Integrability criteria for differential equations on the projective plane.

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CRM-2744
May 2001

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Abstract

In this article we prove two new criteria for the existence of rational general integral of an algebraic differential equation (cf. [4]) on the complex projective plane. These results are stated in terms of divisors and zero-cycles of a projective variety and are built by using intersection numbers of projective curves and multiplicities of singularities. We also prove a new result giving new sufficient conditions for the existence of a Darboux general integral for quadratic differential equations over the projective plane.

Résumé

Dans cet article nous démontrons deux nouveaux critères d'existence d'une intégrale générale rationnelle d'une équation différentielle algébrique (voir [4]) sur le plan projectif complexe. Ces résultats sont énoncés en termes de diviseurs et de zéro-cycles d'une variété projective et ils sont construits à l'aide de nombres d'intersection de courbes projectives et de multiplicités de singularités. Nous démontrons aussi un nouveau résultat donnant des conditions nécessaires et suffisantes pour l'existence d'une intégrale générale de Darboux pour des équations différentielles sur le plan projectif.

1 Introduction

In a Memoir published in 1878 (cf.[4]), Darboux initiated the study of algebraic differential equations on the complex projective plane and the theory of integrability via particular algebraic solutions. Darboux's Memoir was described by Poincaré in [8] as being "admirable" and in [9] as "oeuvre magistrale" (masterly, brilliant work). Poincaré considers in [9] the problem of algebraic integrability of planar differential equations with polynomial coefficients. By the problem of *algebraic integrability* Poincaré meant the problem of determining necessary and sufficient conditions for the existence of a rational general integral for such equations. Poincaré notes that after the work of Darboux, this problem was neglected and that it was only when, twenty years later, the "Académie de Sciences de Paris" proposed the Grand Prize of Mathematical Sciences on this topic that attention was again drawn to this problem. The prize was won by Painlevé [7] and an honorable mention was awarded to Autonne [1]. Poincaré later wrote two articles on this problem, [9] and [10]. After almost a century and a quarter since the publication of Darboux's memoir, this problem is still unsolved and it was mostly during the late 1970's and 1980's that work on this theme began catching a certain momentum, see for instance [6].

In this work we present three new results, two of them of a general character giving sufficient conditions for algebraic integrability and the third one giving conditions for Darboux integrability for quadratic equations in the projective plane.

2 Basic concepts and statement of results

We consider here the complex projective plane i.e. $P_2(\mathbb{C}) = (\mathbb{C}^3\backslash\{(0,0,0)\})/\sim$ where the equivalence relation is $(X,Y,Z) \sim (X',Y',Z')$ if and only if there exists $\lambda \in \mathbb{C}\backslash\{0\}$ such that $(X',Y',Z') = \lambda(X,Y,Z)$. We denote the equivalence class in $P_2(\mathbb{C})$ of the triple (X,Y,Z) by $[X:Y:Z]$ (or by $[(X,Y,Z)]$), i.e. $[X:Y:Z] = \{\lambda(X,Y,Z) \mid \lambda \in \mathbb{C}\backslash\{0\}\}$.

We consider differential equations on $P_2(\mathbb{C})$. By a *first order differential equation* on $P_2(\mathbb{C})$ we mean an equation of the form:

$$PdX + QdY + RdZ = 0 \tag{1}$$

where P, Q, R are homogeneous polynomials in X, Y, Z with coefficients in $\mathbb{C}$, of the same degree $m+1$, subject to the condition:

$$XP + YQ + ZR = 0. \tag{2}$$

Observation. For brevity, from now on, whenever we talk about a differential equation (1), we shall always mean an equation (1) subject to condition (2).

In this work we shall always assume that $(P,Q,R) = 1$, by this meaning that there is no non–constant polynomial which is a common divisor of P, Q and R. By the *degree* of equation (1) we mean the number m such that the common value of the degrees P, Q and R is $m+1$.

The class of all such equations is acted upon, on the right, by group $PGL(3,\mathbb{C})$ of projective transformations $T: P_2(\mathbb{C}) \to P_2(\mathbb{C})$, i.e. transformations defined by 3×3 matrices over $\mathbb{C}$ by the rule $T([X:Y:Z]) = [A(X,Y,Z)^t]$ where $(\ldots)^t$ denotes the transpose.

Darboux showed that an equation (1) subject to the condition (2) can always be written in the form

$$L(YdZ - ZdY) + M(ZdX - XdZ) + N(XdY - YdX) = 0 \tag{3}$$

where L, M, N are such that

$$P = MZ - NY, \quad Q = NX - LZ, \quad R = LY - MX, \tag{4}$$

the polynomials L, M, N not being uniquely determined by P, Q, R. Indeed, Darboux showed that along with (3) we have a family of equations of the same kind but where L, M, N could be replaced by L', M', N' such that $L' = L + AX$, $M' = M + AY$, $N' = N + AZ$ where A is any homogeneous polynomial with $\deg A = m-1$. When the differential equation is in this form (4) (called the Clebsch form), the degree m of the differential equation (3) is the common value of the degrees of L, M, N.

Proposition 1 *Let $T_A : P_2(\mathbb{C}) \to P_2(\mathbb{C})$ be a projective transformation given by an invertible matrix A over $\mathbb{C}$, i.e. $T_A([X:Y:Z]) = [A(X,Y,Z)^t]$. Consider an equation (3) defined by polynomials L, M, N over $\mathbb{C}$, which are relatively prime, i.e., their common factors are constants. Then the equation induced by T_A from (3) is of the same* *form and is defined by polynomials L', M', N' in u,v,w over $\mathbb{C}$, with L',M',N' relatively coprimes and satisfying the equation:*

$$\begin{pmatrix} L' \\ M' \\ N' \end{pmatrix} = (\det A)A^{-1} \begin{pmatrix} L \\ M \\ N \end{pmatrix} \circ A \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

or what is equivalent:

$$(L',M',N') = \{(L\circ T_A, M\circ T_A, N\circ T_A)((u,v,w))\}((\det A)A^{-1})^t.$$

Proof. We consider the projective transformation $T_A : P_2(\mathbb{C}) \to P_2(\mathbb{C})$ defined by an invertible matrix A over $\mathbb{C}$, i.e. $T_A([X:Y:Z]) = [A(X,Y,Z)^t]$ and $(dX,dY,dZ)^t = A\cdot(du,dv,dw)^t$. We have:

$$\begin{array}{rcl} YdZ-ZdY &=& A_{11}(vdw-wdv)-A_{12}(wdu-udw)+A_{13}(udv-vdu),\\ ZdX-XdZ &=& -A_{21}(vdw-wdv)+A_{22}(wdu-udw)-A_{23}(udv-vdu),\\ XdY-YdX &=& A_{31}(vdw-wdv)-A_{32}(wdu-udw)+A_{33}(udv-vdu). \end{array}$$

Then, we have:

$$\begin{array}{rcl} \omega &=& L(YdZ-ZdY)+M(ZdX-XdZ)+N(XdY-YdX)\\ &=& L'(vdw-wdv)+M'(wdu-udw)+N'(udv-vdu) \end{array}$$

where

$$(L',M',N')^t = (\det A)A^{-1}\{(L\circ T_A, M\circ T_A, N\circ T_A)(u,v,w)\}^t.$$

From the above relation we have that L',M',N' are relatively coprime. ∎

Definition 1 (Darboux [4]) *An* algebraic solution, *or algebraic particular integral, of a differential equation (1) with polynomial coefficients satisfying condition (2) is the set of zeroes in the complex projective space of an homogeneous polynomial $F\in\mathbb{C}[X,Y,Z]$ of positive degree, irreducible over $\mathbb{C}$, which is such that there is an homogeneous polynomial $K\in\mathbb{C}[X,Y,Z]$ satisfying the identity:*

$$L\frac{\partial F}{\partial X}+M\frac{\partial F}{\partial Y}+N\frac{\partial F}{\partial Z}=KF. \tag{5}$$

Remark 1 *If F is an algebraic solution for (3) with respect to the triple L, M, N with cofactor K and $L'=L+Ax$, $M'=M+Ay$, $N'=N+Az$, where A is an homogeneous polynomial of degree $\deg L-1$, then F is also an algebraic solution of (3) with cofactor $K+(\deg F)A$.*

Curves which are not necessarily defined by irreducible polynomials over $\mathbb{C}$ are also needed. We also need to consider the components of such curves with multiplicities. For this reason we shall use the terminology of algebraic geometers (cf.[5]). Let F be an homogeneous polynomial of positive degree over $\mathbb{C}$. We first note that for every $\lambda\in\mathbb{C}\backslash\{0\}$ the zero-set of λF coincides with the zero-set of F. It is therefore convenient to consider the equivalence relation $F\sim\lambda F$ for $F\in\mathbb{C}[X,Y,Z]$ and $\lambda\in\mathbb{C}\backslash\{0\}$. A *formal curve* over $\mathbb{K}$, where $\mathbb{K}$ is a field, will be an equivalence class of non–constant homogeneous polynomials under this equivalence relation. We shall make an abuse of language and speak about the curve F when talking about its equivalence class. Affine formal curves are defined analogously, i.e. equivalence classes of non–constant polynomials over the defining field $\mathbb{K}$.

Definition 2 *A formal curve F over $\mathbb{C}$ is an* invariant algebraic curve *of differential equation (1) if and only if, F satisfies the identity (5) for some homogeneous polynomial K.*

Proposition 2 *A formal curve $F=F_1^{e_1}F_2^{e_2}\dots F_k^{e_k}$, where F_i's are irreducible homgeneous polynomials over $\mathbb{C}$ and e_i's are positive integers, is an* invariant algebraic curve *of (1) if, and only if, each one of its irreducible factors F_i is an algebraic solution.*

When considering an invariant algebraic curve $F=F_1^{e_1}F_2^{e_2}\dots F_k^{e_k}$ of (1) or, equivalently, of (3) we shall say that e_i is the *multiplicity* of its i–th component F_i.

For each curve F over $\mathbb{C}$, we consider its associated *algebraic set* $V(F)=\{p\in P_2(\mathbb{C})\mid F(p)=0\}$. In general, if I is an ideal of homogeneous polynomials in $\mathbb{C}[X,Y,Z]$, then we define its associated *algebraic set* by $V(I)=\{p\in P_2(\mathbb{C})\mid F(p)=0,\forall F\in I\}$. An algebraic set $V(I)$ is a *variety* if, and only if, I is a prime ideal.

Definition 3 *Let* V *be a (irreducible) variety over* $\mathbb{C}$*. An* $r-cycle$ *over* V *is a formal sum* $\sum n_i W_i$*, where* $n_i's$ *are integers, only a finite number of them non–zero and* W_i*'s are (irreducible) subvarieties of codimension* r *of* V*. A codimension* 1*–cycle over* V *is called a* divisor *of* V*. The number* $\sum n_i$ *is called the* degree *of the* r*–cycle* $\sum n_i W_i$*.*

Definition 4 *Let* F *be a curve over* $\mathbb{C}$*,* $F = F_1^{e_1} F_2^{e_2} \ldots F_k^{e_k}$*, where* F_j*'s are irreducible polynomials and* e_j *are positive integers, for* $j = 1, 2, \ldots, k$*. If for a factor* F_j *we have* $e_j = 1$*, we say that* F_j *is a* simple factor*. Otherwise, if* $e_j > 1$*, we say that* F_j *is a* multiple factor*. We define the* divisor associated to F *by* $\sum_{j=1}^{k} e_j \mathrm{V}(F_j)$*.*

To introduce our specific divisors we recall the notion of a multiple point of a curve (cf.[5]). Let $f(x, y)$ be an affine curve over a field $\mathbb{K}$ and let $p = (a, b)$. p is a *multiple* point of f (or *singular*) if $f(p) = \frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = 0$. By using an affine transformation we may assume $p = (0, 0)$. We can write $f = f_s + f_{s+1} + \ldots + f_n$ where f_k is a form of degree k over $\mathbb{K}$. If p is a multiple point then $s > 1$ and s is called the *multiplicity* of f at p, and we denote it by $m_p(f)$. To define the multiplicity of the point $p = [p_1 : p_2 : p_3] \in P_2(\mathbb{K})$ of a projective curve F over the field $\mathbb{K}$, we may assume that $p_3 = 1$ and we consider $f(x, y)$ the affine curve defined by $f(x, y) = F(x, y, 1)$, then $m_p(F) = m_p(f)$. The notion of multiplicity of a curve F at point p, $m_p(F)$ is an affine and a projective invariant (see [5]).

We also need to recall the notion of intersection numbers (cf.[5]). Let $\mathbb{K}$ be a closed algebraic field and let $P_2(\mathbb{K})$ be the projective space of dimension 2 over $\mathbb{K}$. We say that F and G, two algebraic curves over $\mathbb{K}$, *intersect properly* at $p \in P_2(\mathbb{K})$ if F and G do not have any common component passing through p. The intersection number for two polynomials f and g at a point $p \in A_2(\mathbb{K})$ is defined by means of seven properties which assure its uniqueness.

Theorem 1 *There exists a unique multiplicity or intersection number* $I_p(f, g)$ *defined for all algebraic curves* f *and* g *and for all point* p *of* $A_2(\mathbb{K})$ *satisfying the following properties:*

(i) $I_p(f, g)$ *is a non–negative integer for all* f*,* g *and* p *when* f *and* g *intersect properly at* p*.* $I_p(f, g) = \infty$ *if* f *and* g *do not intersect strictly at* p*.*

(ii) $I_p(f, g) = 0$ *if and only if* p *is not a common point to* f *and* g*.* $I_p(f, g)$ *depends only on the components of* f *and* g *which pass through* p*.*

(iii) If T *is a change of coordinates and* $T(p) = q$*, then* $I_q(T(f), T(g)) = I_p(f, g)$*.*

(iv) $I_p(f, g) = I_p(g, f)$*.*

(v) $I_p(f, g) \geq m_p(f)m_p(g)$*, the equality holds if and only if* f *and* g *do not have common tangents at* p*.*

(vi) $I_p(f, g_1 g_2) = I_p(f, g_1) + I_p(f, g_2)$*.*

(vii) $I_p(f, g) = I_p(f, (g + af))$ *for any polynomial* $a(x, y)$*.*

Furthermore, $I_p(f, g) = \dim_{\mathbb{K}} (\mathrm{O}_p(A_2(\mathbb{K}))/(f, g))$ *where* $\mathrm{O}_p(A_2(\mathbb{K}))$ *is the local ring of the affine plane* $A_2(\mathbb{K})$ *at the point* p*, i.e.* $\mathrm{O}_p(A_2(\mathbb{K}))$ *is the set of rational functions on* $A_2(\mathbb{K})$ *which are defined at* p*.*

This theorem is proved in [5].

We consider I an ideal of the ring of polynomials $\mathbb{K}[x, y]$, where $\mathbb{K}$ is an algebraically closed field, with the property that the set $\{p \in P_2(\mathbb{K}) : f(p) = 0, \ \forall f \in I\}$ is finite. We write I_p, $p \in P_2(\mathbb{K})$, for the ideal of the local ring $O_p(A_2(\mathbb{K})$ generated by I. The following theorem gives us a way of generalize the intersection number for more than two polynomials. For any ideal I we may generalize the notion of intersection index in the following way.

Definition 5 *It is easy to see that* $\mathrm{O}_p/\mathrm{I}_p$ *is a vector space with finite dimension over* $\mathbb{K}$ *and we define the multiplicity or intersection index of* I *at* p *as*

$$I_p(\mathrm{I}) = \dim_{\mathbb{K}} \mathrm{O}_p/\mathrm{I}_p.$$

We may also consider the local ring of the projective plane $P_2(\mathbb{K})$ at the point $p \in P_2(\mathbb{K})$ and we notice that if $p = [X : Y : 1]$ then $O_p(P_2(\mathbb{K})$ is canonically isomorfic to $O_p(A_2(\mathbb{K}))$, where Z is the straight line at infinity. In [5], it is proved that the intersection number of two curves over $\mathbb{K}$ at a point p, only depends on the local ring $O_p(P_2(\mathbb{K})$, so we can define the intersection number of two algebraic curves F and G over $\mathbb{K}$ at a point p of $P_2(\mathbb{K})$ by

$$I_p(F, G) = \dim_{\mathbb{K}} O_p(P_2(\mathbb{K}))/ \langle f, g \rangle \, ,$$

where f and g are defined by $f(x, y) = F(x, y, 1)$ and $g(x, y) = G(x, y, 1)$, if $p = [X : Y : 1]$ and $\langle f, g \rangle$ is the ideal generated by the polynomials f and g. The number $I_p(F, G)$ does not depend on the way we define f and g, that is, the way we take local charts. Analogously to the affine plane, we can also generalize the intersection number for more than two polynomials in the projective plane.

The following theorem is the most important way of computing the intersection index for two polynomials without common components.

Theorem 2 (Bézout's theorem) *Let F and G be two algebraic curves of $P_2(\mathbb{K})$ of degrees r and s respectively without common components. Then*

$$\sum_{p \in P_2(\mathbb{K})} I_p(F, G) = rs\,.$$

Let $p \in P_2(\mathbb{K})$, we notice that any projectivity (change of coordinates) induces an isomorfism on the local ring $O_p(P_2(\mathbb{C})$, and, therefore the definition of intersection index at p is invariant under projectivities, that is the notion of intersection number is a projective invariant.

Definition 6 *The* total multiplicity *or intersection index of the ideal* I*, denoted by $I(\mathrm{I})$, is the degree of the 0–cycle in $P_2(\mathbb{C})$:*

$$\sum_p I_p(\mathrm{I})p.$$

One of the most important results used in this work is based on a Lemma of Darboux. This lemma gives a relationship among the degrees of 6 polynomials and some of their intersection numbers. As stated by Darboux [4], this relation does not always hold as counterexamples show, (cf.[3]). We give below a new version of Darboux's lemma whose proof appeared in [3].

Definition 7 *We consider the triple of homogeneous polynomials $V = [A, B, C]$ in $\mathbb{C}[X, Y, Z]$. We say that V is* irreducible *if the polynomials A, B and C are relatively coprimes (i.e.* no exists *$d : d \mid A,\ d \mid B,\ d \mid C$).*

We notice that in this case the ideal $\langle A, B, C\rangle$ verifies $\{p \in P_2(\mathbb{C}) : F(p) = 0,\ \forall F \in \langle A, B, C\rangle\}$ is finite.

Definition 8 *We consider $V = [A, B, C]$ and $W = [A', B', C']$ two triples of homogeneous polynomials in $\mathbb{C}[X, Y, Z]$ with $l = \deg A$, $m = \deg B$, $n = \deg C$, $l' = \deg A'$, $m' = \deg B'$ and $n' = \deg C'$. V and W are* orthogonal *if*

$$AA' + BB' + CC' = 0.$$

Observation: If $V = [A, B, C]$ and $W = [A', B', C']$, two triples of homogeneous polynomials in $\mathbb{C}[X, Y, Z]$ with $l = \deg A$, $m = \deg B$, $n = \deg C$, $l' = \deg A'$, $m' = \deg B'$ and $n' = \deg C'$, are orthogonal, then it is easy to see that $l + l' = m + m' = n + n' = r$.

Lemma 1 (Darboux's Lemma) *We consider two triples of homogeneus polynomials $V = [A, B, C]$ and $W = [A', B', C']$ in $\mathbb{C}[X, Y, Z]$ with $l = \deg A$, $m = \deg B$, $n = \deg C$, $l' = \deg A'$, $m' = \deg B'$ and $n' = \deg C'$. Assume that*

(i) V and W are irreducible,

(ii) V and W are orthogonal,

Consider $h = I(A, B, C)$ and $h' = I(A', B', C')$. Then

$$h + h' \geq \frac{lmn + l'm'n'}{r}. \tag{6}$$

If, moreover,
(iii) $\mathrm{V}(A, B, C, A', B', C') = \emptyset$ *in $P_2(\mathbb{C})$*
then the equality in (6) holds.

The proof of this lemma is given in [3].

We shall need the following divisors and zero–cycles in $P_2(\mathbb{C})$. We consider the following *intersection zero–cycle* of $P_2(\mathbb{C})$ associated to equation (1); $D(P, Q, R) = \sum_{p \in P_2(\mathbb{C})} I_p(P, Q, R)p$. If F is a formal irreducible curve over $\mathbb{C}$, we define the *singularities divisor* of F by $\sum_{p \in P_2(\mathbb{C})} I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right)p$.

Definition 9 *We say that a line L is a* tangent *of an algebraic curve F at $p \in P_2(\mathbb{C})$ if $I_p(L, F) > m_p(F)$*

Definition 10 *Let F be an algebraic curve in $P_2(\mathbb{C})$ and $p \in P_2(\mathbb{C})$. We say that p is a* singular point *of F if $m_p(F) > 0$. We say that p is an* ordinary singular point *of F if F has $m_p(F)$ distinct tangents at p.*

Proposition 3 *The degree of the intersection cycle $D(P, Q, R)$ is $I(P, Q, R) = m^2 + m + 1$, where $m + 1 = \deg P = \deg Q = \deg R$.*

The proof of this proposition is an easy application of Darboux's Lemma 1 to the polynomials P, Q, R, X, Y, Z because of condition (2) and the fact that there is no point in $P_2(\mathbb{C})$ satisfying all $X = 0$, $Y = 0$ and $Z = 0$.

Definition 11 *We say that the equation (1) or, equivalently, (3), is* Darboux integrable *if and only if there exist $F_1, F_2, \ldots, F_s$ irreducible invariant algebraic curves, with degrees $n_1, n_2, \ldots, n_s$ and cofactors $K_1, K_2, \ldots, K_s$ respectively, and complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$, not all of them null, such that*

(i) $\sum_{i=1}^{s} \alpha_i n_i = 0$,

(ii) $\sum_{i=1}^{s} \alpha_i K_i = 0$.

We consider the function $H := F_1^{\alpha_1} F_2^{\alpha_2} \ldots F_s^{\alpha_s}$, which is defined in all $P_2(\mathbb{C})$ except for the zero–set on an algebraic curve. For all the points at which H is defined we say that $H = C$, where C is an arbitrary constant, is a Darboux general integral *for equation (1) (or equivalently (3)).*

Following Poincaré, we shall say that (1) (or (3)) is algebraically integrable *if it is Darboux integrable and for α_i all integer numbers, $i = 1, \ldots, s$. In this case, we consider $H := F_1^{\alpha_1} F_2^{\alpha_2} \ldots F_s^{\alpha_s}$ and for all the points at which it is defined we say that $H = C$, where C is an arbitrary constant, is a* rational general integral *for equation (1) (or equivalently (3)).*

Remark 2 *If $\alpha_1, \ldots, \alpha_s$ are rational numbers, we can consider for $i = 1, \ldots, s$ $\beta_i = lcd(\alpha_1, \ldots, \alpha_s)\alpha_i$, with $lcd(\alpha_1, \ldots, \alpha_s)$ is the lowest common divisor of $\alpha_1, \ldots, \alpha_s$. Then, $H = F_1^{\beta_1} F_2^{\beta_2} \ldots F_s^{\beta_s}$ is a rational integral for equation (1) (or equivalently (3)).*

If $\alpha_1, \ldots, \alpha_s \in \mathbb{C}$ and there exists $\alpha \in \mathbb{C}$ such that $\beta_i = \alpha_i/\alpha$ is a rational number for $i = 1, \ldots, s$, then we always have a rational integral for equation (1) (or equivalently (3)).

The main objectives of this work are to give conditions for algebraic integrability taking into account Darboux's lemma.

Theorem 3 (Darboux [4], Jouanolou [6]) *Consider an equation (1) or, equivalently, (3) and F_i, $i = 1, \ldots, q$, distinct invariant algebraic curves.*

(i) If $q \geq m(m+1)/2 + 1$, equation (1) or, equivalently, (3) has a Darboux general integral.

(ii) If $q \geq m(m+1)/2 + 2$, equation (1) or, equivalently, (3) has a rational general integral.

In this article we prove the following three results. The first one generalizes for the projective plane a theorem proved for the affine case in [2], whose proof is completely different from the one given for the affine case.

Theorem 4 *Let F be an algebraic solution of degree $n > 1$ for equation (1) (or equivalently (3)) with cofactor K and let*

$$h = I\left(L - \frac{KX}{n}, M - \frac{KY}{n}, N - \frac{KZ}{n}\right).$$

Let m be the degree of equation (3), that is $m = \deg L = \deg M = \deg N$. If $h = m^2$, then the equation (3) is algebraically integrable.

Theorem 5 *Let F be an algebraic solution of degree $n > 1$ for equation (1) (or equivalently (3)) whose irreducible factors are all simple. Let h' be the degree of the singularities divisor of F, that is $h' = \sum_p I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right)$. Let m be the degree of the equation (3), that is $m = \deg L = \deg M = \deg N$. We have*

(i) $h' \geq (n-1)(n-m-1)$,

(ii) if $h' = (n-1)(n-m-1)$, then equation (3) is algebraically integrable.

The following theorem stands for Darboux's integrability in the quadratic case.

Theorem 6 *We consider the differential equation (1) (or equivalently (3)) with degree $m = 2$ and let F be an invariant algebraic curve of degree $n > 4$ for equation (1) (or (3)) whose irreducible factors are all simple. Let $h' = I\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right)$. If the singularities zero–cycle of F is such that, for all $p \in P_2(\mathbb{C})$, $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \leq 1$, then (3) is Darboux integrable.*

This article is organized as follows: In next section we give some necessary definitions and results to prove theorems (4), (5) and (6). These proofs are given in section 4.

3 Invariant algebraic curves

Next proposition is an application of Darboux's Lemma.

Proposition 4 *Let F be an invariant algebraic curve of degree n for equation (1) or, equivalently, (3) with cofactor K and let*

$$h = I\left(L - \frac{KX}{n}, M - \frac{KY}{n}, N - \frac{KZ}{n}\right)$$

and h' be the degree of the zero–cycle of $P_2(\mathbb{C})$ attached to F. Let m be the degree of equation (3), that is $m = \deg L = \deg M = \deg N$. Then,

$$h + h' \geq \frac{(n-1)^3 + m^3}{m + n - 1} = (n-1)^2 - m(n-1) + m^2. \tag{7}$$

Proposition 5 *We may always assume that the cofactor of an invariant algebraic curve of degree n for equation (3) is null, because, otherwise, we can change polynomials L, M, N to $L' = L - KX/n$, $M' = M - KY/n$ and $N' = N - KZ/n$ without changing the form of equation (3) and for L', M', N' the cofactor K is null.*

The proof is immediate (cf. remark 1).

Remark 3 *Note that, since we are assuming $V = [P, Q, R] = [MZ - NY, NX - LZ, LY - MX]$ is irreducible, we have that $W = [L, M, N]$ is irreducible.*

Lemma 2 *We can construct a matrix D for which the projectivity*

$$T_D : P_2(\mathbb{C}) \to P_2(\mathbb{C})$$

defined by D transforms equation (3) to another one verifying $(L, M) = 1$, $(L, N) = 1$ and $(M, N) = 1$.

Proof. Note that since $(P, Q, R) = 1$ we can easily show that $(L, M, N) = 1$.

Let us consider the matrix

$$A = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8}$$

where $k \in \mathbb{C}$. Under the transformation T_A, the equation

$$L(YdZ - ZdY) + M(ZdX - XdZ) + N(XdY - YdX) = 0$$

takes the form:

$$(L + kN)(vdw - wdv) + M(wdu - udw) + N(udv - vdu) = 0$$

with L, M, N in variables u, v, w. Assume that $(L + kN, M) \neq 1$ for all $k \in \mathbb{C}$. Since the number of divisors of M is finite ($\deg M < \infty$), there are at least two values $k_1 \neq k_2$ such that a divisor d (non–constant) divides both $L + k_1 N$ and $L + k_2 N$, and M. Then, substracting one expression from the other we conclude that d divides N and analogously d divides L. Then, d is a divisor of L, M and N contrary to the hypothesis. Therefore $L + kN$ and M are coprimes for some $k \in \mathbb{C}$.

We may assume, therefore, that $(L, M) = 1$ and we can consider the projectivity T_B, given by the matrix

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix}, \tag{9}$$

where $k \in \mathbb{C}$. By this projectivity, the polynomials L and M do not change, so they are still coprime and we may find some $k \in \mathbb{C}$ for which $(M, N + kL) = 1$, reasoning analogously as before.

Hence, we may assume that $(L, M) = 1$ and $(M, N) = 1$ and we can consider the projectivity T_C given by the matrix

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix}, \tag{10}$$

where $k \in \mathbb{C}$. By this projectivity, the polynomials L and M do not change, so they are still coprime. If a non–constant polynomial d divides both M and $N + kM$ for a fixed $k \in \mathbb{C}$, then d divides M and N contrary to our hypothesis. So, after applying the projectivity, the equation verifies $(L, M) = 1$ and $(M, N) = 1$. Moreover, reasoning as before, we may find a value of $k \in \mathbb{C}$ for which the new L and N verify $(L, N) = 1$. ∎

Proposition 6 *(i) For all triple of homogeneous polynomials L, M, N, we have that $I(L, M, N) \leq I(L, M)$.*

(ii) Consider equation (3) with m the degree of L,M and N and let $h = I(L, M, N)$. The following inequality holds $h \leq m^2$.

Proof. Since the ideal (L, M, N) is contained into the ideal (L, M) we deduce that $I(L, M, N) \leq I(L, M)$. From lemma 2 we may assume that $(L, M) = 1$, which by Bézout's Theorem yields: $I(L, M) = m^2$. Hence, $h = I(L, M, N) \leq m^2$. ∎

We first give some important results which are used in the proofs of the main theorems.

Proposition 7 *If F is an invariant algebraic curve for equation (3) whose singular points are double and ordinary, then $\deg F \leq 2m$.*

This proposition is proved in [2].

Lemma 3 *Let P, Q, R be homogeneous polynomials with degree $m + 1$ and r a line such that $V = [P, Q, R]$ is irreducible. Then $I(r, P, Q, R) \leq m + 1$.*

Proof. If $I(r, P, Q, R) > m + 1$, since

$$I(r, P, Q, R) \leq \min\{I(r, P), I(r, Q), I(r, R)\}, \tag{11}$$

then $I(r, P) > m+1$, $I(r, Q) > m+1$ and $I(r, R) > m+1$. Hence, r divides P, Q and R in contradiction to the fact that $V = [P, Q, R]$ is irreducible. ∎

Proposition 8 *A line r is an algebraic solution line for equation (1) if, and only if, $I(r, P, Q, R) = m + 1$.*

Proof. We may assume $r = Z$ without loss of generality. Then, from (11), we deduce $I(r, P) \geq m + 1$ and $I(r, Q) \geq m + 1$. Equivalently, $I(Z, NY) \geq m + 1$ and $I(Z, NX) \geq m + 1$. Since X, Y and Z do not have any common point, then $I(Z, N) \geq m + 1$. Therefore Z divides N and r is an invariant algebraic line.

Conversely, if r is an invariant algebraic line, we may assume that $r = Z$ and its cofactor is null (by proposition 5), then $N = 0$. We have $P = MZ$, $Q = -LZ$ and $R = LY - MX$. Let us consider $p \in P_2(\mathbb{C})$ a point in $Z = 0$, then $I_p(Z, MZ, -LZ, LY - MX) = I_p(Z, LY - MX)$ because MZ and $-LZ$ belong to the ideal generated by Z and $LY - MX$. Hence, $I(Z, MZ, -LZ, LY - MX) = I(Z, LY - MX)$. If Z divides $LY - MX$ then Z divides P, Q and R in contradiction to our hypothesis, so, by Bézout's theorem, $I(Z, LY - MX) = m + 1$. ∎

Lemma 4 *Let P, Q, R be homogeneous polynomials with degree $m+1$ such that $V = [P, Q, R]$ is irreducible satisfying $PX + QY + RZ = 0$ and u an irreducible polynomial with degree $k > 1$. Then $I(u, P, Q, R) < (m + 1)k$.*

Proof. If $I(u, P, Q, R) \geq (m + 1)k$. We consider the net of curves $C_\lambda = \alpha P + \beta Q + \gamma R = 0$ where $\lambda = [\alpha, \beta, \gamma]$, $[\alpha : \beta : \gamma] \in P_2(\mathbb{C})$. We have that $I(C_\lambda, u) \geq (m + 1)k$, $\forall \lambda$. We choose different λ_i for $i = 1, 2$ such that the curve C_{λ_i} passes through another point in $V(u)$ not contained in $V(u, P, Q, R)$. Then $I(C_{\lambda_i}, u) \geq (m + 1)k + 1$. Then, since u is irreducible, by Bézout's theorem u divides C_{λ_i}, that is $C_{\lambda_i} = u \cdot \Omega_i$, for $i = 1, 2$. We consider another curve of the net different from C_{λ_i} for $i = 1, 2$ which we call C_{λ_3}. We have that there exist $a_j, b_j, c_j \in P_2(\mathbb{C})$ for $j = 1, 2, 3$ such that

$$\begin{array}{rclcl} P & = & a_1 C_{\lambda_1} + b_1 C_{\lambda_2} + c_1 C_{\lambda_3} & = & a_1 u\Omega_1 + b_1 u\Omega_2 + c_1 C_{\lambda_3}, \\ Q & = & a_2 C_{\lambda_1} + b_2 C_{\lambda_2} + c_2 C_{\lambda_3} & = & a_2 u\Omega_1 + b_2 u\Omega_2 + c_2 C_{\lambda_3}, \\ R & = & a_3 C_{\lambda_1} + b_3 C_{\lambda_2} + c_3 C_{\lambda_3} & = & a_3 u\Omega_1 + b_3 u\Omega_2 + c_3 C_{\lambda_3}. \end{array} \tag{12}$$

Since $PX + QY + RZ = 0$ then $u[\Omega_1(a_1X + a_2Y + a_3Z) + \Omega_2(b_1X + b_2Y + b_3Z)] + C_{\lambda_3}(c_1X + c_2Y + c_3Z) = 0$. Then u divides $C_{\lambda_3}(c_1X + c_2Y + c_3Z)$, but u is an irreducible polynomial of degree > 1, so u divides C_{λ_3}. From (12) u divides P, Q, R in contradiction with the hypothesis. ∎

A very useful theorem concerning intersection of curves was proved by Max Noether. The theorem states that if three curves F, G, H satisfy certain conditions, then there exist homogeneous polynomials A, B such that $H = AF + BG$. The theorem has various forms depending on the restrictions one puts on the singularities of F and G.

Definition 12 *Let us consider three curves in $P_n(\mathbb{C})$, F, G and H, such that F and G do not have any common component. We say that* Noether's conditions are satisfied at $p \in P_n(\mathbb{C})$ with respect to F, G and H) *if $H_* \in (F_*, G_*)$, where (F_*, G_*) is the ideal of O_p. Here, $F_*(x_1, \ldots, x_n) = F(1, x_1, x_2, \ldots, x_n)$ (we are assuming that $x_0(p) \neq 0$).*

Theorem 7 (Max Noether's Fundamental Theorem) *Let F, G, H be projective curves. Assume F and G have no common components. Then there is an equation $H = AF + BG$, with A, B homogeneous polynomials of degrees $\deg H - \deg F, \deg H - \deg G$, respectively if and only if Noether's conditions are satisfied at every point p in both $F = 0$ and $G = 0$.*

4 Proofs of the main theorems

Proof of theorem 4. From proposition 5 we may assume, without loss of generality, that the cofactor K is null. From lemma 2, we can make a variable change in order to have $(L, M) = 1$ and by Bézout's theorem $I(L, M) = m^2$. For all $p \in P_2(\mathbb{C})$ we have that $I_p(L, M, N) \leq I_p(L, M)$, so

$$m^2 = I(L, M, N) = \sum_p I_p(L, M, N) \leq \sum_p I_p(L, M) = I(L, M) = m^2.$$

So, for all p the inequality is an equality, $I_p(L, M, N) = I_p(L, M)$, which means that $O_p \cap (L, M, N) = O_p \cap (L, M)$. We can apply Noether's theorem because $(L, M) = 1$ and for all p satisfying $L = 0$ and $M = 0$ we have that N_* belongs to (L_*, M_*). Therefore, $N = k_1 L + k_2 M$ with $\deg k_i = m - m = 0$, for $i = 1, 2$. Then $v = k_1 X + k_2 Y - Z = 0$ is an invariant algebraic line for equation (3), which is different from F because $\deg F > 1$, with null cofactor. Hence, $H = \frac{F}{u^n}$ is a rational integral for equation (3). ∎

Proof of theorem 5. From proposition 5 we may assume, without loss of generality, that the cofactor K, for F, is null and let $h = I(L, M, N)$. Applying Darboux's Lemma to $L\frac{\partial F}{\partial X} + M\frac{\partial F}{\partial Y} + N\frac{\partial F}{\partial Z} = 0$ we have

$$h + h' \geq \frac{m^3 + (n-1)^3}{m + n - 1} = m^2 - m(n-1) + (n-1)^2. \tag{13}$$

Since $h \leq m^2$ from proposition 6 we deduce $h' \geq (n-1)(n-m-1)$. We have proved part *(i)*. If $h' = (n-1)(n-m-1)$ then, from (13) we deduce $h \geq m^2$ and then $h = m^2$. We can apply theorem 4 and we deduce that equation (3) is algebraically integrable. ∎

We give some results which were stated by Darboux in [4].

Theorem 8 *If F and G are two different invariant algebraic curves for equation (3) and $p \in P_2(\mathbb{C})$ is a point satisfying both $F = 0$ and $G = 0$, then p is a singular point for equation (3).*

Theorem 9 *Let m be the degree of equation (3), that is $m = \deg L = \deg M = \deg N$ and let F be an invariant algebraic curve of degree 4 for equation (3) whose irreducible factors all are simple. Let h' be the degree of the singularities divisor of F, that is $h' = I\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right)$. If the singularities divisor of F is such that, for all $p \in P_2(\mathbb{C})$, $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \leq 1$, then the existence of a Darboux integral with the given form is assured. More precisely, we have:*

(i) *$F = F_1 F_2$, with F_1, F_2 two conics; the Darboux integral is $H = \frac{F_1}{F_2}$.*

(ii) *$F = F_1 F_2 F_3$, with F_1, F_2 two lines and F_3 a conic; the Darboux integral is $H = F_1^{\alpha_1} F_2^{\alpha_2} F_3^{\alpha_3}$, for some $\alpha_i \in \mathbb{C}$ for $i = 1, 2, 3$.*

(iii) *$F = F_1 F_2 F_3 F_4$, with F_i four lines for $i = 1, \ldots, 4$; the Darboux integral is $H = F_1^{\alpha_1} F_2^{\alpha_2} F_3^{\alpha_3} F_4^{\alpha_4}$, for some $\alpha_i \in \mathbb{C}$ for $i = 1, 2, 3, 4$.*

Proof of theorem 6. Let us consider the components $F_1, F_2, \ldots, F_s$ of F, that is the irreducible factors of F.

We know that the number of singular points of (3) is 7, that is $I(LY - MX, LZ - NX, MZ - NY) = 7$, by proposition (3). Particularly, we can have at most 7 different singular points.

If $s = 1$, F has only one component with degree ≥ 4, using proposition 7 we deduce that F is an invariant quartic for equation (3) because if, for all $p \in P_2(\mathbb{C})$, $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \leq 1$, then the singular points of F must all be, in case they exist, double and ordinary. We may assume that the cofactor K for the quartic F is null by proposition 5. With the same notation used in proposition 4 we have $h + h' \geq 7$. Since, for all $p \in P_2(\mathbb{C})$, $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \leq 1$, we have that h' equals the number of singular points of the curve F therefore $h' \leq 3$ (an irreducible quartic has at most 3 singular points). This gives $h \geq 4$ and by proposition 6 we have $h \leq 4$, then $h = I(L, M, N) = 4$. Using theorem 4 we have a rational first integral for equation (3).

If $s = 2$ we have F_1 and F_2 the components of F. Using Bézout theorem, we have that $I(F_1, F_2) = \deg F_1 \cdot \deg F_2$. Since, for all $p \in P_2(\mathbb{C})$, $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \leq 1$, we have $\deg F_1 \cdot \deg F_2$ different points of intersection of F_1 and F_2. Since each point of intersection of F_1 and F_2 is a singular point for equation (3) (theorem 8) we deduce that $\deg F_1 \cdot \deg F_2 \leq 7$. Moreover, we know that if r is an invariant algebraic line for equation (3) then, by lemma 3 the number of singular points of equation (3) which verify $r = 0$ is < 4 and if u is an irreductible invariant algebraic curve of degree k, with $k > 1$, then the such a number is $< 3k$ by lemma 4. Then, since $\deg F = \deg F_1 + \deg F_2 \geq 4$, the only possibilities for F_1 and F_2 to satisfy the hypothesis are *(i)* $\deg F_1 = \deg F_2 = 2$,

(ii) $\deg F_1 = 1$ and $\deg F_2 = 3$.

(i) We have two conics F_1 and F_2, and $F = F_1F_2$, such that, for all $p \in P_2(\mathbb{C})$, $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \leq 1$. Therefore $H = F_1/F_2$ is a first integral for equation (3), because of theorem 9.

(ii) We have a line F_1 and a cubic F_2 algebraic solutions of (3). As before, we may assume that the cofactor of F_2 is null by proposition 5 and using the same notation as in proposition 4 we have that for F_2 $h + h' \geq 4$. The number of singular points for F_2, which must be double and ordinary if they exist, is h' because, for all $p \in P_2(\mathbb{C})$, $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \leq 1$, where $F = F_1F_2$. Then $h' \leq 1$ (because a cubic has at most one singular point) and therefore, $h = I(L, M, N) \geq 3$. After a variable change we may assume that the curve F_1 is Z and then $N = AZ$ where A is a lineal polynomial and since this line and the cubic must transversally cut we deduce that the homogeneous terms of F_2 which do not contain Z, say f_2, factorize in simple factors.

If $A \neq 0$, since $W = [L, M, N]$ is irreducible, it is not possible that A divides both L and M. Then $I(L, M, A) \leq 2$ and therefore $I(L, M, Z) \geq 1$. Then $L = 0$, $M = 0$ and $Z = 0$ have at least one point in common, whose last coordinate is null. Let l and m be the homogeneous terms of L and M respectively which do not contain Z. Since l and m must be zero when evaluated in p, they have a common divisor: $l = d\bar{l}$ and $m = d\bar{m}$. Since F_2 is an invariant algebraic curve with null cofactor, we have $L\frac{\partial F_2}{\partial X} + M\frac{\partial F_2}{\partial Y} + N\frac{\partial F_2}{\partial Z} = 0$. Evaluating last equality in $Z = 0$ and simplifying by d we deduce $\bar{l}\frac{\partial f_2}{\partial X} + \bar{m}\frac{\partial f_2}{\partial Y} = 0$. These relations stand for $\frac{\partial f_2}{\partial x}$ and $\frac{\partial f_2}{\partial y}$ share a factor and then this factor divides twice f_2 in contradiction with the hypothesis.

The only possibility is $A = 0$ and, hence, $N = 0$. Then Z is an invariant curve with null cofactor and $H = F_2/Z^3$ is a rational first integral of equation (3).

If $s = 3$, the only possible case which we have not studied previously is when F's irreducible factors are two lines and one conic. In this case, we have a Darboux integral by theorem 9.

If $s = 4$, the only case to study is when F's irreducible factors are four lines. In this case, the existence of a Darboux first integral is proved in theorem 9. ∎

Corollary 1 *Under the hypothesis of theorem 6, $F = F_1F_2 \ldots F_s$, unless*

(i) $s = 3$, $\deg F_1 = \deg F_2 = 1$ *and* $\deg F_3 = 2$, *or*

(ii) $s = 4$, $\deg F_i = 1$ *for* $i = 1, \ldots 4$

the Darboux integral found in theorem 6 is rational.

The proof of this corollary comes from the proof of theorem 6. We give an example for which equation (3) has 4 invariant lines and it is not rationally integrable. Let $L = X(aY + Z)$, $M = Y(-aX + Z)$ and $N = -Z(X + Y)$. The corresponding equation has X, Y, Z and $X + Y + Z$ as invariant lines. The point $p = [0 : -1 : 1]$ is an intersection point of X and $X + Y + Z$ and, therefore, is a singular point for equation (3). The eigenvalues associated to p are $(-1, -a)$ and its quotient may not be rational if a is not rational. In this case, the equation is not rationally integrable.

Corollary 2 *Under the hypothesis of theorem 6 and if $\deg F \geq 5$, then equation (3) is rationally integrable.*

Proof. Taking into account corollary 1 we have that the only possibilities in which $\deg F \geq 5$ and the rational integrability is not assured are if there are 3 invariant lines and a conic or if there are 5 invariant lines. Both cases are impossible since the number of singular points is ≤ 7 and, for all $p \in P_2(\mathbb{C})$, $I_p\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right) \leq 1$. ∎

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