Lie Symmetries of Multidimensional Difference Equations

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Abstract

A method is presented for calculating the Lie point symmetries of a difference system, consisting of a scalar difference equation in two independent variables and four supplementary equations, defining a two-dimensional lattice. The symmetry transformations act on the equations and on the lattice. They take solutions into solutions and can be used to perform symmetry reduction. The method generalizes one presented in a recent publication for the case of ordinary difference equations. In turn, it can easily be generalized to difference systems involving an arbitrary number of dependent and independent variables.

Résumé

Nous présentons une méthode pour calculer les symétries ponctuelles de Lie pour un sytème aux différences constitué d'une équation scalaire aux différences, à deux variables indépendantes, et quatre équations supplémentaires définissant un réseau deux-dimensionnel. Les transformations de Lie agissent sur l'équations et sur son réseau. Elles transforment une solution en une autre solution et peuvent être utilisées pour faire de la réduction par symétrie. La méthode est une généralisation d'un article récent pour le cas des équations aux différences ordinaires. Cette méthode peut facilement être généralisée aux systèmes avec un nombre arbitraire de variables dépendantes et indépendantes.

1 Introduction

A recent article [1] was devoted to Lie point symmetries, acting on ordinary difference equations and lattices, while leaving their set of solutions invariant. The purpose of this article is to extend the previously obtained methods and results to the case of partial difference equations, i.e. equations involving more than one independent variable.

Algebraic techniques, making use of Lie groups and Lie algebras, have proved themselves to be extremely useful in the theory of differential equations [2].

When applying similar algebraic methods to difference equations, several decisions have to be made.

The first decision is a conceptual one. One can consider difference equations and lattices as given objects to be studied. The aim then is to provide tools for solving these equations, simplifying the equations, classifying equations and their solutions, and identifying integrable, or linearizable difference equations [1, 3, ..., 12]. Alternatively, one can consider difference equations and the lattices on which they are defined, to be auxiliary objects. They are introduced in order to study solutions of differential equations, numerically or otherwise. The question to be asked in this is: how does one discretize a differential equation, while preserving its symmetry properties [13, ..., 16].

In this article we take the first point of view: the equation and the lattice are *a priori* given. The next decision to be made is a technical one: which aspect of symmetry to pursue. For differential equations one can look for point symmetries, or generalized ones. When restricting to point symmetries, and constructing the Lie algebra of the symmetry group, one can use vector fields acting on dependent and independent variables. Alternatively and equivalently, one can use evolutionary vector fields, acting only on dependent variables.

For difference equations, these two approaches are in general not equivalent and may lead to different results, both of them correct and useful.

In this article we shall consider point symmetries only and use vector fields acting on all variables. A general formalism for determining the symmetry algebra is presented in Section 2. It generalizes the algorithm presented earlier [1] for ordinary difference equations to the case of several independent variables. In Section 3 we apply the algorithm to a discrete linear heat equation which we consider on several different lattices, each providing its own symmetries. Section 4 is devoted to difference equations on lattices that are invariant under Lorentz transformations. In Section 5 we discuss two different discrete Burgers equations, one linearizable, the other not. The lattices are the same in both cases, the symmetry algebras turn out to be different. Section 6 treats symmetries of differential-difference equations, i.e. equations involving both discrete and continuous variables. Some conclusions are drawn in the final Section 7.

2 General symmetry formalism

2.1 The difference scheme

For clarity and brevity, let us consider one scalar equation for a continuous function of two (continuous) variables: $u = u(x,t)$. A lattice will be a set of points P_i, lying in the plane $\mathbb{R}^2$ and stretching in all directions with no boundaries. The points P_i in $\mathbb{R}^2$ will be labeled by two discrete labels $P_{m,n}$. The Cartesian coordinates of the point $P_{m,n}$ will be $(x_{m,n}, t_{m,n})$ with $-\infty < m < \infty$, $-\infty < n < \infty$ (we are of course not obliged to use Cartesian coordinates). The value of the dependent variable in the point $P_{m,n}$ will be denoted $u_{m,n} = u(x_{m,n}, t_{m,n})$.

A difference scheme will be a set of equations relating the values of $\{x,t,u\}$ in a finite number of points. We start with one 'reference point' $P_{m,n}$ and define a finite number of points $P_{m+i,n+j}$ in the neighborhood of $P_{m,n}$. They must lie on two different curves, intersecting in $P_{m,n}$. Thus, the difference scheme will have the form

$$\begin{gathered} E_a\Big(\ \{x_{m+i,n+j}, t_{m+i,n+j}, u_{m+i,n+j}\}\ \Big) = 0 \qquad 1 \le a \le 5 \\ -i_1 \le i \le i_2 \qquad -j_1 \le j \le j_2 \quad i_1, i_2, j_1, j_2 \in \mathbb{Z}^{\ge 0}. \end{gathered} \tag{1}$$

The situation is illustrated on Figure 1. It corresponds to a lattice determined by 6 points. Our convention is that x increases as m grows, t increases as n grows (i.e. $x_{m+1,n} - x_{m,n} \equiv h_1 > 0$, $t_{m,n+1} - t_{m,n} \equiv h_2 > 0$). The scheme on Figure 1 could be used e.g. to approximate a differential equation of third order in x, second in t.

Of the above five equations in (1), four determine the lattice, one the difference equation. If a continuous limit exists, it is a partial differential equation in two variables. The four equations determining the lattice will reduce to identities (like $0 = 0$).

The system (1) must satisfy certain independence criteria. Starting from the reference point $P_{m,n}$ and a given number of neighboring points, it must be possible to calculate the values of $\{x,t,u\}$ in all points. This requires a minimum of five equations: to be able to calculate the (x,t) in two directions and u in all points. For instance, to be able to move upward and to the right along the curves passing through $P_{m,n}$ (with either m, or n fixed) we impose a condition on the Jacobian

$$|J| = \left| \frac{\partial(E_1, E_2, E_3, E_4, E_5)}{\partial(x_{m+i_2,n}, t_{m+i_2,n}, x_{m,n+j_2}, t_{m,n+j_2}, u_{m+i_2,n+j_2})} \right| \neq 0. \tag{2}$$

As an example of difference scheme, let us consider the simplest and most standard lattice, namely a uniformly spaced orthogonal lattice and a difference equation approximating the linear heat equation on this lattice. Equations (1) in this case are:

$$x_{m+1,n} - x_{m,n} = h_1 \qquad t_{m+1,n} - t_{m,n} = 0 \tag{3}$$

$$x_{m,n+1} - x_{m,n} = 0 \qquad t_{m,n+1} - t_{m,n} = h_2 \tag{4}$$

$$\frac{u_{m,n+1} - u_{m,n}}{h_2} = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{(h_1)^2} \tag{5}$$

where h_1 and h_2 are constants.

The example is simple and the lattice and the lattice equations can be solved explicitly to give

$$x_{m,n} = h_1 m + x_0 \qquad t_{m,n} = h_2 n + t_0. \tag{6}$$

The usual choice is $x_0 = t_0 = 0$, $h_1 = h_2 = 1$ and then x is simply identified with m, t with n. We need the more complicated two index notation to describe arbitrary lattices and to formulate the symmetry algorithm (see below).

The example suffices to bring out several points:

1. Four equations are needed to describe the lattice.

2. Four points are needed for equations of second order in x, first in t. Only three figure in the lattice equation, namely $P_{m+1,n}$, $P_{m,n}$ and $P_{m,n+1}$. To get the fourth point, $P_{m-1,n}$, we shift m down by one unit in equations (3).

3. The independence condition (2) is needed to be able to solve for $x_{m+1,n}$, $t_{m+1,n}$, $x_{m,n+1}$, $t_{m,n+1}$ and $u_{m,n+1}$.

2.2 Symmetries of the difference scheme

We are interested in point transformations of the type

$$\tilde{x} = F_\lambda(x,t,u) \quad \tilde{t} = G_\lambda(x,t,u) \quad \tilde{u} = H_\lambda(x,t,u) \tag{7}$$

where λ is a group parameter, such that when (x,t,u) satisfy the system (1) then $(\tilde{x},\tilde{t},\tilde{u})$ satisfy the same system. The transformation acts on the entire space (x,t,u), at least locally, i.e. in some neighborhood of the reference point $P_{m,n}$, including all points $P_{m+i,n+j}$ figuring in equation (1). That means that the same functions F, G and H determine the transformation of all points. The transformations (7) are generated by the vector field

$$\hat{X} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u. \tag{8}$$

We wish to find the symmetry algebra of the system (1), that is the Lie algebra of the local symmetry group of local point transformations. To do this we must prolong the action of the vector field $\hat{X}$ from the reference point $(x_{m,n}, t_{m,n}, u_{m,n})$ to all points figuring in the system (1). Since the transformations are given by the same functions F, G and H at all points, the prolongation of the vector field (8) is obtained simply by evaluating the functions ξ, τ and ϕ at the corresponding points.

In order words, we can write

$$\begin{aligned} pr\,\hat{X} = \textstyle\sum_{m,n} \Big[& \xi(x_{m,n}, t_{m,n}, u_{m,n})\partial_{x_{m,n}} + \tau(x_{m,n}, t_{m,n}, u_{m,n})\partial_{t_{m,n}} \\ & +\phi(x_{m,n}, t_{m,n}, u_{m,n})\partial_{u_{m,n}} \Big], \end{aligned} \tag{9}$$

where the summation is over all points figuring in the system (1). The invariance requirement is formulated in terms of the prolonged vector field as

$$\mathrm{pr}\hat{X}\, E_a\,|_{E_b=0} \qquad 1 \le a, b \le 5. \tag{10}$$

Just as in the case of ordinary difference equations, we can turn equation (10) into an algorithm for determining the symmetries, i.e. the coefficients in vector field (8).

The procedure is as follows:

1. Use the original equations (1) and the Jacobian condition (2) to express five independent quantities in terms of the other ones, e.g.

$$\begin{gathered} v_1 = x_{m+i_2,n} \qquad v_2 = t_{m+i_2,n} \qquad v_3 = x_{m,n+j_2} \\ v_4 = t_{m,n+j_2} \qquad v_5 = u_{m+i_2,n+j_2} \end{gathered} \tag{11}$$

as

$$\begin{gathered} v_a = v_a(x_{n+i,m+j}, t_{n+i,m+j}, u_{n+i,m+j}) \\ -i_1 \le i \le i_2 - 1 \qquad -j_1 \le j \le j_2 - 1. \end{gathered} \tag{12}$$

2. Write the five equations (10) explicitly and replace the quantities v_a using equation (12). We obtain five functional equations for the functions ξ, τ and ϕ, evaluated at different point of the lattice. Once the functions v_a are substituted into these equations, each value of $x_{i,k}, t_{i,k}$ and $u_{i,k}$ is independent. Moreover, it can only figure via the corresponding $\xi_{i,k}, \tau_{i,k}$ and $\phi_{i,k}$ (with the same values of i and k) via the functions v_a, or explicitly via the functions E_a.

3. Assume that the dependence of ξ, τ and ϕ on their variables is analytic. Convert the obtained functional equations into a system of differential equations by differentiating with respect to the variables $x_{i,k}, t_{i,k}$ and $u_{i,k}$. This provides an overdetermined system of linear partial differential equations which we must solve.

4. The solutions of the differential equations must be substituted back into the functional ones and these in turn must be solved.

The above algorithm provides us with the function $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$ figuring in equation (8). The finite transformations of the (local) Lie symmetry group are obtained in the usual manner, by integrating the vector field (8):

$$\begin{array}{lll} \frac{d\tilde{x}}{d\lambda}=\xi(\tilde{x},\tilde{t},\tilde{u}) & \frac{d\tilde{t}}{d\lambda}=\tau(\tilde{x},\tilde{t},\tilde{u}) & \frac{d\tilde{u}}{d\lambda}=\phi(\tilde{x},\tilde{t},\tilde{u}) \\ \tilde{x}\,|_{\lambda=0}=x & \tilde{t}\,|_{\lambda=0}=t & \tilde{u}\,|_{\lambda=0}=u. \end{array} \tag{13}$$

3 Discrete heat equation

The heat equation in one-dimension

$$u_t = u_{xx} \tag{14}$$

is invariant under a six-dimensional Lie group, corresponding to translations in x and t, dilations, Galilei transformations, multiplication of u by a constant and expansions. It is also invariant under an infinite dimensional pseudo-group, corresponding to the linear superposition principle.

Symmetries of the discrete heat equation have been studied, using different methods and imposing different restrictions on the symmetries [7, 11, 13].

Here we will use the discrete heat equation to illustrate the methods of Section 2 and to show the influence of the choice of the lattice.

3.1 Fixed rectangular lattice

The discrete heat equation and a fixed lattice were given in equation (5) and (3), (4), respectively. Applying the operator (9) to the lattice, we obtain

$$\xi(x_{m+1,n}, t_{m+1,n}, u_{m+1,n}) = \xi(x_{m,n}, t_{m,n}, u_{m,n}) \tag{15}$$

$$\xi(x_{m,n+1}, t_{m,n+1}, u_{m,n+1}) = \xi(x_{m,n}, t_{m,n}, u_{m,n}). \tag{16}$$

The values $u_{m+1,n}, u_{m,n+1}, u_{m,n}$ are not related by equation (5) (since it also contains $u_{m-1,n}$). Hence if we differentiate equations (15), (16), e.g. with respect to $u_{m,n}$, we find that ξ is independent of u. We have $t_{m+1,n} = t_{m,n}$ so equation (15) implies that ξ does not depend on x. Similarly, equation (16) implies that ξ does not depend on t. Hence ξ is constant. Similarly, we obtain that $\tau(x,t,u)$ is also constant. Applying the prolongation $pr\hat{X}$ to equation (5) we obtain the functional equation

$$\phi_{m,n+1} - \phi_{m,n} = \frac{h_2}{(h_1)^2}(\phi_{m+1,n} - 2\phi_{m,n} + \phi_{m-1,n}) \tag{17}$$

with e.g. $\phi_{m,n} \equiv \phi(x_{m,n}, t_{m,n}, u_{m,n})$.

In $\phi_{m,n+1}$ we replace $u_{m,n+1}$, using equation (5). We then differentiate with respect to $u_{m+1,n}$ and again with respect to $u_{m-1,n}$. We obtain

$$\phi_{m,n} = A(x_{m,n}, t_{m,n})u_{m,n} + B(x_{m,n}, t_{m,n}). \tag{18}$$

Substituting (18) into equation (17), using (5) again and setting the coefficient of $u_{m+1,n}, u_{m-1,n}, u_{m,n}$ and 1 equal to zero separately we find that A must be constant and B must be a solution of equation (5). Thus, the symmetry algebra of the heat equation on the lattice (3), (4) is given by

$$\hat{P}_0 = \partial_t \quad \hat{P}_1 = \partial_x \quad \hat{W} = u\partial_u \quad \hat{S} = S(x,t)\partial_u \tag{19}$$

with S a solution of the equation itself. Thus, the only symmetries are those due to the fact that the equation is linear and autonomous.

3.2 Lattices invariant under dilations

There are at least two ways of making the discrete heat equation invariant under dilations.

A) Five point lattice

We replace the system of equations (3), (4) and (5) by

$$x_{m+1,n} - 2x_{m,n} + x_{m-1,n} = 0 \qquad x_{m,n+1} - x_{m,n} = 0 \tag{20}$$

$$t_{m+1,n} - t_{m,n} = 0 \qquad t_{m,n+1} - 2t_{m,n} + t_{m,n-1} = 0 \tag{21}$$

$$\frac{u_{m,n+1} - u_{m,n}}{t_{m,n+1} - t_{m,n}} = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{(x_{m+1,n} - x_{m,n})^2}. \tag{22}$$

Applying $pr\hat{X}$ of equation (8) to (20) and substituting for $x_{m+1,n}, t_{m+1,n}, t_{m,n+1}$ and $x_{m,n+1}$ from the equations (20), (21) we obtain

$$\begin{aligned} &\xi(2x_{m,n} - x_{m-1,n}, t_{m,n}, u_{m+1,n}) - 2\xi(x_{m,n}, t_{m,n}, u_{m,n}) \\ &+\xi(x_{m-1,n}, t_{m-1,n}, u_{m-1,n}) = 0 \end{aligned} \tag{23}$$

$$\xi(x_{m,n}, 2t_{m,n} - t_{m,n-1}, u_{m,n+1}) = \xi(x_{m,n}, t_{m,n}, u_{m,n}). \tag{24}$$

Since $u_{m,n+1}$ and $u_{m,n}$ are independent a differentiation of (24) with respects to say $u_{m-1,n}$ (contained on the left hand side via $u_{m,n+1}$) implies that ξ does not depend on u. Differentiating (24) with respect to $t_{m,n-1}$ we find that ξ cannot depend on t either. Putting $\xi = \xi(x)$ into equation (23) and taking the second derivative with respect to $x_{m-1,n}$ and $x_{m,n}$, we obtain that ξ is linear in x. Similarly, invariance of equation (21) restrict the form of $\tau(x,t,u)$. Finally the lattice (20), (21) is invariant under the transformation generated by $\hat{X}$ with

$$\xi = \alpha x + \beta \qquad \tau = \gamma t + \delta. \tag{25}$$

Now let us apply $pr\hat{X}$ to equation (22). We obtain

$$\frac{\phi_{m,n+1} - \phi_{m,n}}{t_{m,n+1} - t_{m,n}} = \frac{\phi_{m+1,n} - 2\phi_{m,n} + \phi_{m-1,n}}{(x_{m+1,n} - x_{m,n})^2} - (2\alpha - \gamma)\frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{(x_{m+1,n} - x_{m,n})^2}. \tag{26}$$

Taking the second derivative $\partial_{u_{m+1,n}}\partial_{u_{m-1,n}}$ of equation (26) after using the equation (22) to eliminate $u_{m,n+1}$, we find $\phi_{m,n} = A_{m,n}(x,t)u_{m,n} + B_{m,n}(x,t)$. Substituting back into equation (26) we obtain $A_{m,n} = A = const.$, and see that $B_{m,n}(x,t)$ must satisfy the original difference system. Moreover, we obtain the restriction $\gamma = 2\alpha$.

Finally, on the lattice (20), (21) the heat equation (22) has a symmetry algebra generated by the operators (19) and the additional dilation operator

$$\hat{D} = x\,\partial_x + 2t\,\partial_t. \tag{27}$$

We mention that the lattice equations (20), (21) can be solved to give $x = am+b$, $t = cn+d$. At first glance this seems to coincide with the lattice (6). The difference is that in equation (6) h_1 and h_2 are fixed constants. Here a, b, c and d are integration constants that can be chosen arbitrarily. In particular, they can be dilated. Hence the additional dilational symmetry.

B) A four point lattice

We only need four points to write the discrete heat equation, so it makes sense to write a four point lattice. Let us define the lattice by the equations

$$x_{m+1,n} - 2x_{m,n} + x_{m-1,n} = 0 \qquad x_{m,n+1} - x_{m,n} = 0 \tag{28}$$

$$t_{m+1,n} - t_{m,n} = 0 \qquad t_{m,n+1} - t_{m,n} - c(x_{m,n+1} - x_{m,n})^2 = 0. \tag{29}$$

On this lattice the discrete heat equation (22) simplifies to

$$u_{m,n+1} - u_{m,n} = c(u_{m+1,n} - 2u_{m,n} + u_{m-1,n}). \tag{30}$$

Applying the same method as above, we find that invariance of the lattice implies $\xi = Ax + B$, $\tau = 2At + C$. Invariance of equation (30) then implies $\phi = Du + S(x,t)$ where A, B, C and D are constants and $S(x,t)$ solves the discrete heat equation. Thus, the discrete heat equation on the four point lattice (28), (29) is invariant under the same group as on the five point lattice (20), (21).

3.3 Exponential lattice

Let us now consider a lattice that is neither equally spaced, nor orthogonal, given by the equations

$$x_{m+1,n} - 2x_{m,n} + x_{m,n-1} = 0 \quad x_{m,n+1} = (1+c)\,x_{m,n} \tag{31}$$

$$t_{m,n+1} - t_{m,n} = h \qquad t_{m+1,n} - t_{m,n} = 0 \tag{32}$$

with $c \neq 0, -1$. These equations can be solved and explicitly the lattice is

$$t = hn + t_0 \qquad x = (1+c)^n\,(\alpha m + \beta) \tag{33}$$

where t_0, α and β are integration constants. Thus while t grows by constant increments, x grows with increments which vary exponentially with time (see Figure 2). Numerically this type of lattice may be useful if we can solve the equation asymptotically for large values of t and are interested in the small t behavior.

The heat equation on lattice (31), (31) can be written as

$$\frac{u_{m,n+1}-u_{m,n}}{h}=\frac{u_{m+1,n}-2u_{m,n}+u_{m-1,n}}{(x_{m+1,n}-x_{m,n})^2}. \tag{34}$$

Applying the symmetry algorithm to the lattice equations (31), (32) we find that the symmetry algebra is restricted to

$$\hat{X}=\left[a\,x+b\,(1+c)^{t/h}\right]\,\partial_x+\tau_0\,\partial_t+\phi(x,t,u)\,\partial_u, \tag{35}$$

where a, b and τ_0 are arbitrary constants (whereas c and h are constants determining the lattice). Invariance of the equation (34) implies $a=0$ in (35) and restricts $\phi(x,t,u)$ to reflect linearity of the equation and nothing more. The resulting symmetry algebra has a basis consisting of

$$\hat{P}_1=(1+c)^{t/h}\,\partial_x \qquad \hat{P}_0=\partial_t \qquad \hat{W}=u\,\partial_u \qquad \hat{S}=S(x,t)\,\partial_u \tag{36}$$

where $S(x,t)$ satisfies the heat equation. We see that the system is no longer invariant under space translations, or rather, that these ‘translations’ become time dependent and thus simulate a transformation to a moving frame.

3.4 Galilei invariant lattice

Let us now consider the following difference scheme

$$\frac{u_{m,n+1}-u_{m,n}}{\tau_2}=\tau_2^2\frac{u_{m+1,n}-2u_{m,n}+u_{m-1,n}}{\zeta^2} \tag{37}$$

$$t_{m+1,n}-t_{m,n}=\tau_1 \qquad t_{m,n+1}-t_{m,n}=\tau_2 \tag{38}$$

$$x_{m+1,n}-2x_{m,n}+x_{m-1,n}=0 \tag{39}$$

$$(x_{m+1,n}-x_{m,n})\tau_2-(x_{m,n+1}-x_{m,n})\tau_1=\zeta \tag{40}$$

where τ_1, τ_2 and ζ are fixed constants.

The lattice equations can be solved and we obtain

$$t_{m,n}=\tau_1\,m+\tau_2\,n+t_0 \qquad x_{m,n}=\sigma\tau_1\,m+\left(\frac{\sigma\tau_1\tau_2-\zeta}{\tau_1}\right)n+x_0 \tag{41}$$

where σ, t_0 and x_0 are integration constants. The corresponding lattice is equally spaced and in general, nonorthogonal (see Figure 3). Indeed, the coordinate curves, corresponding to $m = const$ and $n = const$, respectively, are

$$\begin{aligned} x-x_0 &= \sigma\,(t-t_0)-\tfrac{\zeta}{\tau_1}\,n \\ x-x_0 &= \tfrac{\sigma\tau_1\tau_2-\zeta}{\tau_1\tau_2}\,(t-t_0)+\tfrac{\zeta}{\tau_2}\,m. \end{aligned} \tag{42}$$

These are two families of straight lines, orthogonal only in the special case $(\sigma^2+1)\tau_1\tau_2=\sigma\zeta$. If we choose

$$\sigma\tau_1\tau_2-\zeta=0 \tag{43}$$

then the second family of coordinate lines in equation (42) is parallel to the x axis.

Invariance of equation (38) implies that in the vector field we have $\tau(x,t,u)=\alpha=const$. From the invariance of equation (39) we obtain $\xi=A(t)\,x+B(t)$ with

$$A(t_{m+1,n}) = A(t_{m,n}) \qquad B(t_{m+1,n}) - 2B(t_{m,n}) + B(t_{m-1,n}) = 0. \tag{44}$$

Finally, invariance of equation (40) implies $A(t) = 0$ and $B(t) = \beta\, t + \gamma$ where β and γ are constants. Now let us apply the prolonged vector field to equation (37). We obtain $\phi = R\, u + S(x,t)$ where $S(x,t)$ satisfies the system (37),...,(40). The symmetry algebra is given by

$$\hat{P}_0 = \partial_t \quad \hat{P}_1 = \partial_x \quad \hat{B} = t\,\partial_x \quad \hat{W} = u\,\partial_u \quad \hat{S} = S(x,t)\,\partial_u. \tag{45}$$

Thus, the system is Galilei invariant with Galilei transformation generated by the operator $\hat{B}$.

Let us now consider the continuous limit of the system (37),...,(40). We use the solution (41) of the lattice equations (38), (39), (40) and for simplicity restrict the constants by imposing equation (43). We have, from equation (41), (43)

$$\begin{gathered} t_{m,n+1} = t_{m,n} + \tau_2 \qquad x_{m,n+1} = x_{m,n} \\ x_{m\pm1,n} = x_{m,n} \pm \sigma\tau_1 \quad t_{m\pm1,n} = t_{m,n} \pm \tau_1. \end{gathered} \tag{46}$$

The continuous limit is obtained by pushing $\tau_1 \ll 1,\ \tau_2 \ll 1,\ \zeta \ll 1$ and expanding both sides of equation (37) into a Taylor series, keeping only the lowest order terms. The LHS of equation (37) gives

$$\begin{aligned} \tfrac{u_{m,n+1}-u_{m,n}}{\tau_2} &= \tfrac{u(x_{m,n},t_{m,n}+\tau_2)-u(x_{m,n},t_{m,n})}{\tau_2} \\ &= u_t + \mathcal{O}(\tau_2) \end{aligned}$$

and the RHS is given by

$$\begin{aligned} &\left(\tfrac{\tau_2}{\zeta}\right)^2 (u_{m+1,n} - 2u_{m,n} + u_{m-1,n}) \\ &= \left(\tfrac{\tau_2}{\zeta}\right)^2 [u(x_{m,n}+\sigma\tau_1, t_{m,n}+\tau_1) - 2u(x_{m,n},t_{m,n}) + u(x_{m,n}-\sigma\tau_1, t_{m,n}-\tau_1)] \\ &= u_{xx} + \tfrac{2}{\sigma} u_{x,t} + \tfrac{1}{\sigma^2} u_{tt} + \mathcal{O}(\tau_1). \end{aligned}$$

The continuous limit of the system (37),..., (40) is

$$u_t = u_{xx} + \frac{2}{\sigma} u_{x,t} + \frac{1}{\sigma^2} u_{tt} \qquad \sigma \neq 0. \tag{47}$$

The symmetry algebra of this equation, for any value of σ, is isomorphic to that of the heat equation. In addition to the pseudo-group of the superposition principle, we have

$$\begin{gathered} \hat{P}_0 = \partial_t \quad \hat{D} = x\,\partial_x + 2t\,\partial_t - \tfrac{1}{2} u\,\partial_u - cx\,\partial_t \\ \hat{K} = tx\,\partial_x + t^2\,\partial_t - \tfrac{1}{2}(t + \tfrac{1}{2}x^2)u\,\partial_u - c(x^2\,\partial_x + xt\,\partial_t - \tfrac{1}{2} xu\,\partial_u) \\ \hat{P}_1 = \partial_x + c\,\partial_t \quad \hat{W} = u\,\partial_u \\ \hat{B} = t\,\partial_x - \tfrac{1}{2} xu\,\partial_u - c(x\,\partial_x - 2t\,\partial_t) - c^2 x\,\partial_t \qquad c \equiv 1/\sigma. \end{gathered} \tag{48}$$

The fact that the commutation relations do not depend on c suggest that equation (47) could be transformed into the heat equation. This is indeed the case and it suffices to put

$$u(x,t) = \mathrm{e}^{\frac{c[(2+c^2)x+ct]}{4(1+c^2)^2}} w(\alpha,\beta)$$
$$\alpha = x + ct \qquad \beta = (1+c^2)(t-cx) \tag{49}$$

to obtain

$$w_\beta = w_{\alpha\alpha}. \tag{50}$$

Notice that while the difference equation (37) on the lattice (38), (39), (40) is Galilei invariant, this invariance is realized in a different manner, than for the continuous limit (47). To see this, compare the operator $\hat{B}$ of equation (45) with that of equation (48).

4 Lorentz invariant equations

The partial differential equation

$$u_{xy} = f(u) \tag{51}$$

is invariant under the inhomogeneous Lorentz group, with its Lie algebra realized as

$$\hat{X}_1 = \partial_x \qquad \hat{X}_2 = \partial_y \qquad \hat{L} = y\partial_x - x\partial_y \tag{52}$$

(for any function $f(u)$). In equation (51) x and y are 'light cone' coordinates. In the continuous case we can return to the usual space-time coordinates $z = x + y$, $t = x - y$, in which we have

$$u_{zz} - u_{tt} = f(u) \tag{53}$$

instead of equation (51) and the Lorentz group is generated by

$$\hat{P}_0 = \partial_t \qquad \hat{P}_1 = \partial_z \qquad \hat{L} = t\partial_z + z\partial_t. \tag{54}$$

Let us now consider a discrete system, namely

$$\frac{u_{m+1,n+1} - u_{m,n+1} - u_{m+1,n} + u_{m,n}}{(x_{m+1,n} - x_{m,n})(y_{m,n+1} - y_{m,n})} = f(u_{m,n}) \tag{55}$$

$$x_{m+1,n} - 2x_{m,n} + x_{m-1,n} = 0 \qquad x_{m,n+1} - x_{m,n} = 0 \tag{56}$$

$$y_{m,n+1} - 2y_{m,n} + y_{m,n-1} = 0 \qquad y_{m+1,n} - y_{m,n} = 0. \tag{57}$$

Applying the operator $pr\hat{X}$ (with t replaced by y) of equation (9) to equations (56), (57) we obtain

$$\xi = Ax + C \qquad \eta = By + D. \tag{58}$$

Requesting the invariance of equation (55) we find that ϕ must be linear

$$\phi = \alpha(x,y)u + \beta(x,y). \tag{59}$$

The remaining determining equations yield $\alpha = \alpha_0 = const.$ and

$$(A+B)\frac{\partial f}{\partial u_{m,n}} + (\alpha_0 u_{m,n} + \beta(x,y))\frac{\partial^2 f}{\partial u_{m,n}^2} = 0. \tag{60}$$

Thus, for any function $f = f(u)$ we obtain the symmetries (52), just as in the continuous case (they correspond to $B = -A$, $\alpha_0 = \beta = 0$). As in the continuous case, the symmetry algebra can be larger for special choices of the function $f(u)$. Let us analyze these cases.

a) Nonlinear interaction

We have $f'' \neq 0$, hence $\beta = \beta_0 = const$. The function must then satisfy

$$(A + B - \alpha_0)f + (\alpha_0 u + \beta)f' = 0. \tag{61}$$

For $\alpha_0 \neq 0$ we take

$$f = u^p \quad p \neq 0, 1 \tag{62}$$

(we have dropped some inessential constants). The system (55), (56), (57) is, in this case, invariant under a four-dimensional group generated by the algebra (52), complemented by dilation

$$\hat{D} = x\partial_x + y\partial_y + \frac{2}{1-p} u\partial_u. \tag{63}$$

For $\alpha_0 = 0$, $\beta \neq 0$ we have

$$f = \mathrm{e}^u. \tag{64}$$

The algebra is again four-dimensional with the additional dilation

$$\hat{D} = x\partial_x + y\partial_y - 2\partial_u. \tag{65}$$

b) Linear interaction $f(u) = u$

The only elements of the Lie algebra additional to (52) are

$$\hat{D} = u\partial_u \quad \hat{S}(\beta) = \beta\partial_u \tag{66}$$

where β satisfies the system(55), (56), (57) with $f(u) = u$. The presence of $\hat{D}$ and $\hat{S}(\beta)$ is just a consequence of linearity.

c) Constant interaction $f(u) = 1$

The additional elements of the Lie algebra are again a consequence of linearity, namely

$$\hat{L} = x\partial_x + y\partial_y + 2u\partial_u \quad \hat{S} = [S_1(x) + S_2(y)]\partial_u \tag{67}$$

where $S_1(x)$ and $S_2(y)$ are arbitrary (because $S_1(x) + S_2(y)$ is the general solution of equation (55) with $f(u) = 0$ on the lattice (56), (57)).

To find a discretization of equation (53), invariant under the group corresponding to (54) is more difficult and we will not go into that here.

5 Discrete Burgers equation

The continuous Burgers equation is written as

$$u_t = u_{xx} + 2uu_x, \tag{68}$$

or in potential form as

$$v_t = v_{xx} + v_x^2 \qquad u \equiv v_x. \tag{69}$$

We shall determine the symmetry groups of two different discrete Burgers equations, both on the same lattice. The lattice is one of those used above for the heat equation, namely the four point lattice (28), (29). Each of the four lattice equations involves at most three points. Hence, for any difference equation on this lattice, involving all four points, the symmetry algebra will be realized by vector fields of the form (8) with

$$\xi = Ax + B \qquad \tau = 2At + D \tag{70}$$

where A, B and D are constants (see section 3.2B).

5.1 Nonintegrable discrete potential Burgers equation

An absolutely straightforward discretization of equation (69) on the lattice (28), (29) is

$$\frac{u_{m,n+1} - u_{m,n}}{t_{m,n+1} - t_{m,n}} = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{(x_{m+1,n} - x_{m,n})^2} + \left(\frac{u_{m+1,n} - u_{m,n}}{x_{m+1,n} - x_{m,n}}\right)^2. \tag{71}$$

Applying the usual symmetry algorithm, we find a four-dimensional symmetry algebra

$$\hat{P}_1 = \partial_x \quad \hat{P}_0 = \partial_t \quad \hat{D} = x\,\partial_x + 2t\,\partial_t \quad \hat{W} = \partial_u. \tag{72}$$

5.2 A linearizable discrete Burgers equation

A different discrete Burgers equation was proposed recently [8]. It is linearizable by a discrete version of the Cole-Hopf transformation. Using the notation of this article, we write the linearizable equation as

$$u_{m,n+1} = u_{m,n} + c\,\frac{(1 + h_x u_{m,n})[u_{m+2,n} - 2u_{m+1,n} + u_{m,n} + h_x u_{m+1,n}(u_{m+2,n} - u_{m,n})}{1 + ch_x[u_{m+1,n} - u_{m,n} + h_x u_{m,n} u_{m+1,n}]} \tag{73}$$

$$h_x \equiv x_{m+1,n} - x_{m,n} \quad h_t \equiv t_{m,n+1} - t_{m,n} = ch_x^2$$

$$t_{m+1,n} - t_{m,n} = 0 \qquad x_{m,n+1} - x_{m,n} = 0.$$

In equation (73) c is a constant, but h_x is a variable, subject to dilations. The determining equation is obtained in the usual manner. It involves the function $\phi_{m,n}$ at all points figuring in equation (73), and also the constant A of equation (70). The equation is too long to be included here, but is straightforward to obtain. The variable that we choose to eliminate using equation (73) is $u_{m,n+1}$. Differentiating twice with respect to $u_{m+2,n}$ we obtain

$$\frac{\partial^2 \phi_{m,n+1}}{\partial u_{m,n+1}^2}\,\frac{\partial u_{m,n+1}}{\partial u_{m+2,n}} = \frac{\partial^2 \phi_{m+2,n}}{\partial u_{m+2,n}^2}. \tag{74}$$

We differentiate (74) with respect to $u_{m,n}$ and then, separately, with respect to $u_{m-1,n}$. We obtain two equations that are compatible for $c(1+c)^2 h_x(1+h_x u_{m,n}) = 0$. Otherwise they imply that ϕ is linear in u: $\phi = \alpha(x,t)\,u + \beta(x,t)$. We have $c \neq 0$, $h_x \neq 0$, but the case $c = -1$ must be considered separately. We first introduce the expression for ϕ into the determining equation and obtain, after a lengthy computation (using MAPLE): $\alpha = -A$, $\beta = 0$. For $c = -1$ we proceed differently, but got the same result. Finally, the Lie point symmetry algebra of the system (73), (28), (29) has the basis

$$\hat{P}_0 = \partial_t \quad \hat{P}_1 = \partial_x \quad \hat{D} = x\,\partial_x + 2t\,\partial_t - u\,\partial_u. \tag{75}$$

This result should be compared with the symmetry algebra of equation (73) on a fixed constant lattice, found earlier [7, 11]. The symmetry algebra found there was five-dimensional. It was inherited from the heat equation, via the discrete Cole-Hopf transformation. It was realized in a 'discrete evolutionary formalism' by flows, commuting with the flow given by the Burgers equation. The symmetries found there were higher symmetries, and cannot be realized in terms of the vector fields of the form considered in this article.

6 Symmetries of differential-difference equations

6.1 General comments

Symmetries of differential-difference equations were discussed in our previous article [1]. Here we shall put them into the context of partial difference equations and consider a further example. As in the case of multiple discrete variables, we will consistently consider the action of vector fields at points in the space of independent and dependent variables. To do this we introduce a discrete independent variable n (or several such variables) and a continuous independent variable α (or a vector variable $\vec{\alpha}$). A point in the space of independent variables will be $P_{n,\alpha}$, its coordinates $\{x_{n,\alpha}, z_{n,\alpha}\}$ where both x and z can be vectors. The form of the lattice is specified by some relations between $x_{n,\alpha}, z_{n,\alpha}$ and $u_{n,\alpha} \equiv u(x_{n,\alpha}, z_{n,\alpha})$.

We shall not present the general formalism here, but restrict to the case of one discretely varying variable $z \equiv z_n$, $-\infty < n < \infty$ and either one continuous (time) variable (t), or two continuous variables (x, y).

For instance a uniform lattice that is time independent can be given by the relations

$$z_{n+1,\alpha} - 2z_{n,\alpha} + z_{n-1,\alpha} = 0 \tag{76}$$

$$z_{n,\alpha} - z_{n,\alpha'} = 0 \tag{77}$$

$$t_{n+1,\alpha} - t_{n,\alpha} = 0. \tag{78}$$

where α' is a different value of the continuous variable α.

Conditions (77), (78) are rather natural. They state that time is the same at each point of the lattice and that the lattice does not evolve in time. They are however not obligatory. Similarly, equation (76) is not obligatory. The solution of equations (76), ..., (78) is of course trivial, namely

$$z_n = h\,n + z_0 \qquad t = t(\alpha) \tag{79}$$

and we can identify t and α ($t = \alpha$, h and z_0 are constants).

The prolongation of a vector field acting on a differential-difference scheme on the lattice (76),..., (78) will have the form

$$pr\,\hat{X} = \sum_n \Big[\tau(z_{n,\alpha}, t_{n,\alpha}, u_{n,\alpha})\,\partial_{t_{n,\alpha}} + \zeta(z_{n,\alpha}, t_{n,\alpha}, u_{n,\alpha})\,\partial_{z_{n,\alpha}} + \phi(z_{n,\alpha}, t_{n,\alpha}, u_{n,\alpha})\,\partial_{u_{n,\alpha}}\Big] + \dots \tag{80}$$

where the dots signify terms acting on time derivatives of u. Since $u_{n,\alpha}$, $u_{n,\alpha'}$ and $u_{n+1,\alpha}$ are all independent, equations (77) and (78) imply

$$\zeta = \zeta(z_n) \qquad \tau = \tau(t). \tag{81}$$

On any lattice satisfying equation (77), (78) we can simplify notation and write

$$\hat{X} = \zeta(z)\,\partial_z + \tau(t)\,\partial_t + \phi(z,t,u)\,\partial_u. \tag{82}$$

Similarly for an equation with one discretely varying independent variable z and two continuous ones (x, y) one can impose

$$z_{n+1,\alpha_1,\alpha_2} - 2z_{n,\alpha_1,\alpha_2} + z_{n-1,\alpha_1,\alpha_2} = 0 \tag{83}$$

$$\begin{aligned} z_{n,\alpha_1',\alpha_2} - z_{n,\alpha_1,\alpha_2} &= 0 \\ z_{n,\alpha_1,\alpha_2'} - z_{n,\alpha_1,\alpha_2} &= 0 \end{aligned} \tag{84}$$

$$\begin{aligned} x_{n+1,\alpha_1,\alpha_2} - x_{n,\alpha_1,\alpha_2} &= 0 \\ y_{n+1,\alpha_1,\alpha_2} - y_{n,\alpha_1,\alpha_2} &= 0. \end{aligned} \tag{85}$$

Invariance of the conditions (84) and (85) then implies that the vector fields realizing the symmetry algebra have the form

$$\hat{X} = \zeta(z)\,\partial_z + \xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y + \phi(z,x,y,u)\,\partial_u. \tag{86}$$

We can again simplify notation identifying $x = \alpha_1$, $y = \alpha_2$ and solving (83) to give $z_n = h\,n + z_0$ (h and z_0 constant).

6.2 Examples

We shall consider here just one example that brings out the role of the lattice equations very clearly. The example is Toda field theory, or the two-dimensional Toda lattice [17, 18, 19]. It is given by the equation

$$u_{n,xy} = \mathrm{e}^{u_{n-1}-u_n} - \mathrm{e}^{u_n - u_{n+1}} \tag{87}$$

with $u_n \equiv u(z_n, x, y)$.

On the lattice (83),..., (85) we start with equation (86) and have

$$pr\,\hat{X} = \xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y + \sum_{k=-1}^{1} \zeta_{n+k}(z)\,\partial_{z_{n+k}} + \sum_{k=-1}^{1} \phi_{n+k}\,\partial_{u_{n+k}} + \phi_n^{xy}\,\partial_{u_{n,xy}} \tag{88}$$

where ϕ_n^{xy} is calculated in the same way as for differential equations [2].

Applying (88) to equations (83) and (87) we find

$$\xi = \xi(x) \qquad \eta = \eta(y) \qquad \zeta_n = A\, z_n + B \qquad \phi_n = \beta_n(x, y, z_n) \tag{89}$$

and we still have two equation to solve, namely

$$\begin{aligned} \beta_{n+1} - \beta_n + \xi_x + \eta_y &= 0 & (90) \\ \beta_{n,xy} &= 0. & (91) \end{aligned}$$

On the lattice (83),..., (85) z_{n+1} and z_n are independent. Hence we can differentiate (90) with respect to z_{n+1} and find that β_{n+1} is independent of z_{n+1} and hence of n. We thus find a symmetry algebra generated by

$$\begin{gathered} \hat{P}_1 = \partial_x \qquad \hat{P}_2 = \partial_y \qquad \hat{L} = x\,\partial_y - y\,\partial_x \qquad \hat{S} = \partial_z \qquad \hat{D} = z\,\partial_z \\ \hat{U}(k) = k(x)\,\partial_u \qquad \hat{V}(h) = h(y)\,\partial_u \end{gathered} \tag{92}$$

where $k(x)$ and $h(y)$ are arbitrary smooth functions. Notice that $\hat{S}$ and $\hat{D}$ act only on the lattice and $\hat{U}(k)$ and $\hat{V}(h)$ generate gauge transformations, acting only on the dependent variables.

If we change the lattice to a fixed, nontransforming one, i.e. replace (83) by

$$z_{n+1,\alpha_1,\alpha_2} - z_{n,\alpha_1,\alpha_2} = h \tag{93}$$

$h = const$, the situation changes dramatically. We loose the dilation $\hat{D}$ of equation (92), however z_{n+1} and z_n are now related by equation (93). The solution of equation (90), (91) in this case is

$$\beta_n = \frac{z}{h}(\xi_x + \eta_y) + k(x) + h(y). \tag{94}$$

On this fixed lattice the Toda field equations are conformally invariant and the invariance algebra is spanned by

$$\begin{gathered} \hat{X}(f) = f(x)\,\partial_x + \tfrac{z}{h} f'(x)\,\partial_u \qquad \hat{Y}(g) = g(y)\,\partial_y + \tfrac{z}{h} g'(y)\,\partial_u \\ \hat{U}(k) = k(x)\,\partial_u \qquad \hat{V}(h) = h(y)\,\partial_u \qquad \hat{S} = \partial_z. \end{gathered} \tag{95}$$

We see that giving more freedom to the lattice (three points z_{n+1}, z_n, z_{n-1} instead of two) may lead to a reduction of the symmetry group, rather than to an enhancement. For the Toda field theory the reduction is a drastic one: the two arbitrary functions $f(x)$ and $g(y)$ reduce to $f = ax + b$, $g = -ay + d$, respectively (and only the element $\hat{D}$ is added to the symmetry algebra).

7 Conclusions and future outlook

The main conclusion is that we have presented an algorithm for determining the Lie point symmetry group of a difference system, i.e. a difference equation and the lattice it is defined on. The algorithm provides us with all Lie point symmetries of the system. In Ref. 1 we considered only one discretely varying independent variable. In this article we concentrated on the case of two such variables. The case of an arbitrary number of dependent and independent variables is completely analogous though it obviously involves more cumbersome notations and lengthier calculations. The problem of finding the symmetry group is reduced to solving linear functional equations. In turn, these are converted into an overdetermined system of linear partial difference equations, just as in the case of differential equations. The fact that the determining equations are linear, even if the the studied equations are nonlinear, is due to the infinitesimal approach.

The symmetry algorithm can be computerized, just as it has been for differential equations.

In previous articles (other than Ref. 1) we considered only one discretely varying variable and a fixed (nontransforming) lattice [4, ..., 10]. The coefficients in the vector fields, realizing the symmetry algebra, depended on variables evaluated at more than one point of the lattice, possibly infinitely many ones. Thus, one obtained generalized symmetries together with point ones. For integrable equations, including linear and linearizable ones, the symmetry structure can be quite rich [7, 8, 9]. In the continuous limit some of the generalized symmetries reduce to point ones [9] and the structure of the symmetry algebra changes.

A detailed comparison of various symmetry methods is postponed to a future article. Applications of Lie point symmetries, as well as generalized symmetries, to the solution of difference equations, will be given elsewhere.

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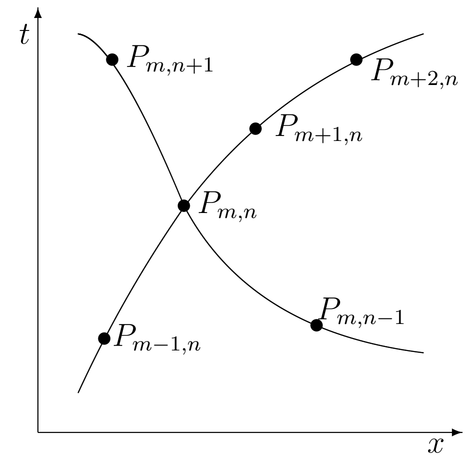

Figure 1: Points on a lattice

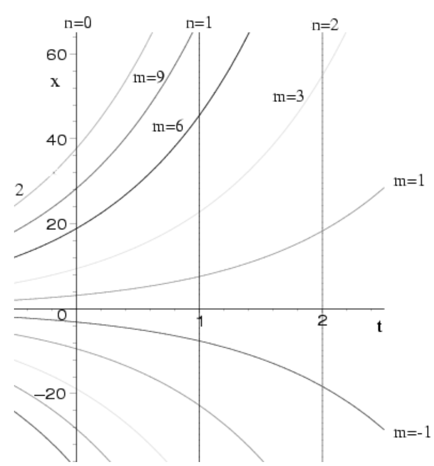

Figure 2: Variables (x, t) as functions of m and n for the lattice equations (31), (32). The parameters and the integration constants are, respectively, $c = \sqrt{2}, h = 1$ and $\alpha = \pi, \beta = 0, t_0 = 0$.

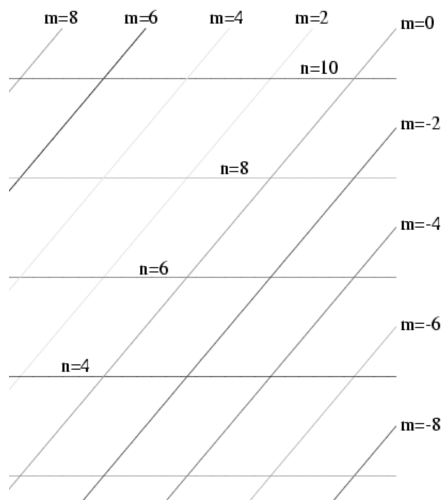

Figure 3: Variables (x, t) as functions of m and n for the lattice equations (38), (39), (40). The parameters and the integration constants are, respectively, $\tau_1 = 1, \tau_2 = 2, \zeta = 2$ and $\sigma = 1, x_0 = 0, t_0 = 0$.