A concise expression for the ODE's of orthogonal polynomials

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Abstract

It is known that orthogonal polynomials obey a 3 terms recurrence relation, as well as a 2×2 differential system. Here, we give an explicit representation of the differential system in terms of the recurrence coefficients. This result is a generalization of an expression of Fokas, Its and Kitaev for the symmetric case.

1 Introduction

Orthogonal polynomials[10] continue to be extensively studied at the moment, partly because of their intrinsic structures they represent, and also because of their relationship with many physical and mathematical applications. For example they are the main tool for studying the large N asymptotics of the spectral statistics of random matrices [9, 8, 2, 3, 6, 4, 11].

Orthogonal polynomials are known to satisfy 3-term recurrence relations, as well as differential equations. These differential equations are very useful for the derivation of the large N asymptotics, and the associated Riemann-Hilbert problem [2].

For any given weight of the form $\mathrm{e}^{-V(x)}$, where $V(x)$ is a polynomial, finding the differential system is easy, but so far, no general expression was known (except when $V(x)$ is an even polynomial [5]). The calculations are straightforward but more and more involved as the degree of V increases. Here, we derive an explicit and concise expression, valid for any polynomial $V(x)$. The expression is very similar to that of [5].

In addition, these orthogonal polynomials satisfy differential equations with respect to the coefficients of $V(x)$, viewed as deformation parameters. Here, we derive an explicit expression for these deformation equations.

Outline of the article:

- In part 2, we introduce the orthogonal polynomials and set the notations.

- In part 3, we give the expression of the differential system satisfied by the orthogonal polynomials.

- In part 4, we give the expression of the deformation equations satisfied by the orthogonal polynomials, and give an alternative proof of the results of part 2.

2 Orthogonal polynomials

Consider the family of monic polynomials $p_n(x) = x^n + \ldots$, orthogonal with respect to the weight $\mathrm{e}^{-V(x)}$:

$$\int_\Gamma \mathrm{d}x\, p_n(x)p_m(x)\ \mathrm{e}^{-V(x)} = h_n\delta_{nm}\ , \tag{1}$$

where $V(x)$ is referred to as the potential. The orthogonal polynomials exist under suitable asumptions on V, and are unique [7, 10].

For simplicity, we will assume that V is a polynomial of even degree, with positive leading coefficient, and the the integration contour Γ is the real axis[3].

Instead of the p_n's, it is more convenient to introduce the following quasi-polynomial functions:

$$\psi_n(x) = \frac{1}{\sqrt{h_n}}\, p_n(x)\, \mathrm{e}^{-\frac{1}{2}V(x)}\ , \tag{2}$$

which are orthonormal:

$$\int \mathrm{d}x\, \psi_n(x)\psi_m(x) = \delta_{nm}\ . \tag{3}$$

[3]Many of these asumptions can easily be lifted, see section 4.

2.0.1 Recursion equation

It is well known [10, 8] that the orthogonal polynomials satisfy a three-term recursion relation, which reads:

$$x\psi_n(x) = \sum_{m=0}^{\infty} Q_{n,m}\psi_m = \gamma_{n+1}\psi_{n+1} + \beta_n\psi_n + \gamma_n\psi_{n-1} \ , \tag{4}$$

where

$$\gamma_n = Q_{n,n-1} = \sqrt{\frac{h_n}{h_{n-1}}} \qquad \text{and} \qquad \beta_n = Q_{n,n} \ . \tag{5}$$

The infinite matrix Q is symmetric ($Q_{n,m} = Q_{m,n}$), and has only 3 bands ($Q_{m,n} = 0$ if $|m-n| > 1$).

$$Q = \begin{pmatrix} \beta_0 & \gamma_1 & 0 & \dots & \dots \\ \gamma_1 & \beta_1 & \gamma_2 & 0 & \\ 0 & \gamma_2 & \beta_2 & \gamma_3 & \\ \vdots & 0 & \ddots & \ddots & \ddots \end{pmatrix} \ . \tag{6}$$

2.0.2 The string equation

The derivative of ψ_n can be expanded on the basis $\{\psi_m\}_{m=0..\infty}$:

$$\psi_n'(x) = \sum_{m=0}^{\infty} P_{n,m}\psi_m(x) \ , \tag{7}$$

Integrating Eq. (3) by parts implies that the infinite matrix P is antisymmetric: $P_{n,m} = -P_{m,n}$, and from $p_n' = np_{n-1} + O(x^{n-2})$, one has:

$$P_{n,m} + \frac{1}{2}\left(V'(Q)\right)_{n,m} = 0 \qquad \text{if } m \geq n \tag{8}$$

$$P_{n,n-1} + \frac{1}{2}\left(V'(Q)\right)_{n,n-1} = \frac{n}{\gamma_n} \ , \tag{9}$$

which implies

$$P_{n,m} = -\frac{1}{2}\left(V'(Q)\right)_{n,m} \qquad \text{if } m \geq n \tag{10}$$

$$P_{n,m} = +\frac{1}{2}\left(V'(Q)\right)_{n,m} \qquad \text{if } m \leq n \ . \tag{11}$$

This can be written:

$$P = -\frac{1}{2}\left(V'(Q)_+ - V'(Q)_-\right) \ , \tag{12}$$

where for any infinite matrix A, A_+ (resp. A_-), denotes the upper (resp. lower) triangular part of A.

In particular, Eq. (10) implies the following equations, known as "string equations":

$$0 = V'(Q)_{n,n} \qquad , \qquad \frac{n}{\gamma_n} = V'(Q)_{n,n-1} \tag{13}$$

2.1 Differential system

The sum in Eq. (7) contains only a finite number of terms and can be written:

$$\psi'_n(x) = -\frac{1}{2}\sum_{k=1}^{\deg V'} \eta_{k,n}\psi_{n+k} - \eta_{k,n-k}\psi_{n-k} \qquad \text{where} \qquad \eta_{k,n} = V'(Q)_{n,n+k} \ . \tag{14}$$

Note that

$$V'(x)\psi_n(x) = \sum_{k=1}^{\deg V'} \eta_{k,n}\psi_{n+k} + \eta_{k,n-k}\psi_{n-k} \ . \tag{15}$$

Using Eq. (4) recursively, one can write any ψ_{n+k} with $k>0$ and any ψ_{n-1-k} with $k>0$ as a linear combination of ψ_n and ψ_{n-1} with coefficients polynomial in x, of degree k. Eq. (14) can thus be rewritten as a 2×2 differential system:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi_{n-1}(x)\\ \psi_n(x)\end{pmatrix} = \mathcal{D}_n(x)\begin{pmatrix}\psi_{n-1}(x)\\ \psi_n(x)\end{pmatrix} \tag{16}$$

where $\mathcal{D}_n(x)$ is a 2×2 matrix with polynomial coefficients of degree at most $d=\deg V'$. It is known that it must have the form [10]:

$$\mathcal{D}_n(x) = \frac{1}{2}V'(x)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}\gamma_n u_n(x) & -\gamma_n v_{n-1}(x)\\ \gamma_n v_n(x) & -\gamma_n u_n(x)\end{pmatrix} \tag{17}$$

where $u_n(x)$ and $v_n(x)$ are polynomials of degrees $\deg u_n(x)\leq d-2$ and $\deg v_n(x)\leq d-1$. What was not known so far, is how to express $u_n(x)$ and $v_n(x)$ in terms of the coefficients γ_n and β_n (i.e. the coefficients of Q), apart form the case where V is an even polynomial [5]. Here, we prove the following theorem:

Theorem 2.1 *The matrix $\mathcal{D}_n(x)$ is:*

$$\mathcal{D}_n(x) = \frac{1}{2}V'(x)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}\left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{n-1,n-1} & \left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{n,n-1}\\ \left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{n,n-1} & \left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{n,n}\end{pmatrix}\begin{pmatrix}0 & -\gamma_n\\ \gamma_n & 0\end{pmatrix} \tag{18}$$

Proof:
One can check from $[\mathrm{d}/\mathrm{d}x, x]=1$ and from the compatibility with the shift $(n\to n+1)$, that $u_n(x)$ and $v_n(x)$ must satisfy the following recurrences:

$$\begin{aligned} 0 &= V'(x) + \gamma_n u_n(x) + \gamma_{n+1}u_{n+1}(x) + (\beta_n - x)v_n(x) & (19)\\ 1 &= (\beta_n - x)(\gamma_{n+1}u_{n+1}(x) - \gamma_n u_n(x)) + \gamma_{n+1}^2 v_{n+1}(x) - \gamma_n^2 v_{n-1}(x) \ . & (20)\end{aligned}$$

and these recurrences uniquely determine $u_n(x)$ and $v_n(x)$ for all n, provided that the initial terms are given.

Now, consider the infinite matrix:

$$R(x) = \left(\frac{V'(Q)-V'(x)}{Q-x}\right) \ . \tag{21}$$

Notice it is symmetric $R_{n,m}=R_{m,n}$.

Computing the diagonal (n,n) term of $(Q-x)R(x)$ and using Eq. (13) gives:

$$V'(Q)_{n,n} - V'(x) = -V'(x) = (\beta_n - x)R_{n,n}(x) + \gamma_n R_{n-1,n}(x) + \gamma_{n+1} R_{n+1,n}(x) \tag{22}$$

and computing the $(n, n\pm 1)$ terms, using Eq. (13) gives:

$$V'(Q)_{n,n+1} = \frac{n+1}{\gamma_{n+1}} = (\beta_n - x)R_{n,n+1}(x) + \gamma_n R_{n-1,n+1}(x) + \gamma_{n+1} R_{n+1,n+1}(x) \tag{23}$$

$$V'(Q)_{n,n-1} = \frac{n}{\gamma_n} = (\beta_n - x)R_{n,n-1}(x) + \gamma_n R_{n-1,n-1}(x) + \gamma_{n+1} R_{n+1,n-1}(x) \tag{24}$$

Using the symmetry, $R_{n-1,n+1} = R_{n+1,n-1}$ can be eliminated:

$$1 = (\beta_n - x)(\gamma_{n+1}R_{n,n+1}(x) - \gamma_n R_{n,n-1}(x)) + \gamma_{n+1}^2 R_{n+1,n+1}(x) - \gamma_n^2 R_{n-1,n-1}(x) \tag{25}$$

We have thus proven that $u_n(x) = R_{n,n-1}(x)$ and $v_n(x) = R_{n,n}(x)$ satisfy the recursion relations Eq. (19),Eq. (20). In order to complete the proof, one must verify that the expression is correct for $n = 1$. This is not difficult, and we don't write it here.

In any case, we will give an alternative proof of this result in the next section,, which does require the $n = 1$ case.

3 Deformations with respect to the potential

The deformation equations determine the variation of the orthogonal polynomials when the potential $V(x)$ is varied. In this section, we don't assume anything about the potential (which need not be a polynomial), or about the integration contour (which can be any (non-connected) path in the complex plane; we don't have to integrate by parts).

For any integer $k > 0$, let the parameter u_k be such that:

$$\frac{\partial}{\partial u_k} V(x) = \frac{1}{k} x^k \ . \tag{26}$$

In particular if $V(x)$ is a polynomial, we have:

$$V(x) = \sum_{k=1}^{\deg V} \frac{1}{k} u_k x^k \ . \tag{27}$$

Expand the variation of $\psi_n(x)$ with respect to u_k on the basis of $\{\psi_n\}$:

$$\frac{\partial}{\partial u_k}\psi_n(x) = \sum_{m=0}^{\infty} (U_k)_{nm}\, \psi_m(x) \ . \tag{28}$$

The infinite matrix U_k is antisymmetric (by differentiating Eq. (3)):

$$(U_k)_{n,m} = -(U_k)_{m,n} \ . \tag{29}$$

It is easy to see that (take the derivative of Eq. (2) w.r.t u_k, use $\partial p_n/\partial u_k = O(x^{n-1})$ and use Eq. (29)):

$$U_k = -\frac{1}{2k}\left(\left(Q^k\right)_+ - \left(Q^k\right)_- \right) \ . \tag{30}$$

This implies that the infinite matrix U_k is finite band: $(U_k)_{n,m} = 0$ if $|n-m| > k$, and this means that Eq. (28) is a finite sum. Notice that the diagonal part yields (from Eq. (2)):

$$\frac{1}{k}\left(Q^k\right)_{n,n} = \frac{\partial \ln h_n}{\partial u_k} \ . \tag{31}$$

U_k can be used to form a differential system (by expressing any ψ_m in terms of ψ_n and ψ_{n-1} with Eq. (4)):

$$\frac{\partial}{\partial u_k}\begin{pmatrix}\psi_{n-1}(x)\\ \psi_n(x)\end{pmatrix} = \mathcal{U}_k(x)\begin{pmatrix}\psi_{n-1}(x)\\ \psi_n(x)\end{pmatrix} \ , \tag{32}$$

where $\mathcal{U}_k(x)$ is a 2×2 matrix with polynomial coefficients of degree $\leq k$. For instance it is easy to compute $\mathcal{U}_1(x)$:

$$\mathcal{U}_1(x) = -\frac{1}{2}\begin{pmatrix}\beta_{n-1}-x & 2\gamma_n\\ -2\gamma_n & x-\beta_n\end{pmatrix} \ . \tag{33}$$

Theorem 3.1 *The expression for $\mathcal{U}_k(x)$ is:*

$$\mathcal{U}_k(x) = \frac{1}{2k}\begin{pmatrix}x^k - Q^k_{n-1,n-1} & 0\\ 0 & Q^k_{n,n}-x^k\end{pmatrix} + \frac{1}{k}\gamma_n\begin{pmatrix}\left(\frac{x^k-Q^k}{x-Q}\right)_{n,n-1} & -\left(\frac{x^k-Q^k}{x-Q}\right)_{n-1,n-1}\\ \left(\frac{x^k-Q^k}{x-Q}\right)_{n,n} & -\left(\frac{x^k-Q^k}{x-Q}\right)_{n,n-1}\end{pmatrix} \tag{34}$$

Notice that:

$$\operatorname{tr}\mathcal{U}_k(x) = \frac{\partial}{\partial u_k}\ln\gamma_n \tag{35}$$

Proof:

Notation: for any infinite matrix A, let A_l denote the l^{th} diagonal above (or below, if $l<0$) the main diagonal. A_+ (resp. A_-) denotes the strictly upper (resp. strictly lower) triangular part of A.

Since the matrix $Q = Q_{-1}+Q_0+Q_1$ has only three non-vanishing diagonals, we have:

$$\begin{aligned}\left(Q^{k+1}\right)_+ &= (QQ^k)_+ = (Q_1Q^k)_+ + (Q_0Q^k)_+ + (Q_{-1}Q^k)_+\\ &= Q_1\left(\left(Q^k\right)_+ + \left(Q^k\right)_0\right) + Q_0\left(Q^k\right)_+ + Q_{-1}\left(\left(Q^k\right)_+ - \left(Q^k\right)_1\right)\\ &= Q\left(Q^k\right)_+ + Q_1\left(Q^k\right)_0 - Q_{-1}\left(Q^k\right)_1 \ .\end{aligned} \tag{36}$$

This combined with a similar calculation for $\left(Q^{k+1}\right)_-$ implies:

$$Q^{k+1}_+ - Q^{k+1}_- = Q(Q^k_+ - Q^k_-) + Q_1(Q^k_0 + Q^k_{-1}) - Q_{-1}(Q^k_1 + Q^k_0) \ , \tag{37}$$

which, componentwise, reads (using Eq. (30)):

$$\begin{aligned}-2(k+1)\frac{\partial}{\partial u_{k+1}}\psi_n(x) &= -2k\frac{\partial}{\partial u_k}x\psi_n(x) + \left(Q^k\right)_{n,n}(\gamma_{n+1}\psi_{n+1} - \gamma_n\psi_{n-1})\\ &\quad + \left(\gamma_n\left(Q^k\right)_{n,n-1} - \gamma_{n+1}\left(Q^k\right)_{n,n+1}\right)\psi_n \ .\end{aligned} \tag{38}$$

Using Eq. (4):

$$\begin{aligned}-2(k+1)\frac{\partial}{\partial u_{k+1}}\psi_n(x) &= -2k\frac{\partial}{\partial u_k}x\psi_n(x) + \left(Q^k\right)_{n,n}((x-\beta_n)\psi_n - 2\gamma_n\psi_{n-1})\\ &\quad + \left(\gamma_n\left(Q^k\right)_{n,n-1} - \gamma_{n+1}\left(Q^k\right)_{n,n+1}\right)\psi_n\end{aligned}$$

$$
\begin{aligned}
-2(k+1)\frac{\partial}{\partial u_{k+1}}\psi_{n-1}(x) &= -2k\frac{\partial}{\partial u_k}x\psi_{n-1}(x) + \left(Q^k\right)_{n-1,n-1}(2\gamma_n\psi_n - (x-\beta_{n-1})\psi_{n-1}) \\
&\quad + \left(\gamma_{n-1}\left(Q^k\right)_{n-1,n-2} - \gamma_n\left(Q^k\right)_{n,n-1}\right)\psi_{n-1}\ ,
\end{aligned} \tag{39}
$$

which can be further simplified using

$$
\begin{aligned}
Q^{k+1}_{n,n} &= \gamma_{n+1}Q^k_{n,n+1} + \beta_n Q^k_{n,n} + \gamma_n Q^k_{n,n-1} \\
Q^{k+1}_{n-1,n-1} &= \gamma_{n-1}Q^k_{n-1,n-2} + \beta_{n-1}Q^k_{n-1,n-1} + \gamma_n Q^k_{n,n-1}\ .
\end{aligned} \tag{40}
$$

We thus get a recursion formula for $\mathcal{U}_k(x)$:

$$
\begin{gathered}
-2(k+1)\mathcal{U}_{k+1}(x) + 2kx\mathcal{U}_k(x) = \\
\begin{pmatrix} -\left(xQ^k - Q^{k+1}\right)_{n-1,n-1} & 0 \\ 0 & \left(xQ^k - Q^{k+1}\right)_{n,n}\end{pmatrix} + 2\gamma_n\begin{pmatrix} -Q^k_{n-1,n} & Q^k_{n-1,n-1} \\ -Q^k_{n,n} & Q^k_{n,n-1}\end{pmatrix}
\end{gathered} \tag{41}
$$

from which the result follows easily.

3.1 Alternative proof of Theorem 2.1

When $V(x)$ is a polynomial, Eq. (12) implies that:

$$
P = \sum_{k=0}^{\deg V'} u_{k+1} k U_k \tag{42}
$$

and therefore

$$
\mathcal{D}_n(x) = \sum_{k=1}^{\deg V'} u_{k+1} k \mathcal{U}_k(x)\ . \tag{43}
$$

Using theorem 3.1, as well as Eq. (13) one immediately gets the proof of theorem 2.1.

4 Conclusion and generalizations

We have found a general expression of the linear differential system $\mathcal{D}_n(x)$, which is very convenient for explicit computations.

The result was derived only for a polynomial V, with even degree and leading coefficient positive. It extends immediately to V of arbitrary degree and leading coefficient (but now the integration path is taken in the complex plane such that the integrals converge), provided that the orthogonal polynomials exist in that case.

It could also be extended to a potential V whose derivative V' is a rational function. The only difference is that $V'(Q)$ is no longer a finite band matrix, and computing $((Q-a)^{-1})_{n,n-1}$ or $((Q-a)^{-1})_{n,n}$ is not easy (other approaches like Geronimus transformations might be more efficient).

Another generalization is when the integration path is not from $-\infty$ to ∞, but has endpoints. Again, a differential system can be found, which has poles at the endpoints. The part without poles is still given by 2.1, but the residues of the poles are not easily found with that method. Other approaches (such as the dressing method of Zakharov Shabat) are more efficient.

Another generalization is for biorthogonal polynomials. Theorems 2.1 and 3.1 can be generalized, and the expressions will be given in a forthcoming article [1].

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