Cyclicity of Isochronous Centers

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Abstract

We investigate the cyclicity of isochronous centers. In particular we address the case of 1-parameter unfoldings, specially for explicitly linearizable centers using a method based on the relative cohomology decomposition of polynomial 1-forms along with a step reduction process.

We determine the higher order multiplicities, majorant of the cyclicities, for the linear 1-form, the reduced Kukles isochrone 1-form, and the Darboux linearizable cubic Hamiltonian center.

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Résumé

Nous étudions la cyclicité des centres isochrones. En particulier nous nous interessons au cas de déploiements a un paramètre, notamment pour des centres explicitement linéarizables. Nous utilisons la méthode de décomposition cohomologique des 1-formes polynomiales combinée avec une méthode de réduction de processus.

Nous déterminons les multiplicités d'ordre supérieur, majorant des cyclicités, pour les cas de 1-forme linéaire, de 1-forme isochrone de Kukles réduit, et de centre cubique Hamiltonian Darboux linéarizable.

1. Introduction

For an analytic planar 1-form unfolding ω_α, $\alpha \in \mathbb{R}^d$, the estimate of the number of limit cycles follows from the decomposition of the displacement function of the unfolding in an ideal of functions, the Bautin Ideal, in the parameter space $\mathbb{R}^d$. For a closed orbit Γ, for instance in a period annulus $\mathbb{A}$, with a given 1-form ω, the upper bound of the number of limit cycles bifurcating from ω at $\alpha = 0$, with $\omega_0 = \omega$, is called the cyclicity of ω_α denoted $Cycl(\omega_\alpha, \Gamma)$. (See [16]). A more formal definition is as follows. Considering the displacement function $D(\rho, \alpha, K)$ as defined in section 2. let $\mathcal{N}(\sigma, \alpha)$ be the number of isolated roots of $D(\rho, \alpha, K) = 0$ on $]-\sigma, \sigma[$, and

$$\mathcal{N}(\sigma, \tau) = sup\{N(\sigma, \alpha)| \quad ||\alpha|| \leq \tau\}.$$

Then we have

$$Cycl(\omega_\alpha, \Gamma) := inf\{\mathcal{N}(\sigma, \tau)| \quad \sigma \longrightarrow 0, \tau \longrightarrow 0\}.$$

The cyclicity is always finite. (see [6,8,16]) Explicitly determining the cyclicity for a given unfolding is an important problem. We address it in the case of a real analytic isochronous 1-form $\omega_0 = \mathcal{I}_0$, starting, in this paper, with a real polynomial 1-parameter unfolding when the linearization of the isochronous center is explicitly known.

Consider an arbitrary n degree autonomous one-parameter family of polynomial 1-forms

$$\omega_\alpha(x, y) = \omega_0(x, y) + \alpha\omega(x, y), \qquad (\mathcal{F}_\alpha)$$

where α a small real parameter, and $\omega(x, y) = g(x, y)dx - f(x, y)dy$ where

$$f(x, y) = \sum_{i=1}^{n}\sum_{k=0}^{i} f_{i-k,k}x^{i-k}y^k; \quad g(x, y) = \sum_{i=1}^{n}\sum_{k=0}^{i} g_{i-k,k}x^{i-k}y^k. \qquad (1\text{-}1)$$

Moreover ω_0 has a nondegenerate center at the origin, that is, it is of the form (See for instance [12])

$$\omega_0(x, y) = (x + \sum_{2\leq i+j\leq d} b_{ij}x^iy^j)dx + (y - \sum_{2\leq i+j\leq d} a_{ij}x^iy^j)dy, \qquad (1\text{-}2)$$

in appropriate coordinates and upon rescaling of time. Recall that the solutions of $\omega_0(x, y) = 0$ are in correspondence with the orbits of the flow of the vector field

$$\mathcal{X}_\alpha = (-y + \sum_{2\leq i+j\leq d} a_{ij}x^iy^j)\partial_x + (x + \sum_{2\leq i+j\leq d} b_{ij}x^iy^j)\partial_y, \qquad (\mathcal{V}_\alpha)$$

or with the solutions of the differential systems

$$\dot{x} = -y + \sum_{2\leq i+j\leq d} a_{ij}x^iy^j; \quad \dot{y} = x + \sum_{2\leq i+j\leq d} b_{ij}x^iy^j. \qquad (\mathcal{D}_\alpha)$$

By Poincaré Normal Form of a Nondegenerate Center (See [12] for Proof and details) there exists an analytic change of coordinates at the origin,

$$u(x, y) = x + o(|(x, y)|); \quad v(x, y) = y + o(|(x, y)|) \qquad (\mathcal{T}_a)$$

and an analytic function Ψ such that

$$\omega_0(u, v) = u(1 + \Psi(u^2 + v^2))du + v(1 + \Psi(u^2 + v^2))dv.$$

Such a center is surrounded by a family of nontrivial cycles called the *period annulus* $\mathbb{A}$. ω_0 is an *isochronous (or linearizable)* 1-form when the coordinate change $(\mathcal{T}_a)$ actually reduces ω_0 to the linear isochrone 1-form $\mathcal{I}_0(u, v) = udu + vdv$. That is, the period annulus consists of cycles of the same constant period. It is an *isochronous period annulus.*

In [12] it has been given a large class of systems with an explicit linearizing transformation in the form of a Darboux function (See Appendix). Here we are interested in isochronous polynomial 1-forms whose linearization is explicitly known and polynomial preserving, i.e., the perturbation polynomial 1-form remains polynomial under the transformation $(\mathcal{T}_a)$. An interesting question is to estimate the number of nontrivial cycles Γ in the isochronous period annulus that survive a k-order, $k \geq 1$, 1-form polynomial perturbation by giving birth to a continuous family Γ_α of limit cycles (isolated cycles) with $\Gamma_0 = \Gamma$ *in the direction of the perturbation.* The cycle Γ is said to be *persistent.* Such a problem, like most of the work on plane polynomial vector fields, is related to the second part of Hilbert's 16*th* problem, i.e., the estimate of the number of limit cycles for a polynomial vector field of arbitrary degree n. (See for instance [1,2,4]). These questions are similar to the more general problem of finding the number of zeros of Abelian integrals over polynomial Hamiltonians. (cf. [7,14,19]). Although still complicated, our version, the estimate of the number of small cycles bifurcating from a linearizable centre in the direction of the perturbation is purely algebraic and algorithmically solvable. Indeed it reduces to the investigation of the simple zeros of a suitable bifurcation function derived from the perturbation Taylor expansion of the displacement function whose isolated zeros correspond to the limit cycles of the perturbed system. (See also [3]).

In section 2 we present a complete analysis based on the relative cohomology decomposition of polynomial 1-forms along with a so-called step-reduction process. This approach yields an algorithmic construction of the *kth* order upper bound $\mathcal{U}_k(N)$ of limit cycles bifurcating in the direction of a polynomial perturbation, as well as of the overall upper bound $\mathcal{U}(N)$.

Moreover, at any order, one may construct in the usual way perturbations with the maximum number, and each limit cycle is asymptotic to a circle whose radius is a simple positive zero of the bifurcation function. In the sequel upper bound refers to the *least upper bound to the number of limit cycles to be born from the linearizable center in the direction of the perturbation.* This is a step towards the estimate of limit cycles from explicitly linearizable centers, i.e., independently from the direction of perturbation. Section 3 is devoted to applications to some Darboux linearizable centers, while section 4 reviews the so-called *Darboux linearizability.*

2. PAM-functions of a linearizable center

Assume that the change of coordinates $\mathcal{T}_a$ is actually a linearizing transformation $\mathcal{T}_l$, explicitly known and locally invertible. Therefore the family $(\mathcal{F}_\alpha)$ is converted into

$$\overline{\omega_\alpha}(u,v) = \mathcal{I}_0(u,v) + \alpha\bar{\omega}(u,v), \qquad (\overline{\mathcal{F}_\alpha})$$

with $\mathcal{I}_0(u,v) = udu + vdv = d\phi$, and $\bar{\omega}(u,v) = G(u,v)du - F(u,v)dv$, where

$$\phi(u,v) = \frac{u^2+v^2}{2}; \quad (F(u,v), G(u,v)) = J(T_l^{-1})(f,g)|_{(u,v)}, \qquad \text{(2-1)}$$

$J(T_l^{-1})$ denotes the Jacobian of the inverse transformation. The corresponding *kth* order perturbation Taylor expansion of the displacement function on a transversal section Σ may be written as

$$D(\rho,\alpha,K) = \alpha^k(D_k(\rho,K) + O(\alpha)) := \alpha^k b_k(\rho,K),$$

where $D_k(\rho,K) = \frac{1}{k!}\frac{\partial^k D(\rho,\alpha,K)}{\partial\alpha^k}|_{\alpha=0}$. $D(\rho,0) \equiv 0$. K is the system coefficients set.

A simple root $\rho_* \in \Sigma$ is a *kth order branch point* of limit cycles of $(\overline{\mathcal{F}_\alpha}.)$ The corresponding periodic orbit $\Gamma(\rho_*)$ is said to *survive* or to *persist* after perturbation.

Moreover it is also known that the displacement function is actually expressed as

$$D(\rho,\alpha,K) = \alpha A_1(\rho,K) + \alpha^2 A_2(\rho,K) + \alpha^3 A_3(\rho,K) + \cdots + \alpha^k A_k(\rho,K) + \cdots \qquad \text{(2-2)}$$

where, by a classic Poincaré formula [5]

$$A_1(\rho,K) = -\int_{\Gamma(\rho)} \bar{\omega}. \qquad \text{(2-3)}$$

It is the first Poincaré-Andronov-Melnikov function along the line $\Gamma(\rho) : \phi = \rho$. In general it is an Abelian/elliptic integral, and any question about its zero is highly nontrivial. See [14,19]. The first order upper bound of the linearizable center is the upper bound of the simple roots of $A_1(\rho,K)$. For system $(\bar{\omega}_\alpha)$, it is clearly computable as

$$A_1(\rho,K) = \int_0^{2\pi} \left(F(\rho\cos t, \rho\sin t)\cos t + G(\rho\cos t, \rho\sin t)\sin t\right) dt. \qquad \text{(2-4)}$$

For $A_1(\rho,K) \equiv 0$ as a function of ρ for a value of the system parameter K, one needs to inspect the next term $A_2(\rho,K)$, the second Poincaré-Andronov-Melnikov function. The functions $A_k(\rho,K)$ are called the *PAM-functions.* The estimate of the zeros of the PAM-function $A_k(\rho,K)$ modulo $A_j(\rho,K) \equiv 0, j < k$ yields the *kth* order upper bound $\mathcal{U}_k(N)$ defined above, and the first non-vanishnig one leads to the overall upper bound $\mathcal{U}(N)$.

We restrict our study to the class of transformations $\mathcal{T}_l$ polynomially preserving $(\mathcal{F}_\alpha)$, i.e., the transformed family of 1-forms $\bar{\omega}_\alpha$ is polynomial of degree N with the unperturbed isochronous period annulus parametrized by $\phi = \rho$. This allows us to make use of the cohomology approach proposed in [5,7] to determine the higher order upper bounds. We first prove

Theorem 2.1. *At first order at most $\mathcal{U}_1(N) = (N-1)/2$, (resp. $(N-2)/2$) for N odd (resp. even) limit cycles bifurcate from the linearizable center in the direction of the perturbation.*

Moreover the limit cycles are asymptotic to the circles whose radii are simple positive roots of the bifurcation function. We can construct small perturbations with the maximum number of limit cycles.

Proof. Set $K = K(N)$ the set of coefficients of the N degree polynomial 1-form $\overline{\omega_\alpha}$, and $\aleph$ the maximum cardinality of $K(N)$. Thus $\aleph = N^2 + 3N$. For $F(u,v)$, $G(u,v)$ polynomials in (u,v) of degree N and coefficients $F_{i-k,k}$, and $G_{i-k,k}$, formula (2-4) yields

$$A_1(\rho, K) = \sum_{i=1}^{N} \rho^i B_i(K), \tag{2-5}$$

with (terms of negative subindex assumed zero) the PAM-coefficients

$$B_i(K) = \sum_{k=0}^{i+1} (F_{i-k,k} + G_{i-k+1,k-1}) \int_0^{2\pi} \cos t^{i-k+1} \sin t^k dt, \tag{2-6}$$

which further reduced to

$$B_i(K) \equiv 0 \quad \text{(resp. } B_i(K) \not\equiv 0\text{) for } i \text{ even (resp. odd).} \tag{2-7}$$

through the well-known rules $\int_0^{2\pi} \cos t^s \sin t^l dt = 0$ for s or l odd. Note that the coefficients $B_i(K)$ are of degree one in the component of $K = K(N)$. They are also linearly independent. For instance

$$B_1(K) = \pi(F_{10} + G_{01}); \quad B_3(K) = \frac{\pi}{4}(3F_{30} + F_{12} + G_{21} + 3G_{03}). \tag{2-8}$$

$A_1(\rho, K) = 0$ reduces to

$$\bar{A}_1(r, K) = B_1(K) + B_3(K)r + \cdots + B_{2\tau_1+1}(K)r^{\tau_1} = 0, \tag{2-9}$$

with $\rho^2 = r$ where $\tau_1 = \frac{N-2}{2}$ (resp. $\frac{N-1}{2}$) for N even (resp. N odd).

As $\alpha \longrightarrow 0$ ($\bar{\omega}_\alpha$) tends to the linear isochrone ($\mathcal{I}_0$) whose solution curves are circles $\Gamma_\rho : u^2 + v^2 = \rho^2$. Therefore the limit cycles are asymptotic to Γ_ρ.

Small perturbations are constructed the usual way by making $B_i(K)B_{i+1}(K) < 0$. □

For instance for $N = 2$ (resp. $N = 3$) the quadratic (resp. cubic) first order upper bound $\mathcal{U}_1$ is zero (resp. one).

We next make use of the relative cohomology decomposition of polynomial one-forms in the plane to analyze the higher order perturbations. Assume $A_1(r,K) \equiv 0$ as a function of ρ, for a certain value $K_1 = K_1(N)$ of $K(N)$. We then need to compute $A_2(\rho, K_1)$, whose positive roots yield the second order upper bound $\mathcal{U}_2(N)$. The relative cohomology decomposition (See for instance [5,7,18]) states that there are polynomials $P_1(u,v)$ and $Q_1(u,v)$ such that

$$\bar{\omega}(u,v) = P_1(u,v)d\phi + dQ_1(u,v). \tag{2-10}$$

yielding the second PAM-function

$$A_2(\rho, K_1) = \int_{\Gamma(\rho)} (P_1\bar{\omega}) \quad \text{(modulo } A_1(\rho, K_1) \equiv 0). \tag{2-11}$$

We recall briefly the construction [5,7]. Let $\Gamma_\alpha(\rho)$ be solution of $0 = \overline{\omega_\alpha} = d\phi + \alpha\bar{\omega}$. From the definitions of A_1 and the displacement function, integrating over Γ_α the equality

$$(1 - \alpha A_1)(d\phi + \alpha\bar{\omega}) = d(\phi + \alpha Q_1) - \alpha^2 P_1 \bar{\omega} \tag{2-12}$$

gives

$$\alpha^2 A_2(\rho, K) + \alpha^2 \int_{\Gamma(\rho)} P_1\bar{\omega} = 0 \quad \text{(modulo } \alpha^3).$$

Hence formula (2-11). Computation of the factor P_1 goes as in the following lemma. (See also [5,7]).

Lemma 2.2. *If $A_1(r, K)$ vanishes identically then the polynomial $P_1(u, v)$ such that*

$$\bar{\omega}(u, v) = P_1(u, v)d\phi + dQ_1(u, v) \tag{2-13}$$

is of maximum degree (N-1) and given by the partial differential equation

$$u\frac{\partial P_1(u, v)}{\partial v} - v\frac{\partial P_1(u, v)}{\partial u} = Div(F, G)(u, v),$$

where $Div(F, G)$ is the divergence of F and G.

The explicit formula of $A_2(\rho, K_1)$ is calculated similarly to (2-4)

$$A_2(\rho, K_1) = \int_0^{2\pi} P_1 \cdot [F(\rho\cos t, \rho\sin t)\cos t + G(\rho\cos t, \rho\sin t)\sin t]\, dt. \tag{2-14}$$

Assume now $A_1(\rho, K_2) = A_2(\rho, K_2) \equiv 0$, for some value K_2 of the parameter K. This entails

$$\bar{\omega}_1(u, v) = P_1\omega(u, v) = P_2(u, v)d\phi + dQ_2(u, v) \tag{2-15}$$

with the cohomology factor $P_2(u, v)$ computed as in lemma 2.2 using P_1F and P_1G. Repeating the above procedure with the identity

$$(1 + \alpha P_1 + \alpha^2 P_2)(d\phi + \alpha\bar{\omega}) = d(\phi + \alpha P_1 + \alpha^2 P_2) - \alpha^3 P_2\bar{\omega} \tag{2-16}$$

yields

$$A_2(\rho, K_2) = \int_{\Gamma(\rho)} (P_2\bar{\omega}). \tag{2-17}$$

Inductively, given

$$\begin{aligned} A_k(\rho, K_{k-1}) &= (-1)^k \int_{\Gamma(\rho)} (P_{k-1}\bar{\omega}) \\ &= (-1)^k \int_0^{2\pi} P_{k-1} \cdot [F(\rho\cos t, \rho\sin t)\cos t + G(\rho\cos t, \rho\sin t)\sin t]\, dt \end{aligned} \tag{2-18}$$

if $A_k(\rho, K_{k-1}) \equiv 0$ (as a function of ρ), there exist polynomials P_k and Q_k such that

$$P_{k-1}\bar{\omega}(u, v) = P_k(u, v)d\phi + dQ_k(u, v), \tag{2-19}$$

and therefore the $(k + 1)th$ order PAM-function is given by

$$A_{k+1}(\rho, K) = (-1)^{k+1} \int_{\Gamma(\rho)} (P_k\bar{\omega}), \tag{2-20}$$

explicitly computed as in (2-19). The $kth, k \geq 1$ cohomology decomposition factor P_k is computed as in lemma 2.2 using the polynomials $P_{k-1}F$ and $P_{k-1}G$.

Consequently to obtain the kth order upper bound $\mathcal{U}_k(N)$ of the linearizable center, it suffices to construct the sequence of cohomology decomposition factors $P_i \in \mathbb{R}[u, v], i = 1, \cdots, k$ as in lemma 2.2 yielding an algorithmic computation of the PAM-function $A_k(\rho, K)$.

We next prove

Theorem 2.3. *The second order upper bound of $(\overline{\mathcal{F}_\alpha})$ is $\mathcal{U}_2(N) = N - 2$ independently of the parity of N.*

Proof. Set $K_1 = K|_{A_i(K)=0}$ the set of system coefficients (F_{ij}, G_{ij}) such that, from (2-9)

$$B_1(K_1) = B_3(K_1) = \cdots = B_i(K_1) = \cdots = B_{2\tau_1+1}(K_1) = 0. \tag{2-21}$$

That is, $A_1(r, K_1) \equiv 0$. Important to our analysis is the fact that every equation $B_i(K_1) = 0$ allows to derive one system coefficient in terms of the remaining. Note that in (2-9) that there are $\frac{N+1}{2}$ (resp. $\frac{N}{2}$) $B_i(K)$ for N odd (resp. N even.) Therefore we have

$$\aleph_1 := \text{card}(K_1) = \begin{cases} N^2 + 3N - \frac{N+1}{2} = \frac{2N^2+5N-1}{2}, & \text{for } N \text{ odd} \\ N^2 + 3N - \frac{N}{2} = \frac{2N^2+5N}{2}, & \text{for } N \text{ even,} \end{cases} \tag{2-22}$$

where again $\aleph_1 = \mathrm{card}(K_1)$ is the number of remaining components F_{ij}, G_{ij} in K_1. Using the relative cohomology decomposition we compute the $(N-1)th$ degree polynomial $P_1(u,v)$ from lemma 2.2. Set

$$P_1(u,v) = \sum_{i=1}^{N-1} \sum_{k=0}^{i} P^1_{i-k,k} u^{i-k} v^k. \tag{2-23}$$

The coefficients $P^1_{i-k,k} = P^1_{i-k,k}(K_1)$ are determined by the relation

$$(k+1)P^1_{i-k-1,k+1} - (i-k+1)P^1_{i-k+1,k-1} = (i-k+1)F_{i-k+1,k} + (k+1)G_{i-k,k+1}. \tag{2-24}$$

Let

$$\begin{aligned} S_i(K_1,t) &= \sum_{k=0}^{i} P^1_{i-k,k} \cos^{i-k} t \sin^k t; \\ T_{i+1}(K_1,t) &= \sum_{k=0}^{i+1} (F_{i-k,k} + G_{i-k+1,k-1}) \cos t^{i-k+1} \sin t^k, \end{aligned} \tag{2-25}$$

and compute the second PAM-function using (2-17). It entails

$$A_2(\rho, K_1) = \sum_{i=2}^{2N-1} \rho^i B_i(K_1), \tag{2-26}$$

with

$$B_i(K_1) = \sum_{k=1}^{i-1} \int_0^{2\pi} S_{i-k}(K_1,t) T_{k+1}(K_1,t) dt, \tag{2-27}$$

terms of negative subindex are assumed zero, $S_j(K_1,t) = 0$ for $j > N-1$, and $T_j(K_1,t) = 0$ for $j > N+1$. Through the above sine/cosine rules it results

$$B_i(K_1) \equiv 0 \quad (\text{resp. } B_i(K_1) \not\equiv 0), \text{ for } i \text{ even (resp. } i \text{ odd)}. \tag{2-28}$$

In particular $B_2(K_1) = 0$, and $B_{2N-1}(K_1) \not\equiv 0$, independently of the parity of N. Hence the claim. □

Next repeat the above process in $s_j, j = 1, \cdots, \sigma_N$ steps after which obtain the first non identically zero PAM-function A_τ and derive the $\tau-$upper bound $\mathcal{U}_\tau(N)$. This procedure is called the *Step Reduction Process.* We prove

Theorem 2.4.

(1) *For N odd (resp. N even), the first odd (resp. even) integer $\tau = \tau(N) = \sigma_N$ determined below by* (2-32) *(resp.* (2-34)*) yields $A_{\tau-1} \not\equiv 0$ (resp. $A_\tau \not\equiv 0$).*

(2) *The overall upper bound is*

$$\mathcal{U}(N) = \mathcal{U}_\tau(N) = \begin{cases} \frac{\tau N - (\tau+2)}{2}, & \text{for } N \text{ odd} \\ \frac{\tau N - (\tau+3)}{2}, & \text{for } N \text{ even} \end{cases}$$

in terms of the degree N of the transformed polynomial perturbation.

(3) *At any arbitrary order $1 \le k \le \tau$ the kth order upper bound $\mathcal{U}_k(N)$ is given by* (2-30).

Proof. At every step s_j we compute the relative cohomology decomposition factor P_k which is a polynomial of degree $k(N-1)th$ for $k = j+1$. At the corresponding coefficients $K_k|_{B_i(K_{k-1})=0}$, the number of bifurcation coefficients $B_i(K_{k-1})$ is

$$\aleph_{k-1} = \mathrm{card}(B_i(K_{k-1})) = \begin{cases} \frac{kN-k}{2}, & \text{for } k \text{ odd, } N \text{ odd} \\ \frac{kN-(k+1)}{2}, & \text{for } k \text{ odd, } N \text{ even.} \end{cases} \tag{2-29}$$

we determine the kth PAM-function $A_k(\rho, K_{k-1})$ that yields a kth order upper bound

$$\mathcal{U}_k(N) = \begin{cases} \frac{kN-(k+2)}{2}, & \text{for } k \text{ even and every } n; \ k \text{ odd and } N \text{ odd.} \\ \frac{kN-(k+3)}{2}, & \text{for } k \text{ odd and } N \text{ even.} \end{cases} \tag{2-30}$$

As above we derive some system coefficients in function of others in solving $B_i(K_{k-1}) = 0$. The PAM-function A_k are in the Noetherian ring $\mathbb{R}[K]$ Thus, from Hilbert's basis theorem the process must stop yielding the overall upper bound. That is, the existence of a uniform bound to the number of steps to have a non identically zero PAM-function. Recall also that the coefficients $B_i(K_{k-1})$ are linearly independent and polynomials of degree k in the components of K_{k-1}. After the last σ_N step the number of remaining system coefficients is less or equal to the number of bifurcation coefficients $B_i(K_{\sigma_N})$. Thus at least the last B_i is necessarily nonzero yielding $A_{\sigma_N} \not\equiv 0$, as illustrated below. We next determine σ_N.

(1) For N odd, after σ_N steps, from (2-22), we have

$$\frac{2N^2+5N-1}{2} \leq \sum_{k=2}^{\sigma_N} \frac{kN-k}{2} \tag{2-31}$$

This leads to σ_N satisfying

$$\sigma_N(\sigma_N+1) \geq 4\frac{N^2+3N-1}{N-1} \tag{2-32}$$

(2) For N even, it amounts to determining $\overline{\sigma}_N = \sigma_N/2$ such that

$$\frac{2N^2+5N}{2} \leq \sum_{k=2}^{\overline{\sigma}_N} \Big(\frac{kN-k}{2} + \frac{(k+1)N-(k+2)}{2} \Big). \tag{2-33}$$

We get

$$\overline{\sigma}_N(\overline{\sigma}_N+1) \geq \frac{2N^2+9N-6}{N-1} \tag{2-34}$$

Hence the result. □

3. Some Illustrations

3.1 The linear 1-form.

We assume that the unperturbed 1-form in $(\mathcal{F}_\alpha)$ is the linear isochrone 1-form $\mathcal{I}_0$ subjected to a polynomial 1-form perturbation. The above cohomology approach is therefore fully applicable. The estimates of the upper bounds are given as in theorem 2.4. For quadratic and cubic perturbations we obtain the followings.

3.1.1 The quadratic perturbation.

We prove

Theorem 3.1. *In a quadratic perturbation of the linear 1-form*

(1) *The overall upper bound $\mathcal{U}(2)$ is three.*
(2) *The relative upper bounds are $\mathcal{U}_k(2)$ are $\mathcal{U}_k(2) = 0$ for $k = 1, 2, 3$, i.e., no limit cycles can bifurcate up to order three.*
(3) *And $\mathcal{U}_k(2) = 1$ for $k = 4, 5$. $\mathcal{U}_6(2) = \mathcal{U}_7(2) = 2$. And $\mathcal{U}_8(2) = 3$.*

Proof. The result is straightforward by taking $N = 2$ in formulas (2-30) and (2-34). We obtain $\sigma_2 \geq 8$. Thus $A_8 \not\equiv 0$. □

Item one in the above theorem confirms results in [3, section 3.1, and Theorem 4.8] whereas items $2, 3$ correct and improve concluding remarks in [9].

3.1.1 The cubic perturbation.

We obtain

Theorem 3.2. *In a cubic perturbation of the linear isochrone*

(1) *The overall upper bound $\mathcal{U}(3)$ is five.*
(2) *For the relative upper bounds $\mathcal{U}_k(3)$ we have $\mathcal{U}_1(3) = \mathcal{U}_2(3) = 1$; $\mathcal{U}_3(3) = 2$; $\mathcal{U}_4(3) = 3$; $\mathcal{U}_5(3) = 4$; $\mathcal{U}_6(3) = 5$.*

Proof. The result follows from $N = 3$ in formulas (2-30) and (2-32). We get $\sigma_3 \geq 5.3$; thus $A_6 \not\equiv 0$. □

From Theorem 2.4 similar corollaries can be formulated for fourth, fifth, $\cdots$, kth order perturbation of the linear 1-form.

3.2 A polynomial preserving case: Darboux linearizable Cubic Hamiltonian Center.

Without loss of generality a cubic Hamiltonian system may be written as [12]

$$\begin{aligned}\varpi_H(x,y) =&(x+3a_6x^2+2a_1xy+a_2y^2+4a_7x^3+3a_4x^2y+2a_5xy^2)dx\\ &+(y+a_1x^2+2a_2xy+3a_3y^2+a_4x^3+2a_5x^2y)dy,\end{aligned} \tag{$\mathcal{H}_3$}$$

$a_i \in \mathbb{R}, i = 1, 2, 3, 4, 5, 6, 7$, and with Hamiltonian function

$$H(x,y) = \frac{x^2+y^2}{2} + a_6x^3 + a_1x^2y + a_2xy^2 + a_3y^3 + a_7x^4 + a_4x^3y + a_5x^2y^2. \tag{3-1}$$

Mardešić et al have established the following characterization in [13].

Theorem 3.3. *The cubic Hamiltonian 1-form* ($\mathcal{H}_3$) *is Darboux linearizable if and only if it is of the form*

$$\omega_{H_i}(x,y) = (x+2Cxy+2C^2x^3)dx + (y+Cx^2)dy \tag{$\mathcal{H}_i$}$$

This system is linearizable through the canonical change of coordinates

$$(u(x,y), v(x,y)) = (x, y+Cx^2). \tag{$\mathcal{T}_l$}$$

Consider a cubic autonomous perturbation ($\mathcal{H}_\alpha$) of system ($\mathcal{H}_i$)

$$\omega_\alpha(x,y) = \omega_i(x,y) + \alpha\omega(x,y) \tag{$\mathcal{H}_\alpha$}$$

where $C \neq 0$, and $\omega(x,y) = g(x,y)dx - f(x,y)dy$ with

$$f(x,y) = \sum_{i=1}^{3}\sum_{k=0}^{i} f_{i-k,k}x^{i-k}y^k, \quad g(x,y) = \sum_{i=1}^{3}\sum_{k=0}^{i} g_{i-k,k}x^{i-k}y^k. \tag{3-2}$$

The system coefficients set is $K = K(3) = (C, f_{ij}, g_{ij}, 1 \le i+j \le 3)$ with $\aleph = \text{card}(K(3)) = 19$. The linearizing change of coordinates ($\mathcal{T}_l$) transforms ($\mathcal{H}_\alpha$) into system

$$\overline{\omega_\alpha}(u,v) = (u+\alpha G(u,v))du + (v-\alpha F(u,v))dv, \tag{$\overline{\mathcal{H}_\alpha}$}$$

with

$$\begin{aligned}F(u,v) =&\sum_{i=1}^{3}\sum_{k=0}^{i} f_{i-k,k}u^{i-k}(v-Cu^2)^k = \sum_{i=1}^{6}\sum_{k=0}^{i} F_{i-k,k}u^{i-k}v^k\\ =&f_{10}u + f_{01}v + (f_{20}-Cf_{01})u^2 + f_{11}uv + f_{02}v^2 + (f_{30}-Cf_{11})u^3 +\\ &(f_{21}-2Cf_{02})u^2v + f_{12}uv^2 + f_{03}v^3 + (c^2f_{02}-Cf_{21})u^4 - 2Cf_{12}u^3v -\\ &3Cf_{03}u^2v^2 + C^2f_{12}u^5 + 3C^2f_{03}u^4v - C^3f_{03}u^6,\\ G(u,v) =&2CuF(u,v) + \sum_{i=1}^{3}\sum_{k=0}^{i} g_{i-k,k}u^{i-k}(v-Cu^2)^k = \sum_{i=1}^{7}\sum_{k=0}^{i} G_{i-k,k}u^{i-k}v^k\\ =&g_{10}u + g_{01}v + (2Cf_{01}+g_{20}-Cg_{01})u^2 + (2Cf_{01}+g_{11})uv + g_{02}v^2 +\\ &(2C(f_{20}-Cf_{01}) + g_{30} - Cg_{11})u^3 + (2Cf_{11}+g_{21}-2Cg_{02})u^2v + (2Cf_{02}+\\ &g_{12})uv^2 + g_{03}v^3 + (2C(f_{30}-Cf_{11}) + C^2g_{02} - Cg_{21})u^4 + (2Cf_{21}-\\ &4C^2f_{02} - 2Cg_{12})u^3v + (2Cf_{12}-3Cg_{03})u^2v^2 + 2Cf_{03}uv^3 + C^2(2Cf_{02}-2g_{21}+\\ &g_{12})u^5 + C^2(-4f_{12}+3g_{03})u^4v - 6C^2f_{03}u^3v^2 + C^3(2f_{12}-g_{03})u^6 +\\ &6C^3f_{03}u^5v - 2C^4f_{03}u^7.\end{aligned} \tag{3-3}$$

Therefore the resulting one-form $\bar{\omega}(u,v) = G(u,v)du - F(u,v)dv$ is polynomial of degree $deg(\bar{\omega}) := max(deg(F), deg(G)) = 7$. Denoting $\bar{K} = \bar{K}(7)$ the system coefficients set after linearization, $\bar{\aleph} = \text{card}(\bar{K}) = 19$. We prove

Theorem 3.4.

(1) *The overall upper bound $\mathcal{U}(3)$ is nine.*
(2) *For the relative upper bounds $\mathcal{U}_k(3)$ we have*

$$U_1(3) = 2; \quad U_2(3) = 3; \quad U_3(3) = 6; \quad U_4(3) = 9 = U(3). \tag{3-4}$$

Proof. Computing the first order bifurcation function $A_1(\rho, \bar{K})$ as in (2-4) yields

$$A_1(\rho, \bar{K}) = \rho\left(B_1(\bar{K}) + B_3(\bar{K})r + B_5(\bar{K})r^2\right), \tag{3-5}$$

with $\rho^2 = r$, and

$$\begin{aligned} &B_1(\bar{K}) = \pi(F_{10} + G_{01}); \quad B_3(\bar{K}) = \frac{\pi}{4}\left(3(F_{30} + G_{03}) + F_{12} + G_{21} - C(F_{11} + 2G_{02})\right); \\ &B_5(\bar{K}) = \frac{\pi}{8}(F_{12} + 3G_{03})C^2. \end{aligned} \tag{3-6}$$

The upper bound $\mathcal{U}_1(3)$ is clearly two.

Next set $\bar{K}_1 = \bar{K}|_{B_i(\bar{K})=0, i=1,3,5}$, that is,

$$F_{10} + G_{01} = F_{12} + 3G_{03} = 3F_{30} + G_{21} - C(F_{11} + 2G_{02}) = 0. \tag{3-7}$$

Thus $A_1(\rho, \bar{K}_1) \equiv 0$. We then analyze the second order perturbation, first determining the relative cohomology decomposition polynomial $P_1(u, v)$ from formula (2-24). We get

$$\begin{aligned} P_1(u, v) = &-(F_{11} + 2G_{02})u + (2F_{20} + G_{11})v + (F_{21} + G_{12})v^2 \\ &- 2C(F_{21} + G_{12})u^2v - 4C(F_{21} + G_{12})v^3. \end{aligned} \tag{3-8}$$

As above the corresponding PAM-function $A_2(\rho, \bar{K}_1)$ reduces to

$$A_2(\rho, \bar{K}_1) = \sum_{i=3, iodd}^{N} \rho^i B_i(\bar{K}_1), \tag{3-9}$$

where the bifurcation coefficients $B_i(\bar{K}_1)$ are computed as in (2-27). We get $N = 9$ leading to $\mathcal{U}_2(3) = (N-3)/2 = 3$.

Then set $\bar{K}_2 = \bar{K}_1|_{B_i(\bar{K}_1)=0, i=3,5,7,9}$, $\aleph_2 = \mathrm{card}(\bar{K}_2) = 12$, and $A_2(\rho, \bar{K}_2) \equiv 0$. It yields the determination of a $8th$ degree relative cohomology decomposition second factor $P_2(u, v)$. We then compute the third PAM-function $A_3(\rho, \bar{K}_2)$ and the PAM-coefficients $B_i(\bar{K}_2), i = 3, 5, 7, 8, 9, 11, 13, 15$ as in (2-27). This entails the third order upper bound $\mathcal{U}_3(3) = 6$. The equations $B_i(\bar{K}_2) = 0, i = 3, 5, 7, 8, 9, 11, 13, 15$ yield a coefficient set $\bar{K}_3 = \bar{K}_2|_{B_i(\bar{K}_2)=0, i=3,5,7,8,9,11,13,15}$ such that $A_3(\rho, \bar{K}_3) \equiv 0$, and $\aleph_3 = \mathrm{card}(\bar{K}_3) = 6$. This leads to compute a $14th$ degree cohomology decomposition factor $P_3(u, v)$, and 10 PAM-coefficients $B_i(\bar{K}_3), i = 3, \cdots, 21; odd$. It entails a $4th$ order bifurcation function non identically zero. We obtain the $4th$ order upper bound $\mathcal{U}_4(3) = 9$ as claimed. □

Remark 3.5. In the previous case, although Theorem 2.4 might have given the estimates of the upper bounds, to obtain the most accurate estimates one must consider the explicit expressions of each resulting polynomial perturbation in the building-up of the combined cohomology decomposition-step reduction process.

3.3 A non-polynomial preserving case: A modified Kukles isochrone.

We consider the reduced Kukles 1-form

$$\varpi_{\mathcal{K}}(x, y) = (x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2)dx + ydy \tag{$\mathcal{K}$}$$

$a_i \in \mathbb{R}, i = 1, 2, 3, 4, 5, 6$. [17] proved the following theorem.

Kukles Isochrone. *([17]) The origin is an isochronous center of $(\mathcal{K})$ if and only if the system is linear or can be brought, through rescaling of (x, y) and t to the form*

$$\omega_{\mathcal{K}_i}(x, y) = (x + 3xy + x^3)dx + ydy. \tag{$\mathcal{K}_i$}$$

Moreover, a rational linearizing change of coordinates of the system $(\mathcal{K}_i)$ *is given by*

$$(u(x,y), v(x,y)) = \left(\frac{x}{x^2+y+1}, \frac{x^2+y}{x^2+y+1}\right). \qquad (\mathcal{T}_l)$$

Consider a one-parameter cubic autonomous perturbation $(\mathcal{K}_\alpha)$ of the Kukles nonlinear isochrone $(\mathcal{K}_i)$ in the form

$$\omega_\alpha(x,y) = \omega_{\mathcal{K}_i}(x,y) + \alpha\omega(x,y), \qquad (\mathcal{K}_\alpha)$$

with $\omega(x,y) = g(x,y)dx - f(x,y)dy$, where

$$f(x,y) = \sum_{i=1}^{3}\sum_{k=0}^{i} f_{i-k,k}x^{i-k}y^k; \quad g(x,y) = \sum_{i=1}^{3}\sum_{k=0}^{i} g_{i-k,k}x^{i-k}y^k. \qquad (3\text{-}10)$$

The linearizing change of coordinates $(\mathcal{T}_l)$ yields

$$x(u,v) = \frac{u}{1-v}; \quad y(u,v) = \frac{v-(u^2+v^2)}{(1-v)^2}. \qquad (3\text{-}11)$$

The system $(\mathcal{K}_\alpha)$ is transformed via $(\mathcal{T}_l)$ into

$$\bar{\omega}_\alpha(u,v) = \mathcal{I}_0(u,v) + \alpha\bar{\omega}(u,v), \qquad (\bar{\mathcal{K}}_\alpha)$$

with $\bar{\omega}(u,v) = G(u,v)du - F(u,v)dv$, and $(F(u,v), G(u,v)) = J(T_l^{-1})(f,g)|_{(u,v)}$, that is,

$$\begin{aligned} F(u,v) &= (1-2u^2-v)f\left(\frac{u}{1-v}, \frac{v-(u^2+v^2)}{(1-v)^2}\right) - u(1-v)g\left(\frac{u}{1-v}, \frac{v-(u^2+v^2)}{(1-v)^2}\right); \\ G(u,v) &= (2u(1-v))f\left(\frac{u}{1-v}, \frac{v-(u^2+v^2)}{(1-v)^2}\right) + (1-v)^2 g\left(\frac{u}{1-v}, \frac{v-(u^2+v^2)}{(1-v)^2}\right) \end{aligned} \qquad (3\text{-}12)$$

Obviously the resulting perturbed 1-form $\bar{\omega}_\alpha(u,v)$ is not polynomial. Therefore we cannot apply the cohomology decompositon approach for the order $k \geq 2$ upper bound. The first order upper bound is computed as in (2-3). To illustrate, we present the case of a modified version of $(\mathcal{K}_\alpha)$ proposed by Iliev [10] as follows. Take $\omega(x,y) = (a_1y + a_2x^2 + a_3y^2 + a_4y^3)dx$, and denote $(\mathcal{KM}_\alpha)$ the corresponding 1-parameter perturbation. Obtain

Theorem 3.6. *The first order upper bound of* $(\mathcal{KM}_\alpha)$ *is* $\mathcal{U}_1(3) = 3$.

Proof. From (2-4) and (3-12), the corresponding first order bifurcation function becomes, with $\rho \in (0,1)$, and $K = (a_i, i = 1,2,3,4)$, ϕ as in section 2,

$$\begin{aligned} A_1(\rho,K) &= \int_{\phi=\rho^2} \mathcal{B}(u,v,K)((1-v)du + udv) \\ &= -\int_{\phi=\rho^2} udv + \int_{\phi=\rho^2}\left(1 - \frac{1-\rho^2}{(1-v)}\right)udv - \int_{\phi=\rho^2}\left(\frac{(1-\rho^2)^2}{(1-v)^3} + \frac{1}{1-v}\right)udv \\ &+ \int_{\phi=\rho^2}\left(-3\frac{(1-\rho^2)^3}{(1-v)^5} + 6\frac{(1-\rho^2)^2}{(1-v)^4} - 3\frac{(1-\rho^2)^2}{(1-v)^3}\right)udv, \end{aligned} \qquad (3\text{-}13)$$

with

$$\mathcal{B}(u,v,K) = \left(a_1\frac{v(1-v)-u^2}{1-v} + a_2\frac{u^2}{1-v} + a_3\frac{(v(1-v)-u^2)^2}{(1-v)^3} + a_4\frac{(v(1-v)-u^2)^3}{(1-v)^5}\right), \qquad (3\text{-}14)$$

using the symmetry of the ovals of integration and integration by parts. Set $r = \sqrt{1-\rho^2}$. Direct calculations of $A_1(\rho,K)$ are carried out through the following recurrence dependencies between integrals

$$I_k = \int \frac{udv}{1-v} = \int \frac{u(1+v)^k dv}{(1-v^2)^k}; \quad I_{jk} = \int \frac{uv^j dv}{(1-v^2)^k}. \qquad (3\text{-}15)$$

For instance

$$\begin{aligned} I_{00} = \rho^2\pi; \quad I_{01} = 2\pi(1-r); \quad I_{02} = \frac{\rho^2\pi}{r}; \quad I_{03} = \frac{\rho^2\pi(4-3\rho^2)}{4(1-\rho^2)r}, \cdots, \\ I_0 = I_{00}; \quad I_1 = I_{01}; \quad I_2 = 2I_{02} - I_{01}; \quad I_3 = 4I_{03} - 3I_{02}. \end{aligned} \qquad (3\text{-}16)$$

Thus

$$A_1(\rho, K) = \frac{\rho^2 \pi}{r(1+r)}[B_0(K) + B_1(K)r + B_2(K)r^2 + B_3(K)r^3] \quad (3\text{-}17)$$

where

$$\begin{aligned} &B_0(K) = -\frac{3}{4}a_4; \quad B_1(K) = (a_2 - a_1 + 2a_3 - \frac{3}{4}a_4); \\ &B_2(K) = (a_2 - a_1 - a_3 + \frac{3}{4}a_4); \quad B_3(K) = (-2a_2 - a_3 + \frac{3}{4}a_4). \end{aligned} \quad (3\text{-}18)$$

The PAM-coefficients $B_i(K), i = 0, 1, 2, 3$ are clearly independent. So the cubic polynomial in (3-16) can possess three real zeros in $(0, 1)$. $\square$

4. Appendix: Some explicitly linearizable Centers

4.1 Darboux Function.

An *invariant algebraic curve* of the polynomial equation

$$\omega(x, y) = T_2(x, y)dx - T_1(x, y)dy = 0 \quad (4\text{-}1)$$

is the n-degree curve $\mathcal{F}(x, y) = 0, \quad \mathcal{F}(x, y) \in \mathbb{C}_n[x, y]$ in the complex plane such that

$$\frac{d\mathcal{F}}{dt} = \frac{\partial \mathcal{F}}{\partial x}T_1(x, y) + \frac{\partial \mathcal{F}}{\partial y}T_2(x, y) = \mathcal{F}(x, y)cof(\mathcal{F})(x, y), \quad (4\text{-}2)$$

with $cof(\mathcal{F}) \in \mathbb{C}_{n-1}[x, y]$ the cofactor of $\mathcal{F}$. A *Darboux function* $Z(x, y)$ is of the form

$$Z(x, y) = \prod_{j=0}^{k} F_j^{\alpha_j}, \quad \alpha_j \in \mathbb{C}, \quad (4\text{-}3)$$

with either $F_j \in \mathbb{C}[z, \bar{z}] = \mathbb{C}[x, y]$ or $F_j = exp(G_j)$, with $G_j \in \mathbb{C}(z, \bar{z})$, for each $j = 0, \cdots, k$. If such a function is a linearizing change of coordinates $\mathcal{T}_l$ for system $(\mathcal{F}_\alpha)$ the system is said to be *Darboux linearizable.* The following has been proved in [13].

Theorem: Darboux Linearizability.

$(\mathcal{F}_\alpha)$ is Darboux linearizable if and only if the analytic invariant manifold $F_0 = 0$, of the form $F_0(z, \bar{z}) = z + o(|(z, \bar{z})|)$, is algebraic and there exists invariant algebraic curves $F_j = 0$ or exponential Darboux factors $F_j = exp(G_j), j \in J$ where J is a finite subset of $\mathbb{N}$ (possibly void) such that $F_j(0, 0) \neq 0$ and

$$K_0 + \sum_{j \in J} \alpha K_j = i = \sqrt{-1}, \quad (4\text{-}4)$$

where K_j is the cofactor of $F_j, j \in J$. Moreover, if the assumptions are satisfied, then a Darboux linearization is given by

$$Z = F_0 \prod_{j=1}^{k} F_j^{\alpha_j}. \quad (4\text{-}5)$$

4.2 Examples. The quadratic (resp. cubic symmetric) 1-form

$$\omega(x, y) = (x + Q_n(x, y))dx + (y - P_n(x, y))dy, \quad (4\text{-}6)$$

is explicitly Darboux linearizable [12], where P_n and Q_n are homogeneous quadratic for $n = 2$ (resp. cubic for $n = 3$) polynomials and at least one is non-vanishing. It also possesses 4 isochronous strata in each case. (cf [11,15]). In addition to the reduced Kukles isochrone and the Hamiltonian cubic isochrone, one can find more explicitly Darboux linearizable centers in [12,13], as well as details on the techniques yielding these linearizations.

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