Linear Quadratic Optimal Control of Linear Time Invariant Systems with Delays in State, Control, and Observation Variables

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CRM-2752

September 2001

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Abstract

The paper deals with the problem of linear quadratic optimal control associated to a coupled mathematical system wich provides a good model for an important families of linear time-invariant hereditary systems. The HUM approach based on the construction of an appropriate hilbert space and the inversion of an isomorphism is used and the optimal control is characterized by the resolution of some algebraic equation.

Résumé

Le papier traite le problème de contrôle linéaire quadratique associé à un système mathématique qui se trouve un bon modèle pour une famille assez large des systèmes hériditaires. L'approche HUM basée sur la construction d'une structure hilbertienne et l'inversion d'un isomorphisme est utilisée et le contrôle optimal est caractérisé par la résolution d'une équation algébrique dans un espace approprié.

1 Introduction

In this paper we study the problem of linear quadratic optimal control associated to a simple mathematical system wich provides a good model for important families of linear time-invariant hereditary systems. It accounts for families of delay differential equations, integro-differential equations, integral equations, functional differential equations of retarded and neutral types, difference equations, etc.

This model has been studied in details by Delfour and Karrakchou [1], [6], and a complete theory was given. It is shown that of the more important particular cases of the considerded model are the system of neutral type and the system with delay on state, control and observation.

The system considered is

$$\begin{cases} Mx_t - Ny_t = B_0 u_t & , y_0 = \phi^1, x_0 = \phi^2 \\ \dot{y}(t) - Hy_t - Lx_t = B_1 u_t & , y(0) = \phi^0, u_0 = \mu, \end{cases}$$

and the control problem is to find a control function minimizing the functional

$$J(u) = Gy(T).y(T) + \int_0^T \{Qy(t).y(t) + Cx(t).x(t) + Ru(t).u(t)\}\, dt,$$

where G, Q, C, are self-adjoint non negative-functions and R coercive.

Classical techniques allowed us to show that the optimal control is characterized by an optimality system using an adjoint system. Then, techniques similar to HUM (Hibert Uniqueness Method) based on the construction of an appropriate Hilbert space and the inversion of an isomorphism (see Lions [12]) are used, and the optimal control is derived from the resolution of some algebraic equation in an appropriate Hilbert space. The idea was used in Karrakchou and Namir [8] and Karrakchou & al [9], [10], [3] for distributed systems with delay on the control in both the continuous and the discrete cases. This approach presents the advantage that classical numerical methods, such as Galerkin method, could easly be applied to approximate the solution.

2 Preliminaries results

Given the integers $n \geq 1, k \geq 1, m \geq 1$, a number $h, 0 < h \leq \infty$, and continuous linear maps

$$\begin{aligned} L &: K(-h,0;IR^k) \longrightarrow IR^n, \qquad M : K(-h,0;IR^k) \longrightarrow IR^k, \\ N &: K(-h,0;IR^n) \longrightarrow IR^k, \qquad H : K(-h,0;IR^n) \longrightarrow IR^n, \\ B_0 &: K(-h,0;IR^m) \longrightarrow IR^n, \qquad B_1 : K(-h,0;IR^m) \longrightarrow IR^k, \end{aligned}$$

where

$$K(-h,0;IR^n) = \begin{cases} C(-h,0;IR^n) & \text{if } h < +\infty \\ C_0(-h,0;IR^n) & \text{if } h = +\infty. \end{cases},$$

We shall consider the following coupled system of equations

$$\begin{cases} Mx_t - Ny_t = g(t) & , y_0 = \phi^1, x_0 = \phi^2 \\ \dot{y}(t) - Hy_t - Lx_t = f(t) & , y(0) = \phi^0 \end{cases}, \tag{1}$$

where

$$\begin{aligned} g &: [0,\infty[\longrightarrow IR^k, \qquad f : [0,\infty[\longrightarrow IR^n, \\ \phi^1 &: I(-h,0) \longrightarrow IR^k, \qquad \phi^2 : I(-h,0) \longrightarrow IR^n, \qquad \phi^0 \in IR^n, \end{aligned}$$

$$x : I(-h,\infty) \longrightarrow IR^k, \qquad y : I(-h,\infty) \longrightarrow IR^n,$$

$I(a,b) = [a,b] \cap IR$ and for all $t > 0$, the function x_t is defined on $I(-h,0) = [-h,0] \cap IR$ by $x_t(\theta) = x(t+\theta)$.

In this section we give two preliminary lemmas, the fundamental theorem of existence, uniqueness and continuity of solutions with respect to data, and we introduce the transposed system and give a very important technical lemma who allows to bind the solution of the original system to that of the adjoint one. For the proof see [1].

2.1 Preliminaries lemmas

Lemma 1 *Fix the length of the memory h, $0 < h \leq \infty$, and $p, 1 \leq p < \infty$.*

(i) There exists an $n \times k$ matrix λ of regular Borel measures such that

$$L\phi = \int_{-h}^{0} d_\theta \lambda \phi(\theta).$$

(ii) Fix a real number $T > 0$. For each x in $C_c(-\infty, T; IR^k)$, the function $\mathcal{L}x$ is defined as follows:

$$\mathcal{L}x(t) = Lx_t, \quad x_t(\theta) = x(t+\theta), \quad \theta \in I(-h,0), t \in [0,T].$$

It is continuous on $[0,T]$ and generates the continuous linear map

$$\mathcal{L} : C_c(-\infty, T; IR^k) \longrightarrow L^p(0,T; IR^n).$$

(iii) The above map has a continuous linear extension from C_c to L^p :

$$\mathcal{L} : L^p(-\infty, T; IR^k) \longrightarrow L^p(0,T; IR^n)$$

such that

$$\|\mathcal{L}x\|_{L^p(0,t)} \leq |\lambda| \, . \, \|x\|_{L^p(-h,t)}, \quad \forall t \in [0,T]$$

where $|\lambda|$ is the total variation of the matrix λ. ■

Similar constructions can be undertaken for the maps M, N, H, B_0 and B_1.

Lemma 2 *Let p, $1 \leq p < \infty$, be an integer and*

$$M : K(-h, 0; IR^k) \longrightarrow IR^k$$

be the continuous linear map to which we associate the $k \times k$ matrix μ of regular Borel measures and the continuous map $\mathcal{M}$:$L^p(-\infty, T; IR^k) \longrightarrow L^p(0,T; IR^k)$ (cf. Lemma1).

For $T > 0$, define the continuous linear operator

$$\mathcal{M}e_+^0 : L^p(0,T; IR^k) \longrightarrow L^p(0,T; IR^k) \quad (\mathcal{M}e_+^0)x = \mathcal{M}(e_+^0 x),$$

where $e_+^0 x$ is the extension by 0 to $I(-\infty, T)$ of the function x.

In an analogous fashion to $\mathcal{M}$, for all $t, 0 < t \leq T$, define the family of continuous linear operators

$$\mathcal{M}^t : L^p(-\infty, t; IR^k) \longrightarrow L^p(0, t; IR^k).$$

(in particular $\mathcal{M}^T = \mathcal{M}$).

(i) For all $t \in]0,T]$ and x in $L^p(0,T; IR^k)$,

$$\mathcal{M}^t e_+^0 (x\,|_{[0.t]}) = (\mathcal{M}e_+^0 x)\,|_{[0.t]}$$

(ii) If for all $t \in]0,T], \mathcal{M}^t e_+^0$ is an isomorphism, then there exists a constant C_T such that

$$\|x\|_{L^p(0,t)} \leq C_T \left\|\mathcal{M}^t e_+^0 x\right\|_{L^p(0,t)}.$$

(iii) If M has an isolated atom at the point 0, that is, if there exists a $k \times k$ invertible matrix M_0 of real numbers and a $k \times k$ matrix μ_0 of regular real Borel measures such that

$$M\phi = M_0 \phi(0) + \int_{-h}^{0} d_\theta \mu \phi(\theta), \quad \forall \phi \in K(-h, 0; IR^k),$$

and

$$\lim_{\epsilon \longrightarrow 0} |\mu_0| \, ([-\epsilon, 0]) = 0,$$

then $\mathcal{M}^t e_+^0$ is an isomorphism of $L^p(0,T; IR^k)$ onto itself for all $t > 0$. ■

Remark 3 *For convenience we shall often use the notation Lx_t instead of $\mathcal{L}x(t)$ for $x \in L^2(-h, T; IR^k)$. It will be the same thing for the maps M, N, H, B_0 and B_1.*

2.2 Existence and uniqueness result

Theorem 4 *(1) Fix an integer $p, 1 \le p < \infty$, and the maps H, L, M and N. Assume that for all $t > 0$, the operator $\mathcal{M}^t e_+^0$ is an isomorphism. Then for all*

$$\begin{array}{rcl} \phi & = & \left(\phi^0, \phi^1, \phi^2\right) \in Z^p = IR^n \times L^p(-h, 0; IR^n) \times L^p(-h, 0; IR^k), \\ f & \in & L^p_{loc}(0, \infty; IR^n) \text{ and } g \in L^p_{loc}(0, \infty; IR^k) \ , \end{array}$$

the system of equations

$$\begin{array}{l} Mx_t - Ny_t = g(t), \quad y_0 = \phi^1, x_0 = \phi^2 \\ \dot{y}(t) - Hy_t - Lx_t = f(t), \quad y(0) = \phi^0, \end{array}$$

has a unique solution (x, y) in $L^p_{loc}(0, \infty; IR^k) \times W^{1,p}_{loc}(0, \infty; IR^n)$. Moreover for each $T > 0$, there exists a constant $C(T) > 0$ such that

$$\|x\|_{L^p(0,T)} + \|y\|_{W^{1,p}(0,T)} \le C(T) \left[\|\phi\|_{Z^p} + \|g\|_{L^p(0,T)} + \|f\|_{L^p(0,T)} \right].$$

(2) For $f = g = 0$, for $t \ge 0$, define the map

$$S(t) : Z^p \longrightarrow Z^p, \quad S(t)\phi = (y(t), y_t, x_t).$$

The family $\{S(t) : t \ge 0\}$ is a strongly continuous semigroup of operators on Z^P of class C_0. Its infinitesimal generator A is defined on

$$D(A) = \left\{ \left(\phi^0, \phi^1, \phi^2\right) \in Z^p \;\middle|\; \begin{array}{c} \phi^1 \in W^{1,p}(-h, 0; IR^n) \\ \phi^2 \in W^{1,p}(-h, 0; IR^k) \\ M\phi^2 = N\phi^1, \phi^0 = \phi^1(0) \end{array} \right\},$$

by

$$A\left(\phi^0, \phi^1, \phi^2\right) = (H\phi^1 + L\phi^2, D\phi^1, D\phi^2). \blacksquare$$

2.3 Transposition and adjoint system

Definition 5 *For each continuous linear map*

$$L : K(-h, 0; IR^k) \longrightarrow IR^n,$$

we define the transposed operator L^T as follows

$$L^T\phi = \int_{-h}^{0} d_\theta \lambda^T \phi(\theta),$$

where λ^T is the transposed (in the usual sense) of the matrix λ.

Definition 6 *Given $q, 1 \le q < \infty$. The 'transposed" system associated with system (1) is defined as follows*

$$\left\{ \begin{array}{ll} M^T z_t - L^T w_t = g(t) & , w_0 = \psi^1, z_0 = \psi^2 \\ \dot{w}(t) - H^T w_t - N^T z_t = f(t) & , w(0) = \psi^0. \end{array} \right. \tag{2}$$

System (2) enjoys exactely the same properties as system (1) and Theorem (4) directly applies. When $f = g = 0$ the solution (z, w) of system (2) also generates a strongly continuous semigroup $\{S^T(t)\}$.

Remark 7 *The 'transposed" semigroup $\{S^T(t)\}$ and its generator A^T are not to be confused with the usual (topological) adjoint semigroup $\{S^*(t)\}$ and its generator A^*. They are completly different, but there is an interesting interwining relation between them (see [6]).*

Definition 8 *Fix $s, 1 < s < \infty$. Given $\psi \in Z^s, f \in L^s(0,T;IR^n)$ and $g \in L^s(0,T;IR^k)$. The adjoint system is defined by the system of equations*

$$\left\{ \begin{array}{l} M^T p^t - L^T q^t = g(t) \quad , q^T = \psi^1, p^T = \psi^2 \\ -[\dot{q}(t) + H^T q^t + N^T p^t] = f(t), \quad q(T) = \psi^0, \end{array} \right. \tag{3}$$

where for any function y in $L^s(0,T+h;IR^l)$ ($l \geq 1$ an integer)

$$y^t : I(-h,0) \longrightarrow IR^l$$

is defined by

$$y^t(\theta) = g(t-\theta).$$

It is readily seen that finding a solution (p,q) to (3) is equivalent to finding a solution (z,w) to the 'transposed system'

$$\left\{ \begin{array}{l} M^T z_t - L^T w_t = \widetilde{g}(t) \quad , w_0 = \psi^1, z_0 = \psi^2 \\ \dot{w}(t) - H^T w_t - N^T z_t = \widetilde{f}(t) \quad , w(0) = \psi^0, \end{array} \right.$$

where $(z,w,\widetilde{f},\widetilde{g})$ are related to (p,q,f,g) by the relations

$$\begin{array}{rcll} p(t) & = & z(T-t), & q(t) = w(T-t), \\ \widetilde{g}(t) & = & g(T-t), & \widetilde{f}(t) = f(T-t). \end{array}$$

Remark 9 *The operator $\mathcal{L}^* : C(0,T+h;IR^k) \longrightarrow L^p(0,T;IR^n)$ defined by $\mathcal{L}^* x(t) = L^T x^t$ has a continuous extension to $L^p(0,T+h;IR^k)$. For convenience, we will note always $L^T y^t$ instead of $\mathcal{L}^* x(t)$. It is the same for maps M, N, H, B_0 and B_1.*

The following Lemma will play an important role in the characterization of the optimality system.

Lemma 10 *Let L be a continuous linear map from $K(-h,0;IR^k)$ to $IR^n, p, 1 < p < \infty$, and $q, q^{-1} + p^{-1} = 1$, be real numbers. Then*

(1) for all z in $L^q(0.T;IR^n)$ and x in $L^p(0.T;IR^k)$,

$$\int_0^T z(T-t).(\mathcal{L}e_0^+ x)(t)dt = \int_0^T (\mathcal{L}^T e_0^+ z)(T-s).x(s)ds,$$

(2) for all p in $L^q(0,T+h;IR^n), p^T = 0$ and x in $L^p(-h,T;IR^k), x_0 = 0$

$$\int_0^T p(t).Lx_t dt = \int_0^T L^T p^t.x(t)dt. \blacksquare$$

3 Optimality system

Let us consider the system

$$(S) \left\{ \begin{array}{l} Mx_t - Ny_t = B_0 u_t \quad , y_0 = \phi^1, x_0 = \phi^2 \\ \dot{y}(t) - Hy_t - Lx_t = B_1 u_t \quad , y(0) = \phi^0, u_0 = \mu, \end{array} \right. \tag{4}$$

and the functional cost

$$J(u) = Gy(T).y(T) + \int_0^T \{Qy(t).y(t) + Cx(t).x(t) + Ru(t).u(t)\} dt, \tag{5}$$

where G, Q, C, are self-adjoint non negative matrix and R is coercive.

The problem considered is to find a control $u^\star \in L^2(0,T;IR^m)$ minimizing $J(u)$, where (x,y) is the solution of system (S) corresponding to the initial conditions $(\phi^0, \phi^1, \phi^2) \in Z^2, \mu \in L^2(-h,0;IR^m)$ and the control function $u \in L^2(0,T;IR^m)$.

Proposition 11 *Let $(\phi^0,\phi^1,\phi^2) \in Z^2 = IR^n \times L^2(-h,0;IR^n) \times L^2(-h,0;IR^k)$ and $\mu \in L^2(-h,0;IR^m)$. Suppose that the operator M satisfies the hypothesis given in Theorem 4. Then the optimal control is given by the optimality system*

$$\begin{cases} (S)\begin{cases} Mx_t - Ny_t = B_0u_t & ,y_0 = \phi^1, x_0 = \phi^2 \\ \dot{y}(t) - Hy_t - Lx_t = B_1u_t & ,y(0) = \phi^0, u_0 = \mu, \end{cases} \\ \\ (S^\star)\begin{cases} -M^Tp^t + L^Tq^t + Cz(t) = 0 & ,q^T = 0, p^T = 0 \\ \dot{q}(t) + H^Tq^t + N^Tp^t + Qy(t) = 0 & ,q(T) = Gy(T), \end{cases} \\ \\ \qquad u^\star(t) = -R^{-1}[B_0^Tp^t + B_1^Tq^t]. \end{cases}$$

*Proof.*Under the hypothesis given in Theorem (4), for all $u \in L^2(0,T;IR^m)$ the solution of system (S) exists and it is unique. Classical techniques (see [4], [11], [5]) allowed us to see that the optimal control $u^\star$ is given by

$$J'(u^\star)v = 0, \qquad \forall v \in L^2(0,T;IR^m).$$

Or,

$$\frac{1}{2}J'(u)v = Gy(T).\widetilde{y}(T) + \int_0^T \{Qy(t).\widetilde{y}(t) + Cx(t).\widetilde{x}(t) + Ru(t).v(t)\}\,dt$$

where $(\widetilde{x},\widetilde{y})$ is the solution of the system

$$\begin{cases} M\widetilde{x}_t - N\widetilde{y}_t = B_0v_t & ,(\widetilde{y}(0),\widetilde{y}_0,\widetilde{x}_0) = 0 \\ \dot{\widetilde{y}}(t) - H\widetilde{y}_t - L\widetilde{x}_t = B_1v_t & ,v_0 = 0 \end{cases}.. \tag{6}$$

Introduce the adjoint system

$$\begin{cases} -M^Tp^t + L^Tq^t + Cx(t) = 0 & ,q^T = 0, p^T = 0 \\ \dot{q}(t) + H^Tq^t + N^Tp^t + Qy(t) = 0 & ,q(T) = Gy(T) \end{cases}, \tag{7}$$

then $J'(u^\star)v = 0$ is equivalent to

$$\begin{array}{c} Gy(T).\widetilde{y}(T) - \int_0^T(\dot{q}(t) + H^Tq^t + N^Tp^t).\widetilde{y}(t)dt + \\ \int_0^T(M^Tp^t - L^Tq^t).\widetilde{x}(t)dt + \int_0^T Ru^\star(t).v(t)dt = 0. \end{array}$$

Using an integration by parts formula, system (6) and the part (2) of Lemma 10, we show that

$$-\int_0^T N\widetilde{y}_t.p(t)dt + \int_0^T M\widetilde{x}_t.p(t)dt + \int_0^T q(t).B_1v_tdt + \int_0^T Ru^\star(t).v(t)dt = 0.$$

Again using system (6), we have

$$\int_0^T B_0v_t.p(t)dt + \int_0^T q(t).B_1v_tdt + \int_0^T Ru^\star(t).v(t)dt = 0.$$

Then

$$\int_0^T [B_0^Tp^t + B_1^Tq^t + Ru^\star(t)].v(t)dt = 0, \forall v \in L^2(0,T;IR^m).$$

Finally the optimal control is given by

$$u^\star(t) = -R^{-1}[B_0^Tp^t + B_1^Tq^t] \blacksquare$$

■

4 Resolution of the problem

The next step consists in decoupling the Hamiltonian system. For this we will use techniques similar to the Hilbert Uniqueness Method, introduced first by Lions [12] to study the exact controllability for hyperbolic systems and generalized or "adapted" by other authors to study differents concepts for different dynamical systems [2], [8], [3], etc. We show that introducing a new norm on the state space, the optimal control is derived from the resolution of some algebraic equation in an appropriate Hilbert space.

Let us define the space
$$F = IR^n \times L^2(0,T;IR^n) \times L^2(0,T;IR^k).$$
For all $\varphi = (\varphi^0, \varphi^1, \varphi^2) \in F$, the system
$$\begin{cases} -M^T p^t + L^T q^t + C^{\frac{1}{2}}\varphi^2(t) = 0 & , q^T = 0, p^T = 0 \\ \dot{q}(t) + H^T q^t + N^T p^t + Q^{\frac{1}{2}}\varphi^1(t) = 0 & , q(T) = G^{\frac{1}{2}}\varphi^0 \end{cases}, \tag{8}$$
has a unique solution $(q,p) \in W^{1,2}(0,T;IR^n) \times L^2(0,T;IR^k)$. It is continuous with respect to data (cf. Theorem 4)

Define now on F the functional $\|.\|_1$ by
$$\|\varphi\|_1^2 = \|\varphi\|^2 + \int_0^T \left| R^{-\frac{1}{2}}(B_0^T p^t + B_1^T q^t) \right|^2 dt.$$
It is a norm on F that is equivalent to the usual one. This is due to the fact that the operators B_0^T and B_1^T are bounded on $L^p(0, T+h)$, and that the solution of the transposed system (8) is continuous with respect to data.
In what follows we will see that this norm can be associated to an inner product on F, and that the space $(F, \|.\|_1)$ is a Hilbert space.

Let Λ be the operator defined on F by:

1- For $\varphi \in F$, we resolve the system (8)

2- we put
$$u(t) = R^{-1}(B_0^T p^t + B_1^T q^t), \tag{9}$$
3- using this control function $u(t)$, we resolve the system with initial conditions equal to zero
$$\begin{cases} Mx_t - Ny_t = B_0 u_t & , y_0 = 0, x_0 = 0 \\ \dot{y}(t) - Hy_t - Lx_t = B_1 u_t & , y(0) = 0, u_0 = 0 \end{cases}, \tag{10}$$
4- we finally take
$$\Lambda\varphi = \varphi + (G^{\frac{1}{2}}y(T), Q^{\frac{1}{2}}y(.), C^{\frac{1}{2}}x(.)), \tag{11}$$
where (x,y) is the solution of system (10). It is easy to see that the operator Λ is symmetric and positive definite.

Proposition 12 *For all $\varphi, \psi \in F$, we have*
1- $\langle \Lambda\varphi, \psi \rangle = \langle \varphi, \Lambda\psi \rangle$,
2- $\langle \Lambda\varphi, \varphi \rangle = \|\varphi\|_1^2$,
where$\langle .,. \rangle$ is the scalar product in the space F.

*Proof.*1- Given $\varphi, \psi \in F$. Let (y,x,p,q,u) be the solution of system (8), (9), (10) corresponding to φ, and $(\widetilde{y}, \widetilde{x}, \widetilde{p}, \widetilde{q}, \widetilde{u})$ be the solution corresponding to ψ. We have
$$\begin{aligned} \langle \Lambda\varphi, \psi \rangle &= \langle \varphi, \psi \rangle + \left\langle (G^{\frac{1}{2}}y(T), Q^{\frac{1}{2}}y(.), C^{\frac{1}{2}}x(.)), \psi \right\rangle \\ &= \langle \varphi, \psi \rangle + G^{\frac{1}{2}}y(T).\psi^0 + \int_0^T Q^{\frac{1}{2}}y(t).\psi^1(t)dt + \int_0^T C^{\frac{1}{2}}x(t).\psi^2(t)dt, \end{aligned}$$
and by equation (8) applied to ψ we have
$$\begin{aligned} \langle \Lambda\varphi, \psi \rangle &= \langle \varphi, \psi \rangle + G^{\frac{1}{2}}y(T).\psi^0 - \int_0^T y(t).(\dot{\widetilde{q}}(t) + H^T\widetilde{q}^t + N^T\widetilde{p}^t)dt \\ &\quad + \int_0^T x(t).(M^T\widetilde{p}^t - L^T\widetilde{q}^t)dt. \end{aligned}$$
By integration by parts and using equation (10) we obtain
$$\begin{aligned} \langle \Lambda\varphi, \psi \rangle &= \langle \varphi, \psi \rangle + \int_0^T \{(Hy_t + Lx_t + B_1u_t).\widetilde{q}(t) - y(t).(H^T\widetilde{q}^t + N^T\widetilde{p}^t)\}dt \\ &\quad + \int_0^T x(t).(M^T\widetilde{p}^t - L^T\widetilde{q}^t)dt. \end{aligned}$$

Then using part (2) of Lemma 10, and the first equation of system (10) we have

$$\begin{aligned}\langle \Lambda\varphi,\psi\rangle &= \langle \varphi,\psi\rangle + \int_0^T u(t).(B_1^T\widetilde{q}^t + B_0^T\widetilde{p}^t)dt \\ &= \langle \varphi,\psi\rangle + \int_0^T u(t).R\widetilde{u}(t)dt.\end{aligned}$$

This proves (1).

2- For $\psi=\varphi$, we have

$$\begin{aligned}\langle \Lambda\varphi,\psi\rangle &= \langle \varphi,\psi\rangle + \int_0^T u(t).Ru(t)dt \\ &= \langle \varphi,\psi\rangle + \int_0^T R^{-1}(B_0^Tp^t + B_1^Tq^t).(B_0^Tp^t + B_1^Tq^t)dt \\ &= \langle \varphi,\psi\rangle + \int_0^T \left|R^{-\frac{1}{2}}(B_0^Tp^t + B_1^Tq^t\right|^2 dt \\ &= \|\varphi\|_1 . \blacksquare\end{aligned}$$

■

It follows from the last proposition that Λ is a continuous, coercive isomorphism. Hence the main result is then given by the following theorem

Theorem 13 *The optimal control solution of the problem (4), (5) is given by*

$$u(t) = R^{-1}(B_0^T\overline{p}^t + B_1^T\overline{q}^t),$$

where $(\overline{q},\overline{p})$ *is given by:*

1- solve the "zero-controlled" system

$$\begin{cases} Mx_t - Ny_t = B_0u_t & ,y_0=\phi^1, x_0=\phi^2, u(t)=0 \text{ for } t>0 \\ \dot{y}(t) - Hy_t - Lx_t = B_1u_t & ,y(0)=\phi^0, u_0=\mu \end{cases}, \tag{12}$$

2- solve the equation

$$\Lambda\varphi = -(G^{\frac{1}{2}}y(T), Q^{\frac{1}{2}}y(.), C^{\frac{1}{2}}x(.)), \tag{13}$$

3- solve the adjoint system

$$\begin{cases} -M^T\overline{p}^t + L^T\overline{q}^t + C^{\frac{1}{2}}\varphi^2(t) = 0 & ,\overline{q}^T=0, \overline{p}^T=0 \\ \dot{\overline{q}}(t) + H^T\overline{q}^t + N^T\overline{p}^t + Q^{\frac{1}{2}}\varphi^1(t) = 0 & ,\overline{q}(T) = G^{\frac{1}{2}}\varphi^0 \end{cases}. \tag{14}$$

Moreover the optimal cost is given by

$$J(u^*) = \|\varphi\|_1^2 .$$

Proof. By Proposition 12, Λ is an isomorphism on F. Then φ is uniquely defined.

Let $(y(t,(\phi,\mu),u), x(t,(\phi,\mu),u))$ be the solution of system (S) coresponding to the initial condition (ϕ,μ) and the control function u. By definition of Λ we have

$$\begin{aligned}&\varphi + (G^{\frac{1}{2}}y(T,0,u), Q^{\frac{1}{2}}y(.,0,u.), C^{\frac{1}{2}}x(.,0,u)) \\ =\ & -(G^{\frac{1}{2}}y(T,(\phi,\mu),0), Q^{\frac{1}{2}}y(.,(\phi,\mu),0), C^{\frac{1}{2}}x(.,(\phi,\mu),0))\end{aligned}$$

where u is given by the equation (9).

Then, by linearity

$$\varphi = -(G^{\frac{1}{2}}y(T), Q^{\frac{1}{2}}y(.), C^{\frac{1}{2}}x(.)),$$

where (y,x) is the solution of system (S).

Equation (14) is then written

$$\begin{cases} -M^T\overline{p}^t + L^T\overline{q}^t - Cx(t) = 0 & ,\overline{q}^T=0, \overline{p}^T=0 \\ \dot{\overline{q}}(t) + H^T\overline{q}^t + N^T\overline{p}^t - Qy(t) = 0 & ,\overline{q}(T) = Gy(T). \end{cases}$$

By unicity of the solution we have $(\overline{q},\overline{p})=-(q,p)$ with (q,p) is the solution of the adjoint system (S^*).

Then

$$u(t)=-R^{-1}(B_0^Tp^t+B_1^Tq^t),$$

and the optimality system is satisfied.

Moreover,

$$\begin{aligned}
J(u^*) &= G^{\frac{1}{2}}y(T).G^{\frac{1}{2}}y(T)+\int_0^T\{Q^{\frac{1}{2}}y(t).Q^{\frac{1}{2}}y(t)+C^{\frac{1}{2}}x(t).C^{\frac{1}{2}}x(t))dt \\
&\quad +\int_0^T Ru(t).u(t)\}dt \\
&= \left|\varphi^0\right|^2+\int_0^T\left|\varphi^1(t)\right|^2dt+\int_0^T\left|\varphi^2(t)\right|^2dt \\
&\quad +\int_0^T(B_0^Tp^t+B_1^Tq^t).R^{-1}(B_0^Tp^t+B_1^Tq^t)dt \\
&= \|\varphi\|^2+\int_0^T\left|R^{-\frac{1}{2}}(B_0^Tp^t+B_1^Tq^t)\right|^2dt \\
&= \|\varphi\|_1^2.
\end{aligned}$$

This completes the proof. ■

Remark 14 *In order to obtain the optimal control u^* one has to solve equation (13). However in general we do not know an explicit form of the operator Λ^{-1}. But since the bilinear continuous form: $(\varphi,\psi)\in F\times F\longrightarrow(\Lambda\varphi,\psi)$ is coercive, the Galerkin method can easly be applied to approximate the solution.*

5 Particular cases:

In this section we give some important cases of delay systems with delay on the control, delay on the control and observation, and systems of neutral type. We will see that our approach can be applied to those systems and the optimal control can be easly characterized by the solution of some algebraic equations.

5.1 Systems with delay on state and control

The problem considered is:

Mnimize on $L^2(0,T;IR^n)$ the functional

$$J(u)=Gy(T).y(T)+\int_0^T\{Qy(t).y(t)+Ru(t).u(t)\}\,dt, \tag{15}$$

where y is the solution of the delayed system

$$\dot{y}(t)-Hy_t=Bu_t \quad ,y(0)=\phi^0,y_0=\phi^1,u_0=\mu. \tag{16}$$

Such problem has been considered by Delfour [5] in the case $G=0$. Using a state approach, the optimal control was characterized by the resolution of some Riccati equation. The fact that $G=0$ was necessary to have the desired regularity about the adjoint state. The approach used here permits us to include the final state in the cost functional.

Let us put $x(t)=y(t)$. Equation (16) can be written as

$$\left\{\begin{array}{ll} Mx_t-Ny_t=0 & ,y_0=\phi^1,x_0=0 \\ \dot{y}(t)-Hy_t=Bu_t & ,y(0)=\phi^0,u_0=\mu \end{array}\right. ,$$

where M and N are defined by $M\phi=N\phi=\phi(0)$. The problem considered here could be then treated as a particular case of the original problem (4), (5) and the optimal control is characterized by the following results

5.1.1 Definition of the operator Λ

Let $F = IR^n \times L^2(0,T;IR^n)$. For $\varphi = (\varphi^0, \varphi^1) \in F$, we define $\Lambda\varphi$ by

$$\Lambda\varphi = \varphi + (G^{\frac{1}{2}}y(T), Q^{\frac{1}{2}}y(.)),$$

where y is given by the resolution of the system

$$\begin{cases} \dot{q}(t) + H^T q^t + Q^{\frac{1}{2}}\varphi^1(t) = 0 \quad , q(T) = G^{\frac{1}{2}}\varphi^0, q^T = 0, \\ u(t) = R^{-1}B^T q^t, \\ \dot{y}(t) - Hy_t = Bu_t \quad , y(0) = 0, y_0 = 0, u_0 = 0 \end{cases}.$$

The operator Λ is an isomorphism on F, and we have

$$\langle \Lambda\varphi, \varphi \rangle = \|\varphi\|_1^2,$$

where $\|.\|_1$ is a norm defined on F by

$$\|\varphi\|_1^2 = \|\varphi\|^2 + \int_0^T \left| R^{-\frac{1}{2}} B^T q^t \right|^2 dt.$$

Now, we can characterize the optimal control by the following theorem.

5.1.2 Theorem

The optimal control solution of the problem (15), (16) is given by the resolution of the algebraic system

$$\begin{cases} \dot{y}(t) - Hy_t = Bu_t \quad , y(0) = \phi^0, y_0 = \phi^1, u_0 = \mu, u(t) = 0 \text{ for } t > 0, \\ \\ \Lambda\varphi = -(G^{\frac{1}{2}}y(T), Q^{\frac{1}{2}}y(.)), \\ \\ \dot{q}(t) + H^T q^t + Q^{\frac{1}{2}}\varphi^1(t) = 0 \quad , q(T) = G^{\frac{1}{2}}\varphi^0, \\ \\ u^*(t) = R^{-1}B^T q^t \end{cases}.$$

5.2 Systems with delay on state, control and observation

Let us consider the problem

Mnimize on $L^2(0,T;IR^n)$ the functional

$$J(u) = Gy(T).y(T) + \int_0^T \{|Ey_t + Du_t|^2 + Ru(t).u(t)\}\, dt \tag{17}$$

where y is the solution of the delayed system

$$\dot{y}(t) - Hy_t = Bu_t \quad , y(0) = \phi^0, y_0 = \phi^1, u_0 = \mu \tag{18}$$

and the operators E and D are linear and continuous.

Define the variable $x(t)$ by

$$x(t) = Ey_t + Du_t.$$

The problem (17), (18) can now be written as

Mnimize on $L^2(0,T;IR^n)$ the functional

$$J(u) = Gy(T).y(T) + \int_0^T \{|x(t)|^2 + Ru(t).u(t)\}\, dt$$

where (x, y) is the solution of the system

$$\begin{aligned} x(t) - Ey_t &= Du_t, \quad x_0 = 0, u_0 = \mu \\ \dot{y}(t) - Hy_t &= Bu_t \quad , y(0) = \phi^0, y_0 = \phi^1. \end{aligned}$$

It is again a particular case of problem (4), (5), with $M\phi = \phi(0).Q = 0$ and $C = I$ Conditions of applications of Theorem 4, and Theorem 13 are all satisfies and the optimal control is characterized by the inversion of an isomorphism in an appropppriate Hilbert space. The results can be then given by the following

5.2.1 Definition of the operator Λ

Let $F = IR^n \times L^2(0,T;IR^k)$. For $\varphi = (\varphi^0, \varphi^1) \in F$, we define $\Lambda\varphi$ by

$$\Lambda\varphi = \varphi + (G^{\frac{1}{2}} y(T), x(.))$$

where (x, y) is the solution of the system

$$\begin{cases} \begin{cases} p(t) = \varphi^1(t), \quad q^T = p^T = 0 \\ \dot{q}(t) + H^T q^t + E^T p^t = 0 \quad , q(T) = G^{\frac{1}{2}}\varphi^0, \end{cases} \\ u(t) = R^{-1}(D^T p^t + B^T q^t), \\ \begin{cases} x(t) - Ey_t = Du_t, \quad x_0 = 0, u_0 = 0 \\ \dot{y}(t) - Hy_t = Bu_t \quad , y(0) = 0, y_0 = 0. \end{cases} \end{cases}$$

The operator Λ is an isomorphism on F, and we have

$$\langle \Lambda\varphi, \varphi \rangle = \|\varphi\|_1^2$$

where $\|.\|_1$ is a norm defined on F by

$$\|\varphi\|_1^2 = \|\varphi\|^2 + \int_0^T \left| R^{-\frac{1}{2}}(D^T p^t + B^T q^t) \right|^2 dt.$$

Now, we can characterize the optimal control by the following theorem

5.2.2 Theorem

The optimal control solution of the problem (17), (18) is given by the resolution of the algebraic system

$$\begin{cases} \begin{cases} x(t) - Ey_t = Du_t, \quad u_0 = \mu, u(t) = 0 \text{ pour } t > 0 \\ \dot{y}(t) - Hy_t = Bu_t \quad , y(0) = \phi^0, y_0 = \phi^1, \end{cases} \\ \Lambda\varphi = -(G^{\frac{1}{2}} y(T), x(.)), \\ \begin{cases} p(t) = \varphi^1(t), \quad q^T = p^T = 0 \\ \dot{q}(t) + H^T q^t + E^T p^t = 0 \quad , q(T) = G^{\frac{1}{2}}\varphi^0, \end{cases} \\ u^\star(t) = R^{-1}(D^T p^t + B^T q^t). \end{cases}$$

5.3 System of neutral type

Let us consider the neutral system

$$\frac{d}{dt} Mx_t = Lx_t + Bu_t, \quad x_0 = \phi, u_0 = \mu, \tag{19}$$

where M is a continuous linear operator with the classical hypothesis that the corresponding matrix of regular Borel measures has an isolated atom at 0.
To this equation we associate the functional cost

$$J(u) = \int_0^T \{Cx(t).x(t) + Ru(t).u(t)\} \, dt \tag{20}$$

Introducing the new variable $y(t) = Mx_t$, equation(19) can be written as the system

$$\begin{array}{l} Mx_t - y(t) = 0, \quad x_0 = \phi \\ \dot{y}(t) - Lx_t = Bu_t, \quad y(0) = M\phi, u_0 = \mu, \end{array}$$

and the problem (19, (20) is again a particular case of the original one (4), (5). Our approach can then be applied and the optimal control is characterized as follows.

5.3.1 Definition of the operator Λ

Let $F = L^2(0,T;IR^k)$. For $\varphi \in F$, we define $\Lambda\varphi$ by

$$\Lambda\varphi = \varphi + C^{\frac{1}{2}}x(.)$$

where x is the solution of the system

$$\begin{cases} -M^T p^t + L^T q^t + C^{\frac{1}{2}}\varphi = 0, \quad q^T = p^T = 0 \\ \dot{q}(t) + p(t) = 0 \quad , q(T) = 0, \end{cases}$$

$$u(t) = R^{-1}B^T q^t,$$

$$\begin{cases} Mx_t - y(t) = 0, \quad x_0 = 0, u_0 = 0 \\ \dot{y}(t) - Lx_t = Bu_t \quad , y(0) = 0. \end{cases}$$

The operator Λ is an isomorphism on F, and we have

$$\langle \Lambda\varphi, \varphi \rangle = \|\varphi\|_1^2$$

where $\|.\|_1$ is a norm defined on F by

$$\|\varphi\|_1^2 = \|\varphi\|^2 + \int_0^T \left| R^{-\frac{1}{2}} B^T q^t \right|^2 dt.$$

Now, we can characterize the optimal control by the following theorem

5.3.2 Theorem

The optimal control solution of the problem (19), (20) is given by the solution of the algebraic system

$$\begin{cases} \begin{cases} Mx_t - y(t) = 0, \quad x_0 = \phi \\ \dot{y}(t) - Lx_t = Bu_t, \quad y(0) = M\phi, u_0 = \mu, u(t) = 0 \text{ for } t > 0, \end{cases} \\ \\ \Lambda\varphi = -C^{\frac{1}{2}}x(.), \\ \\ \begin{cases} -M^T p^t + L^T q^t + C^{\frac{1}{2}}\varphi(t) = 0, \quad q^T = p^T = 0 \\ \dot{q}(t) + p(t) = 0 \quad , q(T) = 0, \end{cases} \\ \\ u^\star(t) = R^{-1}B^T q^t. \end{cases}$$

References

[1] A. Bensoussan, G. Da Prato, M. Delfour & S. Mitter, "Representation and Control of Infinite Dimensional Systems", Vol. 1, Birkhäuser, Boston, 1992.

[2] A. Belfekih & A. El jai, Exacte contrôlabilité et contrôle optimal des systèmes paraboliques. RAIRO Automat.-Prod. Inform. Ind. 24 , no. 4 (1990), pp. 357–376.

[3] L.Chraibi, J.Karrakchou, A.Ouansafi & M.Rachik , Exact controllability and optimal control for distributed systems with a discrete delayed control. Journal of the Franklin Institute, vol 337 (2000), pp. 499-514.

[4] R. Curtain & A.J.Pritchard, Infinite dimensional linear systems theory. Lecture Notes in Control and Information Sciences, 8, 1978.

[5] M.C.Delfour, The linear quadratic optimal control problem with delays in state and control variables: A state space approach. SIAM J. Control and Optimization. Vol 24, No 5 (1986), pp.835-883.

[6] M.C.Delfour & J.Karrakchou, State space theory of linear time invariant systems with delays in state, control and observation variables, J. Math. Ana. and App. Vol 125, 2, (1987). Part I, pp. 361-399. Part II, pp.400-452.

[7] J.K. Hale, "Theory of Functional Differential Equations", Spring-Verlag, New York, 1977.

[8] J.Karrakchou & A.Namir, A new approach for a linear quadratic optimal-control problem of distributed systems with delays in the control. IMA Journal of Mathematical control and Information, 9 (1992), pp. 221-230.

[9] J.Karrakchou & M.Rachik, Optimal control of discrete distributed systems with delays in the control: the finite horizon case. Archives of Control Sciences, vol4(XL),1-2 (1995), pp. 37-53.

[10] J.Karrakchou , R.Rabah & M.Rachik, Optimal control of discrete distributed systems with delay in state and control: State space theory and HUM approaches. Systems Analysis, Modelling Simulation. Vol.30 (1998), pp.225-245.

[11] J.L.Lions, Contrôle optimal des systèmes gouvernés par des équations aux dérivées partielles. Dunod et Gauthiers-Villars, Paris, 1986.

[12] J.L.Lions, Exact controllability, stabilization and perturbation for distributed systems. SIAM Reviews, 30, pp.1-86, 1988.

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