Some Geometric Aspects of CP^2 Maps

A. M. Grundland[*] W. J. Zakrzewski[†]

CRM-2754

September 2001

[*]Centre de recherches mathématiques, Université de Montréal, C. P. 6128, Succ. Centre-ville, Montréal, (QC) H3C 3J7, Canada; grundlan@crm.umontreal.ca

[†]Department of Mathematical Sciences,University of Durham, Durham DH1 3LE, UK; w.j.zakrzewski@durham.ac.uk

Abstract

A generalisation of the Weierstrass system of equations corresponding to CP^2 harmonic maps is given. This generalisation allows us to study two-dimensional surfaces immersed in a flat space R^8 with Euclidean metric. We use this system to suggest a possible geometrical interpretation of CP^2 harmonic maps.

Résumé

Nous donnons une géneralisation du système d'équations de Weierstrass qui correspond aux applications harmoniques CP^2. Cette généralisation nous permet d'étudier les surfaces á deux dimensions plongées dans un espace R^8 avec une metrique euclidienne. Nous utilisons ce système pour sugérer une interpretation géometrique des applications CP^2 harmoniques.

1 Introduction

Sigma models in two spatial dimensions are integrable and have been studied for variety of reasons. They are low dimensional analogues of four-dimensional Yang-Mills theories which play a pivotal role in particle physics, they arise in some areas of condensed-matter physics *etc* and ... they are also interesting from a purely mathematical point of view. Moreover, they arise naturally in differential geometry in the investigation of immersion and deformations of surfaces.

Of course, there are many classes of σ models; amongst them particularly important are the so-called CP^{N-1} σ models. These models, all, possess topological properties and, as such, lead to the appearance of "topological solitons".

The models are a generalisation of the, perhaps the simplest, σ model, namely the S^2 model - also called the vector $SO(3)$ model. The CP^{N-1} models involve maps from R^2, or S^2 if one wants to have nontrivial topology, to CP^{N-1}. It is easiest to define them in terms of the Lagrangian density[1]

$$L = \tfrac{1}{4}(D_\mu z)^\dagger \cdot D_\mu z, \tag{1}$$

where z is a vector field of N components, $z = (z^1, ..., z^N)$, which satisfies

$$z^\dagger \cdot z = 1. \tag{2}$$

The differential operator D_μ acts on $\psi : S^2 \to CP^{N-1}$ according to the formula:

$$D_\mu \psi = \partial_\mu \psi - \psi\,(z^\dagger \cdot \partial_\mu z). \tag{3}$$

Here $\mu = 1, 2$, of course, and denotes the space coordinates x and y.

The total Lagrangian is given by

$$\mathcal{L} = \int L\, dx\, dy \tag{4}$$

and, if the model is defined over S^2, we require that $\mathcal{L}$ is finite.

As is well known, the CP^1 sigma model ($N = 2$), can be equivalently described in terms of a three component real vector field $\vec{\phi}$ defined by:

$$\vec{\phi} = z^\dagger \vec{\sigma} z, \tag{5}$$

where $\vec{\sigma}$ is a vector of Pauli's σ matrices.

Then the Lagrangian density of the CP^1 model (1) becomes

$$L = \partial_\mu \vec{\phi} \cdot \partial_\mu \vec{\phi} \tag{6}$$

together with the constraint

$$\vec{\phi} \cdot \vec{\phi} = 1, \tag{7}$$

i.e. $\vec{\phi}$ lies on a unit sphere S^2.

For a model defined over S^2 all harmonic maps, *i.e.* solutions of the Euler Lagrange equations which follow from (4) are well known [1]. They come in three separate classes: those which are holomorphic, those which are antiholomorphic and the mixed ones. The CP^1 model has only holomorphic and antiholomorphic harmonic maps but for CP^N $N \geq 2$ there are also mixed maps. These solutions have very different properties; *e.g.* holomorphic solutions are stable, as they are minima of the total Lagrangian, while the mixed solutions are not, as they are only saddle points.

Recently, there has been a lot of interest in relating CP^1 maps to the solutions of the Weierstrass problem[3, 5, 6].

In this case, according to [6] one considers a set of first order equations for complex fields ϕ and ψ of z and $\bar{z}$, given by:

$$\partial\,\psi = p\,\phi, \qquad \bar{\partial}\phi = -p\,\psi, \qquad p = |\phi|^2 + |\psi|^2 \tag{8}$$

where

$$\partial = \frac{\partial}{\partial(x+iy)} = \frac{\partial}{\partial\zeta}, \qquad \bar{\partial} = \frac{\partial}{\partial\bar{\zeta}}. \tag{9}$$

From the Weierstrass system (8) one constructs a geometric coordinate system of 3 real variables X_i, $i = 1, 2, 3$ and, treating $X_i(x, y)$ as a map of R^2 into R^3 discusses the geometry of these surfaces.

In this paper we address the question of the generalisation of these ideas to higher CP^N. Thus we consider the case of CP^2. What is the corresponding Weierstrass system and how to construct the corresponding real quantities X_i? In the next section we collect together various results on the CP^1 system. In the following section we discuss various formulations of the CP^2 model and racall some of its solutions. Next we present our generalised Weierstrass system and discuss some of its properties. In the following section we present our ideas how to construct a set of real-valued functions $X_i(x, y)$ and then show how they describe the geometry of surfaces.

2 CP^1 model

2.1 General properties

For the CP^1 sigma model it is convenient to introduce

$$W = \frac{z_2}{z_1} = \frac{\phi^1 + i\phi^2}{1+\phi^3}. \tag{10}$$

In terms of W the Lagrangian becomes

$$\mathcal{L} = \int \frac{|\partial W|^2 + |\partial \bar{W}|^2}{(1+|\partial W|^2)^2}\, d\zeta\, d\bar{\zeta} \tag{11}$$

and its Euler Lagrange equations take the form

$$\partial\bar{\partial}W - 2\bar{W}\frac{\partial W \bar{\partial} W}{|W|^2+1} = 0. \tag{12}$$

In [5] it was shown that the relation between CP^1 maps and the ϕ, ψ fields of the Weierstrass problem is given by (up to an overall multiplication of ϕ and ψ by a factor -1)

$$\psi = W\frac{(\bar{\partial}\bar{W})^{\frac{1}{2}}}{1+|W|^2}, \qquad \phi = \frac{(\partial W)^{\frac{1}{2}}}{1+|W|^2}, \tag{13}$$

where

$$W = \frac{\psi}{\bar{\phi}}. \tag{14}$$

As is well known [4] the CP^1 equations can be written as a compatibility condition for a set of two linear spectral equations for a two component auxiliary vector Ψ

$$\begin{aligned} \partial\Psi &= \frac{2}{1+\lambda}\,[\partial P,\, P]\,\Psi \\ \bar{\partial}\Psi &= \frac{2}{1-\lambda}\,[\bar{\partial} P,\, P]\,\Psi, \end{aligned} \tag{15}$$

where the 2 by 2 matrix P is a projector given by

$$P = \frac{1}{A}\begin{pmatrix} 1 & \bar{W} \\ W & |W|^2 \end{pmatrix}, \qquad A = 1 + |W|^2. \tag{16}$$

The compatibility conditions for (15) are, clearly,

$$[\partial\bar{\partial}P,\, P] \,=\, 0 \tag{17}$$

which, as can be easily checked, are equivalent to (12).

Note that (17) can be written in the form of a conservation law

$$\partial\,[\bar{\partial}P,\, P] \,+\, \bar{\partial}\,[\partial P,\, P] \,=\, 0 \tag{18}$$

or, equivalently, using the traceless matrices K and M,

$$\partial K \,+\, \bar{\partial}M \,=\, 0, \tag{19}$$

where the matrices K and M are given by

$$K \,=\, \frac{1}{A^2}\begin{pmatrix} \bar{W}\,\bar{\partial}W \,-\, W\,\bar{\partial}\bar{W} & \bar{\partial}\bar{W} \,+\, \bar{W}^2\bar{\partial}W \\ -\bar{\partial}W \,-\, W^2\bar{\partial}\bar{W} & W\,\bar{\partial}\bar{W} \,-\, \bar{W}\,\bar{\partial}W \end{pmatrix} \tag{20}$$

and

$$M \,=\, \frac{1}{A^2}\begin{pmatrix} \bar{W}\,\partial W \,-\, W\,\partial\bar{W} & \partial\bar{W} \,+\, \bar{W}^2\partial W \\ -\partial W \,-\, W^2\partial\bar{W} & W\,\partial\bar{W} \,-\, \bar{W}\,\partial W \end{pmatrix}. \tag{21}$$

To proceed further it is worth recalling the existence of various quantities of the CP^{N-1} model which are holomorphic [1]. One of them is

$$T \,=\, (\bar{D}z)^{\dagger}\cdot Dz \tag{22}$$

where Dz denotes the covariant derivative (3) involving ∂ (*i.e.* with respect to $\zeta = x + iy$). As is easy to check, T satisfies

$$\bar{\partial}\,T \,=\, 0 \tag{23}$$

and so is a function of $\zeta = x + iy$ only.

In the CP^1 case, the function T expressed in terms of W is given by

$$T \,=\, 2\,\frac{\partial W\,\partial\bar{W}}{(1+|W|^2)^2}. \tag{24}$$

In fact, for the solutions of (12) which describe fields defined on S^2, W is any holomorphic or antiholomorphic function and so $T = 0$.

2.2 Weierstrass system

If we introduce the complex fields ϕ and ψ by (8) *ie*

$$\partial\,\psi \,=\, p\,\phi, \qquad \bar{\partial}\phi \,=\, -p\,\psi, \qquad p \,=\, |\phi|^2 + |\psi|^2, \tag{25}$$

then, when we reexpress the quantity T in terms of them using (13) we get

$$T \,=\, \bar{\psi}\partial\phi \,-\, \phi\partial\bar{\psi}. \tag{26}$$

Moreover, we have also

$$p \,=\, A\,|\phi|^2, \qquad A = 1 + |W|^2 \tag{27}$$

and we can express the first derivatives of W in terms of ψ and ϕ:

$$\partial W \,=\, A^2\phi^2, \qquad \bar{\partial}W \,=\, -\bar{T}(\phi)^{-2}. \tag{28}$$

This allows us to derive the explicit form of matrices K and M in terms of ϕ and ψ. So, we have

$$K \,=\, \begin{pmatrix} -(\psi\bar{\phi} + \bar{R}\bar{\psi}\phi) & \bar{\phi}^2 - \bar{R}\bar{\psi}^2 \\ -\psi^2 + \bar{R}\phi^2 & \psi\bar{\phi} + \bar{R}\bar{\psi}\phi \end{pmatrix} \tag{29}$$

and

$$M = \begin{pmatrix} (\bar\psi\phi + R\psi\bar\phi) & \bar\psi^2 - R\bar\phi^2 \\ -\phi^2 + R\psi^2 & -(\bar\psi\phi + R\psi\bar\phi), \end{pmatrix} \tag{30}$$

where we have introduced the following notation

$$R = \frac{T}{p^2}.$$

As a consequence of (18) we find that the system (25) possesses, at least, three further conservation laws

$$\begin{aligned} -\,\partial(\psi\bar\phi + \bar R\bar\psi\phi) \,+\, \bar\partial(\bar\psi\phi + R\psi\bar\phi) \;&=\; 0 \\ \partial(-\psi^2 + \bar R\phi^2) \,+\, \bar\partial(-\phi^2 + R\psi^2) \;&=\; 0 \\ \partial(\bar\phi^2 - \bar R\bar\psi^2) \,+\, \bar\partial(\bar\psi^2 - R\bar\phi^2) \;&=\; 0. \end{aligned} \tag{31}$$

These formulae differ slightly from the conservation laws given in [6], as they contain additional terms involving R. However, if we put $R = 0$ in (30) we recover the expressions of [6].

As a result of conservation laws (30) we can introduce three real-valued functions $X_i(\zeta, \bar\zeta)$, $i = 1, 2, 3$ given by

$$\begin{aligned} X_1 &= i\int_\gamma [\bar\psi^2 + \phi^2 - R(\psi^2 + \bar\phi^2)]d\zeta \,-\, [\psi^2 + \bar\phi^2 - \bar R(\bar\psi^2 + \phi^2)]d\bar\zeta, \\ X_2 &= \int_\gamma [\bar\psi^2 - \phi^2 + R(\psi^2 - \bar\phi^2)]d\zeta \,+\, [\psi^2 - \bar\phi^2 + \bar R(\bar\psi^2 - \phi^2)]d\bar\zeta, \\ X_3 &= -2\int_\gamma [\bar\psi\phi + R\psi\bar\phi^2]d\zeta \,+\, [\psi\bar\phi + \bar R\bar\psi\phi]d\bar\zeta, \end{aligned} \tag{32}$$

where γ is any curve from a fixed point to ζ.

The conservation laws (31) guarantee that quantities X_i do not depend on the choice of the curve γ in the complex plane C (but only its endpoints). This is because X_i can be rewritten as

$$X_i \;=\; \int_\gamma F_i(\zeta,\bar\zeta)\,d\zeta \,+\, \bar F_i(\zeta,\bar\zeta)\,d\bar\zeta, \qquad i = 1, 2, 3 \tag{33}$$

where F_i satisfy the following conditions:

$$\bar\partial F_i \;=\; \partial\bar F_i, \tag{34}$$

which shows that the integrands are total derivatives.

The functions $X_i(\zeta, \bar\zeta)$ can be considered as components of a radius vector

$$\vec r(\zeta,\bar\zeta) \;=\; \big(X_1(\zeta,\bar\zeta),\, X_2(\zeta,\bar\zeta),\, X_3(\zeta,\bar\zeta)\big) \tag{35}$$

of an orientable, simply connected, surface (locally parametrised by ζ and $\bar\zeta$ immersed in R^3).

This allows us to calculate the tangent vectors to the surface *i.e.*

$$\partial\vec r \;=\; \big(i[\bar\psi^2 + \phi^2 - R(\psi^2 + \bar\phi^2)],\, [\bar\psi^2 - \phi^2 + R(\psi^2 - \bar\phi^2)],\, -2(\bar\psi\phi + R\psi\bar\phi)\big) \tag{36}$$

and

$$\bar\partial\vec r \;=\; \big(-i[\psi^2 + \bar\phi^2 - \bar R(\bar\psi^2 + \phi^2)],\, [\psi^2 - \bar\phi^2 + \bar R(\bar\psi^2 - \phi^2)],\, -2(\psi\bar\phi + \bar R\bar\psi\phi)\big)\,. \tag{37}$$

These expressions allow us to calculate the induced metric. We find the following expressions for the components of the induced metric (written in holomorphic components):

$$g_{\zeta\zeta} \;=\; (\partial\vec r, \partial\vec r) \;=\; 4Rp^2, \qquad g_{\bar\zeta\bar\zeta} \;=\; (\bar\partial\vec r, \bar\partial\vec r) \;=\; 4\bar R p^2 \tag{38}$$

and

$$g_{\zeta\bar{\zeta}} = (\partial\vec{r}, \bar{\partial}\vec{r}) = 2(p^2 + |R|^2 p^2). \tag{39}$$

Hence, for the harmonic maps, $R = 0$ and the only nonvanishing component of the metric is $g_{\zeta\bar{\zeta}} = 2p^2$. In this case solutions of the system (24) are represented by expressions (10), where $W(\zeta)$ is an arbitrary holomorphic function. As is well known [1] finiteness of the energy restricts $W(\zeta)$ to being a rational function. Geometrically, such functions parametrise an immersed sphere $S^2 \in R^3$.

3 CP^2 Model

3.1 General Properties

Now we consider a more general situation when the Lagrangian is given by (1) with a three component vector $z = (z_1, z_2, z_3)$ (z_i, which satisfies $\bar{z} \cdot z = \sum_i \bar{z}_i z_i = 1$). We can define two complex fields W_i, $i = 1, 2$ through

$$W_1 = \frac{z_1}{z_3}, \qquad W_2 = \frac{z_2}{z_3} \tag{40}$$

and find that the Euler Lagrange equations take the form

$$\begin{aligned} \bar{\partial}\partial W_1 - \frac{2\bar{W}_1}{A}\partial W_1 \bar{\partial} W_1 - \frac{\bar{W}_2}{A}(\partial W_1 \bar{\partial} W_2 + \bar{\partial} W_1 \partial W_2) &= 0, \\ \bar{\partial}\partial W_2 - \frac{2\bar{W}_2}{A}\partial W_2 \bar{\partial} W_2 - \frac{\bar{W}_1}{A}(\partial W_1 \bar{\partial} W_2 + \bar{\partial} W_1 \partial W_2) &= 0, \end{aligned} \tag{41}$$

and their respective complex conjugate equations and where

$$A = 1 + |W_1|^2 + |W_2|^2.$$

Clearly, when, say, $W_2 = 0$ (*i.e.* $z_2 = 0$) the model, and its equations, reduce to the CP^1 case.

Like in the CP^1 case we still have an auxiliary spectral problem given by (15) but this time the auxiliary vector Φ has 3 components and the 3×3 projector P is given by

$$P = \frac{1}{A}\begin{pmatrix} 1 & \bar{W}_1 & \bar{W}_2 \\ W_1 & |W_1|^2 & W_1\bar{W}_2 \\ W_2 & \bar{W}_1 W_2 & |W_2|^2 \end{pmatrix}. \tag{42}$$

As in the CP^1 case the compatibility condition for the two equations in (15) gives the equations of motion (16) which are equivalent to (41). Similarly as in the previous case the system (40) possesses a conservation law

$$\partial K + \bar{\partial} M = 0 \tag{43}$$

with

$$K = [\bar{\partial} P, P], \qquad M = [\partial P, P] \tag{44}$$

The explicit forms of 3×3 traceless matrices K and M are rather complicated expressions so we shall not write them explicitly here.

In the CP^2 case the quantity T can be written in terms of W_1 and W_2 as

$$T = \frac{\partial W_1 \partial \bar{W}_1 + \partial W_2 \partial \bar{W}_2 + (\bar{W}_1 \partial \bar{W}_2 - \bar{W}_2 \partial \bar{W}_1)(W_1 \partial W_2 - W_2 \partial W_1)}{(1 + |W_1|^2 + |W_2|^2)^2}. \tag{45}$$

All solutions of the CP^2 model are well known[2]. They fall into three classes; those described by analytic fields (*i.e.* $W_i = W_i(\zeta)$), antianalytic ($W_i = W_i(\bar{\zeta})$) and mixed ones.

A mixed solutions can be determined from either the holomorphic or the antiholomorphic fields by the following procedure. Take arbitrary holomorphic functions $f_i = f_i(\zeta)$ and define

$$F_{ij} \;=\; f_i \partial f_j - f_j \partial f_i, \qquad i,j = 1,2,3. \tag{46}$$

Next introduce new complex valued functions:

$$g_i = \sum_{k \neq i} \bar{f}_k \, F_{ki}. \tag{47}$$

Then we can determine W_i as ratios of the components of g_i, *i.e.*

$$W_1 \;=\; \frac{g_2}{g_1}, \qquad W_2 \;=\; \frac{g_3}{g_1}. \tag{48}$$

Then it can be shown [1] that all mixed solutions correspond to W_i constructed in this way from some $f_i(\zeta)$. An alternative approach starts with antiholomorphic f_i *i.e.* $f_i = f(\bar{\zeta})$ and constructs g_i in the same way but using $\bar{\partial}$ instead of ∂ in the definition of F_{ij}.

3.2 Generalised Weierstrass system

Now, according to the discussion of the previous section we can introduce two pairs of complex functions ψ_i, ϕ_i $i = 1,2$, which have to satisfy

$$W_i \;=\; \frac{\psi_i}{\bar{\phi}_i}, \qquad i = 1,2. \tag{49}$$

The aim of this section is to find a system of first order equations which is a generalisation of (25) and which are in a one-to-one correspondence with the equations of the CP^2 sigma model (42).

Let us note that a possible set of equations for functions ψ_i and ϕ_i is given by

$$\bar{\partial}\phi_1 = -\tfrac{1}{2}\Bigg[(A+|W_1|^2)\phi_1\bar{\phi}_2\psi_2 + (A+1+|W_1|^2)|\phi_1|^2\psi_1 + \frac{\phi_2}{\phi_1}\bar{\psi}_2\psi_1^2 + (1+|W_2|^2)\frac{|\phi_2|^4}{|\phi_1|^2}\psi_1\Bigg], \tag{50}$$

$$\bar{\partial}\phi_2 = -\tfrac{1}{2}\Bigg[(A+|W_2|^2)\phi_2\bar{\phi}_1\psi_1 + (A+1+|W_2|^2)|\phi_2|^2\psi_2 + \frac{\phi_1}{\phi_2}\bar{\psi}_1\psi_2^2 + (1+|W_1|^2)\frac{|\phi_1|^4}{|\phi_2|^2}\psi_2\Bigg], \tag{51}$$

$$\partial\psi_1 = -\tfrac{1}{2}\Bigg[(A+|W_1|^2)\phi_2\bar{\psi}_2\psi_1 + (A+1+|W_1|^2)|\psi_1|^2\phi_1 + \frac{\bar{\phi}_2}{\bar{\phi}_1^2}\bar{\psi}_1\psi_2|\psi_1|^2\Bigg] - \tfrac{1}{2}(1+|W_2|^2)\frac{|\phi_2|^4}{\bar{\phi}_1|\phi_1|^2}|\psi_1|^2 + A(1+|W_1|^2)|\phi_1|^2\phi_1 + A\phi_2\psi_1\bar{\psi}_2, \tag{52}$$

$$\partial\psi_2 = -\tfrac{1}{2}\Bigg[(A+|W_2|^2)\phi_1\bar{\psi}_1\psi_2 + (A+1+|W_2|^2)|\psi_2|^2\phi_2 + \frac{\bar{\phi}_1}{\bar{\phi}_2^2}\bar{\psi}_2\psi_1|\psi_2|^2\Bigg] - \tfrac{1}{2}(1+|W_1|^2)\frac{|\phi_1|^4}{\bar{\phi}_2|\phi_2|^2}|\psi_2|^2 + A(1+|W_2|^2)|\phi_2|^2\phi_2 + A\phi_1\psi_2\bar{\psi}_1,$$

$$A = 1 + |W_1|^2 + |W_2|^2, \tag{53}$$

and their respective complex conjugate equations, where we can use equation (49) to express the above expressions in terms of ψ_i and ϕ_i's.

It is easy to check that if we put $\phi_2 = W_2 = 0$ then the system of equations (49)-(52) reduces to equations (24) and the system (40) reduces to the CP^1 model (13). These limits characterize some properties of solutions of the system of first order equations (49)-(52).

Moreover, it is possible, although somewhat tedious, to show that equations (49)-(52) are equivalent to (40). Thus eq. (49-52) can be thought of as being CP^2 analogues of (24); (like (24) they involve only first derivatives and only $\bar{\partial}\phi_i$ and $\partial\psi_i$ are given).

Here we can use the fact (the generalisation of (10) that

$$\begin{aligned} A^2\phi_1^2 &= (1+|W_2|^2)\partial W_1 - W_1\bar{W}_2\partial W_2 \\ A^2\phi_2^2 &= (1+|W_1|^2)\partial W_2 - W_2\bar{W}_1\partial W_1. \end{aligned} \tag{54}$$

Next we address the question of the existence of real valued functions X_i's. To construct them we note that we can exploit the matrices K and M given by (42) and (43). Then the CP^2 analogue of matrices (28) and (29) become

$$K = K_1 - \frac{1}{A^2}K_2, \tag{55}$$

where

$$K_1 = \begin{pmatrix} -(\psi_1\bar{\phi}_1 + \psi_2\bar{\phi}_2) & \bar{\phi}_1^2 & \bar{\phi}_2^2 \\ -W_1(\psi_1\bar{\phi}_1 + \psi_2\bar{\phi}_2) & +\psi_1\bar{\phi}_1 & +\frac{\psi_1}{\phi_1}\bar{\phi}_2^2 \\ -W_2(\psi_1\bar{\phi}_1 + \psi_2\bar{\phi}_2) & +\frac{\psi_2}{\phi_2}\bar{\phi}_1^2 & +\psi_2\bar{\phi}_2 \end{pmatrix} \tag{56}$$

and

$$K_2 = \begin{pmatrix} -(\bar{W}_1\bar{\partial}W_1 + \bar{W}_2\bar{\partial}W_2) & -\bar{W}_1(\bar{W}_1\bar{\partial}W_1 + \bar{W}_2\bar{\partial}W_2) & -\bar{W}_2(\bar{W}_1\bar{\partial}W_1 + \bar{W}_2\bar{\partial}W_2) \\ \Phi_1 & \bar{W}_1\Phi_1 & \bar{W}_2\Phi_1 \\ \Phi_2 & \bar{W}_1\Phi_2 & \bar{W}_2\Phi_2 \end{pmatrix} \tag{57}$$

where we have defined the following expressions

$$\begin{aligned} \Phi_1 &= (1+|W_2|^2)\bar{\partial}W_1 - W_1\bar{W}_2\bar{\partial}W_2 \\ \Phi_2 &= (1+|W_1|^2)\bar{\partial}W_2 - W_2\bar{W}_1\bar{\partial}W_1 \end{aligned} \tag{58}$$

in order to abbreviate expressions (56) and (59).

Similarily, matrix M is given by $M = M_1 - \frac{1}{A^2}M_2$, where

$$M_1 = \begin{pmatrix} (\bar{\psi}_1\phi_1 + \bar{\psi}_2\phi_2) & \bar{W}_1(\bar{\psi}_1\phi_1 + \bar{\psi}_2\phi_2) & +\bar{W}_2(\bar{\psi}_1\phi_1 + \bar{\psi}_2\phi_2) \\ -\phi_1^2 & -\bar{\psi}_1\phi_1 & -\frac{\bar{\psi}_2}{\phi_2}\phi_1^2 \\ -\phi_2^2 & -\frac{\bar{\psi}_1}{\phi_1}\phi_2^2 & -\bar{\psi}_2\phi_2 \end{pmatrix} \tag{59}$$

and

$$M_2 = \begin{pmatrix} -(\bar{W}_1\bar{\partial}W_1 + \bar{W}_2\bar{\partial}W_2) & \bar{\Phi}_1 & \bar{\Phi}_2 \\ -W_1(W_1\partial\bar{W}_1 + W_2\partial\bar{W}_2) & W_1\bar{\Phi}_1 & W_1\bar{\Phi}_2 \\ -W_2(W_1\partial\bar{W}_1 + W_2\partial\bar{W}_2) & W_2\bar{\Phi}_1 & W_2\bar{\Phi}_2 \end{pmatrix}. \tag{60}$$

Note that matrices K_2 and M_2 involve expressions which involve $\bar{\partial}W_i$ which are not known. In the holomorphic case $W = W_i(\zeta)$ and so $K_2 = M_2 = 0$. In fact, this is also true in general; when the equations of motion are satisfied we can set $M_2 = K_2 = 0$ and the conservation laws are still satisfied. Hence, in our search of real-valued functions X_i we can restrict our attention to only K_1 and M_1.

As both matrices (K_1 and M_1) are traceless we can use them to define eight new conservation laws. These in turn allow us to define nine real quantities $X_i(\zeta, \bar{\zeta})$, eight of which are linearly independent. X_i $i = 1, 2, 3$ are constructed by taking diagonal entries of matrices M and K:

$$\begin{aligned} X_1 &= -\int_\gamma [\bar{\psi}_1\phi_1 + \bar{\psi}_2\phi_2]\,d\zeta + [\psi_1\bar{\phi}_1 + \psi_2\bar{\phi}_2]d\bar{\zeta} \\ X_2 &= \int_\gamma \bar{\psi}_1\phi_1\,d\zeta + \psi_1\bar{\phi}_1\,d\bar{\zeta} \end{aligned} \tag{61}$$

$$X_3 = \int_\gamma \bar\psi_2\phi_2\, d\zeta \,+\, \psi_2\bar\phi_2\, d\bar\zeta,$$

which satisfy

$$X_1 \,+\, X_2 \,+\, X_3 \,=\, 0. \tag{62}$$

The off-diagonal entries of matrices K and M when combined with the property that $K^\dagger = -M$) give us a further 6 real quantities X_i, $i = 4, ..9$ *i.e.*

$$\begin{aligned}
X_4 \,+\, iX_5 \,&=\, \int_\gamma \left[-\mu\phi_1^2 \,+\, \bar\mu\left(\bar\psi_1^2 + \bar\psi_1\bar\psi_2\frac{\phi_2}{\phi_1}\right)\right] d\zeta \,+\, \left[-\bar\mu\bar\phi_1^2 \,+\, \mu\left(\psi_1^2 + \psi_1\psi_2\frac{\bar\phi_2}{\bar\phi_1}\right)\right] d\bar\zeta \\
X_6 \,+\, iX_7 \,&=\, \int_\gamma \left[-\nu\phi_2^2 \,+\, \bar\nu\left(\bar\psi_2^2 + \bar\psi_1\bar\psi_2\frac{\phi_1}{\phi_2}\right)\right] d\zeta \,+\, \left[-\bar\nu\bar\phi_2^2 \,+\, \nu\left(\psi_2^2 + \psi_1\psi_2\frac{\bar\phi_1}{\bar\phi_2}\right)\right] d\bar\zeta \\
X_8 \,+\, iX_9 \,&=\, -\int_\gamma \left[\alpha\phi_2^2\frac{\bar\psi_1}{\phi_1} + \bar\alpha\phi_1^2\frac{\bar\psi_2}{\phi_2}\right] d\zeta \,+\, \left[\bar\alpha\bar\phi_2^2\frac{\psi_1}{\bar\phi_1} + \alpha\bar\phi_1^2\frac{\psi_2}{\bar\phi_2}\right] d\bar\zeta.
\end{aligned} \tag{63}$$

Here μ, ν and α are arbitrary constants. As in the CP^1 case the integrals in the definition of X_i do not depend on the trajectory of the curve γ (but are only on its endpoint ζ) in C since the conditions (33) hold. So we can then consider function X_i as components of an eight-dimensional vector $\vec r$ and use it to construct and investigate two-dimensional surfaces immersed in R^8. Thus, putting it all together, we see that if the complex functions ψ_i and ϕ_i, $i = 1, 2$ are solutions of the system of first order equations (49)-(52), the generalised Weierstrass formulae given by (60)-(62) determine the conformal immersion of a surface into R^8.

The geometrical aspects of surfaces obtained from the first order equations (49-50) will described in more detail in future work.

4 Conclusions and Further Comments

In this paper we have shown how to generalise the old ideas of Weierstrass from the CP^1 to the CP^2 case. We have found that the corresponding first order equations have much more complicated form. We have also started discussing the geometry of the associated surfaces leaving details to a future publication. We are currently looking at the generalisation of our results to CP^N.

5 ACKNOWLEDGMENTS

This work was started when one of us (WJZ) visited the CRM, Université de Montreal in March 2000. He would like to thank the LMS and P. Winternitz for the support of his stay at the CRM. The work was continued when WJZ spent a sabbatical term at the Center for Theoretical Physics, MIT and when AMG visited the University of Durham. WJZ would like to thank the Center for the support of his stay at MIT and its hospitality. AMG would like to thank the University of Durham for the award of Alan Richards fellowship to him that allowed him to spend a term in Durham. Partial support for AMG's work was provided also by a grant from NSERC of Canada.

References

[1] see eg W.J. Zakrzewski - Low Dimensional Sigma Models, Adam Hilger, Bristol (1989)

[2] A. D'Adda, P. Di Vecchia and M. Lüscher - Nucl. Phys. **B 146**, 63 (1978)

[3] see *e.g.* B. Konopelchenko and G. Landolfi, Stud. Appl. Maths. **104**, 129-169 (1999);

[4] V.E. Zakharov and A.V. Mikhailov - Zh. Eksp. Theor. Fiz. **74**, 1953 (1978); (Sov. Phys. JETP **47**, 1017 (1979))

[5] P. Bracken and A.M. Grundland, J. Math. Phys. **40**, 3379-3403 (1999).

[6] B. Konopelchenko, Stud. Appl. Maths. **96**, 9-51 (1996).

[7] P. Bracken and A.M. Grundland, J. Math. Phys. **42**, 1250-1282 (2001).

[8] R. Carrol and B. Konopelchenko, Int. J. Mod. Phys. **A 11**, 11-83-? (1996).

[9] K. Kenmotsu, Math. Ann. **245**, 89-99 (1979).

[10] A.V. Mikhailov; Integrable Magnetic Models in Solitons ed. E. Trullinger et al - North Holland, Amsterdam 623-690 (1981).

[11] A. Bobenko, Surfaces in terms of 2x2 matrices; R.S. Ward, Sigma Models in 2+1 dimensions; both in Harmonic Maps and Integrable Systems, Ed. A. Fordy and J.C. Wood; Vieweg, (1990).