Invariants of the Nilpotent and Solvable Triangular Lie Algebras

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Abstract

Invariants of the coadjoint representation of two classes of Lie algebras are calculated. The first class consists of the nilpotent Lie algebras $T(M)$, isomorphic to the algebras of upper triangular $M \times M$ matrices. The Lie algebras $T(M)$ is shown to have $[M/2]$ functionally independent invariants. They can all be chosen to be polynomials and they are presented explicitly. The second class consists of the solvable Lie algebras $L(M, f)$ with $T(M)$ as their nilradical and f additional linearly nilindependent elements. Some general results on the invariants of $L(M, f)$ are given and the cases $M = 4$ and $f = 1$, or $f = M - 1$ are treated in detail.

Résumé

Nous considérons les invariants de la représentation coadjointe pour deux classes d'algèbres de Lie. La première classe étudiée sont les algèbres de Lie nilpotentes $T(M)$, isomorphes à l'algèbre des matrices $M \times M$ triangulaires supérieures. Nous montrons que les algèbres de Lie $T(M)$ possèdent $[M/2]$ invariants fonctionnellement indépendants. Ceux-ci peuvent être choisis comme des polynômes et sont présentés explicitement. La deuxième classe considérée sont les algèbres de Lie résolubles $L(M, f)$ avec $T(M)$ comme nilradical et f éléments linéairement nilindépendants. Certains résultats généraux sur les invariants de $L(M, f)$ sont présentés et les cas $M = 4$ ainsi que $f = 1$, ou $f = M - 1$ sont considérés en détails.

1 Introduction

The purpose of this paper is to present some results on the invariants of two classes of Lie algebras, over the field of complex or real numbers ($K = \mathbb{C}$ or $\mathbb{R}$). The first class are the finite triangular nilpotent Lie algebras $T(M)$ of dimension $M(M-1)/2$. By triangular nilpotent Lie algebra, we mean the nilpotent Lie algebra isomorphic to the Lie algebra of strictly upper triangular $M \times M$ matrices. The second class of algebras studied below are the finite solvable triangular Lie algebras $L(M, f)$ that have $T(M)$ as their nilradicals (maximal nilpotent ideals) and contain f further nonnilpotent elements. For the algebras $L(M, f)$, use will be made of a recent article [1] in which we obtained a classification of such Lie algebras and presented the general form of the commutation relations.

In physics, invariant operators of the symmetry group of a physical system and its subgroups provide quantum numbers. Indeed, the eigenvalues of the invariant operators of the entire symmetry group will be the quantum numbers, characterizing the system as such (e.g., the particle mass and spin in the case of the Poincaré group). The invariant operators of subgroups will then characterize states of the system (its energy, linear or angular momentum etc.) [2].

In other applications, invariant operators of dynamical groups provide mass formulas [3, 4], energy spectra [5, 6] and in general characterize specific properties of physical systems.

Let us stress here that in this context the concept of an invariant need not mean a Casimir operator. Indeed, the problem of finding invariants will be reduced to that of solving a certain set of linear first order partial differential equations [7, 8]. These may have polynomial solutions, giving rise to Casimir operators. They may also have rational solutions, giving rise to rational invariants. Finally, the equations may have more general solutions, including transcendental functions of various types, leading to general invariants.

Casimir operators are polynomials in the enveloping algebra of a Lie algebra that commute with all elements of the Lie algebra. In other words, a Casimir operator of a Lie algebra is an element of the centre of the enveloping algebra. For a Lie algebra L, the Casimir operators can be calculated directly. Namely, we impose that a general polynomial in the enveloping algebra commutes with all basis elements X_i of the Lie algebra L. However, more efficiently, they can be calculated as invariants of the coadjoint representation of the corresponding Lie algebra [9, 10].

The Casimir operators of semisimple Lie algebras are well known. Their number p is equal to the rank of the considered Lie algebra [11, . . . ,17]. Moreover, for semisimple Lie algebra, all invariants of the coadjoint representation can be expressed as functions of p homogeneous polynomials.

For solvable Lie algebras, the situation is less clear. Neither the specific type of functions, nor the number of functionally independent invariants is known.

One method, for calculating the polynomial and other invariants for arbitrary Lie algebras, is an infinitesimal one. This method has been presented in [7] and applied to low dimensional Lie algebras [18, 19], to subalgebras of the Poincaré Lie algebra [20] and to solvable Lie algebras with Heisenberg or Abelian nilradicals [21, 22].

From a mathematical point of view, in the representation theory of solvable Lie algebras, polynomial and non-polynomial invariants in the coadjoint representation appear on the same footing: they characterize irreducible representations. Casimir operators in the enveloping algebra correspond to polynomial invariants. The functions of the infinitesimal operators, corresponding to the non-polynomial invariants, will be called 'generalized Casimir operators'. In the study of the integrability of classical Hamiltonian systems, integrals of motion do not have to be polynomials in the dynamical variables [23, 24].

In Section 2 we formulate the problem of calculating the invariants of the coadjoint representation. Section 3 is devoted to the nilpotent algebras $T(M)$. We calculate the invariants explicitly. There are $[M/2]$ functionally independent invariants, all of them polynomials. In Section 4 we calculate the invariants of the solvable Lie algebras $L(M, f)$. We first treat the case $M = 4$ in detail, then present results and conjectures for $L(M, M-1)$ and $L(M, 1)$.

2 General results and formulation of the problem

Let us consider a N-dimensional Lie algebra given by the basis $\{Y_1, \ldots, Y_N\}$ and the commutation relations

$$[Y_i, Y_j] = \sum_{k=1}^{N} C_{ij}^k \, Y_k \qquad 1 \leq i, j, k \leq N. \tag{2.1}$$

In order to calculate the invariants of the Lie algebra L, we shall work on the dual of L. We consider smooth functions $F: \ (y_1, \ldots, y_N) \rightarrow K$ where the variables y_i are ordinary (commuting) variables on the space L^*, dual of L, and K is the field of complex or real numbers ($K = \mathbb{C}$ or $\mathbb{R}$). The generators Y_i are given in the coadjoint representation by the differential operators

$$\hat{Y}_i = \sum_{j,k} C_{ij}^k \, y_k \frac{\partial}{\partial y_j} \,. \tag{2.2}$$

We can verify easily that the differential operators $\hat{Y}_i$ satisfy the commutation relations (2.1).

The function F will be an invariant of the coadjoint representation of L if it satisfies the linear first order partial differential equations

$$\hat{Y}_i \cdot F = 0 \qquad i = 1, \ldots, N \tag{2.3}$$

which, one hopes, can be solved by standard methods.

Our aim is to find a complete set of functionally independent solutions to equation (2.3), for nilpotent and solvable triangular Lie algebras. If the solutions are polynomials, we obtain Casimir operators by replacing the variables y_i by the generators Y_i and symmetrizing, whenever necessary. The number of independent solutions n_I, i.e. the number of functionally independent invariants, is equal to

$$n_I = N - rank(M) \tag{2.4}$$

where M is the antisymmetric matrix with elements

$$M_{ij} = \sum_{k=1}^{N} C_{ij}^k \, y_k \tag{2.5}$$

(see Ref. [7]).

3 Invariants of nilpotent triangular Lie algebras

3.1 Structure of the nilpotent Lie algebra $T(M)$ and its realization by differential operators

Let us consider the finite triangular Lie algebra $T(M)$ over the field K of complex or real numbers. A basis for this algebra is

$$\begin{gathered}\{N_{ik} \ \mid \ 1 \leq i < k \leq M\} \\ (N_{ik})_{ab} = \delta_{i,a} \ \delta_{k,b} \qquad \dim \ T(M) = \tfrac{1}{2}M(M-1) \equiv r\end{gathered} \tag{3.1}$$

with $M > 3$. The Lie algebra $T(2)$ is trivial and the Lie algebra $T(3)$ is isomorphic to the Heisenberg algebra $H(1)$. The dimension $M = 3$ is the only case for which there is an isomorphism between the triangular and the Heisenberg Lie algebras.

The commutation relations of $T(M)$ are given by

$$[N_{ik}, N_{ab}] = \delta_{k,a} N_{ib} - \delta_{b,i} N_{ak}. \tag{3.2}$$

This basis can be represented by the standard basis of the strictly upper triangular $M \times M$ matrices.

The differential operators $\hat{N}_{ik}$ realizing the coadjoint representation of $T(M)$, are

$$\hat{N}_{ik} = \sum_{b=k+1}^{M} n_{ib} \frac{\partial}{\partial n_{kb}} - \sum_{a=1}^{i-1} n_{ak} \frac{\partial}{\partial n_{ai}}. \tag{3.3}$$

Note that $\hat{N}_{1M} \equiv 0$ in (3.3), since N_{1M} commutes with all the elements of $T(M)$.

We shall realize the coadjoint representation of $T(M)$ in a space of differentiable functions of r variables, i.e.

$$F = F(n_{12}, n_{23}, \ldots, n_{(M-1)M}, n_{13}, n_{24}, \ldots, n_{(M-2)M}, \ldots, n_{1M}). \tag{3.4}$$

The function F will be an invariant of the coadjoint representation of $T(M)$, if it satisfies the linear first order partial differential equations

$$\hat{N}_{ik} \cdot F = 0 \qquad 1 \leq i < k \leq M. \tag{3.5}$$

3.2 Definitions and results

Let us consider the set of strictly upper triangular $M \times M$ matrices $Q = Q(M)$ over the field K i.e.

$$Q_{ik} = \begin{cases} n_{ik} & for \ \ k - i \geq 1 \\ 0 & otherwise. \end{cases} \tag{3.6}$$

We define the determinant $Z_\mu = Z_\mu(M)$ constructed from the $\mu \times \mu$ right upper corner sub-matrix of the matrix Q, i.e.

$$Z_\mu = \begin{vmatrix} n_{1(M-\mu+1)} & n_{1(M-\mu+2)} & \cdots & n_{1M} \\ n_{2(M-\mu+1)} & n_{2(M-\mu+2)} & \cdots & n_{2M} \\ \vdots & \vdots & & \vdots \\ n_{\mu(M-\mu+1)} & n_{\mu(M-\mu+2)} & \cdots & n_{\mu M} \end{vmatrix} \qquad 1 \leq \mu \leq \left[\frac{M}{2}\right] \tag{3.7}$$

where we shall use the standard notation $[x]$ for the entire part of a positive number. In particular,

$$p = \left[\frac{M}{2}\right] = \begin{cases} \frac{M}{2} & for \ \ M = 2p \\ \frac{M-1}{2} & for \ \ M = 2p+1. \end{cases} \tag{3.8}$$

Theorem 1 *The triangular Lie algebra $T(M)$ defined by equations (3.1) and (3.2) has exactly $[M/2]$. functionally independent invariants. A basis of invariants is given by*

$$I_\mu = Z_\mu \qquad \mu = 1, \ldots, \left[\frac{M}{2}\right] \tag{3.9}$$

where Z_μ is the determinant function given by eq.(3.7).

Proof. Let us first consider the cases M odd, i.e. $M = 2p+1$ for $p = 2, 3, \ldots$

We begin by applying the set of $p(p+2)$ differential operators of eq.(3.3), given by

$$\begin{array}{cccc} \hat{N}_{1(p+1)} & \hat{N}_{1(p+2)} & \cdots & \hat{N}_{1M} \\ \hat{N}_{2(p+1)} & \hat{N}_{2(p+2)} & \cdots & \hat{N}_{2M} \\ \vdots & \vdots & & \vdots \\ \hat{N}_{p(p+1)} & \hat{N}_{p(p+2)} & \cdots & \hat{N}_{pM} \\ 0 & \hat{N}_{(p+1)(p+2)} & \cdots & \hat{N}_{(p+1)M} \end{array} \tag{3.10}$$

on the functions (3.4). The action of all these operators eliminates the dependence on the $p(p+1)$ variables n_{ik} for

$$1 \leq i \leq p \qquad i+1 \leq k \leq p+1 \tag{3.11}$$

and

$$p+1 \leq i \leq M-1 \qquad i+1 \leq k \leq M\ . \tag{3.12}$$

The p^2 remaining variables are

$$\begin{array}{cccc} n_{1(p+2)} & n_{1(p+3)} & \cdots & n_{1M} \\ n_{2(p+2)} & n_{2(p+3)} & \cdots & n_{2M} \\ \vdots & \vdots & & \vdots \\ n_{p(p+2)} & n_{p(p+3)} & \cdots & n_{pM} \end{array} \tag{3.13}$$

and the $p(p-1)$ remaining differential operators $\hat{N}_{ik}$ of eq.(3.3) are given by

$$\hat{N}_{ik} = \sum_{b=p+2}^{M} n_{ib} \frac{\partial}{\partial n_{kb}} \qquad 1 \leq i \leq p-1 \qquad i+1 \leq k \leq p \tag{3.14}$$

$$\hat{N}_{ik} = -\sum_{a=1}^{p} n_{ak} \frac{\partial}{\partial n_{ai}} \qquad p+2 \leq i \leq M-1 \qquad i+1 \leq k \leq M. \tag{3.15}$$

These differential operators are linearly independent. Therefore, the number of invariants for $T(2p+1)$ is p, i.e. the difference between the number of remaining variables and the number of remaining independent differential operators.

At this stage of the proof it is sufficient to verify that the remaining differential operators (3.14) and (3.15) annihilate determinants $Z_1, \ldots, Z_p$, i.e. $\hat{N}_{ik} \cdot Z_\alpha = 0$ for $\alpha = 1, \ldots, p$.

Let us first consider the set of differential operators (3.14). A given differential operator $\hat{N}_{ik}$ (i and k fixed) of (3.14) annihilates the determinant $Z_1, \ldots, Z_{k-1}$, since the variables n_{kb} ($p+2 \leq b \leq M$) do not figure in these determinants. It is therefore sufficient to look how $\hat{N}_{ik}$ acts on $Z_k, \ldots, Z_p$.

The determinant Z_β $\beta \in \{k, k+1, \dots, p\}$ can be expanded in terms of its k^{th} row

$$Z_\beta = \sum_{b=2p+2-\beta}^{M} n_{kb}\, C_{kb}^{(\beta)} \tag{3.16}$$

where $C_{kb}^{(\beta)}$ is the cofactor of the $\beta \times \beta$ square matrix associated with the determinant Z_β. Hence, the differential operator $\hat{N}_{ik}$ applied on these determinants gives

$$\hat{N}_{ik} \cdot Z_\beta = \sum_{b=2p+2-\beta}^{M} n_{ib}\, C_{kb}^{(\beta)}. \tag{3.17}$$

The right hand side of eq.(3.17) vanishes, since it corresponds to the expansion of determinant in terms of the cofactors of a different row. This gives the determinant of a matrix with two identical rows, hence zero.

The procedure is very similar for the set of differential operators (3.15). An operator $\hat{N}_{ik}$ of this set annihilates the determinants $Z_1, \dots, Z_{M-i}$ since the operator acts only on variables not figuring in the determinants. Let us consider the action of N_{ik} in (3.15) for the determinants Z_γ, where $\gamma \in \{(M-i+1), (M-i+2), \dots, p\}$.

We can write the determinants Z_γ as

$$Z_\gamma = \sum_{a=1}^{\gamma} n_{ai}\, C_{ai}^{(\gamma)} \tag{3.18}$$

and the action of the differential operators $\hat{N}_{ik}$ on these determinants is given by

$$\hat{N}_{ik} \cdot Z_\gamma = -\sum_{a=1}^{\gamma} n_{ak}\, C_{ai}^{(\gamma)}. \tag{3.19}$$

Hence, we obtain a determinant with two identical columns. More precisely, the action of differential operator $\hat{N}_{ik}$ in (3.15) on determinants (3.18) is the following: the column n_{ai} in determinants Z_γ is replaced by the column $-n_{ak}$, for $1 \le a \le \gamma$. Therefore, by the property of determinants, this action annihilates Z_γ.

The proof for the even case is very similar to the odd case and we omit it. $\square$

4 Invariants of the solvable triangular Lie algebras

4.1 Structure of the solvable triangular Lie algebra $L(M, f)$

In this section we sum up the main results of Ref. [1] to make this article self-contained.

Let us extend the algebra $T(M)$ to an indecomposable solvable Lie algebra $L(M, f)$ of dimension $d = \frac{1}{2}M(M-1) + f$ having $T(M)$ as its nilradical. In other words, we add f further linearly nilindependent elements to $T(M)$. Let us denote them $\{X^1, \dots, X^f\}$.

Definition 1 • *A set of elements $\{X^\alpha\}$ of a Lie algebra L is linearly nilindependent if no nontrivial linear combination of them is a nilpotent element.*

• *A set of matrices $\{A^\alpha\}_{\alpha=1,\dots,n}$ is linearly nilindependent if no nontrivial linear combination of them is a nilpotent matrix, i.e. if*

$$\left(\sum_{i=1}^{n} c_i\, A^i\right)^k = 0 \tag{4.1}$$

for some $k \in \mathbb{Z}^+$, implies $c_i = 0\ \ \forall i$.

The results on the structure of the Lie algebras $L(M, f)$ that we have obtained in [1] can be summed up as follows.

Each Lie algebra $L(M, f)$ can be transformed to a canonical basis $\{X^\alpha, N_{ik}\}$, $\alpha = 1, \dots, f$, $1 \le i < k \le M$ with commutation relations (3.2) and

$$[X^\alpha, N_{ik}] \;=\; \sum_{p<q} A^\alpha_{ik\,,\,pq}\; N_{pq} \tag{4.2}$$

$$[X^\alpha, X^\beta] \;=\; \sigma^{\alpha\beta} N_{1M} \tag{4.3}$$

$$1 \le \alpha, \beta \le f \qquad A^\alpha_{ik\,,\,pq}, \quad \sigma^{\alpha\beta} \in K.$$

The commutation relations (4.2) can be rewritten as

$$\begin{gathered} [X^\alpha, N] = A^\alpha N \\ N \equiv (N_{12}\ N_{23} \dots N_{(M-1)M}\ N_{13} \dots N_{(M-2)M} \dots N_{1M})^T \\ A^\alpha \in K^{r\times r} \qquad N \in K^{r\times 1} \end{gathered} \tag{4.4}$$

where the superscript T indicates transposition. We mention that the vector N introduces an order in lines (columns) of the matrices A^α, where each line (column) is represented by two numbers. The matrices $A^\alpha = \{A^\alpha_{ik\,,\,pq}\}$ have the following canonical form.

(i) They are upper triangular.

(ii) The only off-diagonal matrix elements that do not vanish identically and cannot be annulled by a redefinition of the elements X^α are:

$$A^\alpha_{12\,,\,2M} \qquad A^\alpha_{j(j+1)\,,\,1M}\ (2 \le j \le M-2) \qquad A^\alpha_{(M-1)M\,,\,1(M-1)}. \tag{4.5}$$

(iii) The diagonal elements $a^\alpha_{i(i+1)}$, $1 \le i \le M-1$ are free. The other diagonal elements satisfy

$$a^\alpha_{ik} = \sum_{p=i}^{k-1} a^\alpha_{p(p+1)} \qquad k > i+1 \tag{4.6}$$

where we have introduced the compact notation $A^\alpha_{ik\,,\,ik} \equiv a^\alpha_{ik}$.

The canonical forms of the structure matrices A^α and the constants $\sigma^{\alpha\beta}$ satisfy the following conditions:

1. The set of matrices A^α have the form specified above and are linearly nilindependent. For $f \geq 2$ they all commute, i.e.

$$\left[A^\alpha, A^\beta\right] = 0. \tag{4.7}$$

2. All constants $\sigma^{\alpha\beta}$ vanish unless we have $a^\gamma_{1M} = 0$ for $\gamma = 1, \ldots, f$ simultaneously for all γ.

3. The remaining off-diagonal elements $A^\alpha_{ik\,,ab}$ also vanish, unless the diagonal elements satisfy $a^\beta_{ik} = a^\beta_{ab}$ for $\beta = 1, \ldots, f$ simultaneously for all β.

4. The maximal number of non-nilpotent elements is $f_{max} = M-1$ and in this case the non-nilpotent elements always commute, i.e.

$$\left[X^\alpha, X^\beta\right] = 0\,. \tag{4.8}$$

Furthermore, the matrices A^α are the diagonal matrices given by

$$a^\alpha_{ik} = \sum_{p=i}^{k-1} \delta_{\alpha,p} \qquad 1 \leq i < k \leq M \qquad 1 \leq \alpha \leq M-1. \tag{4.9}$$

5. For $f = 1$ the matrix A has at most $M-2$ off-diagonal elements that can be normalized to $+1$ for $K = \mathbb{C}$ and to $+1$, or -1 for $K = \mathbb{R}$.

4.2 Differential operators and the system of equations

Using the preceding results, we can construct (as in Section 2) the differential operators realizing a basis for the coadjoint representation of the Lie algebras $L(M, f)$:

$$\hat{N}_{ik} = \sum_{b=k+1}^{M} n_{ib}\frac{\partial}{\partial n_{kb}} - \sum_{a=1}^{i-1} n_{ak}\frac{\partial}{\partial n_{ai}} - \sum_{\alpha=1}^{f} (a^\alpha_{ik} n_{ik} + \Gamma^\alpha_{ik})\,\frac{\partial}{\partial x^\alpha} \tag{4.10}$$

$$\hat{X}^\alpha = \sum_{i<k} (a^\alpha_{ik} n_{ik} + \Gamma^\alpha_{ik})\,\frac{\partial}{\partial n_{ik}} + \sum_{\beta=1}^{f} \left(\sigma^{\alpha\beta} n_{1N}\right)\frac{\partial}{\partial x^\beta}. \tag{4.11}$$

We have introduced the notation

$$\begin{aligned} \Gamma^\alpha_{12} &\equiv A^\alpha_{12,2M}\, n_{2M} \\ \Gamma^\alpha_{j(j+1)} &\equiv A^\alpha_{j(j+1)\,,1M}\, n_{1M} \qquad j = 2, 3, \ldots, M-2 \\ \Gamma^\alpha_{(M-1)M} &\equiv A^\alpha_{(M-1)M\,,1(M-1)}\, n_{1(M-1)} \\ \Gamma^\alpha_{lm} &\equiv 0 \qquad m - l \geq 2. \end{aligned} \tag{4.12}$$

In the generic case the differential operators (4.11) will not contain the second summation since $\sigma^{\alpha\beta} = 0$ unless $a^\alpha_{1M} = 0$ for $\gamma = 1, \ldots, f$.

Equation (2.3) determining the invariants in our case amounts to the system of equations

$$\hat{N}_{ik} \cdot F(n_{12}, n_{23}, \ldots, n_{1M}, x^1, \ldots, x^f) = 0 \qquad 1 \leq i < k \leq M \tag{4.13}$$

$$\hat{X}^\alpha \cdot F(n_{12}, n_{23}, \ldots, n_{1M}, x^1, \ldots, x^f) = 0 \qquad \alpha = 1, \ldots, f. \tag{4.14}$$

It is useful to construct linear combinations of these operators that involve only x derivatives. These linear combinations are not elements of the Lie algebras $L(M, f)$, since they have variable coefficients. This is permitted since we are now treating equations (4.13) and (4.14) simply as a system of linear partial differential equations.

Let us associate a differential operator $\hat{Z}_\mu$ with each invariant Z_μ of the nilpotent Lie algebras $T(M)$ (see eq.(3.7)). For each Z_μ we take a sum of μ determinants of the form (3.7) and in each of them we replace one column of scalars by a column of operators $\hat{N}_{ik}$. For examples, we have

$$\hat{Z}_1 = \hat{N}_{1M} \qquad \hat{Z}_2 = \begin{vmatrix} \hat{N}_{1(M-1)} & n_{1M} \\ \hat{N}_{2(M-1)} & n_{2M} \end{vmatrix} + \begin{vmatrix} n_{1(M-1)} & \hat{N}_{1M} \\ n_{2(M-1)} & \hat{N}_{2M} \end{vmatrix} \tag{4.15}$$

and in general, we have the formula

$$\hat{Z}_\mu = \sum_{j=1}^{\mu} \begin{vmatrix} n_{1(M-\mu+1)} & n_{1(M-\mu+2)} & \cdots & \hat{N}_{1(M-\mu+j)} & \cdots & n_{1M} \\ n_{2(M-\mu+1)} & n_{2(M-\mu+2)} & \cdots & \hat{N}_{2(M-\mu+j)} & \cdots & n_{2M} \\ \vdots & \vdots & & \vdots & & \vdots \\ n_{\mu(M-\mu+1)} & n_{\mu(M-\mu+2)} & \cdots & \hat{N}_{\mu(M-\mu+j)} & \cdots & n_{\mu M} \end{vmatrix} \qquad 1 \leq \mu \leq \left[\frac{M}{2}\right] . \tag{4.16}$$

It is a straightforward calculation to prove that we have

$$\hat{Z}_\mu = \sum_{\alpha=1}^{f} f_\alpha(n_{ik}) \frac{\partial}{\partial x^\alpha} \tag{4.17}$$

i.e. that all the n_{ik} derivatives drop out. For example, in the cases of diagonal matrices A^α, we obtain the formula

$$\hat{Z}_j = -Z_j \left(\sum_{\alpha=1}^{f} \sum_{\mu=1}^{j} a^\alpha_{\mu(M-\mu+1)} \frac{\partial}{\partial x^\alpha} \right) . \tag{4.18}$$

Remark: For non-diagonal matrices A^α, this formula is generic for odd M. However, for even M, off-diagonal terms will appear.

We can construct $[M/2]$. such operators; at most f of them are linearly independent.

4.3 Examples: Invariants of $L(4, f)$

Let us now illustrate the procedure to obtain the functionally independent invariants for the solvable Lie algebras $L(4, f)$, $f = 1, 2$ or 3. For each algebra $L(4, f)$ we will state results concerning the form and the number of invariants. For each Lemma, the strategy that we will adopt to prove it is the following.

We will separate the proof in two parts:

(A) We find the invariants depending only on the variables n_{ab} , $1 \leq a < b \leq 4$.

(B) We find the invariants which are dependent on variables n_{ik} **and** x^α , $\alpha = 1, \ldots, f$.

In each of these cases, we will apply the differential operators $\hat{N}_{ik}$ and $\hat{X}^\alpha$ of the coadjoint representation of $L(4, f)$, on the functions $F = F(\{n_{ab}\}, \{x^\alpha\})$. However in the case (A), since we postulate that the functions F only depend on the variables n_{ab}, the differential operators $\hat{N}_{ik}$ will be the same as the operators of the nilpotent Lie algebra $T(4)$ (the x derivatives do not act on F).

Therefore, by using the results of Theorem 1, we will only have to apply the differential operators X^α on functions of the type

$$F = F(Z_1, Z_2) \tag{4.19}$$

where $Z_1 = n_{14}$ and $Z_2 = n_{13}\, n_{24} - n_{23}\, n_{14}$.

In the case (B), we will begin by imposing

$$\hat{Z}_j \cdot F(n_{ab}\,,\, x^\alpha) = 0 \qquad j = 1, 2 \tag{4.20}$$

such that the dependence on the x^α variables is preserved in F. Then we will apply all the differential operators (4.10) and (4.11) of the coadjoint representation of $L(4, f)$.

4.3.1 The Lie algebras $L(4, 1)$

The characteristic matrix A of these Lie algebras $L(4, 1)$ has the form [1]

$$A = \begin{pmatrix} a_{12} & 0 & 0 & 0 & \lambda_1 & 0 \\ & a_{23} & 0 & 0 & 0 & \lambda_2 \\ & & a_{34} & \lambda_3 & 0 & 0 \\ & & & a_{13} & 0 & 0 \\ & & & & a_{24} & 0 \\ & & & & & a_{14} \end{pmatrix} \tag{4.21}$$

where we have at most 2 non-zero off-diagonal elements λ_i and by eq.(4.6) a_{13}, a_{24} and a_{14} are determined in terms of a_{12}, a_{23} and a_{34}.

Lemma 1 *A solvable triangular Lie algebra of the type $L(4, 1)$ has either 3 invariants, or 1 invariant.*

1) *Three invariants exist iff the conditions*

$$a_{14} = a_{23} = \lambda_2 = 0 \tag{4.22}$$

are satisfied. In this case the algebra can be characterized by $a_{12} = -a_{34} = 1\ ,\ a_{23} = 0\ ,\ \lambda_1 = \lambda_2 = \lambda_3 = 0$ in (4.21). A basis for the invariants is:

$$I_1 = Z_1 \tag{4.23}$$
$$I_2 = Z_2 \tag{4.24}$$
$$I_3 = (n_{12}\, n_{24} + n_{13}\, n_{34}) + n_{14}\, x\ . \tag{4.25}$$

Otherwise there exists precisely one invariant. Two types of Lie algebras occur.

2) $(a_{12} + a_{34}, a_{23}) \not\equiv (0, 0)$ *and* $\lambda_2 = 0$ *in (4.21). The invariant is:*

$$I = \frac{(Z_2)^{a_{14}}}{(Z_1)^{a_{14}+a_{23}}}\ . \tag{4.26}$$

3) $a_{12}+a_{34}=0,\ \lambda_2=1\ ,\ a_{23}$ *is a free parameter in* (4.21) *and the invariant is:*

$$I=a_{23}\frac{Z_2}{(Z_1)^2}-\ln Z_1\ . \tag{4.27}$$

Proof.

(A) We impose that the differential operator $\hat{X}$ of eq.(4.11) should annihilates the functions $F=F(Z_1,Z_2)$, i.e.

$$\begin{aligned}\hat{X}\cdot F &= \Big[(a_{12}\,n_{12}+\lambda_1\,n_{24})\frac{\partial}{\partial n_{12}}+(a_{23}\,n_{23}+\lambda_2\,n_{14})\frac{\partial}{\partial n_{23}}\\ &\quad +(a_{34}\,n_{34}+\lambda_3\,n_{13})\frac{\partial}{\partial n_{34}}+a_{13}\,n_{13}\frac{\partial}{\partial n_{13}}+a_{24}\,n_{24}\frac{\partial}{\partial n_{24}}\\ &\quad +a_{14}\,n_{14}\frac{\partial}{\partial n_{14}}\Big]\,F\\ &= a_{14}\,Z_1\,\frac{\partial F}{\partial Z_1}+\left[(a_{14}+a_{23})\,Z_2-\lambda_2\,(Z_1)^2\right]\,\frac{\partial F}{\partial Z_2}=0.\end{aligned} \tag{4.28}$$

We first note that if we have $a_{14}=a_{23}=\lambda_2=0$, i.e. conditions (4.22) which implies $a_{12}+a_{34}=0$ from eq.(4.6), then both Z_1 and Z_2 are invariants. Also, the matrix A can, with no loss of generality [1], be diagonalized and set equal to

$$A=\mathrm{diag}(1,\quad 0,\quad -1,\quad 1,\quad -1,\quad 0). \tag{4.29}$$

In all other cases eq.(4.28) implies that just one invariant of this type exists. We obtain it using the method of characteristics.

Two cases arise:

(i) $\lambda_2=0$: The invariant is then given by (4.26), with $(a_{12}+a_{34},a_{23})\not\equiv(0,0)$.

(ii) $\lambda_2\not\equiv 0$: From our previous article [1], we know that in this case we can normalize λ_2 to 1 and we necessarily have $a_{23}=a_{14}$, which implies $a_{12}+a_{34}=0$. Hence, we obtain the invariant (4.27), where a_{23} is a free parameter.

(B) In this case we impose $\hat{Z}_j\cdot F=0$, where $F=F(n_{12},n_{23},n_{34},n_{13},n_{24},n_{14},x)$ and we have

$$\hat{Z}_1 \equiv \hat{N}_{14}=-a_{14}\,Z_1\frac{\partial}{\partial x} \tag{4.30}$$

$$\begin{aligned}\hat{Z}_2 &\equiv n_{13}\,\hat{N}_{24}-n_{23}\,\hat{N}_{14}+n_{24}\,\hat{N}_{13}-n_{14}\,\hat{N}_{23}\\ &= \left[-(a_{14}+a_{23})\,Z_2+\lambda_2\,(Z_1)^2\right]\,\frac{\partial}{\partial x}.\end{aligned} \tag{4.31}$$

Hence the required dependence on x will survive only if we have $a_{14}=a_{23}=\lambda_2=0$. This coincides with eq.(4.22), the condition for Z_1 and Z_2 to be invariants. Furthermore, we can normalize a_{12} to 1 and cancel λ_1 and λ_3 by transformations [1].

We now apply all the differential operators of the coadjoint representation of $L(4,1)$ and the final result is that we obtain two invariants (4.23) and (4.24) independent of x and one invariant (4.25) depending on x. □

4.3.2 The Lie algebras $L(4,2)$

The Lie algebras $L(4,2)$ have the following characteristic matrices [1]:

$$A^1 = \begin{pmatrix} a_{12} & & & & & \\ & a_{23} & & & & \\ & & a_{34} & & & \\ & & & a_{13} & & \\ & & & & a_{24} & \\ & & & & & a_{14} \end{pmatrix} \quad A^2 = \begin{pmatrix} b_{12} & 0 & 0 & 0 & \lambda_1 & 0 \\ & b_{23} & 0 & 0 & 0 & \lambda_2 \\ & & b_{34} & \lambda_3 & 0 & 0 \\ & & & b_{13} & 0 & 0 \\ & & & & b_{24} & 0 \\ & & & & & b_{14} \end{pmatrix} \tag{4.32}$$

where we have at most one off-diagonal element in A^2 and a_{ik}, b_{ik} satisfy the eq.(4.6). Furthermore, the coefficient σ^{12} in eq.(4.3) is in the generic case zero (i.e. the two non-nilpotent elements commute). However, for the particular case $a_{14} = 0 = b_{14}$, we can have $\sigma^{12} \not\equiv 0$ in eq.(4.3).

Lemma 2 *A solvable triangular Lie algebra of the type $L(4,2)$ has either 2 invariants or none. Two invariants exist iff the conditions*

$$b_{23}\,(a_{12}+a_{34}) - a_{23}\,(b_{12}+b_{34}) = 0 \tag{4.33}$$

$$a_{14}\,\lambda_2 = 0 \tag{4.34}$$

are satisfied simultaneously. They lead to the following algebras and invariants.

1) $a_{12} = -a_{34} = b_{23} = \lambda_2 = 1$ *and* $a_{23} = b_{12} = b_{34} = \lambda_1 = \lambda_3 = \sigma^{12} = 0$ *in* (4.32) *and a basis for the invariants is:*

$$I_1 = \frac{Z_2}{(Z_1)^2} + \ln Z_1 \tag{4.35}$$

$$I_2 = \frac{n_{12}\,n_{24} + n_{13}\,n_{34}}{n_{14}} + x^1 \,. \tag{4.36}$$

2a) $a_{12} = -a_{34} = b_{23} = 1$, $a_{23} = b_{12} = \lambda_1 = \lambda_2 = \lambda_3 = \sigma^{12} = 0$ *and* b_{34} *a free parameter in* (4.32),

2b) $a_{12} = b_{34} = 1$ *and* $a_{23} = a_{34} = b_{12} = b_{23} = \lambda_1 = \lambda_2 = \lambda_3 = \sigma^{12} = 0$ *in* (4.32)

In both cases we have the invariants:

$$I_1 = \frac{(Z_2)^{a_{14}}}{(Z_1)^{a_{14}+a_{23}}} \tag{4.37}$$

$$I_2 = (a_{34}b_{13} - b_{34}a_{13}) \left(\frac{n_{12}n_{24} + n_{13}n_{34}}{n_{14}} \right) + a_{14}x^2 - b_{14}x^1 \,. \tag{4.38}$$

3) $a_{12} = -a_{34} = b_{23} = -b_{34} = 1$ *and* $a_{23} = b_{12} = \lambda_1 = \lambda_2 = \lambda_3 = 0$ *in* (4.32) *and the invariants are:*

$$I_1 = Z_1 \tag{4.39}$$

$$I_2 = n_{12}\, n_{24} + n_{13}\, n_{34} + Z_1\, x^1 + \sigma^{12}\, (Z_1)^2 \, \ln Z_2 \, . \tag{4.40}$$

Otherwise, there is no invariant.

Proof.

(A) We first apply differential operators $\hat{X}_1$ and $\hat{X}_2$ on functions $F = F(Z_1, Z_2)$. We obtain a system of two linear partial differential equations given by

$$\begin{pmatrix} \hat{X}^1 \cdot F \\ \hat{X}^2 \cdot F \end{pmatrix} = \begin{pmatrix} a_{14}\, Z_1 & (a_{14} + a_{23})\, Z_2 \\ b_{14}\, Z_1 & (b_{14} + b_{23})\, Z_2 - \lambda_2\, (Z_1)^2 \end{pmatrix} \begin{pmatrix} \dfrac{\partial F}{\partial Z_1} \\ \dfrac{\partial F}{\partial Z_2} \end{pmatrix} = 0. \tag{4.41}$$

The rank of the 2×2 matrix in eq.(4.41) cannot be zero, since then matrices A^1 and A^2 would not be linearly nilindependent. Also, if the rank is 2 there is no invariant that depends only on Z_1 and Z_2. However, solution exist if the rank of the matrix is 1 for all values of Z_1 and Z_2. This gives conditions (4.33) and (4.34).

Let us now assume that the condition (4.33) is respected. We consider the diagonal and the non-diagonal cases separately.

(i) $\lambda_2 = 0$: In this case, we obtain the invariant (4.26) for $(a_{12} + a_{34}, a_{23}) \not\equiv (0, 0)$.

(ii) $\lambda_2 \not\equiv 0$, $a_{14} = 0$: Since λ_2 is non-zero in A^2, we necessarily have $b_{23} = b_{14}$, i.e. $b_{12} + b_{34} = 0$ which gives the condition $a_{23}\, b_{23} = 0$ by (4.33). Two cases are possible under these condition.

One case gives the invariant (4.35) for $a_{12} = -a_{34} = b_{23} = \lambda_2 = 1$ and $a_{23} = b_{12} = b_{34} = \lambda_1 = \lambda_3 = 0$.

In the other case, we simply obtain the invariant $I = Z_1$ for the Lie algebra characterized by $a_{23} = b_{12} = -b_{34} = \lambda_2 = 1$, $b_{23} = \lambda_1 = \lambda_3 = 0$ and $a_{34} = -(a_{12} + 1)$ (with a_{12} a free parameter).

Remark. The case $a_{23} = 0 = b_{23}$ gives two nildependent matrices A^1, A^2 and is therefore not considered.

(B) In this case, we begin by applying the differential operators $\hat{Z}_1$, $\hat{Z}_2$ on functions $F = F(n_{12}, n_{23}, n_{34}, n_{13}, n_{24}, n_{14}, x^1, x^2)$, i.e.

$$\begin{pmatrix} \hat{Z}_1 \cdot F \\ \hat{Z}_2 \cdot F \end{pmatrix} = \begin{pmatrix} -a_{14}\, Z_1 & -b_{14}\, Z_1 \\ (a_{23} + a_{14})\, Z_2 & (b_{23} + b_{14})\, Z_2 - \lambda_2\, (Z_1)^2 \end{pmatrix} \begin{pmatrix} \dfrac{\partial F}{\partial x^1} \\ \dfrac{\partial F}{\partial x^2} \end{pmatrix} = 0 \, . \tag{4.42}$$

The dependence on x^1 and x^2 can exist only if the determinant of the 2×2 matrix in (4.42) is zero. This again imposes the conditions (4.33) and (4.34).

Let us again assume that the condition (4.33) is satisfied. We separate the problem into three distinct cases.

(i) $(a_{14}, b_{14}) \not\equiv (0,0)$, $\lambda_2 \not\equiv 0$: The condition $\lambda_2 \not\equiv 0$ implies two consequences. First we have from (4.34) that $a_{14} = 0$ and therefore $b_{14} \not\equiv 0$. Second, we necessarily have $b_{23} = b_{14}$ which implies from (4.34) that $b_{23}\, a_{23} = 0$

In this case, the invariants are (4.35) and (4.36) and the Lie algebra $L(4,2)$ satisfies $a_{12} = -a_{34} = b_{23} = \lambda_2 = 1$, $a_{23} = b_{12} = b_{34} = \lambda_1 = \lambda_3 = \sigma^{12} = 0$.

(ii) $(a_{14}, b_{14}) \not\equiv (0,0)$, $\lambda_2 = 0$: In this case, two triangular solvable Lie algebras are associated with the invariants (4.37) and (4.38). One Lie algebra is characterized by the parameters $a_{12} = -a_{34} = b_{23} = 1$, $a_{23} = b_{12} = \lambda_1 = \lambda_2 = \lambda_3 = \sigma^{12} = 0$ and b_{34} a free parameter. The other Lie algebra is characterized by $a_{12} = b_{34} = 1$ and $a_{23} = a_{34} = b_{12} = b_{23} = \lambda_1 = \lambda_2 = \lambda_3 = \sigma^{12} = 0$.

(iii) $(a_{14}, b_{14}) = (0,0)$: In this case, we see that conditions (4.33) and (4.34) are automatically respected. Also, we can have a non-zero σ^{12} in eq.(4.11).

Since $a_{14} = 0 = b_{14}$, we can substitute a_{34} by $-(a_{12}+a_{23})$ and b_{34} by $-(b_{12}+b_{23})$ in the structure matrices (4.32). However, by imposing the commutativity (4.7) and the nilindependence of the matrices A^1 and A^2, we obtain $a_{12} = -a_{34} = b_{23} = -b_{34} = 1$ and $a_{23} = b_{12} = \lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence, we obtain the two invariants (4.39) and (4.40). $\Box$

4.3.3 The Lie algebra $L(4,3)$

For the Lie algebra $L(4,3)$, we have diagonal characteristic matrices given by

$$\begin{aligned} A^1 &= \mathrm{diag}\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \; A^2 = \mathrm{diag}\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \\ A^3 &= \mathrm{diag}\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} . \end{aligned} \tag{4.43}$$

Furthermore, the non-nilpotent elements commute, i.e. $\sigma^{\alpha\,\beta} = 0,\ \alpha,\, \beta = 1,2,3$ (see eqs. (4.3) and (4.8))

Lemma 3 *The triangular solvable Lie algebra $L(4,3)$ has precisely 1 invariant given by*

$$I = \frac{n_{12}\, n_{24} + n_{13}\, n_{34}}{n_{14}} + (x^1 - x^3)\ . \tag{4.44}$$

Proof.

(A) In this case, it is easy to demonstrate that after we have applied the differential operator $\hat{X}^1$ on functions $F = F(Z_1, Z_2)$, we obtain the quotient of Z_2 over Z_1. However, when we apply operator $\hat{X}^2$ on functions $\tilde{F} = \tilde{F}(I)$ with $I = Z_2/Z_1$, we obtain

$$\begin{aligned} 0 &= \hat{X}^2 \cdot \tilde{F} = \left(n_{23}\, \frac{\partial}{\partial n_{23}} + n_{13}\, \frac{\partial}{\partial n_{13}} + n_{24}\, \frac{\partial}{\partial n_{24}} + n_{14}\, \frac{\partial}{\partial n_{14}} \right) \tilde{F} \\ &= I \frac{\partial}{\partial I} \tilde{F}\ . \end{aligned} \tag{4.45}$$

Therefore, there is no invariant in this case.

(B) We first impose that the differential operators $\hat{Z}_1$ and $\hat{Z}_2$ annihilate the functions $F = F(n_{12}, n_{23}, n_{34}, n_{13}, n_{24}, n_{14}, x^1, x^2, x^3)$, where

$$\hat{Z}_1 = -Z_1\left(\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^2}+\frac{\partial}{\partial x^3}\right) \tag{4.46}$$

$$\hat{Z}_2 = -Z_2\left(\frac{\partial}{\partial x^1}+2\frac{\partial}{\partial x^2}+\frac{\partial}{\partial x^3}\right). \tag{4.47}$$

Since the Lie algebra $L(4,3)$ has no parameters, these conditions are not on the parameters of the algebra (as before) but on the x dependence of the invariant(s). Hence, the new functions on which we will apply all the differential operators of the coadjoint representation of $L(4,3)$ are $\tilde{F} = \tilde{F}(n_{12}, n_{23}, n_{34}, n_{13}, n_{24}, n_{14}, x^1 - x^3)$. We then obtain the invariant (4.44) by imposing that the operators of the coadjoint representation of $L(4,3)$ annihilate $\tilde{F}$. $\square$

4.4 General results

Proposition 1 *The triangular solvable Lie algebra $L(M, M-1)$ has precisely $\left[\frac{M-1}{2}\right]$ functionally independent invariants. A basis is given by*

$$I_\mu = \frac{(-1)^{\mu+1}}{Z_\mu}\left(\sum_{\rho=1}^{M-2\mu} W_\rho^{(\mu)}\right) + (x^\mu - x^{M-\mu}) \tag{4.48}$$

for $\mu = 1, \dots, \left[\frac{M-1}{2}\right]$. The function Z_μ is the determinant given by eq.(3.7) and $W_\rho^{(\mu)}$ is also a determinant function given by the determinant of the $(\mu+1)\times(\mu+1)$ matrix:

$$W_\rho^{(\mu)} = \begin{vmatrix} n_{1(\rho+\mu)} & n_{1(M-\mu+1)} & n_{1(M-\mu+2)} & \cdots & n_{1M} \\ n_{2(\rho+\mu)} & n_{2(M-\mu+1)} & n_{2(M-\mu+2)} & \cdots & n_{2M} \\ \vdots & \vdots & \vdots & & \vdots \\ n_{\mu(\rho+\mu)} & n_{\mu(M-\mu+1)} & n_{\mu(M-\mu+2)} & \cdots & n_{\mu M} \\ 0 & n_{(\rho+\mu)(M-\mu+1)} & n_{(\rho+\mu)(M-\mu+2)} & \cdots & n_{(\rho+\mu)M} \end{vmatrix}. \tag{4.49}$$

Proposition 2 *A diagonal solvable Lie algebra of the type $L(M,1)$ has $\left[\frac{M}{2}\right] \pm 1$ functionally independent invariants.*

1) $\left[\frac{M}{2}\right]+1$ *invariants exist iff the conditions*

$$a_{i(i+1)} + a_{(M-i)(M-i+1)} = 0 \qquad i = 1, \dots, \left[\frac{M}{2}\right] \tag{4.50}$$

are satisfied. A basis is given by $[M/2]$. invariants independent of x and one invariant depending on x:

$$I_\mu = Z_\mu \qquad \mu = 1, \dots, \left[\frac{M}{2}\right] \tag{4.51}$$

$$I_{\left[\frac{M}{2}\right]+1} = \sum_{\mu=1}^{[(M-1)/2]}\sum_{\rho=1}^{M-2\mu} \frac{(-1)^{\mu+1}}{Z_\mu} a_{\mu(\mu+1)} W_\rho^{(\mu)} + x \tag{4.52}$$

where the function Z_μ and $W_\rho^{(\mu)}$ are determinant functions given by the equations (3.7) and (4.49), respectively.

2) *Otherwise there exist precisely* $\left[\frac{M}{2}\right] - 1$ *invariants, all independent of* x*. A basis is given by*

$$I_\mu = \frac{(Z_{\mu+1})^\alpha}{(Z_1)^\beta} \qquad \mu = 1, \ldots, \left[\frac{M}{2}\right] - 1 \tag{4.53}$$

with

$$\frac{\alpha}{\beta} = \frac{a_{1M}}{\sum_{k=1}^{\mu+1} a_{k(M+1-k)}} \tag{4.54}$$

where the function Z_μ *is the determinant function given by eq.*(3.7)*.*

By *diagonal* solvable Lie algebra of the type $L(M,1)$ in Proposition 2, we mean that the structure matrix A of (4.2) is diagonal.

Propositions 1 and 2 each contains two types of information on the invariants: They give the form of the invariant functions and the number of functionally independent invariants. It is an easy calculation to prove that the functions I_μ of Proposition 1 and Proposition 2 are annihilated by the coadjoint representation (4.10), (4.11) of the Lie algebras $L(M, M-1)$ and $L(M,1)$, respectively. However, it is much more difficult to establish the number of functionally independent invariants for Proposition 1 and Proposition 2. The difficulty is to prove that no further invariants exists. One way of doing that is to calculate the rank of the antisymmetric matrix $S = S(L(M, M-1))$ and $S = S(L(M,1))$ of the commutation relations for the corresponding Lie algebra. The number of invariants is then given by the difference between the dimension of the solvable Lie algebra and the rank of the matrix S (see eq.(2.4)).

For the Lie algebra $L(M, M-1)$ of dimension $\frac{1}{2}(M-1)(M+2)$, S is the antisymmetric matrix given by the elements

$$\begin{gathered} S = \{[N_{ik}, N_{ab}] \;\; [N_{ik}, X^\alpha]\} \\ 1 \le i < k \le M \qquad 1 \le a < b \le M \qquad \alpha = 1, \ldots, M-1 \end{gathered} \tag{4.55}$$

and for the Lie algebra $L(M,1)$ of dimension $\frac{1}{2}(M^2 - M + 2)$, the matrix S is given by the elements

$$\begin{gathered} S = \{[N_{ik}, N_{ab}] \;\; [N_{ik}, X]\} \\ 1 \le i < k \le M \qquad 1 \le a < b \le M. \end{gathered} \tag{4.56}$$

For example, the antisymmetric matrix S of the 7-dimensional Lie algebra $L(4,1)$ is given by

$$S = \begin{pmatrix} 0 & n_{13} & 0 & 0 & n_{14} & 0 & -a_{12}n_{12} \\ -n_{13} & 0 & n_{24} & 0 & 0 & 0 & -a_{23}n_{23} \\ 0 & -n_{24} & 0 & -n_{14} & 0 & 0 & -a_{34}n_{34} \\ 0 & 0 & n_{14} & 0 & 0 & 0 & -a_{13}n_{13} \\ -n_{14} & 0 & 0 & 0 & 0 & 0 & -a_{24}n_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_{14}n_{14} \\ a_{12}n_{12} & a_{23}n_{23} & a_{34}n_{34} & a_{13}n_{13} & a_{24}n_{24} & a_{14}n_{14} & 0 \end{pmatrix} \tag{4.57}$$

where the parameters a_{13}, a_{24} and a_{14} are given in terms of a_{12}, a_{23} and a_{34} by the relation (4.6). Hence, it is easy to calculate that

$$rank(S) = \begin{cases} 4 & for\ a_{14} = a_{23} = 0 \\ 6 & otherwise \end{cases} \tag{4.58}$$

giving, respectively, three and one invariants (in accordance with Proposition 2 and Lemma 1).

We have calculated the ranks of the matrices $S(L(M, M-1))$ and $S(L(M, 1))$ for $M \leq 13$ and $M \leq 8$, respectively, using the symbolic package MAPLE. We conjecture that Proposition 1 and 2 hold for all M.

5 Conclusions

The problem of finding all invariants of the coadjoint representation of the triangular nilpotent algebras $T(M)$ is solved completely by Theorem 1. A basis for the invariants consists of polynomials and provides Casimir operators in the enveloping algebra of $T(M)$.

The situation with the solvable triangular Lie algebras $L(M, f)$ is more complicated. We have provided guidelines for calculating the invariants for all values of M, but presented comprehensive results only for $M = 4$. We have also presented conjectures concerning the invariants of $L(M, M-1)$ and $L(M, 1)$ for all values of M (and verified them for a large range of values of M).

The results for $M = 4$ show that all invariants are polynomial only in special cases. In general, rational, irrational and logarithmic type invariants must be allowed in any basis of invariants.

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