there exist $g \in GL(2, \mathbb{R})$, $g(x, y) = (u, v)$, such that the transformed binary form $gC_2(a, x, y) = C_2(a, g^{-1}(u, v))$ is of one of the following canonical forms

$$xy(x-y); \qquad x(x^2+y^2); \qquad x^2y; \qquad x^3. \tag{4.2}$$

Each such canonical form corresponds to the values of the divisors $D_S(C^*, Z)$ indicated in Table 4.1. On the other hand, according to the Definition 4.1 of a GL-comitant, for $C_2(a, x, y)$ whose weight $\chi = -1$, we have

$$C_2(r_g(a),\, u, v) = \det(g)\, C_2(a,\, x, y) = \det(g)\, C_2(a,\, g^{-1}(u, v)).$$

Hence, there exists the following relation between the form gC_2 and the GL-comitant C_2 for the transformed system gS:

$$C_2(r_g(a),\, u, v) = \lambda C_2(a,\, g^{-1}(u, v)), \quad \lambda \in \mathbb{R},$$

where we may consider $\lambda = 1$ ($\lambda \neq 0$ can be brought to take by transformation $x = x_1/\lambda,\ y = y_1/\lambda$.

Thus, for the first canonical form in (4.2) we have

$$-b_{20}x^3 + (a_{20} - 2b_{11})x^2y + (2a_{11} - b_{02})xy^2 + a_{02}y^3 = xy(x-y),$$

and this yields the following canonical system:

$$\begin{cases} \dfrac{dx}{dt} &= k + cx + dy + 2gx^2 + 2(h-1)xy, \\[2ex] \dfrac{dy}{dt} &= l + ex + fy + 2(g-1)xy + 2hy^2. \end{cases} \tag{S_1}$$

In the same manner canonical systems (S_2), (S_3) and (S_4) can be constructed, respectively:

$$\begin{cases} \dfrac{dx}{dt} &= k + cx + dy + 2gx^2 + 2(h+1)xy, \\ \dfrac{dy}{dt} &= l + ex + fy - 2x^2 + 2gxy + 2hy^2; \end{cases} \qquad (S_2)$$

$$\begin{cases} \dfrac{dx}{dt} &= k + cx + dy + 2gx^2 + 2hxy, \\ \dfrac{dy}{dt} &= l + ex + fy + 2(g-1)xy + 2hy^2; \end{cases} \qquad (S_3)$$

$$\begin{cases} \dfrac{dx}{dt} &= k + cx + dy + 2gx^2 + 2hxy, \\ \dfrac{dy}{dt} &= l + ex + fy - x^2 + 2gxy + 2hy^2. \end{cases} \qquad (S_4)$$

5 Classification of the quadratic systems according to the values of the multiplicity divizor $\mathcal{D}_S$

A specific type of a divisor D_S yields a class of quadratic systems (3.1). We want to list all possible types of the divisors D_S and for each specific type to determine the subset of Σ where D_S has this type. We want to give this subset in terms of algebraic invariants and comitants so as to be able to check these conditions $\forall$ system (3.1) in any chart.

In order to construct other necessary invariant polynomials let us consider the differential operator $\mathcal{L} = x \cdot \mathrm{L}_2 - y \cdot \mathrm{L}_1$ acting on $\mathbb{R}[a, x, y]$ constructed in [4], where

$$\begin{aligned} \mathrm{L}_1 &= 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}}; \\ \mathrm{L}_2 &= 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}} \end{aligned}$$

as well the classical differential operator $(f, \varphi)^{(2)}$ acting on $\mathbb{R}[a, x, y]$ which is called *transvectant* of the second index (see, for example, [10]):

$$(f, \varphi)^{(2)} = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 \varphi}{\partial y^2} - 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 \varphi}{\partial x^2}.$$

Here $f(x, y)$ and $\varphi(x, y)$ are polynomials in x and y of degrees greater than or equal to 2.

In [4] it is shown that if a polynomial $P \in \mathbb{R}[a, x, y]$ is a comitant of system (3.1) with respect to the group $GL(2, \mathbb{R})$ then $\mathcal{L}(P)$ is also a GL-comitant. The same is true for the operator transvectant with respect to two comitants, f and φ.

So, by using these operators and the comitants $\mu_0(a)$, $M(a, x, y)$ and $K(a, x, y)$ we shall construct the following polynomials:

$$\begin{aligned} &\mu_i(a, x, y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0), \ \ i = 1, .., 4, \\ &\kappa(a) = \frac{1}{2}(M, K)^{(2)}, \qquad \kappa_1(a) = \frac{1}{2}(M, C_1)^{(2)}, \end{aligned} \qquad (5.1)$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$.

These polynomials are in fact comitants of system (3.1) with respect to the group $GL(2, \mathbb{R})$.

To reveal the geometrical meaning of the comitants $\mu_i(a, x, y),\ i = \overline{0,4}$ we use the following resultants whose calculation yield:

$$Res_X(P^*, Q^*) = \mu_0 Y^4 + \mu_{10} Y^3 Z + \mu_{20} Y^2 Z^2 + \mu_{30} Y Z^3 + \mu_{40} Z^4; \tag{5.2}$$

$$Res_Y(P^*, Q^*) = \mu_0 X^4 + \mu_{01} X^3 Z + \mu_{02} X^2 Z^2 + \mu_{03} X Z^3 + \mu_{04} Z^4, \tag{5.3}$$

where $\mu_{ij} = \mu_{ij}(a) \in \mathbb{R}[a_{00}, \ldots, b_{02}]$.

On the other hand for μ_i, $i = \overline{0,4}$ we have

$$\begin{array}{lcc}
\mu_0(a) & = & \mu_0(a); \\
\mu_1(a,x,y) & = & \mu_{10}x + \mu_{01}y; \\
\mu_2(a,x,y) & = & \mu_{20}x^2 + \mu_{11}xy + \mu_{02}y^2; \\
\mu_3(a,x,y) & = & \mu_{30}x^3 + \mu_{21}x^2y + \mu_{12}xy^2 + \mu_{03}y^3; \\
\mu_4(a,x,y) & = & \mu_{40}x^4 + \mu_{31}x^3y + \mu_{22}x^2y^2 + \mu_{13}xy^3 + \mu_{04}y^4.
\end{array}$$

We observe that the leading coefficients of the comitants μ_i, $i = \overline{0,4}$ with respect to x (respectively y) are the corresponding coefficients in (5.2) (respectively (5.3)).

We draw the attention to the fact, that if the comitant $\mu_i(a, x, y)$ $(i = \overline{1,4})$ is not equal to zero then we may assume that its leading coefficients are both non zero, as this can be obtained by applying a rotation of the phase plane of the system (3.1). From here and (5.2), (5.3) and the above values of μ_i, $i = \overline{0,4}$we have:

Lemma 5.1 *The system $P^*(X,Y,Z) = Q^*(X,Y,Z) = 0$ possesses m $(= \Delta_S)$ $(1 \leq m \leq 4)$ solutions $[X_i : Y_i : Z_i]$ with $Z_i = 0$ $(i = \overline{1,m})$ (considered with multiplicities) if and only if $\forall i \in \{0, 1, \ldots, m-1\}$ $\mu_i(a, x, y) = 0$ in $\mathbb{R}$ and $\mu_m(a, x, y) \neq 0$.*

Remark 5.1 *The following identity holds*

$$\mu_4(a, X, Y) = Res_Z(P^*(X,Y,Z), Q^*(X,Y,Z)).$$

Hence, it can easily be concluded, that for any solution $[X_0 : Y_0 : Z_0]$ (including those with $Z_0 = 0$) of the system of equations $P^*(X,Y,Z) = Q^*(X,Y,Z) = 0$, the following relation is satisfied: $\mu_4(a, X_0, Y_0) = 0$.

We give below our theorem of classification of the types of all divisors D_S occuring in quadratic systems and associate to each type the necessary and sufficient conditions in terms of algebraic invariants and comitants. The computation of these invariants and comitants can be programmed using symbolic manipulations and implemented on computers. Thus for any specific system (3.1) we can calculate explicitly its divisor type in whatever chart (3.1) is given.

Theorem 5.1 *We consider here the family of all quadratic differential systems (3.1) such that $yP_2(x,y) - xQ_2(x,y)$ is not identically zero. We list in the first column of Table 7.1, all possible values which could be taken by Δ_S for such systems (3.1). For each value of Δ_S we list in the second column all possibilities we have for M_{C^*}. For each combination (Δ_S, M_{C^*}) all the possibilities we have for the value of D_S are those indicated in the third column. For a specified (Δ_S, M_{C^*}), the necessary and sufficient conditions to have the values of D_S as indicated in the third column are those indicated in the corresponding fourth column. (We recall that I_j are the conditions indicated in Table 4.1. In the last column of Table 7.1 we denote by Σ_i the class of all quadratic systems which possess (Δ_S, M_{C^*}, D_S) as indicated in thefirst three columns).*

Table 5.1:

Δ_S	M_{C^*}	Value of D_S	Necessary and sufficient conditions on the comitants	Σ_i
0	1	$\binom{0}{1}p+\binom{0}{1}q+\binom{0}{1}r$	$\mu_0\neq 0,\ (\mathcal{I}_1)$	Σ_1
		$\binom{0}{1}p+\binom{0}{1}q^c+\binom{0}{1}r^c$	$\mu_0\neq 0,\ (\mathcal{I}_2)$	Σ_2
	2	$\binom{0}{1}p+\binom{0}{2}q$	$\mu_0\neq 0,\ (\mathcal{I}_3)$	Σ_3
	3	$\binom{0}{3}p$	$\mu_0\neq 0,\ (\mathcal{I}_4)$	Σ_4
1	1	$\binom{1}{1}p+\binom{0}{1}q+\binom{0}{1}r$	$\mu_0=0,\ \mu_1\neq 0,\ (\mathcal{I}_1)$	Σ_5
		$\binom{1}{1}p+\binom{0}{1}q^c+\binom{0}{1}r^c$	$\mu_0=0,\ \mu_1\neq 0,\ (\mathcal{I}_2)$	Σ_6
	2	$\binom{1}{1}p+\binom{0}{2}q$	$\mu_0=0,\ \mu_1\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_3)$	Σ_7
		$\binom{0}{1}p+\binom{1}{2}q$	$\mu_0=0,\ \mu_1\neq 0,\ \kappa=0,\ (\mathcal{I}_3)$	Σ_8
	3	$\binom{1}{3}p$	$\mu_0=0,\ \mu_1\neq 0,\ (\mathcal{I}_4)$	Σ_9
2	1	$\binom{2}{1}p+\binom{0}{1}q+\binom{0}{1}r$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_1)$	Σ_{10}
		$\binom{1}{1}p+\binom{1}{1}q+\binom{0}{1}r$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa=0,\ (\mathcal{I}_1)$	Σ_{11}
		$\binom{2}{1}p+\binom{0}{1}q^c+\binom{0}{1}r^c$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_2)$	Σ_{12}
		$\binom{0}{1}p+\binom{1}{1}q^c+\binom{1}{1}r^c$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa=0,\ (\mathcal{I}_2)$	Σ_{13}
	2	$\binom{2}{1}p+\binom{0}{2}q$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_3)$	Σ_{14}
		$\binom{1}{1}p+\binom{1}{2}q$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa=0,\ L=0,\ (\mathcal{I}_3)$	Σ_{15}
		$\binom{0}{1}p+\binom{2}{2}q$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ \kappa=0,\ L\neq 0,\ (\mathcal{I}_3)$	Σ_{16}
	3	$\binom{2}{3}p$	$\mu_{0,1}=0,\ \mu_2\neq 0,\ (\mathcal{I}_4)$	Σ_{17}
3	1	$\binom{3}{1}p+\binom{0}{1}q+\binom{0}{1}r$	$\mu_{0,1,2}=0,\ \mu_3\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_1)$	Σ_{18}
		$\binom{2}{1}p+\binom{1}{1}q+\binom{0}{1}r$	$\mu_{0,1,2}=0,\ \mu_3\neq 0,\ \kappa=0,\ (\mathcal{I}_1)$	Σ_{19}
		$\binom{3}{1}p+\binom{0}{1}q^c+\binom{0}{1}r^c$	$\mu_{0,1,2}=0,\ \mu_3\neq 0,\ (\mathcal{I}_2)$	Σ_{20}
	2	$\binom{3}{1}p+\binom{0}{2}q$	$\mu_{0,1,2}=0,\ \mu_3\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_3)$	Σ_{21}
		$\binom{2}{1}p+\binom{1}{2}q$	$\mu_{0,1,2}=0,\mu_3\neq 0,\kappa=L=0,\kappa_1\neq 0,(\mathcal{I}_3)$	Σ_{22}
		$\binom{1}{1}p+\binom{2}{2}q$	$\mu_{0,1,2}=0,\mu_3\neq 0,\kappa=L=0,\kappa_1=0,(\mathcal{I}_3)$	Σ_{23}
		$\binom{0}{1}p+\binom{3}{2}q$	$\mu_{0,1,2}=0,\ \mu_3\neq 0,\ \kappa=0,\ L\neq 0,\ (\mathcal{I}_3)$	Σ_{24}
	3	$\binom{3}{3}p$	$\mu_{0,1,2}=0,\ \mu_3\neq 0,\ (\mathcal{I}_4)$	Σ_{25}
4	1	$\binom{4}{1}p+\binom{0}{1}q+\binom{0}{1}r$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_1)$	Σ_{26}
		$\binom{3}{1}p+\binom{1}{1}q+\binom{0}{1}r$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa=0,\ K_1\neq 0,\ (\mathcal{I}_1)$	Σ_{27}
		$\binom{2}{1}p+\binom{2}{1}q+\binom{0}{1}r$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa=0,\ K_1=0,\ (\mathcal{I}_1)$	Σ_{28}
		$\binom{4}{1}p+\binom{0}{1}q^c+\binom{0}{1}r^c$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_2)$	Σ_{29}
		$\binom{0}{1}p+\binom{2}{1}q^c+\binom{2}{1}r^c$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa=0,\ (\mathcal{I}_2)$	Σ_{30}
	2	$\binom{4}{1}p+\binom{0}{2}q$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa\neq 0,\ (\mathcal{I}_3)$	Σ_{31}
		$\binom{3}{1}p+\binom{1}{2}q$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa=L=0,\ \kappa_1\neq 0,\ (\mathcal{I}_3)$	Σ_{32}
		$\binom{2}{1}p+\binom{2}{2}q$	$\mu_{0,1,2,3}=0,\mu_4\neq 0,\kappa=L=\kappa_1=0,K_1=0,(\mathcal{I}_3)$	Σ_{33}
		$\binom{1}{1}p+\binom{3}{2}q$	$\mu_{0,1,2,3}=0,\mu_4\neq 0,\kappa=L=\kappa_1=0,K_1\neq 0,(\mathcal{I}_3)$	Σ_{34}
		$\binom{0}{1}p+\binom{4}{2}q$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ \kappa=0,\ L\neq 0,\ (\mathcal{I}_3)$	Σ_{35}
	3	$\binom{4}{3}p$	$\mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ (\mathcal{I}_4)$	Σ_{36}

Proof.

We need to examine the four distinct cases corresponding to the canonical forms $(S_1)-(S_4)$, respectively.

5.1 Systems of type (S_1)

For systems (S_1) we have $\mu_0 = 16gh(g+h-1)$ and for $\mu_0 \neq 0$ according to Lemma 5.1 we have $\Delta_S = 0$ and, hence, we obtain a system of the class Σ_1 (see Table 7.1).

Let consider now $\mu_0 = 0$. In this case we have $gh(g+h-1) = 0$ and without loss of generality we may assume $g = 0$. Indeed, if $h = 0$ (respectively, $g+h-1 = 0$) we can apply the linear transformation which will replace the straight line $y = 0$ with $x = 0$ (respectively, $y = 0$ with $y = x$). Let $g = 0$. By using the shift transformation $x = x_1 + (f+eh)/2$, $y = y_1 + e/2$ we obtain the relations $e = f = 0$. In this way the system (S_1) will be brought to the following canonical form:

$$\dot{x} = k + cx + dy + 2(h-1)xy, \quad \dot{y} = l - 2xy + 2hy^2, \tag{5.4}$$

for which we have

$$\mu_1 = 8ch(1-h)y, \quad \kappa = 16h(1-h), \quad K = 2h(h-1)y^2.$$

For $\mu_1 \neq 0$, from Lemma 5.1 we obtain $\Delta_S = 1$ which leads us to the case Σ_5.

Considering $\mu_1 = 0$ we shall examine two cases: $\kappa \neq 0$ and $\kappa = 0$.

5.1.1 Case $\kappa \neq 0$

As condition $\kappa \neq 0$ is equivalent to condition $K \neq 0$, according to the Remark 4.1 we conclude that Supp$D_S(P^*, Q^*; Z)$ contains exactly one point $p = [1:0:0]$ since $\gcd(p_2, q_2) = y$. By Lemma 5.1 its multiplicity$I_p(P^*, Q^*)$ depends of the number of vanishing comitants $\mu_i(a, x, y)$. In this way we obtain that a quadratic system belongs to the set Σ_{10} (respectively Σ_{18}; Σ_{26}) for $\mu_{0,1} = 0$, $\mu_2 \neq 0$ (respectively for $\mu_{0,1,2} = 0, \mu_3 \neq 0$; $\mu_{0,1,2,3} = 0$, $\mu_4 \neq 0$). We use the compact notation $\mu_{0,1,2} = 0$ for $\mu_0 = \mu_1 = \mu_2 = 0$.

5.1.2 Case $\kappa = 0$

In this case $h(h-1) = 0$ and analogously to the previous case, without loss of the generality we may assume $h = 0$. Thus, for system (5.4) we obtain:

$$\begin{aligned}
&\mu_0 = \mu_1 = 0, \qquad \mu_2 = -4cdxy, \qquad \mu_3 = 4(k-l)(dy-cx)xy,\\
&\mu_4 = -2xy[lc^2x^2 - 2(k-l)^2xy + 2lcdxy + ld^2y^2], \quad K_1 = -2xy(cx+dy).
\end{aligned}$$

So, if $\mu_2 \neq 0$ taking into consideration Remark 5.1 and the value of the comitant μ_4, we obtain the case Σ_{11} in Table 7.1.

If $\mu_2 = 0$ and $\mu_3 \neq 0$ then $cd = 0$, $c^2 + d^2 \neq 0$ and clearly we arrive at the case Σ_{19}.

Let us now suppose that the conditions $\mu_2 = \mu_3 = 0$ hold.

5.1.2.1 $K_1 \neq 0$.

Then $c^2 + d^2 \neq 0$ and from $\mu_3 = 0$ we obtain $k = l$ which yields either $\mu_4 = -2ld^2xy^3$ (for $c = 0$) or $\mu_4 = -2lc^2x^3y$ (for $d = 0$). Both these cases lead us to the case Σ_{27} in Table 7.1.

5.1.2.2 $K_1 = 0$.

In this case it follows at once that $c = d = 0$ and, hence, $\mu_4 = 4(k-l)^2x^2y^2$. Thus taking into consideration the Remark 5.1 we obtain the case Σ_{28}.

5.2 Systems of type (S_2)

For a canonical system (S_2) we obtain

$$\mu_0 = -16h[g^2 + (h+1)^2], \quad \kappa = -16\left[g^2 + (h+1)(1-3h)\right],$$
$$K = 2(g^2 + h + 1)x^2 + 4ghxy + 2h(h+1)y^2$$

and for $\mu_0 \neq 0$ according to Lemma 5.1 we have $\Delta_S = 0$. Thus we obtain the case Σ_2 in Table 7.1.

Let us consider now $\mu_0 = 0$, i.e. $h[g^2 + (h+1)^2] = 0$.

5.2.1 Case $\kappa \neq 0$

In this case we have $h = 0$ and since accordingto Remark 4.1 the condition $\kappa \neq 0$ is equivalent to the condition $K \neq 0$, Supp$D_S(P^*, Q^*; Z)$ contains only one point, namely the real one. By Lemma 5.1 its multiplicity depends of the number of the vanishing comitants μ_i. Therefore the quadratic system belongs to the set Σ_6 (respectively Σ_{12}; Σ_{20}; Σ_{29}) for $\mu_1 \neq 0$ (respectively for $\mu_1 = 0, \mu_2 \neq 0$; $\mu_{1,2} = 0, \mu_3 \neq 0$; $\mu_{1,2,3} = 0$, $\mu_4 \neq 0$).

5.2.2 Case $\kappa = 0$

The conditions $\mu_0 = \kappa = 0$ yield $g = 0$, $h = -1$ and translating the origin of coordinates at the point $(e/4, f/4)$ the system (S_2) will be brought to the form

$$\dot{x} = k + cx + dy, \qquad \dot{y} = l - 2x^2 - 2y^2, \tag{5.5}$$

for which

$$\mu_0 = \mu_1 = 0, \qquad \mu_2 = 4(c^2 + d^2)(x^2 + y^2),$$
$$\mu_4 = 2\left(x^2 + y^2\right)\left[(2k^2 - c^2 l)x^2 - 2\,cdlxy + (2k^2 - d^2 l)y^2\right].$$

Thus, according to the Remark 5.1, for $\mu_2 \neq 0$ we obtain the case Σ_{13}.

Let us admit that condition $\mu_2 = 0$ is satisfied. Then $c = d = 0$ and for systems (5.5) we have $\mu_3 = 0$, $\mu_4 = 4k^2(x^2 + y^2)^2$. This leads us to the case Σ_{30}.

5.3 Systems of type (S_3)

For canonical systems (S_3) one can calculate

$$\mu_0 = 16gh^2, \quad \kappa = -16h^2, \quad K = 2\left[g(g-1)x^2 + 2ghxy + h^2y^2\right].$$

It is quite clear that for $\mu_0 \neq 0$ we have $\Delta_S = 0$ and this leads us to the case Σ_3.

Let us suppose $\mu_0 = 0$ and let us examine the two cases: $\kappa \neq 0$ and $\kappa = 0$.

5.3.1 Case $\kappa \neq 0$

In this case $h \neq 0$ which yields $g = 0$ and thus for a system (S_3) we have $\gcd(p_2, q_2) = y$. So, taking into consideration the Remark 5.1 and the fact that for a system (S_3) the polynomial $C_2(x, y) = 2x^2y$ we obtain the case Σ_7 if $\mu_1 \neq 0$.

On the other hand the condition $h \neq 0$ implies $K \neq 0$. Hence, by Remark 4.1 and Lemma 5.1, Supp$D_S(P^*, Q^*; Z)$ contains exactly one point $[1:0:0]$ of the multiplicity $(\Delta_S, 1)$.

Consequently we conclude that the quadratic system belongs to the set Σ_{14} (respectively, Σ_{21}; Σ_{31}) for $\mu_1 = 0, \mu_2 \neq 0$ (respectively, $\mu_{1,2} = 0, \mu_3 \neq 0$; $\mu_{1,2,3} = 0$, $\mu_4 \neq 0$).

5.3.2 Case $\kappa = 0$

Then $h = 0$ and for systems (S_3) we have

$$\mu_0 = 0, \quad \mu_1 = 8dg(g-1)^2 x, \quad L = 4gx^2, \quad K = 2g(g-1)x^2,$$

and $\gcd(p_2, q_2) = x$. So, by Lemma 5.1 for $\mu_1 \neq 0$ the quadratic systems belong to the set Σ_8.

Supposing $\mu_1 = 0$ we shall consider two subcases: $L \neq 0$ and $L = 0$.

5.3.2.1 Subcase $L \neq 0$. Then $g \neq 0$ and the condition $\mu_1 = 0$ yields $d(g-1) = 0$.

If $g \neq 1$ then $K \neq 0$ and by the Remark 4.1 and Lemma 5.1, Supp$D_S(P^*, Q^*; Z)$ contains exactly one point $[0 : 1 : 0]$ whose multiplicity depends of the number of vanishing comitants $\mu_i(a, x, y)$. Therefore we conclude that the quadratic systems belong to the set Σ_{16} (respectively, Σ_{24}; Σ_{35}) for $\mu_2 \neq 0$ (respectively, $\mu_2 = 0, \mu_3 \neq 0$; $\mu_{2,3} = 0$, $\mu_4 \neq 0$).

Let us admit now that $g = 1$. Then for systems (S_3) we obtain $\mu_1 = 0$, $\mu_2 = 4f^2x^2$ and the following factorization of the polynomial μ_4 occurs: $\mu_4 = x^2 W_2(x, y)$, where W_2 is a homogeneous polynomial of degree 2. Hence, for $\mu_2 \neq 0$ we again obtain the case Σ_{16} in Table 7.1.

The condition $\mu_2 = 0$ yields $f = 0$ and then

$$\mu_3 = 2de^2x^3, \quad \mu_4 = 2x^3\left[(2l^2 + e^2k - cel)x - dely\right].$$

By the Remark 5.1 we observe that the system under consideration belongs to the class Σ_{24} for $\mu_3 \neq 0$ and to the class Σ_{35} for $\mu_3 = 0, \mu_4 \neq 0$.

5.3.2.2 Subcase $L = 0$**.** If this is the case for systems (S_3) we have $g = 0$ and applying the translation of the phase plane (to obtain $e = f = 0$) this system can be brought to the form

$$\dot{x} = k + cx + dy, \qquad \dot{y} = l - 2xy. \tag{5.6}$$

For the system (5.6) we have:

$$\begin{aligned}
&\mu_0 = \mu_1 = 0, \quad \mu_2 = 4cdxy, \quad \mu_3 = -4kxy(cx - dy), \quad \kappa_1 = -8d,\\
&\mu_4 = -2xy\left[c^2lx^2 + 2(cdl - k^2)xy + d^2ly^2\right], \quad K_1 = 2xy(cx + dy).
\end{aligned}$$

So, if $\mu_2 \neq 0$ by the Remark 5.1 and Lemma 5.1 the system (5.6) belongs to the class Σ_{15}.

Let us suppose that the condition $\mu_2 = 0$ holds.

5.3.2.2.1 If $\kappa_1 \neq 0$ then $d \neq 0$ which implies $c = 0$. Then $\mu_3 = 4kdxy^2$ and taking into consideration the factorization of the comitant μ_4, we obtain the case Σ_{22} for $\mu_3 \neq 0$ and the case Σ_{32} for $\mu_3 = 0$, $\mu_4 \neq 0$. .

5.3.2.2.2 Let us suppose $\kappa_1 = 0$. Then $d = 0$ and for the system (5.6) we obtain

$$\mu_3 = -4ckx^2y, \qquad \mu_4 = -2x^2y(c^2lx - k^2y), \qquad K_1 = 2cx^2y.$$

Therefore, if $\mu_3 \neq 0$ by the Remark 5.1 and Lemma 5.1 the system (5.6) belongs to the class Σ_{23}. If $\mu_3 = 0$ we obtain $ck = 0$ and we need to distinguish two cases: $K_1 \neq 0$ and $K_1 = 0$.

The condition $K_1 \neq 0$ yields $c \neq 0$ and, hence, $k = 0$. This leads us to the case Σ_{34}. If $K_1 = 0$ then $c = 0$ and we obtain the case Σ_{33}. .

5.4 Systems of type (S_4)

Note that for systems of the type (S_4) we have $D_S(C^*, Z) = 3p$. So, $\mathrm{Supp} D_S(P^*, Q^*; Z)$ could contain only the point $[0 : 1 : 0]$. By Lemma 5.1 its multiplicity depends of the number of the vanishing comitants μ_i. Therefore we obtain that the quadratic system belongs to the set Σ_4 (respectively Σ_9; Σ_{17}; Σ_{25}; Σ_{36}) for $\mu_0 \neq 0$ (respectively for $\mu_0 = 0, \mu_1 \neq 0$; $\mu_{0,1} = 0, \mu_2 \neq 0$; $\mu_{0,1,2} = 0$, $\mu_3 \neq 0$; $\mu_{0,1,2,3} = 0$, $\mu_4 \neq 0$).

As all cases are examined Theorem 5.1 is proved.■

6 Divisors encoding the topology of singulariries at infinity

We now need to consider the topological types of the singularities at infinity of quadratic systems. For this we shall introduce a third divisor at infinity:

Definition 6.1 *We call index divisor on the real line at infinity of* $\mathbb{R}^2$*, associated to a real system* (S) *such that* $Z \nmid C^*$*, the expression* $\sum i(p)p$ *where* p *is a singular point on the line at infinity* $Z = 0$ *of the system* (S) *and* $i(p)$ *is the topological index (see definition in [12]) of the real foliation on* $\mathrm{P}^2(\mathbb{R})$ *associated to* (S)*.*

Remark 6.1 *This is a well defined divisor which could be extended trivially to a divisor* $\sum j(p)p$*,* $p \in \{Z = 0\}$ *on the line at infinity* $Z = 0$ *of* $\mathbb{C}^2$ *by letting*

$$j(p) = \begin{cases} i(p) & if \quad p \in \mathrm{P}^2(\mathbb{R}) \\ 0 & if \quad p \in \mathrm{P}^2(\mathbb{C}) \setminus \mathrm{P}^2(\mathbb{R}), \end{cases}$$

where we identify $\mathrm{P}^2(\mathbb{R})$ *with its image via the inclusion* $\mathrm{P}^2(\mathbb{R}) \hookrightarrow \mathrm{P}^2(\mathbb{C})$*.*

Notation 6.1 *Let us denote by* $I(S)$ *the divisor* $I(S) = \sum j(p)p$ *as in Remark* 6.1 *and let*

$$\inf\, I(S) = \inf_{p \in \{Z=0\}} \{j(p)\}.$$

Notation 6.2 *We denote by* $N_{\mathbb{C}}(S)$ *(respectively, by* $N_{\mathbb{R}}(S)$*) the total number ofdistinct singular points, be they real or complex (respectively, real), on the line at infinity* $Z = 0$ *of the complex (respectively, real) foliation with singularities associated to* (S)*.*

We need to see how the divisor $I(S) = \sum j(p)p$ and the divisors $D_S(P^*, Q^*; Z) = \sum I_p(P^*, Q^*)p$ and $D_S(C^*, Z) = \sum I_p(C^*, Z)p$ constructed in Section 3 are combined. For this we shall fuse these three divisors on the complex line at infinity into just one but with the values in Abelian group Z^3:

Notation 6.3 *Let us consider the following divisor with the value in* Z^3 *on* $Z = 0$*:*

$$\mathcal{D}_S = \sum_p \begin{pmatrix} j(p) \\ I_p(P^*, Q^*) \\ I_p(C^*, Z) \end{pmatrix} p$$

where p *belongs to the line* $Z = 0$ *of the complex projective plane.*

The phase portrait of a system $S(\lambda)$ for the parameter value λ gives the picture of what occurs at the point λ in the parameter space and not in its neighbourhood. In particular we cannot detect the multiplicities of the singularities at infinity from just the phase portrait of $S(\lambda)$.

On the other hand $\mathcal{D}_{S(\lambda)}$ has dynamic qualities since it gives us some information about what could happen to the phase portraits in the neighbourhood of λ. For example if $I_p(P^*, Q^*) = 2$ for $S(\lambda_0)$, then we know that in the neighbourhood of λ_0 the phase portraits of $S(\lambda)$ will have 2 finite points arising from p in the neighbourhood of p.

Let us denote by

$$H_{\geq 0} = \left\{ x^2 + y^2 + z^2 = 1 \middle|\ z \geq 0 \right\}, \quad H_{+} = \left\{ x^2 + y^2 + z^2 = 1 \middle|\ z > 0 \right\}.$$

Proposition 6.1 *The maximum number of distinct sectors on $H_{\geq 0}$ of any singular point on the equator $S^1 = \{x^2 + y^2 + z^2 = 1|\ z = 0\}$ of any quadratic differential system is five.*

Proof : This follows easily from the pictures in [14].

Let $\sigma(S)$ be the set of all $n_\infty = 2N_R(S)$ real singular points at infinity considered on the equator S^1 of the Poincaré sphere of a system (S) satisfying the hypothesis of Theorem 5.1.

Let us consider a function $n_{sect} :\ \sigma(S) \longrightarrow\ \{1, 2, 3, 4, 5\}$ where $n_{sect}(p)$ is the number of distinct sectors of the point p on H_+.

Let $p \in \sigma(S)$ and let $\rho(S) = (p_1, p_2, ..., p_{n_\infty})$ be the ordered sequence of singularities of S on S^1, enumerated when S^1 is described in the positive sense and such that $p_1 = p$.

Let $O_S(p) = (n_{sect}(p_1), n_{sect}(p_2), ..., n_{sect}(p_{n_\infty}))$. We note that

$$O_S(p_i) = (n_{sect}(p_i), n_{sect}(p_{i+1}), ..., n_{sect}(p_{n_\infty}), n_{sect}(p_1), ..., n_{sect}(p_{i-1})).$$

Notation 6.4 *We denote by $O(S)$ anyone of the sequences $O_S(p)$.*

Definition 6.2

$$D_{sep}(S) = \sum_{p \in S^1} n(p)p$$

where n_p = the number of the distinct local separatrices on H_+ bordering a hyperbolic sector in H_+ of p.

This is a well defined divisor whenever $Z \nmid C^*$ which yields two integer-valued invariants:

Notation 6.5 $\qquad N^t_{sep}(S) = \deg D_{sep}(S)$.

Let us denote by $N^h_{sep}(S)$ the total number of local separatrices on H_+ bordering 2 adjacent hyperbolic sectors.

7 Classification by geometric and algebraic invariants of quadratic differential systems according to their behaviour in the neighbourhood of infinity

We begin by introducing the following

Notation 7.1

$$\mathcal{I}(S) = (\, N_{\mathbb{C}}(S),\ \deg I(S),\ N_{\mathbb{R}}(S),\ \inf I(S)\,),$$
$$\mathcal{J}(S) = \big(\, O(S),\ type\mathcal{D}_S,\ N^t_{sep}(S),\ N^h_{sep}(S), \big)$$

Remark 7.1 *We observe that the $\mathcal{I}(S)$, $\mathcal{J}(S)$ are integer-valued affine invariants.*

The study of the geometry of the systems yields a simpler set of algebraic invariants than those used in [14]. We refine here the invariants which appeared in [14] so as to reveal much more of the geometry of the systems.

We now need to relate the geometrical invariants defined in the previous section to their algebraic counterparts, i.e. the comitants and algebraic invariants.

To do this we construct below the GL-comitants which shall need, by using the following intermediates ones, two of which, namelly D_0 and J_3, are even invariants:

$$\begin{aligned}
&C_i = yp_i(x,y) - xq_i(x,y),\ i = 0,1,2,\\
&D_i = \frac{\partial}{\partial x}p_i(x,y) + \frac{\partial}{\partial y}q_i(x,y),\ i = 1,2, \quad J_1 = Jacob(C_0, D_2),\\
&J_2 = Jacob(C_0, C_2), \quad J_3 = Discrim(C_1), \quad J_4 = Jacob(C_1, D_2).
\end{aligned}$$

Then taking into consideration the comitants (4.1) and (5.1) constructed in Section 3 we shall construct the following new polynomials:

$$\begin{aligned}
&N = K - H, \quad L_1 = L + 4\,K, \quad \kappa_2 = -J_1, \quad \xi = M - K,\\
&L_2 = 4\,Jacob(J_2, \xi) + 3\,Jacob(C_1, \xi)D_1 - \xi(16J_1 + 3J_3 + 3D_1^2),\\
&L_3 = C_2^2(2J_1 - 3J_3) + C_2(6C_0K - C_1J_4) + K_1(C_1D_2 + 3K_1).
\end{aligned}$$

All these polynomials are GL-comitants, being obtained from simpler GL-comitants.

In the statement of the next Theorem, we use the same notations as in [14]: $Fig\,j,\ \ j = 1, ..., 40$ here, will denote phase portraits in the vicinity of infinity for quadratic systems.

Theorem 7.1 (The classification theorem) *We consider here the family of all nondegenerate quadratic systems (S), i.e.,* $\gcd(P,Q) = 1$ *and such that* $yP_2 - xQ_2$ *is not identically zero.*

A. *The values of the affine invariant* $\mathcal{I}(S) = (\,N_{\mathbb{C}}(S), \deg I(S), N_{\mathbb{R}}(S), \inf I(S)\,)$ *given in the Diagram* 7.1 *yield a partition of the family* ***QS*** *as indicated in that diagram and furthermore, they determine uniquely a phase portrait for all systems considered, with the exeption of those yielding the subclasses* $QS_i,\ i = \{\overline{1,9}\}$*. For these subclasses we use the additional affine invariant* $\mathcal{J}(S) = \big(\,O(S),\ type\mathcal{D}_S,\ N^t_{sep}(S),\ \ N^h_{sep}(S),\big)$ *which classifies each one of these subclasses according to the Diagram 7.2.*

B. *The geometry in the neighbourhood of infinity of the quadratic systems given in part* ***A*** *is expressed in terms of algebraic invariants and comitants as indicated in Diagram 7.3, which also contains the full information regarding multiplicities and indices of the singularities at infinity for all quadratic differential systems.*

Diagram 7.1:

$$
N_{\mathbb{C}}(S) = \begin{cases}
3 \text{ and } \deg I(S) = \begin{cases}
3 \iff \text{Fig. 2;} \\
2 \iff \text{Fig. 4;} \\
1 \text{ and } N_{\mathbb{R}}(S) = \begin{cases}
3 \text{ and } \inf I(S) = \begin{cases} -1 \iff \text{Fig. 1;} \\ 0 \iff QS_1; \end{cases} \\
1 \iff \text{Fig. 30;}
\end{cases} \\
0 \text{ and } N_{\mathbb{R}}(S) = \begin{cases} 3 \iff \text{Fig. 5;} \\ 1 \iff \text{Fig. 32;} \end{cases} \\
-1 \text{ and } N_{\mathbb{R}}(S) = \begin{cases} 3 \iff \text{Fig. 3;} \\ 1 \iff \text{Fig. 31;} \end{cases}
\end{cases} \\
2 \text{ and } \deg I(S) = \begin{cases}
3 \iff \text{Fig. 18;} \\
2 \text{ and } \inf I(S) = \begin{cases} 1 \iff QS_2; \\ 0 \iff \text{Fig. 26;} \end{cases} \\
1 \text{ and } \inf I(S) = \begin{cases} 0 \iff QS_3; \\ -1 \iff \text{Fig. 20;} \end{cases} \\
0 \text{ and } \inf I(S) = \begin{cases} -1 \iff QS_4; \\ 0 \iff QS_5; \end{cases} \\
-1 \text{ and } \inf I(S) = \begin{cases} -1 \iff QS_6; \\ -2 \iff \text{Fig. 19;} \end{cases}
\end{cases} \\
1 \text{ and } \deg I(S) = \begin{cases}
2 \iff \text{Fig. 38;} \\
1 \iff QS_7; \\
0 \iff QS_8; \\
-1 \iff QS_9.
\end{cases}
\end{cases}
$$

Diagram 7.2:

$$QS_1: \qquad O(S) = \begin{cases} (1,2,1,1,1,2) \iff \text{Fig. 6;} \\ (1,2,2,1,1,1) \iff \text{Fig. 7;} \end{cases}$$

$$QS_2: \qquad O(S) = \begin{cases} (1,3,1,1) \iff \text{Fig. 12;} \\ (1,3,1,2) \iff \text{Fig. 23;} \end{cases}$$

$$QS_3: \qquad O(S) = \begin{cases} (1,2,1,2) \text{ and } \mathcal{D}_S = \begin{pmatrix} 0 \\ i \\ 2 \end{pmatrix} p + \begin{pmatrix} 1 \\ j \\ 1 \end{pmatrix} q,\ (i,j) = \begin{cases} \{(0,0),(0,2),(0,4),(4,0)\} \iff \text{Fig. 8;} \\ (2,0) \iff \text{Fig. 21;} \end{cases} \\ (2,3,1,1) \iff \text{Fig. 15;} \\ (1,2,1,1) \iff \text{Fig. 16;} \\ (1,1,1,1) \iff \text{Fig. 22;} \\ (1,3,1,3) \text{ and } N^t_{sep}(S) = \begin{cases} 4 \iff \text{Fig. 27;} \\ 2 \iff \text{Fig. 28;} \end{cases} \\ (2,2,1,3) \Longrightarrow \text{Fig. 29;} \end{cases}$$

$$QS_4 \qquad O(S) = \begin{cases} (1,3,1,1) \iff \text{Fig. 11;} \\ (2,3,2,1) \iff \text{Fig. 13;} \\ (2,3,2,3) \iff \text{Fig. 14;} \\ (1,3,1,2) \iff \text{Fig. 24;} \end{cases}$$

$$QS_5: \qquad \mathcal{D}_S = \begin{pmatrix} 0 \\ i \\ 2 \end{pmatrix} p + \begin{pmatrix} 0 \\ j \\ 1 \end{pmatrix} q,\ (i,j) = \begin{cases} \{(0,1),(0,3)\} \iff \text{Fig. 10;} \\ (2,1) \iff \text{Fig. 25;} \end{cases}$$

$$QS_6: \qquad O(S) = \begin{cases} (2,2,2,2) \iff \text{Fig. 9;} \\ (2,3,2,3) \iff \text{Fig. 25;} \end{cases}$$

$$QS_7: \qquad O(S) = \begin{cases} (1,1) \iff \text{Fig. 30;} \\ (4,1) \iff \text{Fig. 34;} \\ (4,4) \iff \text{Fig. 39;} \\ (5,2) \iff \text{Fig. 40;} \end{cases}$$

$$QS_8: \qquad O(S) = \begin{cases} (2,1) \iff \text{Fig. 32;} \\ (3,1) \text{ and } N^h_{sep}(S) = \begin{cases} 1 \iff \text{Fig. 33;} \\ 0 \iff \text{Fig. 37;} \end{cases} \\ (4,3) \iff \text{Fig. 36;} \end{cases}$$

$$QS_9: \qquad O(S) = \begin{cases} (2,2) \iff \text{Fig. 31;} \\ (3,3) \iff \text{Fig. 35;} \end{cases}$$

Diagram 7.3:

$$\text{Fig. 1:}\qquad \mathcal{D}_S=\begin{cases}\begin{pmatrix}1\\ j\\ 1\end{pmatrix}p+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}q+\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}r, \quad j=\begin{cases}0 \iff \eta>0,\ \mu_0>0;\\ 2 \iff \begin{cases}\eta>0, \mu_{0,1}=0,\\ \mu_2>0,\ \kappa<0;\end{cases}\\ 4 \iff \begin{cases}\eta>0,\ \mu_{0,1,2,3}=0,\\ \mu_4\neq 0,\ \kappa<0;\end{cases}\end{cases}\\ \begin{pmatrix}-1\\ m\\ 1\end{pmatrix}p+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}q+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}r, \quad m=\begin{cases}2 \iff \begin{cases}\eta>0,\ \mu_{0,1}=0,\\ \mu_2>0,\ \kappa>0;\end{cases}\\ 4 \iff \begin{cases}\eta>0,\ \mu_{0,1,2,3}=0,\\ \mu_4\neq 0,\ \kappa>0;\end{cases}\end{cases}\\ \begin{pmatrix}-1\\ 2\\ 1\end{pmatrix}p+\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}q+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}r \iff \begin{cases}\eta>0,\ \mu_{0,1,2,3}=0,\\ \mu_4\neq 0,\ \kappa=0,\ K_1=0;\end{cases}\end{cases}$$

$$\text{Fig. 2:}\qquad \mathcal{D}_S=\begin{pmatrix}1\\ j\\ 1\end{pmatrix}p+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}q+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}r, \quad j=\begin{cases}0 \iff \eta>0, \mu_0<0, \kappa>0;\\ 2 \iff \begin{cases}\eta>0,\ \mu_{0,1}=0,\\ \mu_2<0,\ \kappa>0;\end{cases}\end{cases}$$

$$\text{Fig. 3:}\qquad \mathcal{D}_S=\begin{pmatrix}-1\\ j\\ 1\end{pmatrix}p+\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}q+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}r, \quad j=\begin{cases}0 \iff \eta>0,\ \mu_0<0,\ \kappa<0;\\ 2 \iff \begin{cases}\eta>0,\ \mu_{0,1}=0,\\ \mu_2<0,\ \kappa<0;\end{cases}\end{cases}$$

$$\text{Fig. 4:}\qquad \mathcal{D}_S=\begin{cases}\begin{pmatrix}0\\ j\\ 1\end{pmatrix}p+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}q+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}r, \quad j=\begin{cases}1 \iff \begin{cases}\eta>0,\ \mu_0=0,\\ \mu_1\neq 0,\ \kappa>0;\end{cases}\\ 3 \iff \begin{cases}\eta>0,\ \mu_{0,1,2}=0,\\ \mu_3\neq 0,\ \kappa>0;\end{cases}\end{cases}\\ \begin{pmatrix}1\\ 2\\ 1\end{pmatrix}p+\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}q+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}r \iff \begin{cases}\eta>0,\ \mu_{0,1,2}=0,\\ \mu_3K_1>0,\ \kappa=0;\end{cases}\end{cases}$$

$$\text{Fig. 5:}\qquad \mathcal{D}_S=\begin{cases}\begin{pmatrix}0\\ j\\ 1\end{pmatrix}p+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}q+\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}r, \quad j=\begin{cases}1 \iff \begin{cases}\eta>0,\ \mu_0=0,\\ \mu_1\neq 0,\ \kappa<0;\end{cases}\\ 3 \iff \begin{cases}\eta>0,\ \mu_{0,1,2}=0,\\ \mu_3\neq 0,\ \kappa<0;\end{cases}\end{cases}\\ \begin{pmatrix}1\\ 2\\ 1\end{pmatrix}p+\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}q+\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}r \iff \begin{cases}\eta>0,\ \mu_{0,1,2}=0,\\ \mu_3K_1<0,\ \kappa=0;\end{cases}\end{cases}$$

$$\text{Fig. 6:}\qquad \mathcal{D}_S=\begin{pmatrix}0\\ j\\ 1\end{pmatrix}p+\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}q+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}r, \quad j=\begin{cases}1 \iff \begin{cases}\eta>0,\ \mu_{0,1}=0,\\ \mu_2L<0,\ \kappa=0;\end{cases}\\ 3 \iff \begin{cases}\eta>0, \mu_{0,1,2,3}=\kappa=0,\\ \mu_4L>0,\ K_1\neq 0;\end{cases}\end{cases}$$

$$\text{Fig. 7:}\qquad \mathcal{D}_S=\begin{pmatrix}0\\ j\\ 1\end{pmatrix}p+\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}q+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}r, \quad j=\begin{cases}1 \iff \begin{cases}\eta>0,\ \mu_{0,1}=0,\\ \mu_2L>0,\ \kappa=0;\end{cases}\\ 3 \iff \begin{cases}\eta>0,\ \mu_{0,1,2,3}=\kappa=0,\\ \mu_4L<0,\ K_1\neq 0;\end{cases}\end{cases}$$

$$\text{Fig. 8:}\quad \mathcal{D}_S=\begin{cases} \begin{pmatrix}0\\0\\2\end{pmatrix}p+\begin{pmatrix}1\\j\\1\end{pmatrix}q, \quad j=\begin{cases} 0 \iff \eta=0,\ M\neq 0,\ \mu_0>0;\\ 2 \iff \begin{cases}\eta=0,\ M\neq 0,\ \kappa\neq 0,\\ \mu_{0,1}=0,\ \mu_2>0;\end{cases}\\ 4 \iff \begin{cases}\eta=0,\ M\neq 0,\ \kappa\neq 0,\\ \mu_{0,1,2,3}=0,\ \mu_4\neq 0;\end{cases}\end{cases}\\ \begin{pmatrix}0\\4\\2\end{pmatrix}p+\begin{pmatrix}1\\0\\1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ L\neq 0,\ \kappa=\kappa_1=0\\ \mu_{0,1,2,3}=0,\ \mu_4\neq 0,\ (K=0,\ L_2=0)\\ \vee(K\neq 0,\ L>0,\ L_1\geq 0)\\ \vee(K=0,\ L_2>0,\ \mu_4>0);\end{cases}\end{cases}$$

$$\text{Fig. 9:}\quad \mathcal{D}_S=\begin{pmatrix}0\\0\\2\end{pmatrix}p+\begin{pmatrix}-1\\j\\1\end{pmatrix}q, \quad j=\begin{cases} 0 \iff \eta=0,\ M\neq 0,\ \mu_0<0;\\ 2 \iff \begin{cases}\eta=0,\ M\neq 0,\ \kappa\neq 0,\\ \mu_{0,1}=0,\ \mu_2<0;\end{cases}\end{cases}$$

$$\text{Fig. 10:}\quad \mathcal{D}_S=\begin{pmatrix}0\\0\\2\end{pmatrix}p+\begin{pmatrix}0\\j\\1\end{pmatrix}q, \quad j=\begin{cases} 1 \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_0=0,\\ \mu_1\neq 0,\ \kappa\neq 0;\end{cases}\\ 3 \iff \begin{cases}\eta=0,\ M\neq 0,\ \kappa\neq 0,\\ \mu_{0,1,2}=0,\ \mu_3\neq 0;\end{cases}\end{cases}$$

$$\text{Fig. 11:}\quad \mathcal{D}_S=\begin{pmatrix}-1\\j\\2\end{pmatrix}p+\begin{pmatrix}1\\0\\1\end{pmatrix}q, \quad j=\begin{cases} 1 \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_0=\kappa=0,\\ \mu_1\neq 0,\ L>0,\ K>0;\end{cases}\\ 3 \iff \begin{cases}\eta=0,\ M\neq 0,\ \kappa_1 L\neq 0,\\ \mu_{0,1,2}=\kappa=0,\ \mu_3 K_1>0;\end{cases}\end{cases}$$

$$\text{Fig. 12:}\quad \mathcal{D}_S=\begin{cases} \begin{pmatrix}1\\j\\2\end{pmatrix}p+\begin{pmatrix}1\\0\\1\end{pmatrix}q, \quad j=\begin{cases} 1 \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_0=\kappa=0,\\ \mu_1\neq 0,\ L>0,\ K<0;\end{cases}\\ 3 \iff \begin{cases}\eta=0,\ M\neq 0,\ \kappa_1 L\neq 0,\\ \mu_{0,1,2}=\kappa=0,\ \mu_3 K_1<0;\end{cases}\end{cases}\\ \begin{pmatrix}1\\1\\2\end{pmatrix}p+\begin{pmatrix}1\\2\\1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1,2}=0,\\ \kappa=L=0,\ \kappa_1\neq 0,\ \mu_3 K_1<0;\end{cases}\end{cases}$$

$$\text{Fig. 13:}\quad \mathcal{D}_S=\begin{pmatrix}1\\1\\2\end{pmatrix}p+\begin{pmatrix}-1\\j\\1\end{pmatrix}q, \quad j=\begin{cases} 0 \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_0=\kappa=0,\\ \mu_1\neq 0,\ L<0,\ N\leq 0;\end{cases}\\ 2 \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1,2}=0,\\ \kappa=L=0,\ \kappa_1\neq 0,\ \mu_3 K_1>0;\end{cases}\end{cases}$$

$$\text{Fig. 14:}\quad \mathcal{D}_S=\begin{pmatrix}1\\j\\2\end{pmatrix}p+\begin{pmatrix}-1\\0\\1\end{pmatrix}q, \quad j=\begin{cases} 1 \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_0=\kappa=0,\\ \mu_1\neq 0,\ L<0,\ N>0;\end{cases}\\ 3 \iff \begin{cases}\eta=0,\ M\neq 0,\ \kappa=\kappa_1=0,\\ \mu_{0,1,2}=0,\ \mu_3\neq 0,\ L<0;\end{cases}\end{cases}$$

$$\text{Fig. 15:}\quad \mathcal{D}_S=\begin{pmatrix}1\\1\\2\end{pmatrix}p+\begin{pmatrix}0\\j\\1\end{pmatrix}q, \quad j=\begin{cases} 1 \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1}=0,\\ \mu_2\neq 0,\ \kappa=L=0;\end{cases}\\ 3 \iff \begin{cases}\eta=0,\ M\neq 0,\ \kappa=L=0,\\ \mu_{0,1,2,3}=0,\ \mu_4\kappa_1\neq 0;\end{cases}\end{cases}$$

Diagram 7.3: (continued)

$$\text{Fig. 16:}\quad \mathcal{D}_S=\begin{pmatrix}0\\ j\\ 2\end{pmatrix}p+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}q,\quad j=\begin{cases}2 \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1}=0,\\ \mu_2\kappa_1\neq 0,\ \kappa=0,\ L\neq 0;\end{cases}\\ 4 \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1,2,3}=0,\\ \mu_4\kappa_1\neq 0,\ \kappa=0,\ L\neq 0;\end{cases}\end{cases}$$

$$\text{Fig. 17:}\quad \mathcal{D}_S=\begin{pmatrix}0\\ 2\\ 2\end{pmatrix}p+\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1}=0,\ \mu_2<0,\\ \kappa=\kappa_1=0,\ L<0,\ L_2>0;\end{cases}$$

$$\text{Fig. 18:}\quad \mathcal{D}_S=\begin{pmatrix}2\\ 2\\ 2\end{pmatrix}p+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1}=0,\ \mu_2<0,\\ \kappa=\kappa_1=0,\ L>0,\ L_2>0,\ K<0;\end{cases}$$

$$\text{Fig. 19:}\quad \mathcal{D}_S=\begin{pmatrix}-2\\ 2\\ 2\end{pmatrix}p+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1}=0,\ \mu_2<0,\\ \kappa=\kappa_1=0,\ L>0,\ L_2>0,\ K>0;\end{cases}$$

$$\text{Fig. 20:}\quad \mathcal{D}_S=\begin{cases}\begin{pmatrix}2\\ j\\ 2\end{pmatrix}p+\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}q,\quad j=\begin{cases}2 \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1}=0,\\ \mu_2\neq 0, \kappa=\kappa_1=0, L<0,\\ (L_2\leq 0)\vee(\mu_2>0, L_2>0);\end{cases}\\ 4 \iff \begin{cases}\eta=0,\ M\neq 0,\ \kappa=\kappa_1=0,\\ \mu_{0,1,2,3}=0, \mu_4K\neq 0, L<0;\end{cases}\end{cases}\\ \begin{pmatrix}2\\ 2\\ 2\end{pmatrix}p+\begin{pmatrix}-1\\ 2\\ 1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1,2,3}=0,\ \mu_4\neq 0,\\ \kappa=\kappa_1=L=K_1=0,\ \kappa_2>0;\end{cases}\end{cases}$$

$$\text{Fig. 21:}\quad \mathcal{D}_S=\begin{pmatrix}0\\ 2\\ 2\end{pmatrix}p+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1}=\kappa=0,\\ \mu_2\neq 0,\ \kappa_1=0,\ L>0,\\ (L_2=0)\vee(\mu_2>0,\ L_2>0);\end{cases}$$

$$\text{Fig. 22:}\quad \mathcal{D}_S=\begin{cases}\begin{pmatrix}0\\ j\\ 2\end{pmatrix}p+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}q,\quad j=\begin{cases}2 \iff \begin{cases}\eta=0,\ M\mu_2\neq 0,\ \mu_{0,1}=0,\\ \kappa=\kappa_1=0,\ L>0,\ L_2<0;\end{cases}\\ 4 \iff \begin{cases}\eta=0, M\neq 0, \kappa=\kappa_1=K=0,\\ \mu_{0,1,2,3}=0,\ \mu_4L\neq 0,\ L_2<0;\end{cases}\end{cases}\\ \begin{pmatrix}0\\ 2\\ 2\end{pmatrix}p+\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1,2,3}=0,\ \mu_4\neq 0,\\ \kappa=\kappa_1=L=K_1=0,\ \kappa_2<0;\end{cases}\end{cases}$$

$$\text{Fig. 23:}\quad \mathcal{D}_S=\begin{pmatrix}1\\ 3\\ 2\end{pmatrix}p+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \kappa=\kappa_1=0,\\ \mu_{0,1,2}=0,\ \mu_3\neq 0,\ L>0,\ K<0;\end{cases}$$

$$\text{Fig. 24:}\quad \mathcal{D}_S=\begin{pmatrix}-1\\ 3\\ 2\end{pmatrix}p+\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \kappa=\kappa_1=0,\\ \mu_{0,1,2}=0,\ \mu_3\neq 0,\ L>0,\ K>0;\end{cases}$$

Diagram 7.3: (continued)

$$\text{Fig. 25:}\quad \mathcal{D}_S=\begin{pmatrix}0\\2\\2\end{pmatrix}p+\begin{pmatrix}0\\1\\1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1,2}=\kappa=0,\\ \kappa_1=L=0,\ \mu_3K_1>0;\end{cases}$$

$$\text{Fig. 26:}\quad \mathcal{D}_S=\begin{pmatrix}0\\2\\2\end{pmatrix}p+\begin{pmatrix}0\\1\\1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1,2}=\kappa=0,\\ \kappa_1=L=0,\ \mu_3K_1<0;\end{cases}$$

$$\text{Fig. 27:}\quad \mathcal{D}_S=\begin{pmatrix}0\\4\\2\end{pmatrix}p+\begin{pmatrix}1\\0\\1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1,2,3}=\kappa=0,\\ \kappa_1=0,\ \mu_4K\neq 0,\ L>0,\ L_1<0;\end{cases}$$

$$\text{Fig. 28:}\quad \mathcal{D}_S=\begin{pmatrix}0\\4\\2\end{pmatrix}p+\begin{pmatrix}1\\0\\1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1,2,3}=\kappa=0,\\ \kappa_1=K=0,\ L\neq 0,\ L_2>0,\ \mu_4<0;\end{cases}$$

$$\text{Fig. 29:}\quad \mathcal{D}_S=\begin{pmatrix}1\\3\\2\end{pmatrix}p+\begin{pmatrix}0\\1\\1\end{pmatrix}q \iff \begin{cases}\eta=0,\ M\neq 0,\ \mu_{0,1,2,3}=0,\\ \kappa=\kappa_1=L=0,\ \mu_4K_1\neq 0;\end{cases}$$

$$\text{Fig. 30:}\quad \mathcal{D}_S=\begin{cases}\begin{pmatrix}1\\j\\1\end{pmatrix}p+\begin{pmatrix}0\\0\\1\end{pmatrix}q^c+\begin{pmatrix}0\\0\\1\end{pmatrix}r^c,\quad j=\begin{cases}0 \iff \eta<0,\ \mu_0>0;\\ 2 \iff \begin{cases}\eta<0,\ \mu_{0,1}=0,\\ \mu_2>0,\ \kappa\neq 0;\end{cases}\\ 4 \iff \begin{cases}\eta<0,\ \mu_4\kappa\neq 0,\\ \mu_{0,1,2,3}=0;\end{cases}\end{cases}\\ \begin{pmatrix}1\\0\\1\end{pmatrix}p+\begin{pmatrix}0\\j\\1\end{pmatrix}q^c+\begin{pmatrix}0\\j\\1\end{pmatrix}r^c,\quad j=\begin{cases}1 \iff \begin{cases}\eta<0,\ \mu_{0,1}=0,\\ \mu_2\neq 0,\ \kappa=0;\end{cases}\\ 2 \iff \begin{cases}\eta<0,\ \mu_4\neq 0,\\ \mu_{0,1,2,3}=\kappa=0;\end{cases}\end{cases}\\ \begin{pmatrix}1\\j\\3\end{pmatrix}p,\quad j=\begin{cases}0 \iff M=0,\ C_2\neq 0,\ \mu_0>0;\\ 2 \iff \begin{cases}M=0,\ C_2\neq 0,\ \mu_{0,1}=0,\ \mu_2\neq 0,\\ (K\neq 0,\ L_2<0)\vee(K=0);\end{cases}\\ 4 \iff \begin{cases}M=0,\ C_2\neq 0,\ \mu_{0,1,2,3}=0,\ \mu_4\neq 0,\\ (K\neq 0,\ L_3>0)\vee(K=K_1=L_3=0)\\ \vee(K=0,\ K_1\neq 0,\ \mu_4>0);\end{cases}\end{cases}\end{cases}$$

$$\text{Fig. 31:}\quad \mathcal{D}_S=\begin{cases}\begin{pmatrix}-1\\j\\1\end{pmatrix}p\ +\begin{pmatrix}0\\0\\1\end{pmatrix}q^c+\begin{pmatrix}0\\0\\1\end{pmatrix}r^c,\quad j=\begin{cases}0 \iff \eta<0,\ \mu_0<0;\\ 2 \iff \begin{cases}\eta<0,\ \mu_{0,1}=0,\\ \mu_2<0,\ \kappa\neq 0;\end{cases}\end{cases}\\ \begin{pmatrix}-1\\0\\3\end{pmatrix}p \iff M=0,\ C_2\neq 0,\ \mu_0<0;\end{cases}$$

Diagram 7.3: (continued)

$$\text{Fig. 32:}\quad \mathcal{D}_S = \begin{cases} \begin{pmatrix}0\\ j\\ 1\end{pmatrix} p + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix} q^c + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix} r^c, \quad j = \begin{cases} 1 \iff \eta < 0, \mu_0 = 0, \mu_1 \neq 0; \\ 3 \iff \begin{cases} \eta < 0,\ \mu_3 \neq 0, \\ \mu_{0,1,2} = 0; \end{cases} \end{cases} \\ \begin{pmatrix}0\\ 3\\ 3\end{pmatrix} p \iff \begin{cases} M = 0,\ C_2 \neq 0,\ \mu_{0,1,2} = 0,\ \mu_3 \neq 0, \\ K = 0,\ (\mu_3 K_1 > 0,\ L_3 > 0) \vee (L_3 = 0); \end{cases} \end{cases}$$

$$\text{Fig. 33:}\quad \mathcal{D}_S = \begin{pmatrix}0\\ j\\ 3\end{pmatrix} p, \quad j = \begin{cases} 1 \iff M = 0,\ C_2 \neq 0,\ \mu_0 = 0,\ \mu_1 \neq 0; \\ 3 \iff \begin{cases} M = 0,\ C_2 \neq 0,\ \mu_{0,1,2} = 0, \\ \mu_3 \neq 0,\ K \neq 0,\ L_3 > 0; \end{cases} \end{cases}$$

$$\text{Fig. 34:}\quad \mathcal{D}_S = \begin{pmatrix}1\\ j\\ 3\end{pmatrix} p, \quad j = \begin{cases} 2 \iff \begin{cases} M = 0,\ C_2 \neq 0,\ \mu_{0,1} = 0,\ \mu_2 K \neq 0, \\ (\mu_2 > 0,\ L_2 > 0) \vee (L_2 = 0); \end{cases} \\ 4 \iff \begin{cases} M = 0,\ C_2 \neq 0,\ \mu_{0,1,2,3} = 0, \\ \mu_4 \neq 0,\ K = K_1 = 0,\ L_3 \neq 0; \end{cases} \end{cases}$$

$$\text{Fig. 35:}\quad \mathcal{D}_S = \begin{pmatrix}-1\\ 2\\ 3\end{pmatrix} p \iff \begin{cases} M = 0,\ C_2 \neq 0,\ \mu_{0,1} = 0, \\ \mu_2 < 0,\ K \neq 0,\ L_2 > 0; \end{cases}$$

$$\text{Fig. 36:}\quad \mathcal{D}_S = \begin{pmatrix}0\\ 3\\ 3\end{pmatrix} p \iff \begin{cases} M = 0,\ C_2 \neq 0,\ \mu_{0,1,2} = 0, \\ \mu_3 \neq 0, K \neq 0,\ L_3 < 0; \end{cases}$$

$$\text{Fig. 37:}\quad \mathcal{D}_S = \begin{pmatrix}0\\ 3\\ 3\end{pmatrix} p \iff \begin{cases} M = 0,\ C_2 \neq 0,\ \mu_{0,1,2} = 0, \\ \mu_3 \neq 0, K = 0,\ L_3 < 0; \end{cases}$$

$$\text{Fig. 38:}\quad \mathcal{D}_S = \begin{pmatrix}2\\ 3\\ 3\end{pmatrix} p \iff \begin{cases} M = 0,\ C_2 \neq 0,\ \mu_{0,1,2} = 0, \\ K = 0,\ L_3 > 0,\ \mu_3 K_1 < 0; \end{cases}$$

$$\text{Fig. 39:}\quad \mathcal{D}_S = \begin{pmatrix}1\\ 4\\ 3\end{pmatrix} p \iff \begin{cases} M = 0,\ C_2 \neq 0,\ \mu_{0,1,2,3} = 0, \\ \mu_4 \neq 0,\ K \neq 0,\ L_3 < 0; \end{cases}$$

$$\text{Fig. 40:}\quad \mathcal{D}_S = \begin{pmatrix}1\\ 4\\ 3\end{pmatrix} p \iff \begin{cases} M = 0,\ C_2 \neq 0,\ \mu_{0,1,2,3} = 0, \\ \mu_4 < 0,\ K = 0,\ K_1 \neq 0; \end{cases}$$

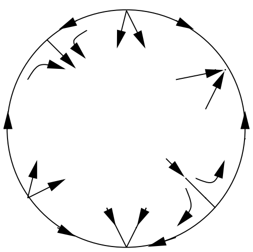


FIG. 1

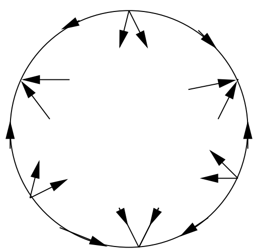


FIG. 2

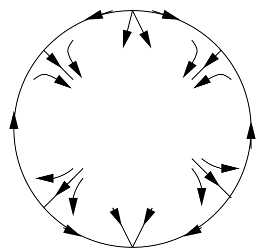


FIG 3

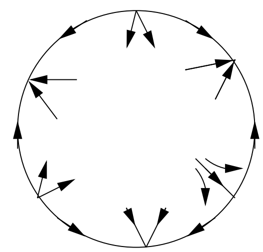


FIG. 4

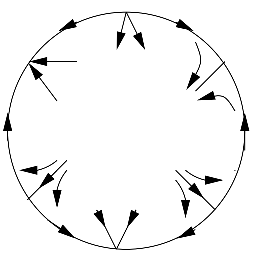


FIG. 5

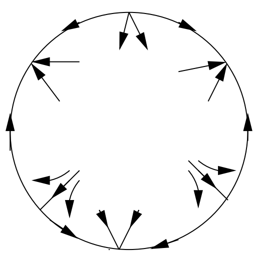


FIG. 6

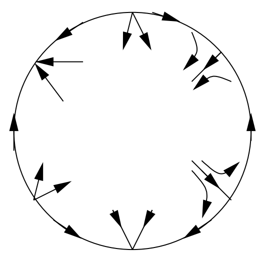


FIG. 7

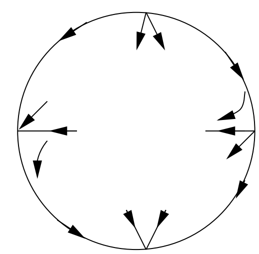


FIG. 8

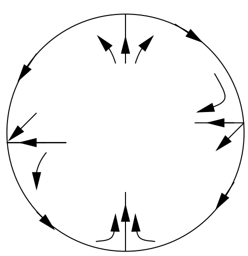


FIG. 9

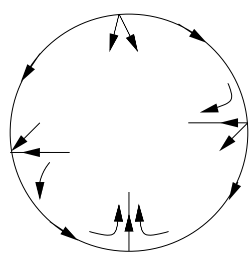


FIG. 10

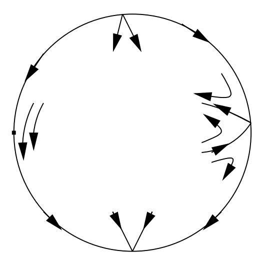


FIG. 11

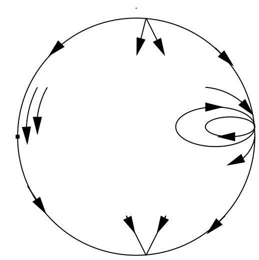


FIG. 12

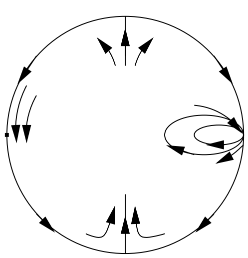


FIG. 13

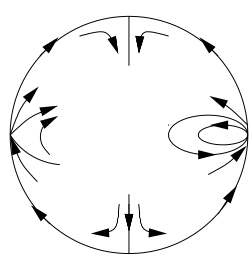


FIG. 14

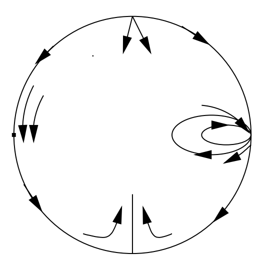


FIG. 15

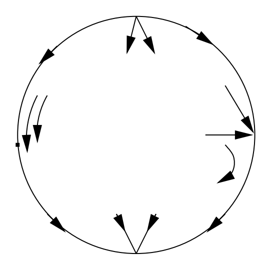


FIG. 16

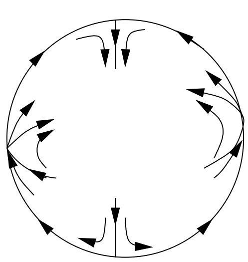
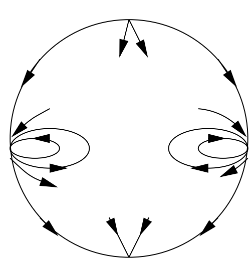


FIG. 17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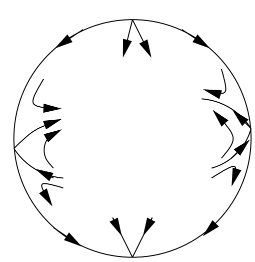
FIG. 19

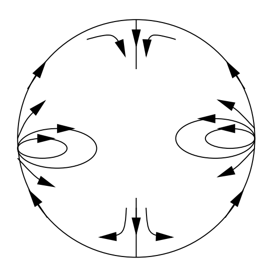
FIG. 20

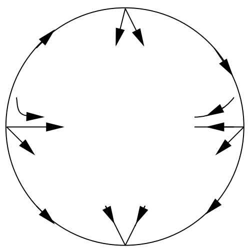

FIG. 21

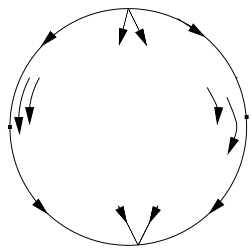

FIG. 22

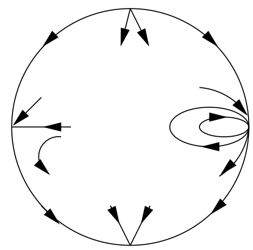

FIG. 23

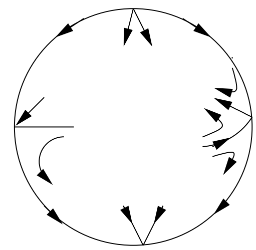

FIG. 24

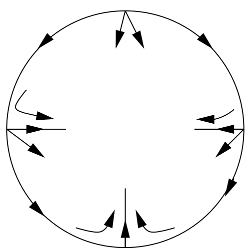

FIG. 25

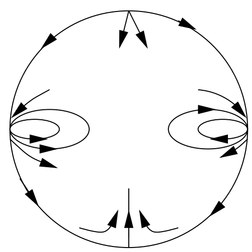

FIG. 26

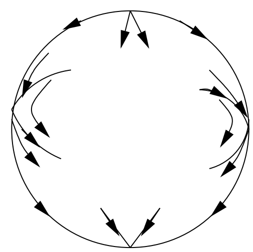

FIG. 27

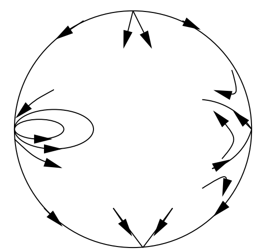

FIG. 28

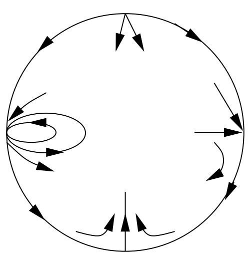

FIG. 29

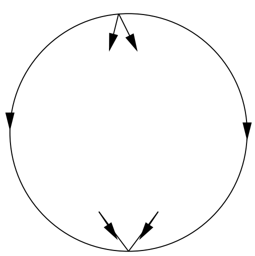

FIG. 30

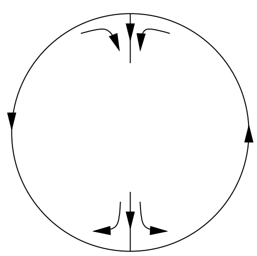

FIG. 31

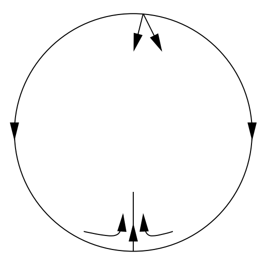

FIG. 32

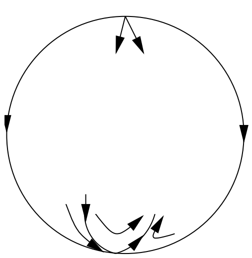

FIG. 33

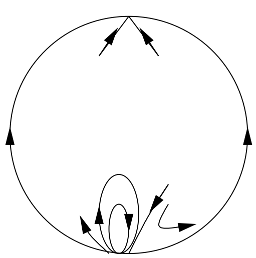

FIG. 34

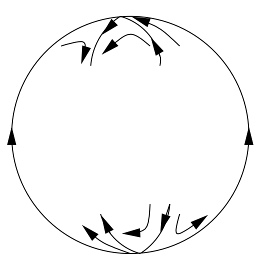

FIG. 35

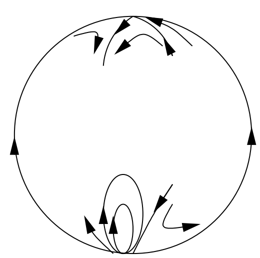

FIG. 36

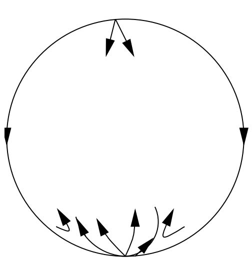

FIG. 37

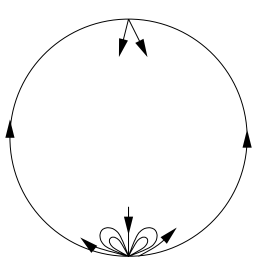

FIG. 38

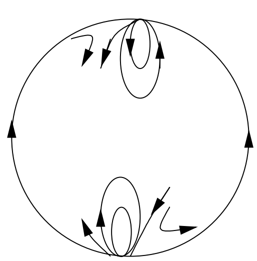

FIG. 39

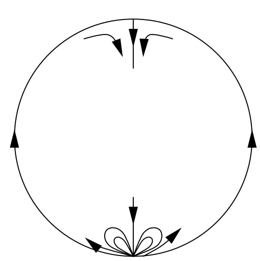

FIG. 40

Proof of the Theorem 7.1. The proof is based on the Theorem 5.1 as well as on the invariant classification of quadratic system at infinity given in [14]. By using some combinations of invariants and comitants given in tensorial form (see Appendix), necessary and sufficient conditions for the existence of eachone of phase portrait of quadratic systems at infinity where established in [14].

It remains to prove the equivalence of the conditions considered in the statement of the Theorem 7.1 with the conditions given in [14] .

In order to do this let us examine the correspondence between the set of the comitants constructed in this article and the set of comitants considered in [14]. Since some letters appear both here and there but with not always the same meaning, we shall use the convention to make here with "tilde" those letters which appear in [14] without "tilde".

First of all we note that the following relations hold for all quadratic differential systems (3.1):

$$\begin{array}{ll} \mu_0(a) = \tilde{\mu}(a); & C_2(a,x,y) = \tilde{L}(a,x,y); \\ \mu_1(a,x,y) = 2\tilde{H}(a,x,y); & M(a,x,y) = \tilde{M}(a,x,y); \\ \mu_2(a,x,y) = \tilde{G}(a,x,y); & \eta(a) = \tilde{\eta}(a); \\ \mu_3(a,x,y) = \tilde{F}(a,x,y); & \kappa(a) = 2\tilde{\theta}(a); \\ \mu_4(a,x,y) = \tilde{V}(a,x,y); & K(a,x,y) = \tilde{N}(a,x,y); \\ L(a,x,y) = \tilde{A}(a,x,y); & K_1(a,x,y) = \tilde{S}_1(a,x,y). \end{array} \tag{7.1}$$

It remains to compare the conditions for the existence of each one of the Figures 1–40 given by Theorem 7.1 with the corresponding conditions given in [14].

7.1 Systems of types (S_1) and (S_2)

For $\eta > 0$ (respectively $\eta < 0$) the phase portraits given by Fig. 1–7 (respectively, Fig. 30–32) are distinguished by the comitants

$$\mu_0(a),\ \mu_i(a,x,y)\ (i=\overline{1,4}),\quad \kappa(a),\quad K_1(a,x,y),\quad L(a,x,y), \tag{7.2}$$

in the statement of Theorem 7.1 and by the comitants

$$\tilde{\mu}(a),\ \tilde{H}(a,x,y),\ \tilde{G}(a,x,y),\ \tilde{F}(a,x,y),\ \tilde{V}(a,x,y),\ \tilde{\theta}(a),\ \tilde{S}_1(a,x,y),\ \tilde{A}(a,x,y) \tag{7.3}$$

in [14]. Thus, due to (7.1) we can conclude, that in both these cases the corresponding conditions given in [14] are equivalent to the conditions in Theorem 7.1, respectively. The following correspondence between the notation used in [14] for a configuration of three singular points and the value of the divisor $\mathcal{D}_S$ considered here holds:

$$R^{i_1}_{j_1k_1}R^{i_2}_{j_2k_2}R^{i_3}_{j_3k_3} \iff \mathcal{D}_S = \begin{pmatrix} i_1 \\ j_1 \\ k_1 \end{pmatrix} p + \begin{pmatrix} i_2 \\ j_2 \\ k_2 \end{pmatrix} q + \begin{pmatrix} i_3 \\ j_3 \\ k_3 \end{pmatrix} r.$$

7.2 Systems of type (S_3)

In this case (i.e. $\eta = 0,\ M \neq 0$) the topological portraits in the vicinity of infinity of quadratic systems is given by one of the Figures 8–29 and we can easily observe from Diagram 7.3 that these classes are distinguished by the comitants (7.2) and

$$K(a,x,y),\quad N(a,x,y),\quad \kappa_1(a),\quad L_1(a,x,y),\quad L_2(a,x,y),\quad \kappa_2(a). \tag{7.4}$$

On the other hand in [14] the same Figures are distinguished by the comitants (7.3) and

$$\tilde{N}(a,x,y),\ \tilde{A}(a,x,y)+\tilde{N}(a,x,y),\ \tilde{\sigma}(a),\ \tilde{A}(a,x,y)+4\tilde{N}(a,x,y),\ \tilde{S}_2(a,x,y),\ \tilde{S}_4(a). \tag{7.5}$$

Taking into consideration the relations (7.1) as well as the relation $L_1 = \tilde{A}+4\tilde{N}$ we have to examine only the conditions given by the comitants N, κ_1, L_2 and κ_2. One can observe, that all comitants (7.4) (respectively, (7.5)) are used for systems (S_3) only in the case when $\kappa = 0$ (respectively, $\tilde{\theta} = 0$). In this case for systems (S_3) the condition $\kappa = -16h^2 = 0$ yields $h = 0$ and we obtain the systems

$$\dot{x} = k + cx + dy + 2gx^2, \qquad \dot{y} = l + ex + fy + 2(g-1)xy, \tag{7.6}$$

for which$L = 4gx^2$ and

$$\begin{array}{ll} \kappa_1 = -8d, & \tilde{\sigma} = -d(5g^2-2g+1), \\ N = (g-1)(g+1)x^2, & \tilde{A}+\tilde{N} = 2g(g+1)x^2. \end{array}$$

As we can observe, the condition $\kappa_1 = 0$ is equivalent to $\tilde{\sigma} = 0$. At the same time it can be observed from Diagram 7.3 that the comitant $N(a,x,y)$ is applied to distinguish phase portraits of system (7.6) only when condition the $L < 0$ is satisfied. Thus, for $L < 0$ we obtain $\operatorname{sgn}(N) = \operatorname{sgn}(\tilde{A}+\tilde{N})$.

Continuing the proof, let us observe that from Diagram 7.3 it follows that the comitant L_2 is applied in the case of systems (S_3) with $\kappa = 0$ (i.e. for systems (7.6)), but with additional conditions: $L \neq 0$ (i.e. $g \neq 0$) and $\kappa_1 = 0$ (i.e. $d = 0$). Therefore for systems (7.6) with $d = 0$ we have

$$L_2 = 24(g^2-g+2)(c^2-8gk)x^2, \qquad \tilde{S}_2 = 4g^2(c^2-8gk)x^2.$$

Hence, L_2 has a well determined sign and since $\forall g\ g^2-g+2 > 0$ from $L \neq 0$ we obtain $\operatorname{sgn}(L_2) = \operatorname{sgn}(\tilde{S}_2)$.

We note thatthe invariant $\kappa_2(a)$ is here used only to distinguish Fig. 20 and Fig. 22 in the case when systems (S_3) belong to the class Σ_{33} in Table 7.1. Since for this class the conditions $\kappa = L = K_1 = 0$ hold for systems (S_3) we obtain respectively $h = g = c^2+d^2 = 0$. So, the systems (S_3) become

$$\dot{x} = k, \qquad \dot{y} = l + ex + fy - 2xy,$$

for which $\kappa_2 = -2k, \quad \tilde{S}_4 = -4k$ and, hence, $\operatorname{sgn}(\kappa_2) = \operatorname{sgn}(\tilde{S}_4)$.

7.3 Systems of type (S_4)

To characterize the Figures 30–40 which can occur for this class of systems according to Diagram 7.3, the comitants

$$(7.2),\quad K(a,x,y),\quad L_2(a,x,y),\quad L_3(a,x,y)$$

were used. In paper [14] the comitants

$$(7.3),\ \tilde{N}(a,x,y),\quad \tilde{S}_2(a,x,y),\quad \tilde{S}_3(a,x,y).$$

were applied for the same purpose.

Taking into consideration the relations (7.1) we have to examine only the conditions given by the comitants L_2 and L_3 (respectively $\tilde{S}_2$ and $\tilde{S}_3$).

We observe that the comitant L_2 is used to distinguish Figures 30, 34 and 35 in the case when system (S_4) belongs to the class Σ_{17} in Table 7.1. Moreover, the condition $K(a,x,y) \neq 0$ is satisfied for this class. As for systems (S_4) we have $\mu_0 = -8h^3$ then the condition $\mu_0 = 0$ yields $h = 0$ and the systems (S_4) become

$$\dot{x} = k + cx + dy + 2gx^2, \quad y` = l + ex + fy - x^2 + 2gxy, \tag{7.7}$$

for which

$$K = 2g^2x^2, \qquad \mu_1 = 8dg^3x.$$

So, from $K \neq 0$ the condition $\mu_1 = 0$ implies $d = 0$ and for systems (7.7) in this case we have:

$$L_2 = 24g^2(c^2 - 8gk)x^2, \qquad \tilde{S}_2 = 4g^2(c^2 - 8gk)x^2.$$

Thus, in the case under consideration the comitant L_2 has a well determined sign and $\operatorname{sgn}(L_2) = \operatorname{sgn}(\tilde{S}_2)$.

At the same time the comitant L_3 is applied for systems (S_4) only in the cases when $\Delta_S \geq 3$, i.e. $\mu_{0,1,2} = 0$. So, we shall consider the systems (7.7) for which $\mu_0 = 0$ and let us examine two sub-cases: $K \neq 0$ and $K = 0$.

7.3.1 Sub-case $K \neq 0$.

Then $g \neq 0$ and for the systems (7.7) the condition $\mu_1 = 0$ gives $d = 0$. Furthermore, by using a shift transformation one may bring the coefficients e, f to satisfy $e = f = 0$. So,the considered system will be brought to the system:

$$\dot{x} = k + cx + 2gx^2, \qquad \dot{y} = l - x^2 + 2gxy, \tag{7.8}$$

for which $\mu_2 = 8g^3kx^2$ and by $g \neq 0$ the condition $\mu_2 = 0$ yields $k = 0$. Then for the system (7.8) we obtain $L_3 = -12g^2lx^6$, $\tilde{S}_3 = -12g^2lx^6$, and, hence L_3 has a well determined sign and $L_3 = \tilde{S}_3$.

7.3.2 Sub-case $K = 0$.

In this case we have $g = 0$ and forthe systems (7.7) we obtain $\mu_1 = 0$, $\mu_2 = d^2x^2$. Thus, the condition $\mu_2 = 0$ yields $d = 0$ and we obtain the following system

$$\dot{x} = k + cx, \qquad \dot{y} = l + ex + fy - x^2,$$

for which $L_3 = 3f(2c - f)x^6$, $\tilde{S}_3 = 3f(2c - f)x^6$ and hence we again obtain that L_3 has a well determined sign and $L_3 = \tilde{S}_3$.

7.4 Affine invariance of the conditions

We draw the attention to the fact that all the constructedinvariant polynomials which were applied in Theorems 3.1 and 5.1 are GL-invariant and GL-comitants.

In fact we are interested in the action of the affine group $Aff(2, \mathbb{R})$ on these systems.

Remark 7.2 *In what follows we shall prove the affine invariance of the conditions given by Theorems* 5.1 *an* 7.1 . *For this it is sufficient to verify the following two facts:*

- *all polynomial invariants used in the statements of these theorems are T-comitants or CT-comitants modulo the respective set of comitants corresponding to each one of the considered cases;*
- *conditions of the type $R(a,x,y) > 0$ (or < 0) are given only by comitants of even weight and even degree in the coeficients of the systems (3.1), and these comiatants have well determined signs on the respective algebraic susbset corresponding to each one of the considered cases.*

Remark 7.3 *The polynomials $\eta(a)$, $\kappa(a)$ and $\mu_0(a)$ are affine invariants:*

$$\eta(r_\tau \cdot a) = \eta(a); \quad \kappa(r_\tau \cdot a) = \kappa(a); \quad \mu_0(r_\tau \cdot a) = \mu_0(a),$$

because these polynomials are GL-invariants which depend only on coefficients of $P_2(x,y)$ and $Q_2(x,y)$.

Remark 7.4 *The polynomials $K(a,x,y)$, $L(a,x,y)$, $L_1(a,x,y)$, $M(a,x,y)$ and $N(a,x,y)$ are T-comitants, because these GL-comitants were constructed only by using the polynomials $P_2(x,y)$ and $Q_2(x,y)$.*

Lemma 7.1 *For every $i = \overline{1,4}$ the GL-comitant $\mu_i(a,x,y) = \mathcal{L}^{(i)}(\mu_0(a))$ is a CT-comitant modulo $\langle \mu_0, \mu_1, , ..., \mu_{i-1} \rangle$, i.e. $\mu_i(a,x,y)$ becomes T-comitant if the conditions $\mu_j = 0$, $j = \overline{0, i-1}$ are satisfied.*

Proof. It is sufficient to verify that whenever $\xi = \tilde{x}\beta - \tilde{y}\alpha$, the following relations occur:

$$\begin{aligned}
\mu_1(r_\tau \cdot a,\ \tilde{x}, \tilde{y}) &= \mu_1(a, \tilde{x}, \tilde{y}) + 4\xi\mu_0(a); \\
\mu_2(r_\tau \cdot a,\ \tilde{x}, \tilde{y}) &= \mu_2(a, \tilde{x}, \tilde{y}) + 3\xi\mu_1(a,\ \tilde{x}, \tilde{y}) + 6\xi^2\mu_0(a); \\
\mu_3(r_\tau \cdot a, \tilde{x}, \tilde{y}) &= \mu_3(a, \tilde{x}, \tilde{y}) + 2\xi\mu_2(a,\ \tilde{x}, \tilde{y}) + 3\xi^2\mu_1(a, \tilde{x}, \tilde{y}) + 4\xi^3\mu_0(a); \\
\mu_4(r_\tau \cdot a,\ \tilde{x}, \tilde{y}) &= \mu_4(a, \tilde{x}, \tilde{y}) + \xi\mu_3(a, \tilde{x}, \tilde{y}) + \xi^2\mu_2(a, \tilde{x}, \tilde{y}) + \xi^3\mu_1(a, \tilde{x}, \tilde{y}) + \xi^4\mu_0(a).
\end{aligned}$$

Lemma 7.2 *The following statements are valid:*

(i) $K_1(a,x,y)$ *is a CT-comitant modulo $\langle K \rangle$;*

(ii) $\kappa_1(a)$ *is a CT-comitant modulo $\langle \eta, \kappa \rangle$;*

(iii) for $M \neq 0$ in $\mathbb{R}[x,y]$ the GL-comitant $L_2(a,x,y)$ is a CT-comitant modulo $\langle \eta, \kappa, \kappa_1 \rangle$ and the GL-invariant $\kappa_2(a)$ is a CT-comitant modulo $\langle \eta, \kappa, L, K_1 \rangle$;

(iv) $L_2(a,x,y)$ *is a CT-comitant modulo $\langle M, \mu_0, \mu_1 \rangle$;*

(v) $L_3(a,x,y)$ *is a CT-comitant modulo $\langle M, \mu_0, \mu_1, \mu_2 \rangle$;*

Proof. *(i)*. Whenever $\xi = \tilde{x}\beta - \tilde{y}\alpha$ the following relation holds:

$$K_1(r_\tau \cdot a,\ \tilde{x}, \tilde{y}) = K_1(a, \tilde{x}, \tilde{y}) - \xi K(a, \tilde{x}, \tilde{y}).$$

(ii). Let us assume that the condition $\eta(a) = 0$ holds and let us examine two cases: *a*) $M(a,\ x, y) \neq 0$ and *b*) $M(a,\ x, y) = 0$.

a) For $M(a,\, x, y) \neq 0$ and $\eta(a) = 0$ by applying a transformation $g \in GL(2, \mathbb{R})$ the systems (3.1) can be brought to the systems (S_3), and then, using the shift transformation $\tau \in T(2, \mathbb{R})$ indicated above: $\tau:\ x = \tilde{x} + \alpha, y = \tilde{y} + \beta$, these systems become:

$$\begin{aligned} \dot{\tilde{x}} = \;& (k + c\alpha + d\beta + 2g\alpha^2 + 2h\alpha\beta) + (c + 4g\alpha + 2h\beta)\tilde{x} \\ & (d + 2h\alpha)\tilde{y} + 2g\tilde{x}^2 + 2h\tilde{x}\tilde{y}, \\ \dot{\tilde{y}} = \;& (b + e\alpha + f\beta + 2(g-1)\alpha\beta + 2h\beta^2) + [e + 2(g-1)\beta]\tilde{x} \\ & +[f + 2(g-1)\alpha + 4h\beta]\tilde{y} + 2(g-1)\tilde{x}\tilde{y} + 2h\tilde{y}^2. \end{aligned} \tag{7.9}$$

For the systems (7.9) we have

$$\eta = 0,\ \kappa = -16h^2,\ \kappa_1 = -8d - 16h\alpha. \tag{7.10}$$

Therefore the condition $\kappa = 0$ yields $h = 0$ and hence the invariant κ_1 becomes independent of the shift transformation τ, i.e. on the algebraic set $V(\eta,\, \kappa)$ we have: $\kappa_1(r_\tau \cdot a) - \kappa_1(a) = 0$.

b) In the case $M(a,\, x, y) = 0$ by applying a linear transformation the systems (3.1) can be brought to the systems (S_4), and then, using the above indicated shift transformation τ we obtain the systems:

$$\begin{aligned} \dot{\tilde{x}} = \;& (k + c\alpha + d\beta + 2g\alpha^2 + 2h\alpha\beta) + (c + 4g\alpha + 2h\beta)\tilde{x} \\ & +(d + 2h\alpha)\tilde{y} + 2g\tilde{x}^2 + 2h\tilde{x}\tilde{y}, \\ \dot{\tilde{y}} = \;& (l + e\alpha + f\beta - \alpha^2 + 2g\alpha\beta + 2h\beta^2) + (e - 2\alpha + 2g\beta)\tilde{x} \\ & +(f + 2g\alpha + 4h\beta)\tilde{y} - \tilde{x}^2 + 2g\tilde{x}\tilde{y} + 2h\tilde{y}^2. \end{aligned} \tag{7.11}$$

For these systems one can calculate: $\eta = \kappa = 0$ and $\kappa_1 = 0$ and we obtain again that κ_1 does not depend of the shift transformation on the set $V(\eta, \kappa)$. Thus, we conclude that the GL-invariant κ_1 is a CT-comitant modulo $\langle \eta, \kappa \rangle$.

(iii) Let us assume that the conditions $M(a, x, y) \neq 0$, $\eta = \kappa = 0$ hold. Then in the same manner as above the systems (S_3) will be brought to the systems (7.9) for which $\eta = 0$.

Let us firstly consider the G-comitant $L_2(a, x, y)$. According to (7.10) the condition $\kappa = 0$ yields $h = 0$ and then by $\kappa_1 = 0$ we have $d = 0$. Hereby for systems (7.9) we obtain $L_2 = 24(g^2 - g + 2)(c^2 - 8kg)x^2$. This tells us that L_2 is a CT-comitant modulo $\langle \eta, \kappa, \kappa_1 \rangle$. Furthemore we can see that L_2 has a well determined sign on the considered algebraic subset $V(\eta, \kappa, \kappa_1)$.

We shall consider now the GL-invariant κ_2. As mentioned above, for $\eta = \kappa = 0$ the systems (S_3) will be brought to the systems (7.9) for which $h = 0$. Then for these systems we have $L = 4g\tilde{x}^2$. So, the condition $L = 0$ yields $g = 0$ and hence for systems (7.9) with $h = g = 0$ we have:

$$K = 0, \qquad K_1 = -2\tilde{x}\tilde{y}(c\tilde{x} + d\tilde{y}), \qquad \kappa_2 = -2k - 2(c\alpha + d\beta).$$

Since according to the statement of this lemma (see (i)) the GL-comitant K_1 is a CT-comitant modulo $\langle K \rangle$ it follows at once, that for $K_1 = 0$ in $R[\tilde{x}, \tilde{y}]$ the GL-invariant κ_2 becomes an affine invariant. Hence κ_2 is indeed a CT-comitant modulo $\langle \eta, \kappa, L, K_1 \rangle$ for $M(a, x, y) \neq 0$.

(iv) Let us assume that the condition $M(a, x, y) = 0$ holds in $\mathbb{R}[x, y]$ and then we need to consider the systems (S_4). Then, using the indicated above shift transformation we obtain the systems (7.11) for which $M = 0$ and $\mu_0 = -8h^3$. The condition $\mu_0 = 0$ yields $h = 0$ and for these systems we obtain

$$\mu_1 = 8dg^3\tilde{x}, \quad L_2 = 24g^2(c^2 - 8kg)\tilde{x}^2 + 24dg^2\tilde{x}\left[(e - 2\alpha - 6g\beta)\tilde{x} + (c + f + 6g\alpha)\tilde{y}\right].$$

Hence for $\mu_1 = 0$ in $\mathbb{R}[\tilde{x}, \tilde{y}]$, the GL-comitant $L_2(a, x, y)$ becomes independent of the shift transformation τ, i.e. it is a CT-comitant modulo $\langle M, \mu_0, \mu_1 \rangle$. Furthemore, as we can see, L_2 has a well determined sign on the algebraic set $V(M, \mu_0, \mu_1)$.

(v) It remains to prove that the GL-comitant $L_3(a,x,y)$ is a CT-comitant modulo $\langle M,\mu_0,\mu_1,\mu_2\rangle$. Taking into consideration the proof given above (see (iv)) we have to examine the values of the comitants μ_2 and L_3 for the systems (7.11) for which the conditions $\mu_0=\mu_1=0$ hold. Hence we obtain $h=dg=0$ and we shall consider two sub-cases: $d=0$ and $g=0$.

Let us firstly assume that $d=0$. In this case for the systems (7.11), due to the conditions $h=d=0$ we obtain

$$\mu_2=4g^2(2kg+f^2-cf)\tilde{x}^2,\quad L_3=3(2cf-4lg^2-4kg-2efg-f^2)\tilde{x}^6+6g(2kg+f^2-cf)\tilde{x}^5\tilde{y}.$$

If $g=0$ then for systems (7.11), because $h=g=0$ we have

$$\mu_2=d^2\tilde{x}^2,\quad L_3=3(2cf+f^2-4de)\tilde{x}^6+3d(8\alpha\tilde{x}^2+2c\tilde{x}\tilde{y}+d\tilde{y}^2)\tilde{x}^4.$$

So, in both cases the coefficients of the GL-comitant $L_3(a,x,y)$ are independent of the shift transformation τ if the condition $\mu_2=0$ holds. Furthemore, in both cases L_3 has a well determined sign on the algebraic subset $V(M,\mu_0,\mu_1,\mu_2)$.

This has completed the proof of the Lemma 7.2. ■

From the proof of Lemma 7.2 it follows at once

Lemma 7.3 *The following assertions are valid:*

- *the CT-comitant $L_2(a,x,y)$ has a well determined sign on the algebraic set $V(M,\mu_0,\mu_1)$ and on the set $V(\eta,\kappa,\kappa_1)$ when $M(a,x,y)\neq 0$;*

- *the CT-comitant $L_3(a,x,y)$ has a well determined sign on the algebraic set $V(M,\mu_0,\mu_1,\mu_2)$.*

Lemma 7.4 *The T-comitants $K(a,x,y)$, $L(a,x,y)$, $L_1(a,x,y)$ and $N(a,x,y)$ have well determined signs on $V(\eta,\mu_0,\kappa)$.*

Proof. It is sufficient to observe, that all the indicated T-comitants are quadratic forms and that the following relations hold:

$$\begin{aligned} Discrim_\gamma(K)&=\mu_0, & Discrim_\gamma(L)&=\eta-4\kappa,\\ Discrim_\gamma(L_1)&=\eta-8\mu_0-4\kappa, & Discrim_\gamma(N)&=-\frac{1}{2}(\mu_0+\kappa)\end{aligned}$$

where $\gamma=x/y$ or $\gamma=y/x$. ■

Lemma 7.5 *For $\eta>0$ the following assertions are valid:*

(i) the CT-comitant $\mu_2(a,x,y)$ has a well determined sign on the algebraic set $V(\mu_0,\mu_1)$ when $\kappa\neq 0$;

(ii) the CT-comitant $\mu_2(a,x,y)L(a,x,y)$ has a well determined sign on the algebraic set $V(\mu_0,\kappa)$;

(iii) the CT-comitant $\mu_3(a,x,y)K_1(a,x,y)$ has a well determined sign on the algebraic set $V(\mu_0,\mu_2,\kappa)$;

(iv) the CT-comitant $\mu_4(a,x,y)L(a,x,y)$ has a well determined sign on the algebraic set $V(\mu_0,\mu_2,\mu_3,\kappa)$ when $K_1\neq 0$.

Proof. As $\eta > 0$ we shall consider the canonical systems (S_1). It was shown in the proof of the Theorem 5.1 (see Section 5, page 13) that by applying an affine transformation, the systems (S_1) under condition $\mu_0 = 0$ can be brought to the systems (5.4) for which

$$\mu_1 = 8ch(1-h)y,\ \kappa = 16h(1-h),\ \mu_2 = 4h[\,(2l(h-1)^2 + 2(h-1) + c^2h + cd\,]y^2 - 4c(ch+d)xy.$$

If $\kappa \neq 0$ then the condition $\mu_1 = 0$ yields $c = 0$ and we obtain that in the considered case the comitant μ_2 has a well determined sign.

Let assume now that the condition $\kappa = 0$ holds.

It was shown in the proof of the Theorem 5.1 (see Section 5, page 13) that for systems (S_1) the conditions $\mu_0 = 0$ and $\kappa = 0$ yield $g = h = 0$. Furthemore, by applying a shift transformation to this case the conditions $e = f = 0$ can be obtained and the systems (S_1) become

$$\dot{x} = k + cx + dy - 2xy, \qquad \dot{y} = l - 2xy, \tag{7.12}$$

for which we have

$$\mu_1 = K = 0,\ \mu_2 = 4cdxy,\ L = 4xy,\ \mu_3 = 4(k-l)(cx-dy)xy,\ K_1 = -2xy(cx+dy).$$

Hence $\mu_2 L = 16cdx^2y^2$ and the assertion of Lemma 7.5 concerning the comitant $\mu_2 L$ is valid.

On the other hand we have $\mu_3 K_1 = 8(k-l)[c^2x^2 - d^2y^2)x^2y^2$ and because $\mu_2 = 0$ (i.e. $cd = 0$) we conclude that $\mu_3 K_1$ has a well determined sign on the algebraic set $V(\mu_0, \mu_2, \kappa)$.

Let us assume now that the conditions $\mu_2 = 0$ and $K_1 \neq 0$ hold. Then $cd = 0$, $c^2 + d^2 \neq 0$ and hence the condition $\mu_3 = 0$ for the systems (7.12) yields $l = k$. Thus for these systems we obtain:

$$\mu_4 = -2c^2kx^3y \quad \text{for} \quad d = 0 \quad \text{or} \quad \mu_4 = -2d^2kxy^3 \quad \text{for} \quad c = 0.$$

We can conclude that under our hypothesis, in both cases the CT-comitant $\mu_4 L$ has a well determined sign. ∎

Lemma 7.6 *For $\eta = 0,\ M \neq 0$ the following assertions are valid:*

(i) the CT-comitant $\mu_2(a, x, y)$ has a well determined sign on the algebraic set $V(\mu_0, \mu_1)$ when $L \neq 0$;

(ii) the CT-comitant $\mu_3(a, x, y)K_1(a, x, y)$ has a well determined sign on the algebraic set $V(\kappa, \mu_1, \mu_2)$ for $\kappa_1 \neq 0$ and on the algebraic set $V(\kappa, \mu_1, \mu_2, \kappa_1, L)$;

(iii) for $L \neq 0$ the CT-comitant $\mu_4(a, x, y)$ has a well determined sign on the algebraic set $V(\kappa, \kappa_1, \mu_2, K)$.

Proof. We shall examine the systems (S_3) and assume that the condition $\kappa = 0$ holds. As for these systems $\kappa = -16h^2$, we obtain $h = 0$ and then we have

$$\mu_0 = 0, \quad \mu_1 = 8dg(g-1)^2x, \quad K = 2g(g-1)x^2, \quad L = 4gx^2, \quad \kappa_1 = -8d. \tag{7.13}$$

7.4.1 $\quad L \neq 0$

In this case $g \neq 0$ and the condition $\mu_1 = 0$ yields $d(g-1) = 0$. Furthemore, because $g \neq 0$, using a shift transformation we can assume $c = 0$.

7.4.1.1 For $\kappa_1 \neq 0$ we obtain $d \neq 0$ and hence $g = 1$. Then for systems (S_3) with $h = c = 0$ and $g = 1$ we have

$$\mu_2 = 4f^2x^2, \quad \mu_3 = 2de^2x^3 + 2f(4lx + dey), \quad K = 0, \quad K_1 = -2(ex + fy)x^2.$$

Thus, μ_2 has a well determined sign and furthermore, for $\mu_2 = 0$ we have $\mu_3 K_1 = -4de^3x^6$, i.e. in this case $\mu_3 K_1$ has also a well determined sign.

7.4.1.2 If $\kappa_1 = 0$ holds (hence, $d = 0$) we obtain $\mu_2 = 4g\,[2k(g-1)^2 - c(g-1) + f^2g]\,x^2$ and therefore the CT-comitant μ_2 has also a well determined sign on $V(\mu_0, \mu_1)$.

As the condition $L \neq 0$ holds, according to the statements of this lemma it remains to examine only the case (iii). Hence we assume $K = 0$ (i.e. $g = 1$) and for systems (S_3) with $c = h = d = g - 1 = 0$ we have:

$$\mu_2 = 4f^2x^2, \quad \mu_3 = 8lfx^3, \quad \mu_4 = 2(e^2k + 2l^2)x^4 + fk(2ex + f^2y)x^2y.$$

Thus, for $\mu_2 = 0$ we have $\mu_3 = 0$, $\mu_4 = 2(e^2k + 2l^2)x^4$ and hence we conclude that in the case under consideration, the $\mu_4(a, x, y)$ has a well determined sign .

7.4.2 $L = 0$

In this case we need to examine only the assertion (ii) of the Lemma 7.6. According to (7.13) condition $L = 0$ yields $g = 0$ and for systems (S_3), by applying the shift transformation $x = \tilde{x} + e/2$, $y = \tilde{y} + f/2$ we obtain the relations $e = f = 0$. Hence, for systems (S_3) with $h = g = e = f = 0$ we have:

$$\mu_0 = \mu_1 = K = 0, \quad \kappa_1 = -8d, \quad \mu_2 = -4cdxy, \quad \mu_3 K_1 = 8k[c^2x^2 - d^2y^2)x^2y^2.$$

Thus in both cases: $\kappa_1 \neq 0$ or $\kappa_1 = 0$, because $\mu_2 = 0$ (i.e. $cd = 0$) we conclude that the $\mu_3 K_1$ has a well determined sign. ∎

Lemma 7.7 *For $M(a, x, y) = 0$ the following assertions are valid:*

(i) the CT-comitant $\mu_2(a, x, y)$ has a well determined sign on the algebraic set $V(\mu_0, \mu_1)$;

(ii) the CT-comitant $\mu_3(a, x, y)K_1(a, x, y)$ has a well determined sign on the algebraic set $V(\mu_0, \mu_1, \mu_2, K)$;

(iii) the CT-comitant $\mu_4(a, x, y)$ has a well determined sign on the algebraic set $V(\mu_0, \mu_1, \mu_2, \mu_3, K)$.

Proof. As $M = 0$ we shall consider the canonical systems (S_4), for which we have $\mu_0 = -8h^3$. Hence, the condition $\mu_0 = 0$ yields $h = 0$ and we obtain

$$\mu_1 = 8dg^3x, \qquad K = 2g^2x^2.$$

Thus the condition $\mu_1 = 0$ yields $dg = 0$ and we obtain either

$$\mu_2 = 4g^2[2gk - cf + f^2]x^2 \quad \text{for} \;\; d = 0 \qquad \text{or} \;\; \mu_2 = d^2x^2 \quad \text{for } g = 0.$$

In both these cases μ_2 has a well determined sign and this proves the validity of the statement (i) of this lemma.

To discuss the statements (ii) and (iii) we shall assume that the condition $K = 0$ holds. Then $g = 0$ and for systems S_4 with $h = g = 0$ we have $\mu_2 = d^2x^2$ and the condition $\mu_2 = 0$ yields $d = 0$. Thus for these systems we obtain

$$\mu_3 = -c^2fx^3, \quad K = 0, \quad K_1 = cx^3, \quad \mu_4 = (k^2 + cek - c^2l)x^4 + cfkx^3y$$

We observe that for $\mu_3 \neq 0$ the CT-comitant $\mu_3 K_1 = -c^3fx^6$ has a well determined sign and for $\mu_3 = 0$ the CT-comitant μ_4 has also a well determined sign. ∎

Lemma 7.8 *For $\eta < 0$ the CT-comitant $\mu_2(a, x, y)$ has a well determined sign on the algebraic set $V(\mu_0, \mu_1)$ when $\kappa \neq 0$.*

Proof. Since $\eta < 0$ we shall consider the systems (S_2), for which we have

$$\mu_0 = -16h[g^2 + (h+1)^2], \quad \kappa = -16\left[g^2 + (h+1)(1-3h)\right].$$

Because $\kappa \neq 0$, the condition $\mu_0 = 0$ yields $h = 0$ and for systems (S_2) we have $\mu_1 = 8(g^2+1)(dg - f)x$. Hence the condition $\mu_1 = 0$ yields $f = dg$ and then we obtain

$$\mu_2 = 4(g^2+1)(d^2g^2 + 2gk - cdg - 2l + d^2 + de)x^2.$$

This completes the proof of this lemma.■

Remark 7.5 *The properties of all constructed invariant polynomials which are used in both the Theorems* 5.1 *and* 7.1 *are indicated in the Table* 7.1*. In the last column are indicated the algebraic sets on which the corresponding GL-comitants are CT-comitants, respectively. According to Remarks* 7.2–7.4 *and Lemmas* 7.1–7.8 *the Table* 7.1 *shows us that all conditions included in the statements of both the Theorems* 5.1 *and* 7.1 *are affine invariant.*

Table 7.1:

GL-comitant	Degree in the coefficients	Degree in the x and y	Weight	Algebraic subset $V(*)$
$\eta(a)$, $\mu_0(a)$, $\kappa(a)$	4	0	2	$V(0)$
$\kappa_1(a)$	3	0	1	$V(\eta, \kappa)$
$\kappa_2(a)$	2	0	0	$V(\eta, \kappa, L, K_1)$
$C_2(a, x, y)$	1	3	-1	$V(0)$
$M(a, x, y)$, $N(a, x, y)$	2	2	0	$V(0)$
$K(a, x, y)$, $L(a, x, y)$	2	2	0	$V(0)$
$L_1(a, x, y)$	2	2	0	$V(0)$
$L_2(a, x, y)$	4	2	0	$V(\eta, \kappa, \kappa_1) \cup V(M, \mu_0, \mu_1)$
$L_3(a, x, y)$	4	6	-2	$V(M, \mu_0, \mu_1, \mu_2)$
$K_1(a, x, y)$	2	3	-1	$V(K)$
$\mu_1(a, x, y)$	4	1	1	$V(\mu_0)$
$\mu_2(a, x, y)$	4	2	0	$V(\mu_0, \mu_1)$
$\mu_3(a, x, y)$	4	3	-1	$V(\mu_0, \mu_1, \mu_2)$
$\mu_4(a, x, y)$	4	4	-2	$V(\mu_0, \mu_1, \mu_2, \mu_3)$

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A Appendix

Let us consider the tensorial form of quadratic system:

$$\frac{dx^j}{dt} = a^j + a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j, \alpha, \beta = 1, 2).$$

The following invariants and comitants, defined by polynomials of J_i, R_i which are tensorially defined GL-comitants, were used in [14] for the classification in the neighbourhood of infinity of quadratic differential systems:

$$\begin{aligned}
2\mu &= J_4, \\
H &= R_{13}, \\
2G &= 2R_1^2 - 2J_2R_3 + 4R_7 + R_8, \\
\sigma &= J_7, \\
2F &= J_2R_5 + 4R_2R_3 + 4R_1R_4, \\
V &= R_4^2 - R_2R_5, \\
2\theta &= J_5, \\
2\eta &= J_4 + 20J_5 - 8J_6, \\
L &= R_{12}, \\
2M &= 9R_3 + 6R_6 - 8R_{11}^2, \\
2A &= 2R_6 - 3R_3, \\
2N &= R_3, \\
S_1 &= R_5, \\
S_2 &= 2J_1^2R_6 + 2J_1R_1^2 - 2J_2R_6 + J_2R_{11}^2 + 8J_3R_3 - 8J_3R_6 - 4R_7 - R_8, \\
S_3 &= R_{12}^2(7J_2 - 6J_1^2 - 8J_3) - R_{12}(10J_1R_5 + 4R_1R_{10} - 6R_3R_9) + 4R_3R_{10}^2 - 4R_5^2, \\
S_4 &= 4J_3 - J_2.
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= a^\alpha_\alpha, \\
J_2 &= a^\alpha_p a^\beta_q \varepsilon_{\alpha\beta}\varepsilon^{pq}, \\
J_3 &= a^\alpha a^\beta_{\alpha\beta}, \\
J_4 &= a^\alpha_{pr} a^\beta_{qk} a^\gamma_{sn} a^\delta_{lm} \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}\varepsilon^{mn}, \\
J_5 &= a^\alpha_{\gamma p} a^\beta_{\delta r} a^\gamma_{qk} a^\delta_{sl} \varepsilon_{\alpha\beta}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_6 &= a^\alpha_{pr} a^\beta_{\alpha q} a^\gamma_{\delta s} a^\delta_{\beta\gamma} \varepsilon^{pq}\varepsilon^{rs}, \\
J_7 &= a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta} \varepsilon^{pq}, \\
R_1 &= x^\alpha a^\beta_q a^\gamma_{p\alpha} \varepsilon_{\beta\gamma}\varepsilon^{pq}, \\
R_2 &= x^\alpha a^\beta a^\gamma_\alpha \varepsilon_{\beta\gamma}, \\
R_3 &= x^\alpha x^\beta a^\gamma_{p\alpha} a^\delta_{q\beta} \varepsilon_{\gamma\delta}\varepsilon^{pq}, \\
R_4 &= x^\alpha x^\beta a^\gamma a^\delta_{\alpha\beta} \varepsilon_{\gamma\delta}, \\
R_5 &= x^\alpha x^\beta x^\gamma a^\delta_\alpha a^\mu_{\beta\gamma} \varepsilon_{\delta\mu},
\end{aligned}$$

$$
\begin{aligned}
R_6 &= x^\alpha x^\beta a^\gamma_{\alpha\beta} a^\delta_{\gamma\delta},\\
R_7 &= x^\alpha x^\beta a^\gamma a^\delta_{\alpha p} a^\mu_{\beta s} a^\nu_{qr} \varepsilon_{\gamma\delta}\varepsilon_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs},\\
R_8 &= x^\alpha x^\beta a^\gamma_\alpha a^\delta_\beta a^\mu_{pr} a^\nu_{qs} \varepsilon_{\gamma\mu}\varepsilon_{\delta\nu}\varepsilon^{pq}\varepsilon^{rs},\\
R_9 &= x^\alpha a^\beta \varepsilon_{\beta\alpha},\\
R_{10} &= x^\alpha x^\beta a^\gamma_\alpha \varepsilon_{\gamma\beta},\\
R_{11} &= x^\alpha a^\beta_{\alpha\beta},\\
R_{12} &= x^\alpha x^\beta x^\gamma a^\delta_{\alpha\beta}\varepsilon_{\delta\gamma},\\
R_{13} &= x^\alpha a^\beta_p a^\gamma_{\alpha r} a^\delta_{qk} a^\mu_{sl} \varepsilon_{\beta\gamma}\varepsilon_{\delta\mu}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl},
\end{aligned}
$$

and

$$
\varepsilon^{11} = \varepsilon^{22} = \varepsilon_{11} = \varepsilon_{22} = 0, \quad \varepsilon^{12} = \varepsilon_{12} = -\varepsilon^{21} = -\varepsilon_{21} = 1,
$$