Lie Point Symmetries and Commuting Flows for Equations on Lattices

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Abstract

Different symmetry formalisms for difference equations on lattices are reviewed and applied to perform symmetry reduction for both linear and nonlinear partial difference equations. Both Lie point symmetries and generalized symmetries are considered and applied to the discrete heat equation and to the integrable discrete time Toda lattice.

Résumé

Deux formalismes différents pour étudier les symétries des équations aux différences finies sur un réseau sont décrits et utilisés pour faire la réduction par symétrie des équations aux différences finies. Les symétries ponctuelles et généralisées sont considérées et appliquées à l'équation de la chaleur linéaire discrète et à un treillis de Toda intégrable en temps discret.

1 Introduction

The purpose of this article is to compare two different approaches to the study of symmetries of difference equations [20, 21, 22, 19, 7, 8, 4, 11, 9, 17, 10, 18, 15, 23, 24, 16, 27, 5, 6, 29, 2, 3, 12, 31, 30, 26, ?, 13, 1, 14]. Both approaches are algebraic, in that they both use an infinitesimal formalism in which certain vector fields realize a Lie algebra: the symmetry algebra. In one approach the vector fields act on the dependent and independent variables and in general on the difference equations and the lattices. In the other approach, use is made of evolutionary vector fields, acting only on the dependent variables. In both approaches, further choices must be made. In particular, the coefficients of the vector fields can depend on the dependent and independent variables at one point of the lattice, on a finite number of points, or on an infinite one. They may also depend on derivatives of the dependent functions, up to some chosen order, or on finite dfferences.

For point symmetries of *differential* equations the two approaches are equivalent and the formalisms of ordinary vector fields and evolutionary ones are related in a simple manner. In particular, the evolutionary formalism provides flows commuting with the original equation. If these flows involve first derivatives only, and they figure linearly, they correspond to point symmetries. The two formalisms provide the same symmetry variables and hence the same symmetry reductions.

For *difference* equations the situation is somewhat different. Also in this case, for point symmetries it is very natural to consider vector fields involving differentiation with respect to both dependent and independent variables. Moreover, it is natural to let the transformations act on the difference equation, and on the lattice itself.

A purely difference equation does not contain derivatives (by definition). Hence the usual evolutionary formalism, with the characteristic Q of the symmetry depending on x, u and derivatives of u, is not a natural tool to use. The natural evolutionary formalism for discrete equations is one in which Q depends on the values of the variables x and u at different points of the lattice. Such evolutionary symmetries are in general not point symmetries and are not related to point symmetries in any simple manner.

One of the most important applications of symmetries of differential equations is to perform symmetry reduction. For partial differential equations this means a reduction of the number of independent variables.

In this article we shall perform symmetry reduction for various partial difference equations and show how different types of symmetries lead to different results.

The general theory is discussed in Section 2, first for differential equations, then for difference ones. Also in Section 2 we introduce the concept of "discrete evolutionary vector fields" and their prolongations. Section 3 is devoted to symmetry reduction for a linear difference equation, namely the discrete heat equation. Translations, as point symmetries, provide a reduced one variable equation that is easily solved. Dilations provide a reduction to a dilation-delay equation. Still in Section 3 we use discrete evolutionary vector fields to obtain solutions of the discrete heat equation, invariant under translations and under dilations. Similarly, in Section 4, we use first point symmetries, then discrete evolutionary vector fields to obtain reductions of an integrable nonlinear partial difference equation: the discrete time Toda lattice.

2 Continuous Symmetries of Difference Schemes

2.1 Equivalence of two symmetry formalisms for differential equations

Let us first recapitulate a well-known result for differential equations, namely that Lie point symmetries (and also generalized symmetries) can be realized in two different ways. The first is by vector fields acting on both independent and dependent variables, the second by evolutionary vector fields, acting on the dependent variable only. This is true for arbitrary systems of differential equations (ordinary or partial, of any order) [28]. For point symmetries the two realizations are entirely equivalent. For generalized symmetries ordinary vector fields can always be replaced by evolutionary ones in a straightforward manner. Let us illustrate this equivalence on the example of a first order ordinary differential equation

$$E \equiv u_x - F(u, x) = 0. \tag{1}$$

Let us assume that

$$\hat{X} = \xi(x, u)\partial_x + \phi(x, u)\partial_u \tag{2}$$

generates a point symmetry, i.e satisfies

$$pr\hat{X}E|_{E=0} = 0. \tag{3}$$

The first prolongation of $\hat{X}$ is

$$pr\hat{X} = \hat{X} + \phi^x \partial_{u_x}, \qquad \phi^x = D_x\phi - (D_x\xi)u_x, \tag{4}$$

where D_x is a total derivative.

The prolonged evolutionary vector field, on the other hand, is

$$pr\hat{X}_e = Q\partial_u + Q^x\partial_{u_x}, \quad Q^x = D_xQ, \tag{5}$$

where Q is the characteristic of the vector field.

The total derivative D_x is itself a generalized symmetry of eq. (1) (and of any equation). Indeed, if we have $E = 0$, then $D_xE = 0$ follows and so does $(\xi D_x)E = 0$. Hence, if $\hat{X}$ satisfies eq. (3) we also have

$$(pr\hat{X} - \xi D_x)E|_{E=0} = 0. \tag{6}$$

In view of eq. (4) we have

$$\begin{aligned} pr\hat{X} - \xi D_x = \quad & \xi\partial_x + \phi\partial_u + [D_x\phi - (D_x\xi)u_x]\partial_{u_x} - \\ & -\xi[\partial_x + u_x\partial_u + u_{xx}\partial_{u_x}] = \\ = \quad & [\phi - \xi u_x]\partial_u + [D_x(\phi - \xi u_x)]\partial_{u_x}. \end{aligned} \tag{7}$$

With

$$Q \equiv \phi - \xi u_x, \tag{8}$$

we obtain eq (5) and we have proved that for first order ODEs condition (3) is equivalent to the condition

$$\hat{X}_e E|_{E=0} = 0, \tag{9}$$

i.e. that the two formalisms are equivalent. The same is true for higher order ODEs and PDEs of any order [28].

Given a vector field (2) we can calculate the corresponding characteristic Q and the evolutionary field $\hat{X}_e$ of eq. (5). Hence flows that commute with the flow given by the considered equation have the form

$$u_\lambda = Q(x, u, u_x, u_{xx}, ...), \tag{10}$$

where λ is a group parameter.

2.2 Point symmetries of difference equations

In two recent articles [17, 18] we presented a method for determining Lie point symmetries of difference systems. In this terminology a "difference system" is a system of relations between a set of points in a $(p+q)$ dimensional space where the p coordinates $x_1, ..., x_p$ represent independent variables and the q coordinates $u_1, ..., u_q$ represent the dependent ones. Let us restrict here to the case of one difference equation for one function u of one variable x. Let us consider a simple case when when three points P, P^+, P^- are involved.

These three points have coordinates (x, u), (x^-, u^-), and (x^+, u^+) , respectively. The difference system consists of two relations

$$E_a(x, x^-, x^+, u, u^-, u^+) = 0, \quad a = 1, 2. \tag{11}$$

This system describes both a difference equation and a lattice. If a continuous limit $x^+ \to x$, $x^- \to x$ exists, then one of these equations goes into a first order differential equation, the other into an identity (like $0 = 0$).

The algorithm for finding the Lie point symmetries of such a system is quite simple [17, 18]. We write a vector field as in the continuous case and prolong it to all points involved in the difference scheme. In the case of system (11) we have

$$\begin{aligned} pr\hat{X} \quad = \quad & \xi(x,u)\partial_x + \phi(x,u)\partial_u + \xi(x^+,u^+)\partial_{x^+} + \\ + \quad & \phi(x^+,u^+)\partial_{u^+} + \xi(x^-,u^-)\partial_{x^-} + \phi(x^-,u^-)\partial_{u^-}. \end{aligned} \tag{12}$$

The algorithm for determining the functions $\xi(x, u)$ and $\phi(x, u)$ is

$$pr\hat{X}E_a|_{E_i=0} = 0, \quad a = 1, 2, \quad i = 1, 2. \tag{13}$$

The algorithm provides functional equations for ξ and ϕ. Solution methods were discussed elsewhere, as were applications [17, 18].

2.3 Commuting flows and evolutionary symmetries for difference equations

Let us again consider a three point ordinary difference system like that of eq. (11), but in a form explicitly solved for u^+ and x^+:

$$E_1 = u^+ - f_1(x, x^-, u, u^-) = 0, \quad E_2 = x^+ - f_2(x, x^-, u, u^-) = 0. \tag{14}$$

A Lie point symmetry (12) can be converted into an evolutionary one in the same way as was done in the continuous case. Indeed we can define an operator $\hat{X}_e$ and its prolongation $pr\hat{X}_e$ by putting

$$\begin{gathered} pr\hat{X}_e \equiv pr\hat{X} - \xi D_x - \xi^+ D_{x^+} - \xi^- D_{x^-} \\ \xi = \xi(x,u), \quad \xi^\pm \equiv \xi(x^\pm, u^\pm), \end{gathered} \tag{15}$$

where D_x, D_{x^+}, and D_{x^-} are total derivatives with respect to x, x^+ and x^- , respectively. This is equivalent to defining an evolutionary symmetry for difference equations in the same way as for differential ones, namely

$$\hat{X}_e = [\phi(x,u) - \xi(x,u)u_x]\partial_u \tag{16}$$

and its prolongation as

$$\begin{aligned} pr\hat{X}_e = (\phi - \xi u_x)\partial_u + (\phi^+ - \xi^+ u^+_{x^+})\partial_{u^+} + \\ +(\phi^- - \xi^- u^-_{x^-})\partial_{u^-}, \end{aligned} \tag{17}$$

where the superscripts + and $-$ correspond to total shifts for the corresponding variables and functions. Using the operator (16) is equivalent to using operator (12) for point symmetries. The operator (16) is not a convenient one for generalizations going beyond point symmetries.

Indeed, for difference equations it is more natural (and more fruitful) to consider evolutionary vector fields with characteristics Q that depend on the independent and dependent variables at different points on the lattice, rather than on derivatives, as in eq. (16) and (17). In this case we have

$$\begin{aligned} \hat{X}_e = & \quad Q(T^k x, T^k u)\partial_u \\ pr\hat{X}_e = & \quad Q(T^k x, T^k u)\partial_u + (TQ)\partial_{u^+} + (T^{-1}Q)\partial_{u^-} \\ & m \le k \le n, \quad m, n \in Z. \end{aligned} \tag{18}$$

Thus Q depends on x and u at a finite, or possibly infinite number of different points. In (18) T is a total shift operator:

$$Tx = x^+, \quad T^{-1}x = x^-, \quad Tu(x) = u(x^+) \equiv u^+, \quad T^{-1}u(x) = u(x^-) \equiv u^- \tag{19}$$

The symmetry algorithm is

$$pr\hat{X}_e E_a|_{E_b=0} = 0, \quad a, b = 1, 2. \tag{20}$$

(together with $E_b = 0$ we must also use all shifted equations like $T^k E_b = 0$).

This is equivalent to requesting that the flow

$$u_\lambda = Q(T^k x, T^k u), \quad x_\lambda = 0 \tag{21}$$

should commute with the flow (14).

To sum up, for difference equations we consider two different algebraic symmetry formalisms. The first uses ordinary vector fields of the form (2), coinciding with those used for differential equations. Their prolongation is different, see eq. (12). The algorithm for determining the coefficients of the vector fields is given in eq. (13). These vector fields can be integrated to provide genuine point transformations taking solutions of difference equations into solutions. The set of Lie point symmetries of a difference system is in general much more restricted than that of a differential one. This is specially true if we consider a difference equation on a fixed lattice, i.e when the lattice equation is just $E_2 = x^+ - x = \sigma$ where σ is a fixed and nontransforming constant.

The second formalism for symmetries of difference equations uses the "discrete" evolutionary vector fields (18) They correspond to generalized symmetries for difference systems. They provide quite general commuting flows. They can be used to perform symmetry reduction. As in the case of differential equations, generalized symmetries are particularly useful for identifying systems that are integrable, that is those for which a discrete Lax pair exists. For linear difference equations these "discrete" evolutionary vector fields provide commuting difference operators [19, 6].

We demonstrate below, in Sections 3 and 4, that both types of symmetries of difference systems are useful for solving difference equations and that they provide different types of results.

3 The Discrete Heat Equation and its Reductions

3.1 Evolutionary and point symmetries for the heat equation

Like the continuous heat equation, the discrete one serves as an excellent example of the application of group theoretical techniques. Here we shall use it to demonstrate the difference between "discrete" evolutionary and point symmetries for linear difference equations.

The discrete heat equation can be written as

$$\Delta_t u - \Delta_{xx} u = 0 \tag{22}$$

The discrete derivatives used in Ref. [19] were defined as

$$\Delta_t = \frac{T_t - 1}{\sigma_t}, \qquad \Delta_{xx} = \frac{T_x^2 - 2T_x + 1}{\sigma_x^2} \tag{23}$$

(a better notation would have been Δ_t^+ and Δ_{xx}^{++}). Here T_t and T_x are shift operators and σ_t and σ_x are the steps in the t and x directions, respectively. The lattice is fixed, uniform and orthogonal. The Lie algebra of "discrete" evolutionary symmetries was found to be six dimensional (after an infinite dimensional subalgebra corresponding to the linear superposition principle was factor out). It is isomorphic to that of the continuous heat equation, and was realized by the "discrete" evolutionary vector fields

$$\begin{aligned}
P_0 &= (\Delta_t u)\partial_u, \quad P_1 = (\Delta_x u)\partial_u, \quad W = u\partial_u, \\
B &= (2tT_t^{-1}\Delta_x u + xT_x^{-1}u + \frac{1}{2}\sigma_x T_x^{-1}u)\partial_u, \\
D &= [2tT_t^{-1}\Delta_t u + xT_x^{-1}\Delta_x u + (1 - \frac{1}{2}T_x^{-1})u]\partial_u, \\
K &= [t^2T_t^{-2}\Delta_t u + txT_t^{-1}T_x^{-1}\Delta_x u + \frac{1}{4}x^2T_x^{-2}u + \\
&\quad + t(T_t^{-2} - \frac{1}{2}T_t^{-1}T_x^{-1})u - \frac{1}{16}\sigma_x^2 T_x^{-2}u]\partial_u.
\end{aligned} \tag{24}$$

We see that the translations P_0 and P_1 involve only discrete derivatives, and the Galilei transformation B, dilation D and expansions K involve explicit shifts to other points of the lattice.

Floreanini et. al. [5, 6] obtained an equivalent result for the heat equation (22) where $\Delta_t \equiv \Delta_t^-$ and $\Delta_{xx} \equiv \Delta_{xx}^{--}$, i.e. they used "down derivatives", with for instance $\Delta_t^- = \frac{1-T_t^{-1}}{\sigma_t}$.

In both cases shifts to a finite number of points are involved. Quite recently it was shown [13], [14] that if symmetric discrete derivatives are used, e.g.

$$\Delta_x^s = \frac{T_x - T_x^{-1}}{2\sigma_x} \tag{25}$$

then the number of points involved in the symmetries, other than translations, will be infinite. Here we will stick with right derivatives, as in (23) and (24).

For Lie point symmetries as used in Ref. [17, 18] the definition of discrete derivative is immaterial. One simply considers relations between points in the lattice. To facilitate the comparison between "discrete" evolutionary and Lie point symmetries, we shall here consider the following heat equation and lattice

$$\frac{u_{m,n+1} - u_{m,n}}{t_{m,n+1} - t_{m,n}} = \frac{u_{m+2,n} - 2u_{m+1,n} + u_{m,n}}{(x_{m+1,n} - x_{m,n})^2} \tag{26}$$

$$\begin{aligned}
x_{m+2,n} - 2x_{m+1,n} + x_{m,n} = 0, &\qquad x_{m,n+1} - x_{m,n} = 0 \\
t_{m+1,n} - t_{m,n} = 0, &\quad t_{m,n+1} - t_{m,n} = c(x_{m+1,n} - x_{m,n})^2,
\end{aligned} \tag{27}$$

where c is a constant. The symmetry algebra coincides with the one obtained earlier [18] using a symmetric derivative on the right hand side of eq.(26). It is spanned by

$$\hat{P}_0 = \partial_t, \quad \hat{P}_1 = \partial_x, \quad \hat{D} = x\partial_x + 2t\partial_t, \quad \hat{W} = u\partial_u, \quad \hat{S} = S(x,t)\partial_u \tag{28}$$

where $S(x,t)$ is a solution of the system (26),(27) and $\hat{S}$ represents the linear superposition principle. The lattice equations (27) can easily be solved and we have

$$x = \sigma_x m + x_0, \qquad t = c\sigma_x^2 n + t_0 \tag{29}$$

where σ_x, x_0 and t_0 are integration constants that are not apriori fixed.

Let us now consider some nontrivial examples.

3.2 Reductions by Lie point symmetries

1. Translationally invariant solutions

A solution of the system (26), (27), invariant under a translation generated by $\hat{P}_0 - a\hat{P}_1$ will have the form

$$u(x,t) = u(z), \qquad z = x + at \tag{30}$$

The lattice equations (27) and the heat equation (26) reduce to

$$z_{m+1,n} - 2z_{m,n} + z_{m-1,n} = 0, \quad z_{m,n+1} - z_{m,n} = ac(z_{m,n} - z_{m-1,n})^2 \tag{31}$$

$$u(z_{m,n+1}) - u(z_{m,n}) = c[u(z_{m+2,n}) - 2u(z_{m+1,n}) + u(z_{m,n})]. \tag{32}$$

The solution of eq.(31) is

$$z_{m,n} = A(m + Aacn) + z_0, \tag{33}$$

so $z_{m,n}$ really depends on just one label $N = m + Aacn$ (A and z_0 are integration constants). If the reduced equation (32) is supposed to be a difference equation on some lattice, the label N must vary over integer values (N simply enumerates different points z_0, $z_{\pm 1}$, $z_{\pm 2}$, ..., independently of their spacing). This implies that the constants A, c and a are constrained by the requirement

$$Aac = k, \qquad k \in Z. \tag{34}$$

Eq.(32) can be written as

$$u(z + acA^2) - u(z) = c[u(z + 2A) - 2u(z + A) + u(z)]. \tag{35}$$

The general solution of eq.(35) can be written in the form

$$u(x,t) = c_1 e^{\alpha z} + c_2, \tag{36}$$

where the constant α is determined by the condition

$$e^{\alpha acA^2} - 1 = c[e^{2\alpha A} - 2e^{\alpha A} + 1]. \tag{37}$$

Eq.(36) also represents a translationally invariant solution of the continuous heat equation for $\alpha = a$. In the continuous limit we have $A = \sigma_x \to 0$ and to order σ_x^2 eq.(37) reduces to $\alpha = a$.

In general eq.(37) is a trascendental equation for α and eq.(35) determines the solution $u(z)$ at a point $z + acA^2$ on the z line in terms of $u(z)$ at three given points. As stated above, this will be a point on the same lattice if we choose $acA^2 = kA$ with k integer. If (34) is satisfied, then eq.(37) is an algebraic one for $v = e^{\alpha A}$:

$$v^k - 1 = c[v^2 - 2v + 1]. \tag{38}$$

In particular, for $k = 1$ eq.(35) is a three point difference equation and we have

$$\alpha A = ln \frac{c+1}{c}. \tag{39}$$

2. Reduction by dilation $\hat{D}$.

A scaling invariant solution will have the form

$$u = u(z), \qquad z = xt^{-\frac{1}{2}}. \tag{40}$$

Eq.(27) imply

$$z_{m+1,n} - 2z_{m,n} + z_{m-1,n} = 0, \quad z_{m,n+1} = \frac{z_{m,n}}{\sqrt{1 + c(z_{m,n} - z_{m-1,n})^2}}. \tag{41}$$

Solving eq.(41), we have

$$z_{m,n} = \frac{m - m_0}{\sqrt{x(n - n_0)}}, \tag{42}$$

where m_0 and n_0 are constants. E.(42) for $z_{m,n} = const.$ determines a parabola in the (m,n)-plane, so $u_{m,n} \equiv u(z_{m,n})$ is constant along each parabola. The discrete heat equation is reduced to the equation (32). Choosing some reference point $z = z_{m,n}$ we obtain an equation that can be written as

$$u(z\frac{\gamma_{n+1}}{\gamma_n}) - u(z) = c[u(z+2\gamma_n) - 2u(z+\gamma_n) + u(z)], \quad \gamma_n = \frac{1}{\sqrt{c(n-n_0)}} \tag{43}$$

Even though we are not able to solve eq.(43) analytically, we see that a reduction has occurred. Eq.(43) involves one independent variable z, rather than two.

For instance we can take $c = 1$, $m_0 = n_0 = 0$ in eq.(42) which then reduces to

$$z_{m,n+1} = z_{m,n}\sqrt{\frac{n}{n+1}}. \tag{44}$$

For any fixed value of m we need to give the values of $u(z)$ in a set of equally spaced points z, $z \pm \gamma_n$, $z \pm 2\gamma_n$, ... Eq.(43)then determines $u(z)$ at irrationally spaced points (44) along the same line.

Thus we have reduced to an equation with one independent variable only, but it is not a difference equation, rather a *difference - delay* one.

3.3 Reduction by discrete evolutionary symmetries

We rewrite eq.(22) as

$$u_{m,n+1} - u_{m,n} = c(u_{m+2,n} - 2u_{m+1,n} + u_{m,n}), \quad c = \frac{\sigma_t}{\sigma_x^2} \tag{45}$$

$$\begin{aligned} x_{m+1,n} - x_{m,n} = \sigma_x, \quad & t_{m+1,n} - t_{m,n} = 0, \\ x_{m,n+1} - x_{m,n} = 0, \quad & t_{m,n+1} - t_{m,n} = \sigma_t \end{aligned} \tag{46}$$

and consider reductions of this system by some of the "discrete" evolutionary symmetries.

1. Translationally invariant solutions

The commuting flow corresponding to a general translation is given by (see P_0 and P_1 in eq.(24)):

$$\frac{du_{m,n}}{d\lambda} = \frac{u_{m,n+1} - u_{m,n}}{\sigma_t} - a\frac{u_{m+1,n} - u_{m,n}}{\sigma_x}, \tag{47}$$

and an invariant solution will satisfy

$$u_{m,n+1} - u_{m,n} = a\frac{\sigma_t}{\sigma_x}(u_{m+1,n} - u_{m,n}). \tag{48}$$

Together with eq.(45) this implies

$$a(u_{m+1,n} - u_{m,n}) = \frac{1}{\sigma_x}(u_{m+2,n} - 2u_{m+1,n} + u_{m,n}). \tag{49}$$

This is a linear three point difference equation in m. Its general solution is

$$u_{m,n} = A(n) + B(n)(1 + a\sigma_x)^m. \tag{50}$$

Substituting back into the heat equation (45) we find

$$A(n+1) = A(n), \qquad B(n+1) = (1 + a^2\sigma_t)B(n). \tag{51}$$

The general solution of the system (45, 48) hence is

$$u_{m,n} = c_1(1 + a^2\sigma_t)^n(1 + a\sigma_x)^m + c_2. \tag{52}$$

Using the lattice conditions (46) we can rewrite this as

$$u(x,t) = c_1(1 + a\sigma_x)^{\frac{x}{\sigma_x}}(1 + a^2\sigma_t)^{\frac{t}{\sigma_t}} + c_2. \tag{53}$$

This is not the same solution (36) obtained using translations as point symmetries. The two translationally invariant solutions only coincide in the continuous limit $\sigma_x \to 0$, $\sigma_t \to 0$.

2. Reduction by dilations

We shall use the operator $D - (1 - \frac{1}{2}T_x^{-1})W$ (see eq.(24)) to perform the reduction. Thus, we solve eq.(45) together with the self-similarity condition

$$2t_{m,n}\frac{u_{m,n} - u_{m,n-1}}{\sigma_t} + x_{m,n}\frac{u_{m,n} - u_{m-1,n}}{\sigma_x} = 0. \tag{54}$$

We solve eq.(54) for $u_{m,n-1}$ and shift eq.(45) down in n, i.e. replace n by $n-1$ everywhere. Substituting for $u_{m+2,n-1}$, $u_{m+1,n-1}$ and $u_{m,n-1}$ we obtain the reduced equation

$$\begin{aligned}2c(n+1)[u_{m+2,n} - 2u_{m+1,n} + u_{m,n}] + m[c(u_{m+2,n} - u_{m+1,n}) - \\ -2c(u_{m+1,n} - u_{m,n}) + (c+1)(u_{m,n} - u_{m-1,n})] = 0.\end{aligned} \tag{55}$$

In eq.(55) m is a variable, n is a parameter. The continuous limit is obtained by putting $x = \sigma_x m$, $t = \sigma_t n$, and taking σ_x and σ_t to zero with $c = \frac{\sigma_t}{\sigma_x^2}$ finite. We obtain

$$2tu_{xx} + xu_x = 0. \tag{56}$$

This is indeed the condition for invariance under dilations generated by $\hat{D}$ of eq. (28). Eq.(56)is easily solved, eq.(55) is more difficult to deal with.

We start by integrating eq.(55) once. The equation involves four values of m. We reduce the number to three by putting

$$v_{m,n} \equiv \frac{u_{m+1,n} - u_{m,n}}{\sigma_x} \tag{57}$$

and obtain

$$2c(n+1)(v_{m+1,n} - v_{m,n}) + m[cv_{m+1,n} - 2cv_{m,n} + (c+1)v_{m-1,n}] = 0 \tag{58}$$

This is a linear ordinary three point difference equation with variable coefficients.

In order to solve it we use a discrete Fourier transform (often also called the Z-tranform), i.e. introduce a generating function

$$G_n(z) = \sum_{m=-\infty}^{\infty} v_{m,n} z^m \tag{59}$$

with

$$v_{m,n} = \frac{1}{2\pi i}\oint_{C_1} \frac{dzG_n(z)}{z^{m+1}}, \tag{60}$$

where the contour C_1 is the unit circle in the complex z plane. Eq.(58) then implies that $G_n(z)$ satisfies

$$2c(n+1)(\frac{1}{z} - 1)G_n + [c(G_{n,z} - \frac{G_n}{z}) - 2czG_{n,z} + (c+1)(z^2G_{n,z} + zG_n)] = 0. \tag{61}$$

Eq.(61) is easily integrated and we obtain

$$G_n(z) = \gamma_n\frac{(z-z_1)^n(z-z_2)^n}{z^{2n+1}}, \tag{62}$$

where γ_n is an integration constant and the complex numbers $z_{1,2}$ are

$$z_{1,2} = \frac{c \pm i\sqrt{c}}{c+1}. \tag{63}$$

Since we have $c > 0$, $z_{1,2}$ lie inside the unit circle and eq.(60) implies

$$v_{n,m} = \frac{1}{2\pi i}\gamma_n\oint_{C_1}\frac{dz(z-z_1)^n(z-z_2)^n}{z^{m+2n+2}}. \tag{64}$$

The dependence on m in eq.(64) is explicit, the dependence of γ_n on n must still be determined. To do this we introduce the notation

$$v_{m,n} = \gamma_n I_{N,n}, \qquad N = m + 2n + 2 \tag{65}$$

and sustitute into eq.(54) (the condition for dilational invariance). We also use $x_{m,n} = \sigma_x m$, $t_{m,n} = \sigma_t n$ and obtain

$$2n(u_{m,n} - u_{m,n-1}) + m(u_{m,n} - u_{m-1,n}) = 0. \tag{66}$$

In order to introduce $v_{m,n}$ into eq.(66) we first take the variation (discrete derivative) with respect to m and then obtain

$$2n(v_{m,n} - v_{m,n-1}) + (m+1)v_{m,n} - mv_{m-1,n} = 0. \tag{67}$$

Substituting the expression (65) for $v_{m,n}$ we obtain an equation for γ_n, namely

$$\gamma_n[(N-1)I_{N,n} - (N-2n-2)I_{N-1,n}] - 2n\gamma_{n-1}I_{N-2,n-1} = 0. \tag{68}$$

Eq.(68)must hold for all values of N and γ_n is independent of N. The quantity $I_{N,n}$ is defined by the integral in eq.(64) and can be evaluated using the residue theorem. For general N and m this is not very illuminating. For low values of N and $n \geq 0$ the results are quite simple, for instance

$$\begin{aligned}
I_{1,n} &= (z_1 z_2)^n = (\tfrac{c}{c+1})^n, \\
I_{0.n} &= 0, \\
I_{-1,n-1} &= 0, \\
I_{2,n} &= -2n(\tfrac{c}{c+1})^n, \\
I_{3,n} &= n(\tfrac{c}{c+1})^{n-1}\tfrac{(2n-1)c-1}{c+1}, \\
I_{4,n} &= -\tfrac{2}{3}n(n-1)(\tfrac{c}{c+1})^{n-1}\tfrac{(2n-1)c+3}{c+1}.
\end{aligned}$$

Substituting these values into eq.(68) we obtain

$$\gamma_n = (c+1)^n \gamma_0, \tag{69}$$

which can be shown to be valid for all integer values of n, both positive and negative. Finally, the m variation $v_{m,n}$ of the dilationally invariant solution of the discrete heat equation (see eq.(57)) satisfies

$$v_{m,n} = \gamma_0 (c+1)^n \frac{1}{2\pi i} \oint_{C_1} \frac{dz(z-z_1)^n(z-z_2)^n}{z^N}, \tag{70}$$

where C_1 is the unit circle and $z_{1,2}$ are defined in eq.(63).

The continuous limit of this self similar solution is

$$v(x,t) = \frac{d}{dx}u(x,t) = \frac{\gamma_0}{\sqrt{t}} e^{\frac{-x^2}{4t}} \tag{71}$$

which satisfies the continuous limit of eq.(54), namely $2tu_t + xu_x = 0$, which implies

$$2tv_x + xv = 0. \tag{72}$$

4 Symmetry Reduction for the Discrete Toda lattice

A recent article was devoted to a hierarchy of nonlinear integrable difference equations associated with a discrete Schrödinger spectral problem [12]. The hierarchy includes the discrete Toda and discrete Volterra lattice equations. All equations in the hierarchy involve two independent variables: discrete space and discrete time.

In this section we shall use Lie point symmetries and "discrete" evolutionary symmetries to perform symmetry reduction for the simplest equation in the hierarchy, namely the Discrete Time Toda Lattice (DTTL) itself:

$$e^{u_{n,m} - u_{n,m+1}} - e^{u_{n,m+1} - u_{n,m+2}} = \alpha^2 (e^{u_{n-1,m+2} - u_{n,m+1}} - e^{u_{n,m+1} - u_{n+1,m}}), \tag{73}$$

(α is a constant).

4.1 Reduction by Lie point symmetries

We complement the DTTL (73) by the lattice equations:

$$\begin{aligned} x_{n+1,m} - 2x_{n,m} + x_{n-1,m} &= 0, & x_{n,m} &= x_{n,m+1}, \\ t_{n,m+1} - 2t_{n,m} + t_{n,m-1} &= 0, & t_{n,m} &= t_{n+1,m}. \end{aligned} \tag{74}$$

The solution of eq.(74) is

$$x_{n,m} = \sigma_x n, \qquad t_{n,m} = \sigma_t m, \tag{75}$$

where the integration constants σ_x and σ_t represent the lattice spacings and we have set the two further "initial value" constants $x_{0,m}$ and $t_{n,0}$ equal to zero.

The Lie algebra of the Lie point symmetry group of the difference system (73) and (74) is spanned by

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad D_0 = t\partial_t, \quad D_1 = x\partial_x, \quad W = \partial_u. \tag{76}$$

Let us look separately at reductions by translations and by dilations.

A. Translationally invariant solutions.

A general translationally invariant solution has the form

$$\begin{aligned} u_{n,m} \equiv u(x_{n,m}, t_{n,m}) &= u(\xi_{n,m}), \\ \xi_{n,m} = x_{n,m} + a t_{n,m} &= \sigma_x n + a\sigma_t m. \end{aligned} \tag{77}$$

Eq.(73) reduces to an equation for $u(\xi)$ evaluated at five points ξ, namely

$$\begin{aligned} \xi_{n,m} \equiv \quad & \xi, \quad \xi_{n,m+1} &= \xi + a\sigma_t, \quad \xi_{n,m+2} = \xi + 2a\sigma_t, \\ \xi_{n+1,m} = \quad & \xi + \sigma_x, \quad \xi_{n-1,m+2} &= \xi - \sigma_x + 2a\sigma_t. \end{aligned} \tag{78}$$

As in the case of the heat equation (see eq. (34,35)) the reduced equation will be an ordinary difference equation on a lattice only if $\xi_{n,m}$ is a function of one discrete label. This occurs if we put

$$a\sigma_t = k\sigma_x, \qquad k \in Z \tag{79}$$

in eq.(77). In particular, if we choose $k = 1$ in eq.(79) we obtain a three point difference equation (since $\xi_{n,m+1} = \xi_{n+1,m} = \xi_{n-1,m+2}$). The right hand side of eq.(73) vanishes identically and the left hand side implies

$$u_{n,m} - 2u_{n,m+1} + u_{n,m+2} = 0. \tag{80}$$

The general solution of eq.(80) is

$$u_{n,m} = f(n)m + g(n). \tag{81}$$

Substituting back into eq.(73) we obtain the general translationally invariant solution of eq.(73) as

$$u_{n,m} = An(n+m) + Bm + Cn + D, \tag{82}$$

where A, B, C and D are constants.

B. Reduction by dilations

A solution invariant under dilations generated by $D_0 - \beta D_1$ will have the form

$$u(x_{n,m}, t_{n,m}) = u(\xi_{n,m}) \quad \xi_{n,m} = x_{n,m} t^{\beta}_{n,m} = \sigma_x \sigma_t^{\beta} n m^{\beta} \equiv \xi. \tag{83}$$

Substituting into the DTTL (73) we obtain a nonlinear equation involving $u(\xi)$ evaluated at 5 points:

$$e^{u(\xi)-u(\xi_1)} - e^{u(\xi_1)-u(\xi_2)} = \alpha^2 (e^{u(\xi_3)-u(\xi_1)} - e^{u(\xi_1)-u(\xi_4)}), \tag{84}$$

with ξ as in (83) and

$$\begin{aligned} \xi_1 = \xi(\frac{m+1}{m})^{\beta}, \quad \xi_2 &= \xi(\frac{m+2}{m})^{\beta}, \\ \xi_3 = \xi(\frac{m+2}{m})^{\beta} - \sigma_x\sigma_t^{\beta}(m+2)^{\beta}, \xi_4 &= \xi + \sigma_x \sigma_t^{\beta} m^{\beta} \end{aligned} \tag{85}$$

Eq. (84) is a *dilation - delay* equation.

4.2 Discrete Evolutionary Symmetries for the DTTL

The DTTL (73) can be written as a system of two equations [12] as

$$a_{n,m+1} - a_{n,m} = \alpha(b_{n,m+1} - b_{n+1,m})\frac{\pi_{n,m+1}}{\pi_{n+1,m}}, \tag{86}$$
$$b_{n,m+1} - b_{n,m} = \alpha(\frac{\pi_{n-1,m+1}}{\pi_{n,m}} - \frac{\pi_{n,m+1}}{\pi_{n+1,m}}). \tag{87}$$

where we have

$$\pi_{n,m} = a_{n,m}\pi_{n+1,m}, \qquad \pi_{n,m} = e^{u_{n,m}}, \tag{88}$$
$$\pi_{n,m} = \Pi_{j=n}^{\infty} a_{j,m}.$$

As in the case of the Toda lattice itself, both isospectral and nonisospectral "discrete" evolutionary symmetries exist [9, 15, 12] and we shall consider both of them.

The simplest nontrivial isospectral (generalized) symmetry is given by

$$\begin{aligned}(a_{n,m})_\epsilon &= a_{n,m}[a_{n-1,m} - a_{n+1,m} + b^2_{n,m} - b^2_{n+1,m}] \\ (b_{n,m})_\epsilon &= a_{n-1,m}[b_{n,m} + b_{n-1,m}] - a_{n,m}[b_{n+1,m} + b_{n,m}].\end{aligned} \tag{89}$$

We assume $a_{n,m} \neq 0$. Symmetry reduction amounts to solving eq.(89) for $(a_{n,m})_\epsilon = (b_{n,m})_\epsilon = 0$. Eq.(89) can be once integrated to yield

$$\begin{aligned}a_{n-1,m} + a_{n,m} + b^2_{n,m} &= A_m \\ a_{n,m}(b_{n+1,m} + b_{n,m}) &= B_m.\end{aligned} \tag{90}$$

We can eliminate $b_{n,m}$ from (90) to obtain

$$a_{n,m}(\sqrt{A_m - a_{n,m} - a_{n+1,m}} + \sqrt{A_m - a_{n-1,m} - a_{n,m}}) = B_m. \tag{91}$$

Eq.(91) can be viewed as a discrete analog of the equation for elliptic functions. It is an integrable difference equation in one variable n with m as a fixed paramether. The functions A_m and B_m must be determined by putting the solution of eq.(90) into (86).

The simplest nonisospectral symmetry is given by

$$(a_{n,m})_\epsilon = a_{n,m}[(2n + 2m + 3)b_{n+1,m} - (2n + 2m - 1)b_{n,m}] \tag{92}$$
$$(b_{n,m})_\epsilon = b^2_{n,m} - 4 + 2[(n + m + 1)a_{n,m} - (n + m - 1)a_{n-1,m}], \tag{93}$$

(see eq. (64) in Ref. [12] with $k = 0$, $\alpha = 1$).

Again, we must solve the equations $(a_{n,m})_\epsilon = (b_{n,m})_\epsilon = 0$. We eliminate $b_{n,m}$ from these two equations and obtain

$$\begin{aligned}&(2n + 2m + 3)^2(n + m + 2)a_{n+1,m} + \\ -[(n + m)(2n + 2m + 3)^2 &+ (n + m + 1)(2n + 2m - 1)^2]a_{n,m} + \\ &+(n + m - 1)(2n + 2m - 1)^2 a_{n-1,m} = 16(2n + 2m + 1).\end{aligned} \tag{94}$$

One solution of the corresponding homogeneous linear ordinary difference equation (in which n is the independent variable, m a parameter) can be guessed, namely

$$a^1_{n,m} = \frac{1}{(n + m)(n + m + 1)}. \tag{95}$$

The general solution of the inhomogeneous equation is constructed in the form

$$a_{n,m} = a^1_{n,m}\beta_{n,m}. \tag{96}$$

We define

$$c_{n,m} = \beta_{n+1,m} - \beta_{n,m}. \tag{97}$$

Substituting (96) into (94) we obtain a first order equation for $c_{n,m}$, namely

$$c_{n+1,m} = \frac{(n + m + 2)(2n + 2m + 1)^2}{(n + m + 1)(2n + 2m + 5)^2}c_{n,m} + 16\frac{(n + m + 2)(2n + 2m + 3)}{(2n + 2m + 5)^2}. \tag{98}$$

A particular solution of the homogeneous part of eq.(98) is

$$c_{n,m} = 2(n+m+1) \tag{99}$$

which implies $\beta_{n,m} = n(n+m+1)$. Finally, the general solution of eq.(94) is

$$a_{n,m} = \frac{1}{(n+m)(n+m+1)}\{A(m) + \frac{B(m)}{(2n+2m+1)^2} + n(n+2m+1)\} \tag{100}$$

where $A(m)$ and $B(m)$ are integration constants. Returning to (93) for $(b_{n,m})_\epsilon = 0$ we find

$$b_{n,m} = \frac{4B(m)}{(2n+2m-1)(2n+2m+1)}. \tag{101}$$

Together, eq.(100) and (101) represent the general solution of the invariance condition $(a_{n,m})_\epsilon = (b_{n,m})_\epsilon = 0$ in eq.(92, 93).

The functions $A(m)$ and $B(m)$ must be determined by substituting $a_{n,m}$ and $b_{n,m}$ into the DTTL of eq. (86) (for $\alpha = 1$). This can be done; the result is:

$$B(m) = 0, \qquad A(m) = m(m+1) \tag{102}$$

and finally

$$a_{n,m} = 1, \qquad b_{n,m} = 0. \tag{103}$$

The result is somewhat disappointing since the invariant solution is one in which the fields are equal to their asymptotic values for all n and m.

In order to obtain more interesting solutions, higher symmetries must be considered. Those will however be nonlocal and that is beyond the scope of the present article.

5 Conclusions

We have shown how to use symmetries of difference equations in two variables, either Lie point or "discrete" evolutionary ones, to construct solutions by carrying out a symmetry reduction.

We are confronted with two different situations. In the case of Lie point symmetries the symmetry variables, as opposed to the case of partial differential equations, do not always reduce the equation to a difference equation in a space of lower dimension. We have to impose a further constraint to be able to do so. Moreover in some cases, among them that of dilations, the system reduces to a dilation - delay equation which is difficult to solve. The situation is different for "discrete" evolutionary symmetries. These symmetries exist only when the discrete system is integrable and the symmetries do not act on the lattice. The symmetry reduction can always be carried out, but the obtained reduced equation can be nonlinear and difficult to solve (this is also true in the continuous case).

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