Differential Dynamics in Terms of Jacobian Loops

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Abstract

We discuss differential dynamics in terms of Jacobian loops, i.e., combination of the signed jacobian entries under permutation of the indices. We analyze conditins of their dynamical roles, necessary, and sufficient conditions for loop stability and instability, as well as conditions for exchange of stability.

1. Introduction

Consider a dynamical system described by the differential system

$$\dot{x} = F(x, \mathcal{K}), \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n, \quad \mathcal{K} = (k_1, k_2, \cdots, k_N) \in \mathbb{R}^N. \tag{1-1}$$

or equivalently

$$\dot{x}_i(t) = \frac{dx_i(t)}{dt} = F_i(x_1, x_2, \cdots, x_n; k_1, k_2, \cdots, k_N). \tag{1-2}$$

The component functions F_i, $i = 1, 2, \cdots, n$ are assumed to be at least $C^1(U)$, that is, differentiable along with their first partial derivatives on U an open set of $\mathbb{R}^n$.

The partial order relation

$$x \leq y \Longleftrightarrow x_i \leq y_i, \quad i = 1, ..., n \tag{$\mathcal{O}$}$$

defines the vector order in $\mathbb{R}^n$.

Solving $F(x, \mathcal{K}_0) = 0$ yields the equilibria $\bar{x}$ of system (1-1) for the parameter value $\mathcal{K}_0$. Let $\phi_t(x)$ be the flow associated with (1-1), that is, $\phi_t(x)$ is a solution of (1-1) based at x. The equilibria are fixed points of the flow. System in (1-1) is said to be *monotone increasing* if the flow preserves the order ($\mathcal{O}$) for positive time, i.e..

$$x \leq y \Rightarrow \varphi_t(x) \leq \varphi_t(y), \quad \text{for } t \geq 0. \tag{1-3}$$

The general Jacobian matrix at $x = (x_1, \ldots, x_n)$ is given by

$$J(x) = DF(x) = \left[\frac{\partial(F_1, \cdots, F_n)}{\partial(x_1, \cdots, x_n)}(x)\right] = \left[\frac{\partial F_i}{\partial x_j}(x)\right]_{1 \leq i,j \leq n} = [J_{ij}]_{1 \leq i,j \leq n}, \tag{1-4}$$

and in general depends on the state variables, except for linear systems.

We recall the following definitions (See also M. Hirsch):

(1) An equilibrium $\bar{x}$ is *stable* if every nearby solution $x(t)$ stays nearby for all future time. If in addition x(t) converges to $\bar{x}$, the equilibrium is said to be *asymptotically stable.*
(2) The equilibrium is *unstable* if there is a neighborhood W such that every neighborhood W_1 in W contains at least a solution $x(t)$ based in W_1 and that does not lie entirely in W.
(3) A *closed orbit* $\gamma(t)$ of (1-1) is a nontrivial periodic solution $\gamma(t)$ of some period T. It is asymptotically stable if every neighborhood W contains a neighborhood W_2 such that $\phi_t(W_2) \subset W_1$ for $t > 0$, and $\lim_{t \longrightarrow \infty} dist(\phi_t(x), \gamma(t)) = 0$.
(4) The equilibrium $\bar{x}$ is *hyperbolic (resp. a sink or a source)* if every eigenvalue λ of $J(\bar{x})$ has a nonzero real part (resp. a negative or a positive real part). If both signs occurs, that is, some eigenvalues with negative real part while others have positive real part, $\bar{x}$ is a *saddle point.*
(5) Equilibria and closed orbits are the *simple dynamics or limit sets* of a system. A trajectory $\gamma(t)$ is stable (resp. asymptotically stable) if it remains (resp. converges) over time in the vicinity of (resp. to) simple dynamics.
(6) The basin of $\bar{x}$, $B(\bar{x})$ is the open set of all solutions curves converging to $\bar{x}$.
(7) The *index of* $\bar{x}$, $ind(\bar{x})$ is the number of eigenvalues λ (counting multiplicities) of $J(\bar{x})$ with $\mathcal{R}e(\lambda) < 0$. A sink has index n, and a source has index 0.

The following characterization theorem is also well-known.

Characterization.

(1) *If the equilibrium $\bar{x}$ is stable then no eigenvalue of $J(\bar{x})$ has positive real part.*
(2) *A hyperbolic point is either unstable or asympotically stable.*
(3) *A saddle point is unstable.*

For some types of systems, simple dynamics or limit sets, that is, equilibria and closed orbits, are essentially all that can occur, e.g., gradients systems and planar systems. In higher dimensions complex dynamics can occur, such as, the existence in a compact region of infinitely many periodic solutions with period tending to infinity.

1.1 Jacobian Loops.

We further assume a region $\mathcal{R}$ in the domain U of the phase space where the terms J_{ij} of the Jacobian have constant sign s_{ij}, not necessarily a neighborhood of an equilibrium.

For $\mathcal{I}_n$ the set of indices $1, \cdots, n$ we denote by $I_k = \{i_1, \cdots, i_k\}$ an ordered subset of k different elements of $\mathcal{I}_n$ and by $\tilde{I}_k = \pi_k(I_k) = \{j_1, \cdots, j_k\}$, with $\pi_k \in \Xi_k$ a permutation of I_k. Recall $Card(\Xi_k) = k!$, i.e., there are $k!$ permutations.

Every permutation π_k may be factored into ν disjoint circular (cyclic) permutations $\sigma_i, i = 1, \cdots, \nu$, that is, $\pi_k = \sigma_1\sigma_2\cdots\sigma_\nu$. The signature of π_k, denoted $sg(\pi_k)$, is $(-1)^\eta$, η the number of inversions in π_k, that is, the number of pairs (j_m, j_n) with $j_m > j_n$ while $i_m < i_n$, for $j_m = \pi_k(i_m)$, and $j_n = \pi_k(i_n)$. The permutation π_k, is even (resp. odd) for an even (resp. odd) η. There are exactly $\frac{k!}{2}$ even and exactly $\frac{k!}{2}$ odd permutations in Ξ_k. We denote Ξ_k^c (resp. Ξ_k^e, Ξ_k^o) the subset of circular (resp. even, odd) permutations.

Definition 1.1.

(1) *The set of nonzero terms J_{ij}, $i \in I_k$, and $j \in \tilde{I}_k$, describes a Jacobian loop associated to the nonzero product*

$$P(\pi_k, J) := \prod_{l=1}^{l=k} J_{i_l\pi_k(i_l)} = J_{i_1\pi_k(i_1)}J_{i_2\pi_k(i_2)}\ldots J_{i_k\pi_k(i_k)} \tag{1-5}$$

called a loop product.

(2) *The loop is a simple Jacobian loop L_k for a circular permutation σ_k. Its sign $sgn(L_k)$ is that of the loop product $P(L_k) := P(\sigma_k, J)$.*
Its length or dimension $l(L_k) = k$ is the number of loop factors $J_{i_l\sigma_k(i_l)}$ involved.

Remarks 1.2.

(1) For $k = n$, and $I_n = \{1, \cdots, n\}$ the product $P(\pi_n, J)$ is associated with the *diagonal of* J denoted $diag(a_{1j_1}, a_{2j_2}, \cdots, a_{nj_n})$, the main diagonal corresponding to $J_n = \{1, \cdots, n\}$.
(2) A simple Jacobian loop L_k is positive (resp. negative) for an even (resp. odd) number of negative loop factors $J_{i_l\sigma_k(i_l)}$ in $P(L_k)$.
(3) A Jacobian loop L_k has the following representation called *loop model* and denoted by $\mathbb{M}_k$: it consists of k distinct vertices $x_i, i = 1, \ldots, k$ and edges $E_{ij} = (x_i, x_j, s_{ij})$. Hence its dimension k corresponds to the number of edges in $\mathbb{M}_k$. A positive (resp. negative) loop is also conveniently denoted $L^+_{x_1x_2\cdots x_k}$ (resp. $L^-_{x_1x_2\cdots x_k}$).

Definition 1.3.

A loop model $\mathbb{M}_k$ is complete if for every $i \neq j$ there is a directed polygonal line connecting x_i to x_j, that is, $\overline{x_ix_{k_1}}$, $\overline{x_{k_1}x_{k_2}}$, $\cdots$, $\overline{x_{k_r}x_j}$.

Therefore the relation $\dot{x}_i(t) = F_i(x_1, \cdots, x_j, \cdots, x_n, \mathcal{K})$ describes how the rate of change in variable x_i is dependent on changes in the variable x_j, according to the values of the Jacobian term J_{ij}. We denote by p_{ij} the probability that such a dependence is not nil, i.e., $J_{ij} \neq 0$.

Definition 1.4.

(1) *A non-circular permutation π_k yields a union of simple Jacobian loops, called a compound loop $\mathcal{L}_k^\nu = \cup_{i=1}^{i=\nu} L_i = (L_1, \ldots, L_\nu)$ of dimension $d(\mathcal{L}_k^\nu) = k$ given by the sum of the lengths of its ν simple components loops, i.e., $k = \sum_{i=1}^{i=\nu} d(L_i) = 1 + \cdots + \nu$.*
A compound loop $\mathcal{L}_k^\nu$ is proper if any two of its component loops are disjoint in the sense they do not share a vertex.
(2) *The sign of a compound loop $\mathcal{L}_k^\nu$ is $sign(\mathcal{L}_k^\nu) = \prod_{l=1}^{l=\nu} sign(L_i) = (-1)^{\nu_-}$, where ν_- is the number of negative simple loops in $\mathcal{L}_k^\nu$.*

Remarks 1.5.

A compound loop $\mathcal{L}_k^\nu$ is positive (resp. negative) for an even (resp. odd) number of its negative simple loops.

To the matrix J whose elements J_{ij} have constant sign s_{ij} in the region $\mathcal{R}$, corresponds a so-called *logical matrix* J_l consisting exclusively of the signs s_{ij}, that is,

$$J_l := [s_{ij}]_{1\leq i,j\leq n} = [sign(J_{ij})]_{1\leq i,j\leq n}. \tag{1-6}$$

The *loop structure (or qualitative structure)*, denoted $\mathbb{L}_\mathcal{R}$, corresponding to the region $\mathcal{R}$, is the set of all Jacobian loops with their signs found in the matrix J. (See examples below).

Main Definitions 1.5.

(1) *For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ $x_l := (sign(x_i), i = 1, \cdots, n)$ is called a logical vector. The corresponding equivalence class is $[x] := \{y \in \mathbb{R}^n / y_l = x_l\}$.*
(2) *A matrix $B = [b_{ij}]_{1\leq i,j\leq n}$ is logically equivalent to $A = [a_{ij}]_{1\leq i,j\leq n}$ in the region $\mathcal{R}$, if B has the same sign pattern as A, i.e., $A_l = B_l$. We denote $A \odot B$.*
(3) *$\langle A\rangle$ denotes the logical equivalence class of matrix A. We further identify the logical matrix A_l with $\langle A\rangle$.*
(4) *A matrix $B = [b_{ij}]_{1\leq i,j\leq n}$ is loop equivalent to $A = [a_{ij}]_{1\leq i,j\leq n}$ in the region $\mathcal{R}$, if B yields the same loop structure $\mathbb{L}_\mathcal{R}$ as A. We denote $A \circlearrowleft B$.*
(5) *$\rangle A\langle$ denotes the loop equivalence class of matrix A.*

1.2 Examples.

Consider in the 3−variables xyz−phase-space the following three distinct matrices.

$$A = \begin{pmatrix} -1 & -3 & -2 \\ +5 & 0 & +8 \\ +1 & -4 & +7 \end{pmatrix}; \quad B = \begin{pmatrix} -8 & -9 & -1 \\ +3 & 0 & +7 \\ +4 & -2 & +5 \end{pmatrix}; \quad M = \begin{pmatrix} -2 & 3 & -6 \\ -9 & 0 & -8 \\ 7 & 5 & 4 \end{pmatrix}.$$

The logical equivalence classes are

$$\langle A\rangle = \langle B\rangle \equiv A_l = B_l = \begin{pmatrix} - & - & - \\ + & 0 & + \\ + & - & + \end{pmatrix}, \quad \langle M\rangle \equiv M_l = \begin{pmatrix} - & + & - \\ - & 0 & - \\ + & + & + \end{pmatrix}.$$

So $A \odot B$. Also $A \circlearrowleft B \circlearrowleft M$. Their loop structure $\mathbb{L}$ contains the following simple loops and their composition: L_x^-; L_z^+; L_{xy}^-; L_{yz}^-; L_{xz}^-; L_{xyz}^+; L_{xzy}^-. But A and M are not logical equivalent, as are not B and M. Logical equivalence obviously implies loop equivalence but not inversely.

1.3 Loop Structures in n-D cases, $n = 2, 3, 4$.

For the sake of clarity we consider successively the $n \times n$ matrix $J_n = [J_{ij}]$ for $n = 2, 3, 4$ respectively for a (x, y), (x, y, z) and (x, y, z, u)-variable system.

(1) For $n = 2$, we have

$$|J_2| = J_{11} \cdot J_{22} - J_{12} \cdot J_{21}. \tag{1-7}$$

The loop product $J_{11} \cdot J_{22}$, given by an even permutation is associated with a proper compound loop $\mathcal{L}_2 = (L_x, L_y)$, whereas $J_{12}J_{21}$ given by an odd permutation is the simple loop L_{xy}.

(2) For $n = 3$, we get

$$|J_3| = |J_3^e| - |J_3^o|, \tag{1-8}$$

with

$$\begin{aligned} |J_3^e| &= J_{11} \cdot J_{22} \cdot J_{33} + J_{12} \cdot J_{23} \cdot J_{31} + J_{13} \cdot J_{21} \cdot J_{32} \\ |J_3^o| &= J_{11} \cdot J_{23} \cdot J_{32} + J_{22} \cdot J_{13} \cdot J_{31} + J_{33} \cdot J_{12} \cdot J_{21}. \end{aligned} \tag{1-9}$$

The corresponding compound loops are respectively

$$(L_x, L_y, L_z); \quad L_{xyz}; \quad L_{xzy} \tag{1-10}$$

for the $\frac{3!}{2}$ even permutations, and

$$(L_x, L_{yz}); \quad (L_{xy}, L_z); \quad (L_{xz}, L_y) \tag{1-11}$$

for the $\frac{3!}{2}$ odd permutations.

(3) We obtain for $n = 4$,

$$|J_4| = |J_4^e| - |J_4^o| \tag{1-12}$$

where

$$\begin{aligned} |J_4^e| = {} & J_{11} \cdot J_{22} \cdot J_{33} \cdot J_{44} + J_{11} \cdot J_{23} \cdot J_{34} \cdot J_{42} + J_{11} \cdot J_{24} \cdot J_{43} \cdot J_{32} + \\ & J_{12} \cdot J_{21} \cdot J_{34} \cdot J_{43} + J_{12} \cdot J_{23} \cdot J_{31} \cdot J_{44} + J_{12} \cdot J_{24} \cdot J_{41} \cdot J_{33} + \\ & J_{13} \cdot J_{32} \cdot J_{21} \cdot J_{44} + J_{13} \cdot J_{34} \cdot J_{41} \cdot J_{22} + J_{13} \cdot J_{31} \cdot J_{24} \cdot J_{42} + \\ & J_{14} \cdot J_{42} \cdot J_{21} \cdot J_{33} + J_{14} \cdot J_{43} \cdot J_{31} \cdot J_{22} + J_{14} \cdot J_{41} \cdot J_{23} \cdot J_{32}. \end{aligned} \tag{1-13}$$

The associated loops are respectively

$$\begin{aligned} & (L_x, L_y, L_z, L_u); \quad (L_x, L_{yzu}); \quad (L_x, L_{yuz}); \quad (L_{xy}, L_{zu}); \\ & (L_u, L_{xyz}); \quad (L_z, L_{xyu}); \quad (L_u, L_{xzy}); \quad (L_y, L_{xzu}); \\ & (L_{xz}, L_{yu}); \quad (L_z, L_{xuy}); \quad (L_y, L_{xuz}); \quad (L_{xu}, L_{yz}). \end{aligned} \tag{1-14}$$

$$\begin{aligned} |J_4^o| = {} & J_{11} \cdot J_{22} \cdot J_{34} \cdot J_{43} + J_{11} \cdot J_{44} \cdot J_{23} \cdot J_{32} + J_{11} \cdot J_{33} \cdot J_{24} \cdot J_{42} + \\ & J_{12} \cdot J_{21} \cdot J_{33} \cdot J_{44} + J_{12} \cdot J_{23} \cdot J_{34} \cdot J_{41} + J_{12} \cdot J_{24} \cdot J_{43} \cdot J_{31} + \\ & J_{13} \cdot J_{34} \cdot J_{42} \cdot J_{21} + J_{13} \cdot J_{32} \cdot J_{24} \cdot J_{41} + J_{13} \cdot J_{31} \cdot J_{22} \cdot J_{44} + \\ & J_{14} \cdot J_{43} \cdot J_{32} \cdot J_{21} + J_{14} \cdot J_{41} \cdot J_{22} \cdot J_{33} + J_{14} \cdot J_{42} \cdot J_{23} \cdot J_{31}, \end{aligned} \tag{1-15}$$

with respectively the compound loops

$$\begin{aligned} & (L_x, L_y, L_{zu}); \quad (L_x, L_u, L_{yz}); \quad (L_x, L_z, L_{xu}); \quad (L_z, L_u, L_{xy}); \\ & (L_x, L_y, L_{zu}); \quad L_{xzuy}; \quad L_{xyuz}; \quad L_{xzuy}; \quad (L_y, L_u, L_{xz}); \\ & L_{xzyu}; \quad (L_y, L_z, L_{xu}); \quad L_{xuzy}; \quad L_{xuyz}. \end{aligned} \tag{1-16}$$

Remarks 1.6.

(1) Clearly the real vector spaces $\mathbb{R}^n$ and the space $\mathcal{M}_n$ of all real matrices are partitioned as

$$\mathbb{R}^n = \cup_{x\in\mathbb{R}}[x], \quad \mathcal{M}_n = \cup_{A\in\mathcal{M}}\langle A\rangle.$$

(2) The classes $[x]$ and $\langle A\rangle$ are convex cones respectively in $\mathbb{R}^n$ and $\mathcal{M}_n$, closed by addition and multiplication by a positive scalar. The cone $[x]$ is solid if $sign(x_i) \neq 0, i = 1, ..., n$. The set $\overline{[x]} := \{y \in \mathbb{R}^n / y_l = x_l, \quad or \quad 0\}$ is the closure of $[x]$. Similarly one defines the solid cone $\langle A\rangle$ and the closure $\overline{\langle A\rangle}$.

(3) The equation $Ax = b$ is *sign solvable* if $B \in \langle A\rangle$, $c \in [b]$, $By = c$ imply $y \in [x]$. This clearly entails A nonsingular and every nonzero term in the expansion of $|A|$ has the same sign.

(4) Denote $\langle xAy\rangle$ the set of matrices which map the set $[x]$ into $[y]$. And $\langle A\rangle_s$ the class such that $B \in \langle A\rangle_s$ implies B has all its eigenvalues with a negative real part, i.e., B is the so-called stable matrix. Of course characterizing $\langle xAy\rangle$ and $\langle A\rangle_s$ is an interesting task.

Determining and analyzing the loop structure of a system, anywhere in the phase space, not only around equilibria, if any, yields some understanding of the local and global dynamics of the systems in terms of stability

2. Dynamical role of Loops

2.1 Review.

A n-degree polynomial with real coefficients

$$p(z) := z^n + a_1 z^{n-1} + \cdots + a_k z^{n-k} + \cdots + a_{n-1} z + a_n. \tag{2-1}$$

is called a *stable (resp. strongly stable) polynomial* if its zeros z_*, i.e., $p(z_*) = 0$ have nonpositive (resp. negative) real parts, that is, $\mathcal{R}_e(z_*) \leq 0$ (*resp.* < 0). The following lemma is a classic.

Lemma 2.1. *A necessary condition for $p(z)$ to be a strongly stable polynomial is that all $a_k > 0, k = 1, \cdots, n$.*

Indeed, if all real parts are negative then we have either the form

$$p(z) = \prod(z - (\alpha + i\beta)) = \prod(z^2 - 2\alpha z + \alpha^2 + \beta^2) = \prod(z^2 + az + b), \quad a > 0, \quad b > 0, \tag{2-2}$$

or

$$p(z) = \prod(z - \alpha) = \prod(z + a), \quad a > 0.$$

By successive multiplication we necessarily obtain $p(z)$ with $a_k > 0$. □

Now consider a Jacobian matrix $J = [J_{ij}]_{1\leq i,j\leq n}$ of system (1-1). Its characteristic polynomial is

$$\begin{aligned} \mathcal{C}_J(\lambda) &= |\lambda I - J| \\ &= \lambda^n + c_1\lambda^{n-1} + \cdots + c_k\lambda^{n-k} + \cdots + c_{n-1}\lambda + c_n, \end{aligned} \tag{2-3}$$

with

$$c_k = \sum(-1)^k m_k, \quad k = 0, \cdots, n-1, \tag{2-4}$$

where the sum extends over all *kth* order principal minors m_k of J. We have

$$\begin{aligned} c_n &= (-1)^n det(J) = (-1)^n |J|, \quad \text{for } k = n, \\ c_1 &= -\sum J_{ii} = -trace(J) = -Tr(J), \quad \text{for } k = 1. \end{aligned} \tag{2-6}$$

From the theory of determinant and permutations we may write

$$\begin{aligned} m_k &= \sum_{\pi_k\in\Sigma_k} (-1)^\eta \prod_{i_l\in\mathcal{I}_k} J_{i_l\pi_k(i_l)} \\ &= \sum_{all\nu}(-1)^{k-\nu}\prod_{i=1}^{i=\nu} P(\sigma_i, J), \\ &= \sum_{all\nu}(-1)^{k-\nu} P(\mathcal{L}_k^\nu), \end{aligned} \tag{2-7}$$

where the permutation $\pi_k \in \Sigma_k$ of the indices $1 \leq< i_1 < i_2 < \cdots < i_k \leq n$ factors into the cyclic permutations $(\sigma_1, \cdots, \sigma_\nu)$ yielding the proper compound loop $\mathcal{L}_k^\nu = (\sigma_1, \cdots, \sigma_\nu)$ with loop product $P(\mathcal{L}_k^\nu)$ as defined in section 1. Consequently we obtain an expression of the characteristic coefficients c_k in terms of the proper compound loops.

Definition 2.2. *We call kth order Feedback the combination, denoted $\mathcal{F}_k$, of proper compound loops $\mathcal{L}_k^\nu$ corresponding to the following feedback product derived from formula* (2-2) *and* (2-3) *in the form*

$$\mathcal{P}_k = \sum_{all\nu} (-1)^{\nu+1} P(\mathcal{L}_k^\nu). \tag{2-3}$$

This product is indeed an expression of the characteristic coefficients c_k. Thus the following remarks. First recall the zeros of the characteristic polynomial are the eigenvalues of J. They are of multiplicity k if $(z-\lambda)^k$ factorizes $\mathcal{C}_J(\lambda)$. For $k=1$ the corresponding eigenvalue is said to be simple, such as when J has n distinct eigenvalues..

Remarks 2.3.

(1) The expression of $\mathcal{P}_k$ in (2-7) entails that the loop factors $J_{i\sigma(i)}$ are the only Jacobian entries contributing to the characteristic equation, and therefore, influence directly the eigenvalues of the matrix, and consequently the dynamics.

(2) It is also known that the coefficients c_k are related to the eigenvalues λ_i by the following formulas:

$$\begin{aligned}
c_1 &= -(\lambda_1 + \cdots + \lambda_n) = -\sum_{i=1}^{i=1} J_{ii}.\\
c_2 &= \sum_{i,j=1, i<j} \lambda_i\lambda_j = \sum_{i,j=1,i<j} (J_{ii}J_{jj} - J_{ij}J_{ji})\\
&= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n.\\
c_3 &= -\sum_{i,j,k=1,i<j<k} \lambda_i\lambda_j\lambda_k\\
&= -(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \cdots + \lambda_{n-2}\lambda_{n-1}\lambda_n).\\
\cdots\\
c_n &= (-1)^n \lambda_1\lambda_2\cdots\lambda_n.
\end{aligned} \tag{2-8}$$

(3) From the theory of determinants, we have

$$\begin{aligned}
det(J) = |J| &= \sum_{\pi_n \in \Xi_n} sg(\pi_n) P(\pi_n, J)\\
&= \sum_{\pi_n^e \in \Xi_n^e} P(\pi_n^e, J) - \sum_{\pi_n^o \in \Xi_n^o} P(\pi_n^o, J).
\end{aligned} \tag{2-9}$$

Every permutation $\pi_n = \sigma_1\sigma_1\cdots\sigma_n$ is associated with a compound loop $\mathcal{L}_n = (L_1, L_2, \cdots L_n)$ with the simple loop L_i defined by the cyclic permutation σ_i. π_n^e (resp. π_n^o) denotes an even (resp odd) permutation. In the expression (2-4) of $|J|$ all the loop product $P(\pi_n^e, J)$ have the same sign opposite to that of $P(\pi_n^o, J)$. In fact if $\mathcal{L}_n^e$ (resp. $\mathcal{L}_n^o$) is the compound loop associated with π_n^e (resp. π_n^o) then it has an even (resp. odd) number ν of components L_i for n even, and it has an odd (resp. even) number ν of components L_i for n odd.

Straightforward from lemma 2.1 we have

Corallary 2.3. *A necessary condition to have all eigenvalues with negative real parts $\mathcal{R}_e < 0$ is that all kth order Feedback $\mathcal{F}_k$ must be positive.*

We also have

Lemma 2.4. *A proper compound loop $\mathcal{L}_k^\nu$ with all ν components negative has a negative contribution to the kth order Feedback.*

Proof. Indeed the term $(-1)^{\nu+1}P(\mathcal{L}_k^\nu)$ has the sign $(-1)^\nu(-1)^{\nu+1} = -1$. Hence the claim. □

Lemma 2.5. *If there is no proper compound loop $\mathcal{L}_k$ of dimension $k \le n$, then the characteristic coefficient $c_k = 0$.*

Moreover at least a proper compound loop $\mathcal{L}_n$ of the system dimension is necessary to have a nonsingular jacobian matrix.

Proof. From formula (2-2) the characteristic coefficient c_k can be written as

$$c_k = \sum (-1)^\nu \prod_{l=1}^{l=r_1} J_{i_l\pi_k(i_l)} \prod_{l=r_1+1}^{l=r_2} J_{i_l\pi_k(i_l)} \cdots \prod_{l=r_{\nu-1}}^{l=r_\nu} J_{i_l\pi_k(i_l)}. \tag{2-10}$$

Terms in the expression of c_k with one circular permutation correspond to $r_1 = k$, those with two circular permutations correspond to $r_1 < k$, $r_2 = k - r_1$, and so on. Therefore, if there is no proper compound loop $\mathcal{L}_k$ of dimension k, then each term of the sum is zero. Hence the claim. For $k = n$ the system dimension, formula (2-1) yields clearly $Det(J) = 0$. □

Theorem 2.6. *If the loop structure $\mathbb{L}$ does contains a $\mathcal{L}_n$, and all the $\mathcal{L}_n$ have the same sign, then the corresponding Jacobian determinant $|J|$ is nonzero. We say that the equivalent class $\langle J \rangle$ is logically nondegenerate.*

Proof. Indeed suppose all the compound loops $\mathcal{L}_n$ of the dimension of the system have the same sign. Then c_n, consisting of nonzero terms of the same sign, is therefore nonzero. Consequently, the Jacobian determinant is nonzero.

Moreover c_n is positive (resp. negative) if all $\mathcal{L}_n$ have an odd (resp. even ν_e) number ν_o of simple loops L_i. □

We also prove

Theorem 2.7.

A positive simple loop in the loop equivalence class is a necessary condition for the Jacobian matrix to have a positive real eigenvalue.

Proof. Recall the characteristic coefficients given in (2-2), that is,

$$c_k = \sum_{\mathcal{L}_k=(L_1,\cdots,L_\nu)} (-1)^\nu P(\sigma_1, J) \cdot P(\sigma_2, J) \cdot \;\cdots\; \cdot P(\sigma_\nu, J), \tag{2-11}$$

where the simple Jacobian loops $L_i, i = 1, \cdots, \nu$ are defined by the circular permutations $\sigma_i, i = 1, \cdots, \nu$. Now assume that the region has a negative loop equivalence class, i.e., there is no positive simple loop in its loop structure. So every simple loop L_i defined by $\sigma_i \in \Xi^c_k$ is negative. Therefore the corresponding nonzero loop product $P(\sigma_i, J)$ is also negative. Then for a compound loop $\mathcal{L}_k = (L_1, \cdots, L_\nu)$ we have

$$sign(-1)^\nu P(\sigma_1, J) \cdot P(\sigma_2, J) \cdot \;\cdots\; \cdot P(\sigma_\nu, J) = sign((-1)^{2\nu}) = +. \tag{2-12}$$

Thus all the characteristic coefficients c_k are positive. This entails a characteristic polynomial of degree n with only positive coefficients. By Descartes' rules of sign it cannot have a positive real root. Hence the claim. □

In the plane this clearly yields

Corollary 2.8.

A positive loop in the jacobian loop structure corresponding to a region in the phase-plane is a necessary condition to have a saddle point in this region.

In fact any dynamic situation requiring a real positive eigenvalue requires a positive loop as well.

2.2 Local Loop Stability.

Denote Λ_J the spectrum of matrix J, i.e., the set of all eigenvalues λ of J or zeros of the characteristic polynomial $\mathcal{C}_J(\lambda)$.

We summarize the classic characterizations of stability from Routh-Hurwitz and Lyapunov theories.

Stability Criteria. *The necessary and sufficient conditions to have all real parts negative are given by:*

1. *(**Lyapunov**) There exist a positive definite symmetric matrix Q such that $QJ + J^tQ$ is a negative definite matrix.*
2. *(**Routh-Hurwitz**) All the Hurwitz determinants H_i are positive, where*

$$\begin{aligned} H_1 &= c_1, \\ &\vdots \\ H_n &= \begin{vmatrix} c_1 & c_3 & c_5 & \cdots & c_{2n-1} \\ 1 & c_2 & c_4 & \cdots & c_{2n-2} \\ 0 & c_1 & c_3 & \cdots & c_{2n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & c_n \end{vmatrix} = c_n H_{n-1}, \quad c_j = 0, \quad j > n. \end{aligned} \tag{2-13}$$

3. *(**Liénard-Chipart**) $c_k > 0$, for all k, and alternate Hurtwitz determinants up to order n are positive.*

A direct consequence is therefore.

Corollary 2.9. *At least one kth order positive Feedback is necessary to have an eigenvalue with a positive real part.*

Definition 2.10.

(1) *The matrix J is stable (resp. asymptotically stable) if its characteristic polynomial $\mathcal{C}_J(\lambda)$ is stable (resp. strongly stable), that is, for every eigenvalue λ of J we have $\mathcal{R}_e(\lambda) \leq 0$ (resp. < 0).*
(2) *J is unstable if it is not stable. In other words, there is at least one eigenvalue λ_0 such that $\mathcal{R}_e(\lambda) > 0$.*

Therefore a sink equilibrium has a asymptotically stable Jacobian, whereas a source has an unstable Jacobian.

Moreover, if $\mathcal{R}_e(\lambda) > 0$ for every $\lambda \in \Lambda_J$ the instability is said to be *strong*, e.g., a source is strongly unstable. A *weak instability* is characterized by $\mathcal{R}_e(\lambda) > 0$, for some eigenvalues, and $\mathcal{R}_e(\lambda) = 0$ for the remainings.

Recall that, as a function of the matrix, Λ_J is neither additive nor multiplicative. Moreover we have the followings:

(1) For a nonsingular matrix J, i.e., $|J| = det(J) \neq 0$, $\Lambda_{J^{-1}} = \{\frac{1}{\lambda}, \quad \lambda \in \Lambda_J\}$. So J and J^{-1} must be stable simultaneously.
(2) $\Lambda_J = \Lambda_{J^t}$, J^t denotes the transpose of matrix J. Thus J and J^t must be stable simultaneously.

Definition 2.11.

The matrix J is logically stable (resp. asymptotically stable) or the logical Jacobian J_l is stable (resp. asymptotically stable) if every matrix in the logical equivalence class $\langle J\rangle$ is stable (resp. asymptotically stable) .

J_l is logically unstable if every matrix in the logical equivalence class $\langle J\rangle$ is unstable.

Examples. A $2-d$ example.

Definition 2.12. *The matrix J is loop stable (resp. asymptotically stable) if every matrix in the loop equivalence class $\rangle J\langle$ is stable (resp. asymptotically stable) . J is loop unstable if every matrix in the loop equivalence class $\rangle J\langle$ is unstable.*

The loop analysis is addressed here for irreducible matrices. Recall a matrix $A = [a_{ij}]_{1\leq i,j\leq n}$ is *irreducible or indecomposable* if there is no simultaneous row-and-column permutation $P_r - P_c$ such that A is similar to

$$P_r A P_c = \begin{pmatrix} B & O \\ C & D \end{pmatrix}. \tag{2-14}$$

where P_r and P_c are respectively the row and column permutation matrices, and B, C are respectively a $p\times p$ and a $q \times q$ block such $p+q=n$, and O a $p\times q$ block of zeros. By a Laplace decomposition the spectrum of A is given by $\Lambda_A = \Lambda_B + \Lambda_D$, thus reducing its eigenvalue analysis to that of the individual diagonal block of lower dimension. Therefore, in the sequel, we will assume all matrices are irreducible without loss of generality.

Actually a $P_r - P_c$ permutation amounts to a renumbering of the system variables, and renumbering should certainly not affect the properties of the system in general, and its asymptotic behavior in particular.

Remarks 2.13.

A $n\times n$ matrix $J = [J_{ij}]$ is irreducible if for every partition of the set $\Lambda = \{1,\ldots,n\}$ into disjoint nonempty subsets Λ_1, Λ_2, for any $i \in \Lambda_1$ there exists $j \in \Lambda_2$ with $J_{ij} \neq 0$. An irreducible system is one with irreducible Jacobian matrices. The corresponding loop model is complete.

Lemma 2.14: sign-stability.

Consider an irreducible matrix A and its logical matrix J_l. If J_l is stable then the loop structure $\mathbb{L}$ has at least one L_1, and at least one proper compound loop $\mathcal{L}_n$ of the dimension of the system, and all simple Jacobian loops $L_k, k\leq 2$ are negative.

Proof. (To be done!) □

Theorem 2.15: Loop Stability.

Consider an irreducible matrix A and its logical matrix J_l. If J_l is stable then the loop structure $\mathbb{L}$ has at least one L_1, and at least one proper compound loop $\mathcal{L}_n$ of the dimension of the system, and all simple Jacobian loops $L_k, k\leq 2$ are negative.

Proof. (To be done!) □

Remarks 2.16.

(1) We note that negative Jacobian loops L_k promote stability whereas positive Jacobian loops L_k promote instability, at least for $k \leq 2$. For a proper compound loop $\mathcal{L}_k = \cup_{i=1}^{k} L_k$ with sign $(-1)^{\nu+1}$ where ν is the number of positive simple loops, the combined effect can be stabilizing for ν even, i.e., $\mathcal{L}_k < 0$, and the combined effect will be destabilizing for ν odd, i.e., $\mathcal{L}_k > 0$. It suffices to observe that a positive proper compound $\mathcal{L}_k^+$ contributes a negative term to the characteristic polynomial $p_J(\lambda)$, whereas as a negative $\mathcal{L}_k^-$ contributes a positive term.
(2) For an arbitrary matrix J, logical stability is obtained if every strong component fulfilled theorem 2.7

Theorem 2.17: Necessary condition for loop instability.

If J_l is logically unstable then the loop structure $\mathbb{L}$ contains no more than $(n-1)$ negative simple 1-dimensional loops L_1.

Proof. (To be done!) $\square$

Comments 2.18.

For instance a $n \times n$ matrix J_n with n negative diagonal elements J_{ii} cannot have a corresponding logically unstable matrix J_l, because setting $J_{ii} = 0$ yields a logically stable matrix.

Recall that by definition, instability for a logical matrix implies that every matrix in its equivalence class admits at least an eigenvalue with positive real roots. Hence

Corollary 2.19.

A necessary condition for the Jacobian matrix to have an eigenvalue with positive real part is the presence of no more than $(n-1)$ negative simple 1-dimensional loops in the loop structure.

Theorem 2.18: Sufficient condition for logical instability.

If there is at least one $k = k_0$ in $[1, n]$ such that

(1) *either $\mathbb{L}$ contains no proper compound loop $\mathcal{L}_{k_0}$ (Weak Instability),*
(2) *or every proper compound loop $\mathcal{L}_k$ of dimension $k = k_0$ is positive (Strong Instability)*

then J_l is logically unstable.

Proof. (To be done!) $\square$

Note that the nonexistence of such a proper compound loop $\mathcal{L}_{k_0}$ implies that every matrix in the equivalence class $\langle J \rangle$ has at least one eigenvalue with nonnegative real part.

Corollary 2.21.

A sufficient condition for logical instability is the existence of at least one strong component $\mathbb{L}_0$ in the loop structure $\mathbb{L}$ satisfying theorem 3.4 with $k \leq dim(\mathbb{L}_0)$. Moreover if there is at least one proper compound loop $\mathcal{L}_k$ in $\mathbb{L}_0$, then the logical instability is strong.

Proof. (To be done!) $\square$

The above sufficient condition is not a necessary one. To wit consider the following equivalent class

$$\langle J \rangle \equiv \begin{pmatrix} - & + & 0 \\ + & 0 & - \\ 0 & + & 0 \end{pmatrix}. \tag{2-15}$$

In the corresponding loop structure, we have one proper compound loop $\mathcal{L}_k$, of respective dimension 1, 2, and 3. Moreover there is no positive proper compound loop. Thus the above sufficient condition is not met, but nonetheless, this matrix is logically unstable as can be established using stability criteria such as Routh-Hurwitz.

Now assume that for the parameter vector $\mathcal{K} = (k_1, \ldots, k_N)$, the sign Jacobian matrix J_l is logically/loop stable for the parameter value $\mathcal{K} = \mathcal{K}_*$, and logically/loop unstable for $\lambda \neq \lambda_*$. The system then undergoes an exchange of stabilities at λ_*. Thus from the previous theorems, it follows.

Theorem 2.22: Sufficient condition for Exchange of stability.

For a system to admit an exchange of stabilities, it is sufficient that, in the loop structure $\mathbb{L}$, all 1-dimensional L_1 are negative and there is at least one simple loop L_k with $k \geq 3$.

Proof. (To be done!) $\square$

Therefore a system admits a Hopf bifurcation only if the corresponding Jacobian matrix admits an exchange of stabilities. Hopf bifurcation ensures existence of limit cycles (sustained oscillations.)

Theorem 2.23: Necessary condition for Hopf bifurcation.

In order to admit a Hopf bifurcation, the logical Jacobian matrix J_l can be neither logically stable nor logically unstable.

Proof. (To be done!) $\square$

3. Concluding Remarks

The loop stability (resp. the sign stability) refers to the invariance of the jacobian spectrum under any variation of entries that leave unchanged its loop structure (resp. its sign structure).

Jacobian loops and their combinations provide valuable information about the stability of a system even when only the signs, not the magnitudes of the Jacobian terms, are known.

As such it is certainly an efficient tool in qualitative modeling of complex systems, in the sense it allows the followings:

(1) Stress qualitative understanding as the primary goal rather than numerical prediction.
(2) See how much to avoid measuring and still understanding the system.
(3) Determine which measurements are necessary.
(4) Deal with large number of variables often belonging to various disciplines, interacting indirectly as well as directly.
(5) Assist in understanding the behavior of partly specified systems.
(6) Guide experimental design and data interpretation.
(7) Supplement the more familiar large scale quantitative methods made possible by improved computer technology.
(8) Include variables difficult or even impossible to measure, e.g., a diabetes model should include measurable variables such as glucose, insulin and other chemicals but also real variables such as anxiety or stress but any attempt to measure stress is itself stress inducing.

Sometimes a qualitative analysis is sufficient for predictions or to guide experimentation. At other times it is a preliminary overview of a problem permitting a more systematic use of simulation methods. In economics, social sciences, as well as in complex physical sciences relevant informations about the underlying dynamics reside in the rules of construct of the system and not in the absolute values.

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