Finite cyclicity of elementary graphics surrounding a focus or center in quadratic systems

F. Dumortier, A. Guzmán, and C. Rousseau

CRM-2708

January 2001

This work was supported by NSERC and FCAR in Canada.

Limburgs Universitair Centrum, Universitaire Campus, B-3590 Diepenbeek, Belgium

Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México, Ciudad Universitaria, México, D.F. 04510, México

Département de mathématiques et de statistique and CRM, Université de Montréal, C.P. 6128, succ. Centre-Ville, Montréal, QC H3C 3J7, Canada

Abstract

In this paper we prove that several elementary graphics surrounding a focus or center in quadratic systems have finite cyclicity. This paper represents an additional step in the large program to prove the existence of a uniform bound for the number of limit cycles of a quadratic vector field which we can call the finiteness part of Hilbert's 16th problem for quadratic vector fields. It nearly finishes the part of the program concerned with elementary graphics. In [3] this problem was reduced to the proof that 121 graphics have finite cyclicity. The graphics considered here are the hemicycles (H_4^3), (H_5^3) and (H_6^3) together with (I_{14a}^2), (I_{15a}^2), (I_{15b}^2) and (I_{27}^2) in the notation of [3] (Figure 1).

Résumé

Dans cet article on prouve que plusieurs graphiques entourant un foyer ou un centre dans les systèmes quadratiques ont cyclicité finie. Cet article représente un pas de plus dans le grand programme permettant de prouver l'existence d'une borne uniforme pour le nombre de cycles limites d'un champ de vecteurs quadratique, que nous appelons la partie finitude du 16$^{\text{e}}$ problème de Hilbert pour les systèmes quadratiques. Il termine presque la partie du programme concernant les graphiques élémentaires. Dans [3] ce problème était réduit à la preuve que 121 graphiques ont cyclicité finie. Les graphiques considérés ici sont les hémicycles (H_4^3), (H_5^3) et (H_6^3) ainsi que les graphiques (I_{14a}^2), (I_{15a}^2), (I_{15b}^2) et (I_{27}^2) dans la notation de [3] (Figure 1).

1. Introduction

This paper is part of a large program namely the proof of the finiteness part of Hilbert's 16th problem [5] for quadratic systems:

Finiteness part of Hilbert's 16-th problem for quadratic systems. *There exists a natural number N such that any quadratic vector field in the plane has at most N limit cycles.*

In studying this problem it is natural to compactify the phase space to the Poincaré disc. The parameter space can be compactified as well. We obtain a family $\{\mathcal{X}_\lambda\}_{\lambda\in\Lambda}$, of analytic vector fields defined on a compact phase space and depending on parameters λ varying in a compact set Λ. Using a compacity argument Roussarie [12] showed that to prove the finiteness part of Hilbert's 16-th problem for quadratic vector fields it is sufficient to prove that any limit periodic set in the family $\mathcal{X}_\lambda$ has finite cyclicity, i.e., can give rise to a uniformly bounded number of limit cycles in any perturbation inside the family $\mathcal{X}_\lambda$.

The following theorem was proved in [3].

Theorem 1.1 ([3]). *The finiteness part of Hilbert's 16-th problem for quadratic vector fields will be proved as soon as the following conjecture is proved.*

Conjecture 1.2. *Any limit periodic set surrounding the origin in the family*

$$\begin{aligned} \dot{x} &= \lambda x - \mu y + a_1x^2 + a_2xy + a_3y^2 \\ \dot{y} &= \mu x + \lambda y + b_1x^2 + b_2xy + b_3y^2, \end{aligned} \tag{1}$$

with $(\lambda, \mu) \in \mathbb{S}^1$ and $(a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{S}^5$ has finite cyclicity inside (1).

The paper [3] then showed that the proof of the Conjecture 1.2 could be reduced to the proof of the finite cyclicity of 121 graphics, (54 of them appearing in families of graphics, yielding a total of 85 pictures). Since 1991, where the conjecture was first stated, there has been an intensive attack to prove the finite cyclicity of the 121 graphics listed in [3], the most difficult ones being of course the graphics surrounding a center. We are now approaching 3/4 of the program with the recent progress on graphics through a nilpotent point ([13] and [14]) and nearly all elementary graphics inside the family are proved to have finite cyclicity.

In this paper we prove the finite cyclicity of three hemicycles (H_4^3), (H_5^3) and (H_6^3) (Figure 1) with four singular points: two hyperbolic saddles with opposite hyperbolicity ratio and two saddle nodes. The proof uses a direct calculation of a displacement map. The first two graphics, (H_4^3), (H_5^3), satisfy genericity conditions while the third one can be perturbed to a hemicycle with only two singular points and surrounding a center. When the hyperbolicity ratios of the saddles are different from one, the generic graphics have cyclicity 2 as soon as some regular transition has a non vanishing second derivative. Because the transition occurs along an invariant line or along the equator we are able to calculate explicitly the required derivative. When the hyperbolicity ratios are equal to one the genericity condition is the non-vanishing of the first saddle quantities. The finite cyclicity of (H_6^3) follows from a reduction to the previous generic cases via a Bautin type argument.

We also prove the finite cyclicity of the graphics (I_{14a}^2), (I_{15a}^2), (I_{15b}^2) and (I_{27}^2) (Figure 1). (In Figure 1 the graphics (I_{ia}^2) are the ones having two central transitions while the graphics (I_{ib}^2) have at least a stable-center or center-unstable transition.) The finite cyclicity of these graphics was announced in [1]. However the given elaboration cannot be considered to be a proof since a connection was incorrectly kept fixed. Here again the proof requires a condition on the higher derivatives of a regular transition.

Together with the recent proof of the finite cyclicity of (I_2^2) by Mourtada [11] this nearly completes the proof of the finite cyclicity of the 58 elementary graphics listed in [3]. The finite cyclicity of the graphic (I_{16a}) is still open and a systematic checking of all proofs remain to be done before anouncing that the part of the program dealing with elementary graphics is finished. For the remaining graphics, 50 of them have a nilpotent point (many of which are treated in [13] and [14]) while the remaining 13 have a line of singular points.

2. Finite cyclicity of graphics with a pair of hyperbolic points on the equator and two semi-hyperbolic points of opposite nature

Theorem 2.1. *We consider a graphic Γ of a polynomial vector field which is a hemicycle. The pair of opposite singular points are two hyperbolic saddles P_1 and P_2 with irrational hyperbolicity ratios satisfying $r_1(\lambda)r_2(\lambda) \equiv 1$. There is one attracting saddle-node P_3 on the equator and one repelling saddle-node P_4 on the other connection. Both connections near P_3 and P_4 are central (Figure 2). In the neighborhood of each singular point we use C^k-coordinates bringing the family to C^k integrable normal forms. We define sections to the graphic as in Figure 2. These sections are parallel to the coordinates axes in the C^k normal coordinates and parametrized by these. Then, under one of the following conditions:*

1. *the regular transition $S_0\colon \tau_2 \to \sigma_3$ has a non-vanishing higher order derivative;*
2. *the regular transition $R_0\colon \tau_4 \to \sigma_2$ has a non-vanishing higher order derivative;*

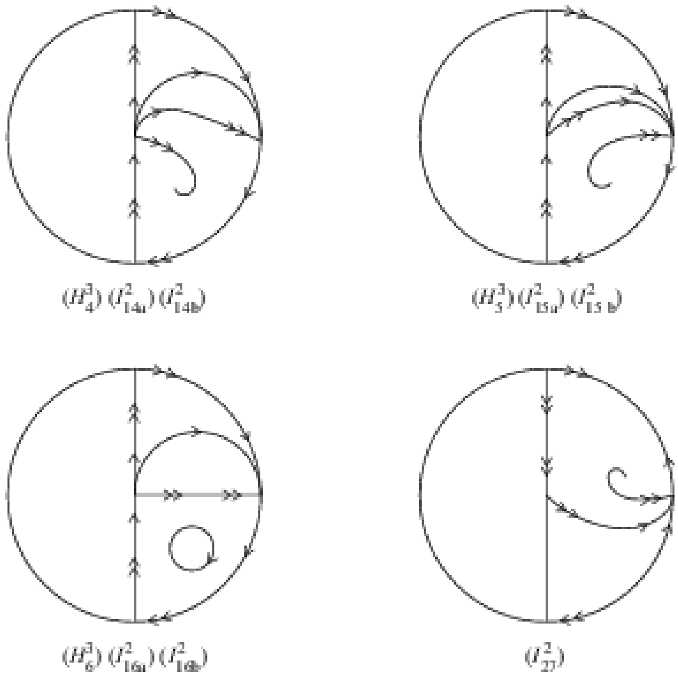

FIG. 1

the graphic Γ *has finite cyclicity inside* X_λ. *The cyclicity of* Γ *is* ≤ 2 *in the two following cases:*

$$\begin{cases} S_0''(0) \neq 0 \text{ and } r < 1 \\ R_0''(0) \neq 0 \text{ and } r > 1. \end{cases}$$

Proof. We take sections σ_i and τ_i as in Figure 2. Let us denote $r_1(\lambda) = \dfrac{1}{r(\lambda)} = s(\lambda)$ and $r_2 = r(\lambda)$. The connections on the equator are unbroken.

The technique is standard: we use C^k-coordinates in the neighborhood of the singular points so that the vector field is in normal form (see for instance [8] or [1]). This allows to calculate the return map and to conclude to finite cyclicity by a standard derivation-division algorithm.

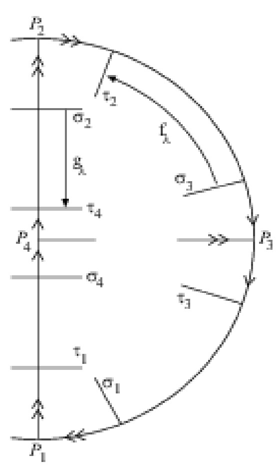

FIG. 2

In the neighborhood of the saddle points the vector field can be linearized so that the Dulac maps have the form

$$D_{i,\lambda}(x_i) = x_i^{r_i(\lambda)}. \tag{2}$$

Also, the normal form of the vector field in the neighborhood of $P_{3,4}$ is

$$\begin{aligned} \dot{x}_i &= x_i^2(1 + A_i(\lambda)x) + \epsilon_i \\ \dot{y}_i &= \pm y_i, \end{aligned} \tag{3}$$

yielding that the Dulac maps near P_3 and P_4 are linear of the form

$$\begin{cases} D_{3,\lambda}(x_3) = m(\lambda)x_3 \text{ with } \lim_{\lambda\to 0} m(\lambda) = 0 \\ D_{4,\lambda}(x_4) = M(\lambda)x_4 \text{ with } \lim_{\lambda\to 0} M(\lambda) = +\infty. \end{cases} \tag{4}$$

Moreover, by the results of [4], the family of C^k-coordinates in which we have the normal form (3) is not unique. Using the freedom in the choice of coordinates it is possible to find normalizing coordinates depending on λ so that the transition map from τ_3 to σ_1 is the identity and so that the transition map from τ_1 to σ_4 is a mere translation $y_1 \mapsto y_1 + \epsilon$. Let S_λ (resp. R_λ) be the regular transition from τ_2 to σ_3 (resp. from τ_4 to σ_2) and let f_λ (resp. g_λ) be its inverse. We have that $f_\lambda(x_3) = x_3 f_{1,\lambda}(x_3) = \sum_{i=1}^k a_i(\lambda)x_3^i + o(x_3^k)$ with $f_{1,\lambda}(0) > 0$. Let us suppose that $g_\lambda(x_2) = \sum_{i=0}^k b_i(\lambda)x_2^i + o(x_2^k)$, with $b_0(0) = 0$ and $b_1(0) \neq 0$.

Limit cycles are given by isolated fixed points of the return map or, equivalently, by isolated solutions of the displacement map $V_\lambda(x_3)$ from σ_3 to τ_4. Then, renaming x_3 by x

$$V_\lambda(x) = M(\lambda)[D_{1,\lambda}(m(\lambda)x) + \epsilon] - g_\lambda \circ D_{2,\lambda}^{-1} \circ f_\lambda(x). \tag{5}$$

Then

$$V_\lambda(x) = M(\lambda)\epsilon + M(\lambda)(m(\lambda))^{s(\lambda)}x^{s(\lambda)} - g_\lambda\left(x^{s(\lambda)}(f_{1,\lambda}(x))^{s(\lambda)}\right) \tag{6}$$

Note that the map $V_\lambda(x)$ has a nice expansion in monomials of the form $x^{is(\lambda)+j}$, allowing to perform the usual derivation-division algorithm. A first derivation kills the constant term. We then divide by $x^{s(\lambda)-1}$. A second derivative removes completely the term with coefficient $M(\lambda)(m(\lambda))^{s(\lambda)}$ and yields $W_\lambda(x) = \left(\frac{V_\lambda'(x)}{x^{s(\lambda)-1}}\right)'$. The remaining part is an expansion in monomials of the form $x^{is(\lambda)+j}$, none of which has an unbounded coefficient (i.e., containing $M(\lambda)$). Here we need to distinguish the two cases:

(1) Let us suppose $f_0^{(n)}(0) \neq 0$. In $V_\lambda(x)$ appears a monomial $x^{n-1+s(\lambda)}$ with nonzero coefficient. After the two derivations and the division it has produced a monomial x^{n-2} with nonzero coefficient. We choose the class of differentiability k sufficiently large so as to be able to continue the classical derivation division algorithm on $W_\lambda(x)$ until we kill all monomials of order of flatness less than $n-2$.

(2) Let us suppose $g_0^{(n)}(0) \neq 0$. Similarly in $V_\lambda(x)$ appears a monomial $x^{ns(\lambda)}$ with nonzero coefficient. After the two derivations and the division it has produced a monomial $x^{(n-1)s(\lambda)-1}$ with nonzero coefficient. We choose the class of differentiability k sufficiently high so as to be able to perform the classical derivation division algorithm on $W_\lambda(x)$ until we kill all monomials of order of flatness less than $(n-1)s(\lambda)-1$.

In the particular case $n = 2$ with the adequate restriction on $r(0)$ the monomial we are exhibiting in $W_\lambda(x)$ is precisely the lower order monomial so $W_\lambda(x)$ divided by this monomial does not vanish. □

Theorem 2.2. *We consider a graphic Γ of a polynomial vector field which is a hemicycle. The pair of opposite singular points are two hyperbolic saddles P_1 and P_2 with rational hyperbolicity ratios satisfying $r_1(\lambda)r_2(\lambda) \equiv 1$. Moreover there is one attracting saddle-node P_3 on the equator and one repelling saddle-node P_4 on the other connection. Both connections near P_3 and P_4 are central (Figure 2). Then, under one of the following conditions:*

(1) *the regular transition from τ_2 to σ_3 has a non-vanishing second order derivative when $r_1(0) > 1$;*
(2) *the regular transition from τ_4 to σ_2 has a non-vanishing second order derivative when $r_1(0) < 1$;*
(3) *the saddle point P_2 has a non-vanishing first saddle quantity when $r(0) = 1$;*

then the graphic Γ has finite cyclicity ≤ 2 inside X_λ.

Proof. The situation is very similar to the study of hemicycles done by El Morsalani in [6], the only difference lying in the presence of the small (resp. large) coefficient $m(\lambda)$ (resp. $M(\lambda)$). We consider the same displacement map (5) as in Theorem 2.1. The difference with Theorem 2.1 is that the Dulac maps $D_{i,\lambda}$ near P_i, $i = 1, 2$ have a more complicated expression. To give it we let $r_1(0) = p/q =$ and $r_2(0) = q/p$ and $\alpha_1(\lambda) = p/q - r_1(\lambda)$. Moreover we introduce the Ecalle-Leontovich-Roussarie compensator

$$\omega(x,\lambda) = \begin{cases} \dfrac{x^{-\alpha_1} - 1}{\alpha_1} & \text{if } \alpha_1 \neq 0 \\ -\ln x & \text{if } \alpha_1 = 0. \end{cases} \tag{7}$$

Then it is shown in [6] (see also [1]) that the maps $D_{1,\lambda}$ and $D_{2,\lambda}^{-1}$ have the form

$$D_{1,\lambda}(x_1) = x_1^{s(\lambda)}\left[1 + \sum_{1\leq j\leq i\leq K(k)} \beta_{ij}(\lambda)x_1^{ip}\omega^j(x_1,\lambda) + \Psi_1(x_1,\lambda)\right], \tag{8}$$

where $\Psi_1(x_1,\lambda)=x_1^k\psi(x_1,\lambda)$ is a C^k-function, k-flat with respect to $x_1=0$ and $\psi(x_1,\lambda)$ satisfies the property I_0^∞ of Mourtada, namely:

$$\forall n\in\mathbb{N}\quad \lim_{x\to 0} x^n\frac{\partial^n\psi}{\partial x^n}(x,\lambda)=0 \tag{9}$$

uniformly for λ in a small neighborhood of the origin in parameter space. Similarly

$$D_{2,\lambda}^{-1}(x_2)=x_2^{s(\lambda)}\left[1+\sum_{1\le j\le i\le K(k)}\gamma_{ij}(\lambda)x_2^{ip}\omega^j(x_2,\lambda)+\Psi_2(x_2,\lambda)\right], \tag{10}$$

with the same compensator ω. Let also $f_\lambda(x_3)=a_1(\lambda)x_3+a_2(\lambda)x_3^2+o(x_3^2)$ with $a_1(0)\neq 0$ and $g_\lambda(x_2)=\epsilon_2+x_2g_{1,\lambda}(x_2)=\epsilon_2+\sum_{i=1}^k b_i(\lambda)x_2^i+o(x_2^k)$, with $g_{1,\lambda}(0)\neq 0$.

All together this yields for $V_\lambda(x)$ an expansion in monomials of the form $x^{is(\lambda)+j}\omega^\ell(x,\lambda)$. Expressions of this type have been studied by El Morsalani in [6] and the general derivation-division algorithm described, once a genericity condition is given. We need only make the connection with his case by identifying the respective coefficients and making sure that the quantities $m(\lambda)$ and $M(\lambda)$ cause no problem.

The first terms of the expansion have the form:

$$V_\lambda(x)=\alpha_{000}(\lambda)+\alpha_{100}(\lambda)x^{s(\lambda)}+\begin{cases} a_{200}x^{2s(\lambda)}+o(x^{2s(\lambda)}) & \text{if } s(0)<1\\ a_{110}x^{s(\lambda)+1}+o(x^{1+s(\lambda)}) & \text{if } s(0)>1\\ a_{111}x^{s(\lambda)+1}\omega(x,\lambda)+O(x^{s(\lambda)+1}) & \text{if } s(0)=1,\end{cases} \tag{11}$$

where

$$\begin{aligned}\alpha_{000}(\lambda)&=M(\lambda)\epsilon-b_0(\lambda)\\ \alpha_{100}(\lambda)&=M(\lambda)(m(\lambda))^{s(\lambda)}-(a_1(\lambda))^{s(\lambda)}b_1(\lambda).\end{aligned} \tag{12}$$

The first part of the algorithm is common to all cases and goes as follows: We start with a first derivation of $V_\lambda(x)$ to kill the constant term. The number of zeros of $V_\lambda(x)$ is at most one plus the number of zeros of $U_\lambda(x)=x^{1-s(\lambda)}V'_\lambda(x)$. A small neighborhood of the origin in parameter space can be divided into three cones:

(i) **the cone** $\dfrac{M(\lambda)m(\lambda)^{s(\lambda)}}{a_1(\lambda)^{p/q}b_1(\lambda)}<1/2$**.** In that region $U_\lambda(x)<0$ for sufficiently small λ yielding at most one root of $V_\lambda(x)$, i.e., at most one limit cycle;

(ii) **the cone** $\dfrac{M(\lambda)m(\lambda)^{s(\lambda)}}{a_1(\lambda)^{p/q}b_1(\lambda)}>2$**.** In that region $U_\lambda(x)>0$ for sufficiently small λ yielding at most one root of $V_\lambda(x)$, i.e., at most one limit cycle;

(iii) **the cone** $1/3<\dfrac{M(\lambda)m(\lambda)^{s(\lambda)}}{a_1(\lambda)^{p/q}b_1(\lambda)}<3$**.** In that case it is clear that $M(\lambda)m(\lambda)^{s(\lambda)}$ just behaves as a regular nonzero quantity. We consider as before $W_\lambda(x)=\left(\dfrac{U_\lambda(x)}{x^{s(\lambda)-1}}\right)'$. In the monomials of the expansion of $W_\lambda(x)$ appear some factors $x^{is(\lambda)}$. We transform them by the rule $x^{s(\lambda)}=x^{p/q}(1+\alpha_1\omega(x,\lambda))$. Then the function $W_\lambda(x)$ has an expansion in monomials of the form $x^{ip/q}\omega^j(x,\lambda)$ with $j\le ip$, which is an expansion exactly of the type studied in [6] (all monomials are well-ordered). We analyse the leading term(s) coming from (11) (in the sequel * always denotes a nonzero constant, possibly depending on λ but bounded away from zero):

$$W_\lambda(x)=\begin{cases}[O(m(\lambda)-*b_2(\lambda)(a_1(\lambda))^{s(\lambda)}]x^{s(\lambda)-1}+o(x^{s(\lambda)-1}) & \text{if } s(0)<1\\ [O(m(\lambda)-*b_1(\lambda)(a_1(\lambda))^{s(\lambda)}a_2(\lambda)]x+o(x) & \text{if } s(0)>1\\ [O(m(\lambda)-*\beta_{11}(\lambda)(a_1(\lambda))^{s(\lambda)}]\omega(x,\lambda)+O(1) & \text{if } s(0)=1.\end{cases} \tag{13}$$

In each case we have that $W_\lambda(x)\neq 0$ for small $x>0$ and λ, yielding that the cyclicity is ≤ 2. □

3. Finite cyclicity of hemicycles among quadratic systems

Theorem 3.1. *The graphics (H_4^3) and (H_5^3) have finite cyclicity (less than or equal to 2) when the hyperbolic saddles at infinity have irrational hyperbolicity ratios.*

Proof. The proof is a direct application of Theorem 2.1 if we show that either the regular transition from τ_2 to σ_3, or the regular transition from τ_4 to σ_2 has a non-vanishing second derivative. In fact we will prove that both transitions simulaneously have non-vanishing second derivative. Let us hence consider a quadratic system having a graphic of type (H_4^3) or (H_5^3). For simplicity in calculation it is better to work in regions where the coordinates are positive. All graphics (possibly modulo some change of sign in y or t) occur inside the family of quadratic systems:

$$\begin{aligned}\dot{x} &= x(1+x+Ay)\\ \dot{y} &= xy+y^2+Bx,\end{aligned}\tag{14}$$

with $A>1$ and $B\in\mathbb{R}$ (see Figure 1) and the center case (H_6^3) corresponds to $B=0$.

The point P_2 is studied in the chart $(v,w)=(x/y,1/y)$ in which the system has the form (after multiplication by w)

$$\begin{aligned}\dot{v} &= (A-1)v+vw-Bv^2w\\ \dot{w} &= -w-vw-Bvw^2.\end{aligned}\tag{15}$$

The point P_3 is studied in the chart $(u,z)=(y/x,1/x)$ in which the system has the form (after multiplication by z)

$$\begin{aligned}\dot{u} &= (1-A)u^2+Bz-uz\\ \dot{z} &= -z-Auz-z^2.\end{aligned}\tag{16}$$

We will show that the second derivatives of the map $R\colon\tau_4\to\sigma_2$ and of the map $S\colon\tau_2\to\sigma_3$ do not vanish for $B(A-2)\neq0$ (the condition $A\neq2$ corresponds to the hyperbolicity ratios of the saddle points being different from 1).

Calculation of the second derivative of the map $R\colon\tau_4\to\sigma_2$:

The sections τ_4 and σ_2 need to be taken parallel to the coordinate axes in the normalizing coordinates. Let us call (X,Y) (resp. (V,W)) the normalizing coordinates around P_4 (resp. P_2). To calculate the second derivative of R we write R as a composition six maps. The derivatives and second derivatives of each of them are calculated using the propositions in the appendix.

(1) the map R_1 from $\{Y=Y_1\}$ to the image of $\{y=y_0\}$ in the (X,Y)-coordinates, where $(X,Y)=(0,Y_1)$ and $(x,y)=(0,y_0)$ represent the same point and where both sections are parametrized by X. Then $R_1'(0)=1$ and $R_1''(0)=-2\dfrac{h_1'(0)}{Y_1^2(1-AY_1)}$, where the section $y=y_0$ has equation $Y=h_1(X)=Y_1+O(X)$;

(2) the map R_2 which is the change of coordinate from X to x on $\{y=y_0\}$. Let $R_2(X)=a_1X+a_2X^2+o(X^2)$, with $a_1>0$;

(3) the map $R_3(x)$ which is the transition from $\{y=y_0\}$ (y_0 small) to $\{y=Y_0\}$ (Y_0 large) in the (x,y)-coordinates;

(4) the map R_4 which is the transition from the coordinate x to the coordinate $v=x/Y_0$ on $\{y=Y_0\}$. Then R_4 is linear $R_4(x)=x/Y_0=xW_0$, where $W_0=1/Y_0$. (Note that the section $\{y=Y_0\}$ becomes the section $\{w=W_0\}$ in the (v,w)-coordinates;

(5) the change of coordinate R_5 from v to V on $\{w=W_0\}$. Let $R_5(v)=b_1v+b_2v^2+o(v^2)$ with $b_1>0$;

(6) the map R_6 from the image of $\{w=W_0\}$ in (V,W)-coordinates to $\{W=W_1\}$, where $(v,w)=(0,W_0)$ and $(V,W)=(0,W_1)$ represent the same point and both sections are parametrized by V in the (V,W)-coordinates. Then $R_6'(0)=1$ and $R_6''(0)=\dfrac{h_2'(0)(A-1)}{W_1}$, where the section $\{w=W_0\}$ has equation $W=h_2(V)=W_1+O(V)$.

Hence

$$R'(0)=\prod_{i=1}^{6}R_i'(0)=\frac{a_1b_1R_3'(0)}{Y_0}>0\tag{17}$$

and

$$R''(0)=R'(0)\left[\frac{-2h_1'(0)}{Y_1^2(1-AY_1)}+2\frac{a_2}{a_1}+a_1\frac{R_3''(0)}{R_3'(0)}+2\frac{a_1b_2}{b_1}W_0R_3'(0)+a_1b_1(A-1)R_3'(0)h_2'(0)\frac{W_0}{W_1}\right].\tag{18}$$

We need calculate the different quantities appearing in (18).

Normal form near P_4:

To bring the system to normal form we first need to diagonalize (14) by means of $(x_1,y_1)=(x,-Bx+y)$, yielding

$$\begin{aligned}\dot{x}_1 &= x_1+(1+AB)x_1^2+Ax_1y_1\\ \dot{y}_1 &= y_1^2+(1-AB+2B)x_1y_1-B^2(A-1)x_1^2.\end{aligned}\tag{19}$$

We then divide the system by $1+(1+AB)x_1+Ay_1$. The normal form coordinates can be taken of the form $(X,Y)=(x,-Bx+y+o(|(x,y)|))$ (see Theorem A.3 of the Appendix). This is sufficient to show that $a_1=1$ and $a_2=0$. For $B\neq0$, $h_1'(0)$ has the sign of $-B$ while for $B=0$ we have $h_1'(0)=-y_0+o(y_0)<0$.

Normal form near P_2:

To bring the system to normal form we first divide the system (15) by $1+v+Bvw$, yielding

$$\begin{aligned}\dot v &= (A-1)v + vw - (A-1)v^2 + o(|(v,w)|^2)\\ \dot w &= -w.\end{aligned} \tag{20}$$

The normalizing change of coordinates has the form $W = w + o(|(v,w)|^3)$ and

$$V = v + v^2 + vw + \frac{v^3}{2} + \left(1 + \frac{AB}{A-2}\right) v^2 w + \frac{vw^2}{2} + o(|(v,w)|^3), \tag{21}$$

yielding $h_2'(0) = 0$ and $\frac{b_2}{b_1} = 1 + \frac{AB}{A-2} W_0 + o(W_0)$. (Remember that $A=2$ corresponds to hyperbolicity ratios of the saddle points being equal to 1, a case not considered here.)

Calculation of $R_3'(0)$ and $R_3''(0)$:

The formulae (A.6) and (A.7) of the Appendix yield

$$R_3'(0) = \exp\left(\int_{y_0}^{Y_0} \frac{1+Ay}{y^2}\,dy\right) = \left(\frac{Y_0}{y_0}\right)^A \exp\left(\frac{1}{y_0} - \frac{1}{Y_0}\right) \tag{22}$$

and

$$\frac{R_3''(0)}{R_3'(0)} = 2y_0^{-A} \exp\frac{A}{y_0} \int_{y_0}^{Y_0} \frac{(1-A)y^2 - (1+AB)y - B}{y^4}\, y^A \exp\left(-\frac{1}{y}\right) dy. \tag{23}$$

The non-vanishing of $R''(0)$ for $B(A-2) \neq 0$:

We now come back to formula (18): in the bracket the second and fifth term vanish. Hence

$$\frac{R''(0)}{R'(0)} = -2\frac{h_1'(0)}{Y_1^2(1-AY_1)} + a_1 \frac{R_3''(0)}{R_3'(0)} + 2a_1 \frac{b_2}{b_1} W_0 R_3'(0). \tag{24}$$

Using that

$$Y_0^{A-1} e^{-1/Y_0} = y_0^{A-1} e^{-1/y_0} + \int_{y_0}^{Y_0} [(A-1)y^{A-2} + y^{A-3}] e^{-1/y} dy \tag{25}$$

we obtain

$$\begin{aligned} y_0^A e^{-1/y_0} \frac{R''(0)}{R'(0)} &= \left(-\frac{2h_1'(0)}{Y_1^2(1-AY_1)} y_0 + 2a_1 \frac{b_2}{b_1}\right) y_0^{A-1} e^{-1/y_0} \\ &\quad + 2a_1 \int_{y_0}^{Y_0} [(A-1)\beta y^{A-2} + (\beta - AB) y^{A-3} - B y^{A-4}] e^{-1/y} dy \end{aligned} \tag{26}$$

where $\beta = \frac{b_2}{b_1} - 1$. The first term is small for y_0 small because of the flatness of e^{-1/y_0} and $Y_1 = y_0 + o(y_0)$. Moreover the integrand is flat at y=0 and the integral diverges to $\text{sgn}(\beta)\infty$ as $Y_0 \to +\infty$. So for small y_0 and large Y_0

$$\text{sgn}(R''(0)) = \text{sgn}(\beta) = \text{sgn}\,\frac{B}{A-2}. \tag{27}$$

The case $B = 0$:

In that case the system has a center and the authors made the complete calculations to verify that, as expected, $R''(0) = 0$. Indeed in that case the first integral

$$H(x,y) = xy^{-A} \exp\left(\frac{x+1}{y}\right) \tag{28}$$

allows to explicitly calculate a normalizing change of coordinates

$$\begin{aligned} Y &= \frac{y}{x+1}\\ X &= x(1+x-Ay)^{-A}\end{aligned} \tag{29}$$

(which does not glue continuously with the one used in the previous case). With the notations introduced before this yields

$$\begin{aligned} a_1 &= (1 - Ay_0)^A \\ a_2 &= A(1 - Ay_0)^{2A-1} \\ h_1'(0) &= -y_0(1 - Ay_0)^A \end{aligned} \tag{30}$$

Similarly the linearizing coordinates near P_2 are given by

$$\begin{aligned} V &= v\exp(v) \\ W &= w\exp\left(\frac{w}{A-1}\right) \end{aligned} \tag{31}$$

yielding $b_1 = b_2 = 1$ and $h_2'(0) = 0$. Moreover using (25) we obtain the explicit expression

$$\frac{R_3''(0)}{R_3'(0)} = -2y_0^{-A}Y_0^{A-1}\exp\left(\frac{1}{y_0} - \frac{1}{Y_0}\right) + \frac{2}{y_0}. \tag{32}$$

Altogether this yields $R''(0) = 0$.

Calculation of the second derivative of the map $S\colon \tau_2 \longrightarrow \sigma_3$:

The calculations are exactly the same as for $R''(0)$. Hence we give less details. The sections τ_2 and σ_2 are taken parallel to the coordinate axes in the normalizing coordinates. Let us call (U, Z) the normalizing coordinates around P_3. To calculate the second derivative of S we write S as a composition six maps whose first derivatives and second derivatives will be calculated as before.

(1) the map S_1 from $\{V = V_0\}$ to the image of $\{v = v_0\}$ in the (V, W)-coordinates where $(V, W) = (V_0, 0)$ and $(v, w) = (v_0, 0)$ represent the same point. As normalizing coordinates (V, W) near P_2 we can choose

$$\begin{aligned} (V, W) = \Bigg(& v + o(|(v, w)|^3), w + \frac{1}{A-1}vw + \frac{1}{A-1}w^2 + \frac{1}{2(A-1)^2}v^2w \\ &+ \left(\frac{1}{(A-1)^2} + \frac{AB}{(A-1)(A-2)}\right)vw^2 + \frac{1}{2(A-1)^2}w^3 + o(|(v, w)|^3)\Bigg), \end{aligned} \tag{33}$$

yielding that $S_1'(0) = 1$ and $S_1''(0) = 0$;

(2) the map S_2 which is the change of coordinate from W to w on $\{v = v_0\}$. If $S_2^{-1}(w) = C_1w + C_2w^2 + o(w^2)$ then $S_2(W) = c_1W + c_2W^2 + o(W^2)$, with $c_1 = 1/C_1$ and $c_2 = -C_2/C_1^3$. Then $c_1 = 1 - \frac{1}{A-1}v_0 + o(v_0)$ and

$$\frac{c_2}{c_1^2} = -\frac{C_2}{C_1} = -\frac{1}{A-1} - \frac{AB}{A-2}v_0 + o(v_0); \tag{34}$$

(3) the map S_3 which is the transition from the coordinate w to the coordinate $z = w/v_0 = U_0z$ on $\{v = v_0\}$. Then S_3 is linear $S_3(w) = w/v_0 = U_0w$, where $U_0 = 1/v_0$. (Note that the section $\{v = v_0\}$ becomes the section $\{u = U_0\}$ in the (u, z)-coordinates;

(4) the map $S_4(x)$ which is the transition from $\{u = U_0\}$ (U_0 large) to $\{u = u_0\}$ (u_0 small) in the (u, z)-coordinates;

(5) the change of coordinate S_5 from z to Z on $\{u = u_0\}$. Let $S_5(z) = d_1z + d_2z^2 + o(z^2)$ with $d_1 > 0$;

(6) the map S_6 from the image of $\{u = u_0\}$ to $\{U = U_1\}$, in the (U, Z)-coordinates, where $(u, z) = (u_0, 0)$ and $(U, Z) = (U_1, 0)$ represent the same point. Then $S_6'(0) = 1$ and $S_6''(0) = \frac{k'(0)}{(A-1)u_0^2(1 - Au_0)}$, where the section $\{u = u_0\}$ has equation $U = k(Z) = U_1 + o(Z)$.

Normal form near P_3:

To bring the system to normal form we first need to diagonalize (16) by means of $(u_1, z_1) = (u + Bz, z)$, yielding

$$\begin{aligned} \dot{u}_1 &= (1 - A)u_1^2 + (AB - 1 - 2B)u_1z_1 + B^2z_1^2 \\ \dot{z}_1 &= -z_1 - Au_1z_1 + (AB - 1)z_1^2. \end{aligned} \tag{35}$$

We then divide the system by $1 + Au_1 - (AB - 1)z_1$. The normal form coordinates hence have the form $(U, Z) = (u + Bz + o(|(u, z)|), z)$. This is sufficient to show that $d_1 = 1$, $d_2 = 0$ and that $k'(0)$ has the sign of B.

Calculation of $S_4'(0)$ and $S_4''(0)$:

The formulae (A.3) and (A.4) of the Appendix yield

$$S_4'(0) = \exp\left(\int_{U_0}^{u_0} \frac{1+Au}{(A-1)u^2}\,du\right) = \left(\frac{u_0}{U_0}\right)^{A/(A-1)} \exp\left(\frac{1}{(A-1)U_0} - \frac{1}{(A-1)u_0}\right) \tag{36}$$

and

$$\frac{S_4''(0)}{S_4'(0)} = 2U_0^{-A/(A-1)} \exp\frac{1}{(A-1)U_0}\int_{U_0}^{u_0} \frac{-u^2+(AB-1)u+B}{(A-1)^2u^4}\,u^{A/(A-1)}\exp\left(-\frac{1}{(A-1)u}\right)du. \tag{37}$$

The non-vanishing of $S''(0)$ for $B \neq 0$:

$$S'(0) = \prod_{i=1}^{6} S_i'(0) = c_1 d_1 S_4'(0) U_0 > 0 \tag{38}$$

and

$$\frac{S''(0)}{S'(0)} = c_1\left(2\frac{c_2}{c_1^2} + U_0\frac{S_4''(0)}{S_4'(0)} + d_1 U_0 S_4'(0) S_6''(0)\right). \tag{39}$$

We use an equivalent of (26) given by

$$\int_{U_0}^{u_0} \frac{u+1}{(A-1)u^3}\,u^{A/(A-1)}\exp\left(-\frac{1}{(A-1)u}\right)du = u^{1/(A-1)}\exp\left(-\frac{1}{(A-1)u}\right)\Big|_{U_0}^{u_0}. \tag{40}$$

As before we work on

$$\begin{aligned}\frac{u_0^{A/(A-1)}e^{-1/[(A-1)u_0]}}{c_1}\frac{S''(0)}{S'(0)} &= u_0^{1/A-1}e^{-1/[(A-1)u_0]}U_0\left(\frac{d_1k'(0)}{(A-1)u_0(1-Au_0)} + 2\frac{c_2}{c_1^2}\right)\\ &\quad + 2U_0\int_{U_0}^{u_0}\left[\gamma u^2 + \left(\gamma + \frac{B}{A-1}\right)u + \frac{B}{A-1}\right]u^{A/(A-1)-4}e^{-1/[(A-1)u]}du, \end{aligned} \tag{41}$$

where $\gamma = -\frac{1}{A-1} - \frac{c_2}{c_1^2} = \frac{ABv_0}{A-2} + o(v_0)$. The first term is small for small u_0 while the second term is large for large U_0 because of the divergence of the integral as $U_0 \to +\infty$. Hence $S''(0)$ has the sign of $-B(A-2)$. □

Theorem 3.2. *The graphics (H_4^3) and (H_5^3) have cyclicity ≤ 2 when the singular points P_1 and P_2 have rational hyperbolicity ratios.*

Proof. We have shown in Theorem 3.1 that both the second derivatives of f and g do not vanish when $B(A-2) \neq 0$. This is sufficient to conclude in the case $r(0) \neq 1$. In the case $r(0) = 1$, i.e., $A = 2$ we calculate the first saddle quantity for system (15). It is equal to $2B \neq 0$. □

Theorem 3.3. *The graphic (H_6^3) has cyclicity ≤ 2 if $r(0) \neq 1$ and ≤ 3 if $r(0) = 1$.*

Proof. The proof is more subtle in this case and we need consider the whole quadratic 5-parameter unfolding (14) where $B = 0$. This family is given by

$$\begin{aligned}\dot{x} &= x(1 + x + Ay) + \delta_1\\ \dot{y} &= xy + y^2 + \delta_2 + \delta_3 x + \delta_4 x^2,\end{aligned} \tag{42}$$

where A is a variable parameter. In the particular case of $r = 1$, i.e., an initial value $A_0 = 2$ we will let $A = 2 + \delta_5$. The center conditions for this family correspond to the existence of three invariant lines (possibly multiple). They are given by

$$\begin{cases} p_1(\delta) = \delta_1 = 0\\ p_2(\delta) = \delta_4 + (A-1)\delta_2 = 0\\ p_3(\delta) = \delta_3 + (A-2)\delta_2 = 0.\end{cases} \tag{43}$$

Moreover the ideal I generated by the three polynomials $p_i(\epsilon)$ is radical. In the particular case $A_0 = 2$ it coincides for $\delta_5 = 0$, i.e., $r = 1$, with the ideal of the first three saddle quantities at the point P_2. (The fact that the ideal is radical is most probably explained by the fact that the system for $\delta = 0$ is not located at the intersection of strata of centers.)

The idea is to perform a derivation-division algorithm together with a Bautin type argument on the displacement map $V_\delta(x)$ from σ_3 to τ_4 defined in (5). This map has the form

$$V_\delta(x) = M(\delta)[D_1(m(\delta)x) + \epsilon(\delta)] - g_\delta \circ D_{2,\delta}^{-1} \circ f_\delta(x). \tag{44}$$

We have to distinguish different cases. In all of them $V_\delta(x)$ vanishes identically under the center conditions.

(1) $r(0) \notin \mathbb{Q}$. Let $s(\delta) = \dfrac{1}{r(\delta)}$. In that case the Dulac maps are simply of the form $y \mapsto y^{s(\delta)}$. Since $f_\delta(0) = 0$ the displacement map has the development

$$V_\delta(x) = a_{00}(\delta) + \sum_{\substack{i+js(\delta)\leq k\\ j>0}} a_{ij}(\delta)x^{i+js(\delta)} + \Psi_\delta(x), \tag{45}$$

where $\Psi_\delta(x) = o(x^k)$.

The whole argument is to show that we can write (45) as

$$V_\delta(x) = \begin{cases} a_{00}(\delta)h_1(x,\delta) + a_{01}(\delta)x^{s(\delta)}h_2(x,\delta) + a_{11}(\delta)x^{1+s(\delta)}h_3(x,\delta) & s(0) > 1 \\ a_{00}(\delta)h_1(x,\delta) + a_{01}(\delta)x^{s(\delta)}h_2(x,\delta) + a_{02}(\delta)x^{2s(\delta)}h_3(x,\delta) & s(0) < 1, \end{cases} \tag{46}$$

with $h_i(0,0) \neq 0$. This is achieved in several steps. We write the rest of the proof in the case $s(0) > 1$, the case $s(0) < 1$ being similar. In the sequel $*$ denotes a nonzero function of δ.

It is clear that $f_\delta(x)$ and $g_\delta(x)$ are linear under the center conditions. Since f_δ and g_δ are smooth their non-affine part can be divided in the ideal I. Hence:

$$W_\delta(x) = V_\delta(x) - a_{00}(\delta) - a_{01}x^{s(\delta)} = p_1(\delta)k_1(x,\delta) + p_2(\delta)k_2(x,\delta) + p_3(\delta)k_3(x,\delta). \tag{47}$$

The rest of the proof consists in three steps:

- show that $p_1(\delta) = \dfrac{*a_{00}(\delta)}{M(\delta)}$: this is done in Lemma 3.4;
- show that $p_2(\delta) = a_{01}(\delta)\gamma(\delta)$ with $\gamma(\delta) = O(\delta)$: this is done in Lemma 3.5;
- show that $a_{11}(\delta) = \xi_1 p_1(\delta) + \xi_2 p_2(\delta) + \xi_3 p_3(\delta)$, with $\xi_3 \neq 0$: this is done in Lemma 3.6.

The discussion is then divided in three cones covering a neighborhood K of the origin. We write it in the case $s(0) > 1$. In the cone $K_1 = \{\delta \in K | a_{00}(\delta) = \max(a_{00}(\delta), a_{01}(\delta), a_{11}(\delta))\}$ we can divide (45) by a_{00} without introducing small denominators. The resulting function does not vanish in K_1 if K is sufficiently small. In $K_2 = \{\delta \in K | a_{01}(\delta) = \max(a_{00}(\delta), a_{01}(\delta), a_{11}(\delta))\}$ we consider $V'_\lambda(x)$ which does not vanish in K_2 if K is sufficiently small. In $K_3 = \{\delta \in K | a_{11}(\delta) = \max(a_{00}(\delta), a_{01}(\delta), a_{11}(\delta))\}$ we consider $\left(\dfrac{V'_\lambda(x)}{x^{s(\delta)-1}}\right)'$ which does not vanish for $x \neq 0$ in K_3 if K is sufficiently small.

(2) $r(0) \in \mathbb{Q}\setminus\{1\}$. We perform a completely similar argument. Under the center conditions the system is Darboux integrable (see for instance [15]) and the first integral yields a first integral near P_2. Hence $D_{1,\delta}(x_1)$ in (8) can be written

$$D_{1,\delta}(x_1) = x_1^{s(\delta)}\left[1 + \sum_{i=1}^{3} p_i(\lambda)h_i(x_1,\lambda)\right] \tag{48}$$

and a similar form for $D_{2,\lambda}^{-1}$. If $s(0) > 1$ (resp. $s(0) < 1$) the first term $x_1^p\omega(x_1,\delta)$ in (8) has order greater than the term with coefficient $a_{11}(\delta)$ (as $p > 1$) (resp. a_{02})). Hence all terms with factors of the form $\omega(x,\delta)$ appear as $o(1)$ in the functions $h_i(x,\delta)$ of expression (46), which is still valid in that case. Substituting (48) in (45) yields an expansion in monomials of the form $x^{i+js(\delta)}\omega^l(x,\delta)$ plus a C^k flat rest, where all monomials of higher order and the rest can be divided in the ideal of the $p_i(\delta)$. We conclude as in the first case.

(3) $r(0) = 1$. The system, localized at P_2 by means of $(v,w) = (x/y, 1/w)$ is given by

$$\begin{aligned} \dot{v} &= v(1+\delta_5) + vz + \delta_1 z^2 - \delta_4 v^3 - \delta_3 v^2 z - \delta_2 v z^2 \\ \dot{w} &= -w - vw - \delta_4 v^2 w - \delta_3 v w^2 - \delta_2 w^3. \end{aligned} \tag{49}$$

The hyperbolicity ratio is precisely one when $\delta_5 = 0$. Under this condition the three first saddle quantities are given by (each L_i is simplified modulo the L_j, $j < i$):

$$\begin{aligned} L_1 &= -\delta_3 \\ L_2 &= -3(\delta_2 + \delta_4) + \delta_1 - 4\delta_1\delta_4 \\ L_3 &= \delta_1(1 - 16\delta_4)(9 - 4\delta_4). \end{aligned} \tag{50}$$

Again we perform a derivation-division on the map $V_\lambda(x)$ given in (45), together with the Bautin argument. Instead of the form (46) we use a form

$$V_\delta(x) = a_0(\delta)h_1(x,\delta) + b(\delta)x\omega(x,\delta) + a_1(\delta)xh_2(x,\delta) + a_2(\delta)x^2\omega(x,\delta)h_3(x,\delta). \tag{51}$$

As before $p_1(\delta) = \frac{*a_0}{M(\delta)}$ and $p_2(\delta) = \gamma(\delta)a_{01}(\delta)$. Also for the same reason as before $a_2(\delta) = \xi_1 p_1(\delta) + \xi_2 p_2(\delta) + \xi_3 p_3(\delta)$ with $\xi_3 \neq 0$. The proof is completely standard and can be done as before in the three cones where each a_i, $i = 0,1,2$, is of maximum absolute value. In the first cone we conclude as before. In the last cone an additional derivation division is necessary to kill the term with coefficient $b(\delta)$ before we can divide by the corresponding $a_i(\delta)$. This yields that the cyclicity is ≤ 3. □

Lemma 3.4. $p_1(\delta) = \dfrac{*a_{00}(\delta)}{M(\delta)}$.

Proof. We need to calculate $a_{00}(\delta) = M(\delta)\epsilon(\delta) - c_0(\delta)$. Clearly both the functions $\epsilon(\delta)$ and $c_0(\delta)$ have to be divisible by δ_1. Here $c_0(\delta)$ represents the translation term in g_δ, so $-c_0(\delta)$ represents the translation term in R_λ. We first calculate the translation term for the transition map $\overline{R}_\lambda$ defined in the usual coordinates. A scaling is then necessary to transform it in the normalizing coordinates. We use Proposition A.2 from the Appendix to show $\dfrac{\partial \overline{R}_\lambda}{\partial \delta_1}(0) > 0$, since all factors in (A.10) are positive. A similar calculation yields $\dfrac{\partial \epsilon}{\partial \delta_1} > 0$. □

Lemma 3.5. $p_2(\delta) = a_{01}(\delta)\gamma(\delta)$ *with* $\gamma(\delta) = O(\delta)$.

Proof. Let us consider

$$a_{01}(\delta) = (m(\delta))^{s(\delta)}M(\delta) - (f_\delta'(0))^{s(\delta)}g_\delta'(0) = (f_\delta'(0))^{s(\delta)}g_\delta'(0)[(m(\delta))^{s(\delta)}M(\delta)((f_\delta^{-1})'(0))^{s(\delta)}(g_\delta^{-1})'(0) - 1]. \tag{52}$$

In Figure 2 let us call $G_\delta\colon \tau_1 \to \sigma_2$ and $F_\delta\colon \tau_2 \to \sigma_1$. The part in brackets in (52) represents the difference $N(\delta) = (F_\delta'(0))^{s(\delta)}G_\delta'(0) - 1$, where the sections σ_1 and τ_2 (resp. τ_1 and σ_2) are chosen symmetric with respect to the x-axis.

For $\delta_2 > 0$, $\delta_4 < 0$ a direct calculation yields

$$\begin{aligned} G_\delta'(0) &= \lim_{Y_0 \to +\infty} \exp\left(\int_{-Y_0}^{Y_0} \frac{1+Ay}{y^2+\delta_2}dy\right) \\ &= \exp\left(\frac{\pi}{\sqrt{\delta_2}}\right). \end{aligned} \tag{53}$$

Similarly

$$F_\delta'(0) = \exp\left(-\pi\sqrt{\frac{1}{-\delta_4(A-1)}}\right). \tag{54}$$

Hence, since $s(\lambda) = A - 1$:

$$N(\delta) = \exp\pi\left(\frac{1}{\sqrt{\delta_2}} - \sqrt{\frac{A-1}{-\delta_4}}\right) - 1 = \pi\left(\frac{1}{\sqrt{\delta_2}} - \sqrt{\frac{A-1}{-\delta_4}}\right)(1+o(1)) = \pi\frac{\sqrt{-\delta_4} - \sqrt{(A-1)\delta_2}}{\sqrt{-\delta_2\delta_4}}(1+o(1)). \tag{55}$$

It easily follows that $p_2(\delta) = N(\delta)\overline{\gamma}(\delta)$ with $\overline{\gamma}(\delta)$ small. As $f_\delta'(0), g_\delta'(0) \neq 0$ it follows that $p_2(\delta) = \gamma(\delta)a_{01}(\delta)$, with γ small. □

Lemma 3.6. $a_{11}(\delta) = \xi_1 p_1(\delta) + \xi_2 p_2(\delta) + \xi_3 p_3(\delta)$, *with* $\xi_3 \neq 0$.

Proof. The proof follows directly from the calculations of Theorem 3.1. There we calculated the second derivatives of f_0 and g_0 for $\delta_1 = \delta_2 = \delta_4$ and got that they were of the form $*\delta_3$, where $*$ could be chosen bounded away from 0 as δ_3 is small. □

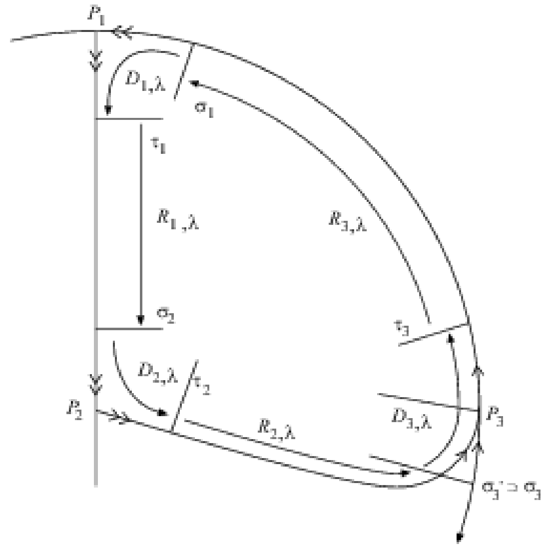

FIG. 3

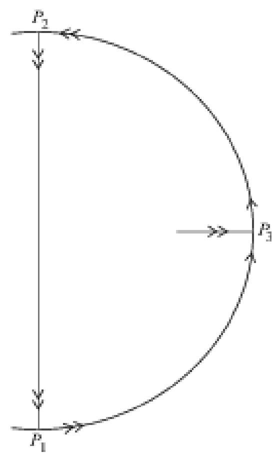

FIG. 4

4. FINITE CYCLICITY OF (I^2_{27})

This graphic was discussed in [1] (Theorem 4.1) and appears in Figures 1 and 3.

When speaking about an "unbroken connection," as seems to occur along the equator in (I^2_{27}) one has to pay attention to what is really meant. Whenever singularities show up in the unfolding of P_3 they have to lie on the equator and the one closest to P_1 will be connected to P_1 since the equator stays invariant. We can say that the connection between P_1 and P_3 remains "unbroken." Whenever the singularities near P_3 disappear this by no means implies that it should be possible to encounter a connection from P_1 to P_2 passing near P_3 on the contrary. Clearly the graphic (I^2_{27}) cannot be approched by graphics with two hyperbolic saddles as in Figure 4. Therefore we necessarily need to consider that the connection between P_1 and P_3 can be broken. The more degenerate (and simpler) case of an unbroken connection has been studied in ([1], Theorem 4.1). The proof for (I^2_{27}) is even simpler as we will show now. We follow the reasoning made in [1].

Theorem 4.1. *The graphic (I^2_{27}) has cyclicity less than or equal to 2.*

Proof. The case $r_1r_2 \neq 1$ was done in [3] and we only need to consider the case $r_1(0)r_2(0) = 1$. The value of $r_1(0)$ will not matter in our proof and we will give different proofs depending on whether $r_2(0) \neq 1$ or $r_2(0) = 1$.

Using the theory of normal forms, we can define three normalized charts around each singularity P_i, $1 \leq i \leq 3$. Near P_3 the vector field $\mathcal{X}_\lambda$ has one of the expressions (3) and the Dulac map is as in the first line of (4). The Dulac maps near P_1 and P_2 have one of the forms (2) or (8). When $r_2(0) = 1$ we will assume that the vector field is in a C^k normal form near P_2 allowing to calculate the Dulac map as in (8). This occurs for λ in some neighbourhood W and for some finite class of differentiability which is fixed by the problem. Let σ_i and τ_i with $1 \leq i \leq 3$ be transverse segments to the vector field $\mathcal{X}_\lambda$ (Figure 3). The essential change with the proof in [1] is to extend the section σ_3 with equation $x = -x_0$ and whose endpoint is chosen to be on the unstable manifold of P_2 to some section σ_3' ending on the equator. We choose a coordinate y on σ_3' so that the equator corresponds to $y = 0$ and the unstable manifold of P_2 to $y = 1$ (this is possible modulo an adequate dilation $y \mapsto B(\lambda)y$ not changing the normal form expression).

For the proof we simply calculate the displacement map. The Dulac maps $D_{i,\lambda}\colon \sigma_i \to \tau_i$ near the hyperbolic saddles P_1 and P_2 have the following expressions:

$$y_i = D_{i,\lambda}(x_i) = x_i^{r_i(\lambda)}(1 + \phi_i(x_i, \lambda)) \tag{56}$$

where x_i and y_i are respectively the parameters on σ_i and τ_i. The functions $\phi_i(x_i, \lambda)$ verify the property I_0^∞ defined in (9). The Dulac maps are invertible and their inverses $d_{i,\lambda}$ have similar expansions:

$$x_i = d_{i,\lambda}(y_i) = y_i^{s_i(\lambda)}(1 + \psi_i(y_i, \lambda)) \tag{57}$$

where $s_i(\lambda) = 1/r_i(\lambda)$ and the function $\psi_i(y_i, \lambda)$ also has the property I_0^∞ defined in (9).

Near the semi-hyperbolic singularity P_3, the Dulac map $D_{3,\lambda}\colon \sigma_3 \to \tau_3$ is linear as in (4):

$$y_3 = D_{3,\lambda}(x_3) = m(\lambda)x_3, \tag{58}$$

with

$$\lim_{\lambda\to 0} m(\lambda) = 0^+. \tag{59}$$

The regular maps $R_{i,\lambda} : \tau_i \to \sigma_{i+1}$, $1 \le i \le 3$ (note that $\sigma_4 = \sigma_1$) are C^k-diffeomorphisms with respective inverses $S_{i,\lambda}$.

A displacement map whose isolated zeros yield the limit cycles close to Γ is defined as the difference of the two transition maps from σ_3 to σ_2.

$$\delta_\lambda(x_3) = R_{1,\lambda} \circ D_{1,\lambda} \circ R_{3,\lambda} \circ D_{3,\lambda}(1 + x_3) - d_{2,\lambda} \circ S_{2,\lambda}(x_3)$$

with x_3 small and $x_3 = 0$ corresponding to the unstable manifold of P_2. The following change of coordinates on σ_3

$$x_3 = R_\lambda(x) = c_1(\lambda)x + o(x), \qquad c_1(\lambda) \neq 0 \tag{60}$$

transforms the regular map $S_{2,\lambda}$ into a genuine translation:

$$S_{2,\lambda}(R_\lambda(x)) = x + b(\lambda) = y \tag{61}$$

with $b(\lambda) = S_{2,\lambda}(0)$. Let us now call $x_3 = x$. Then

$$\overline{\delta}_\lambda(x) = \delta_\lambda(R_\lambda(x)) = b_1(\lambda) + M(\lambda)(*(1 + x) + \phi(x, \lambda)) - y^{s_2(\lambda)}(1 + \psi(y, \lambda)). \tag{62}$$

where $M(\lambda) = m(\lambda)^{r_1(\lambda)}$, $*$ is a nonzero function of the parameter λ, the function $\psi(y, \lambda)$ has the property I_0^∞ and $\phi(x, \lambda)$ is of class C^k.

(1) $r_2(0) \neq 1$. A derivation of δ with respect to x yields:

$$\overline{\delta}'_\lambda(x) = M(\lambda)(* + \phi_1(x, \lambda)) - s_2 y^{s_2-1}(1 + \psi_1(y, \lambda)) \tag{63}$$

where the function $\psi_1(y, \lambda)$ has the property I_0^∞ and $\phi_1(x, \lambda)$ is of class C^{k-1}.

The equation $\overline{\delta}'_\lambda(x) = 0$ is equivalent to

$$N(\lambda)(* + \phi_2(x, \lambda)) = y(1 + \psi_2(y, \lambda)) = Y, \tag{64}$$

where $N(\lambda) = \left(\frac{M(\lambda)}{s_2(\lambda)}\right)^{1/(s_2(\lambda)-1)}$. Using the results of Mourtada in [10], the function

$$Y = y(1 + \psi_2(y, \lambda)). \tag{65}$$

is invertible, with inverse:

$$y = Y(1 + \psi_3(Y, \lambda)), \tag{66}$$

where ψ_3 has property I_0^{k-1}. The equation (64) is equivalent to:

$$y = x + b(\lambda) = N(\lambda)(* + \phi_2(x, \lambda))(1 + \psi_3(Y(x, \lambda), \lambda)). \tag{67}$$

If $s_2(0) > 1$, then $N(\lambda) \to 0$ as $\lambda \to 0$. And (67) has the form

$$x + b(\lambda) - N(\lambda)(* + \phi_3(x, \lambda)) = 0 \tag{68}$$

with ϕ_3 having property I_0^{k-1}. The derivative with respect to x is strictly positive, yielding at most one solution of (68).

If $s_2(0) < 1$, then $N(\lambda) \to +\infty$ for $\lambda \to 0$. Going back to (64), keeping in mind that Y may be taken bounded, and dividing by $N(\lambda)$, we see that (64) has no solution.

(2) $r_2(0) = 1$. In that case we go back to (62) and we let

$$s_2(\lambda) = 1 - \alpha_1(\lambda) \tag{69}$$

with $\alpha_1(0) = 0$. We introduce the compensator $\omega(x, \lambda)$ as in (7). Equation (62) can be rewritten as

$$\begin{aligned} \delta_\lambda(y) &= b_1(\lambda) + M(\lambda)(*(1+x) + \phi(x,\lambda)) \\ &- \alpha_1(\lambda)y\omega(1 + \xi_1(y,\lambda)) - y(1 + \xi_2(y,\lambda) \end{aligned} \tag{70}$$

where we can write $x = y - b(\lambda)$, keeping both y and x close to zero. Knowing that $M(\lambda) \to 0$ for $\lambda \to 0$ this equation clearly has at most two solutions by a classical derivation-division algorithm. □

5. Finite cyclicity of (I^2_{14a}), (I^2_{15a}) and (I_{15b})

Theorem 5.1. *Graphics (I^2_{14a}) and (I^2_{15a}) have finite cyclicity*

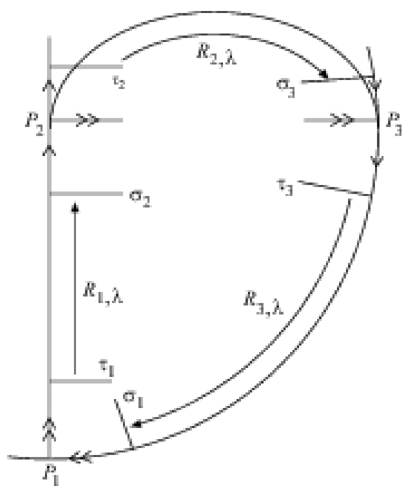

Fig. 5

Proof. The two graphics have the same configuration of singular points as in Figure 5. We choose the variable $x \in \tau_2$ (instead of y_2) so that $\{x = 0\}$ corresponds to the connection for $\lambda = 0$. We choose the normal form coordinates near P_3 so that the invariant line at infinity corresponds to $\{y_3 = 0\}$. Moreover by a λ-dependent scaling in y_3 we take care to have $R_{2,\lambda}(0) \equiv 1$, for sufficiently small λ. The displacement mapping from τ_2 to σ_2 is given by

$$V(x,\lambda) = R_{1,\lambda} \circ D_{1,\lambda} \circ R_{3,\lambda} \circ D_{3,\lambda} \circ R_{2,\lambda}(x) - D^{-1}_{2,\lambda}(x) \tag{71}$$

and its isolated zeros give the limit cycles and graphics. Using adequate normalizing change of coordinates as in [4] it is possible to assume that $R_{1,\lambda}$ is an affine map:

$$x_2 = R_{1,\lambda}(y_1) = b_0(\lambda) + b_1(\lambda)y_1,$$

with $b_0(0) = 0$, $b_1(0) > 0$, and $R_{3,\lambda}$ a linear map:

$$x_1 = R_{3,\lambda}(y_3) = a(\lambda)y_3 \tag{72}$$

with $a(0) > 0$. Moreover we let

$$x_3 = R_{2,\lambda}(x) = 1 + S_\lambda(x) \tag{73}$$

where a good choice of x allows to have $S'_\lambda(0) = 1$ and $S_\lambda(0) = 0$. We also have $y_3 = D_{3,\lambda}(x_3) = m(\lambda)x_3$ and $x_2 = D^{-1}_{2,\lambda}(x) = M(\lambda)x$ where $m(\lambda), M(\lambda) \to 0$ as $\lambda \to 0$. Finally

$$Y_1 = D_{1,\lambda}(x_1) = x_1^{r(\lambda)}(1 + \phi(x_1,\lambda)) \tag{74}$$

where ϕ has the property I^k_0.

Hence

$$V(x,\lambda) = b_0(\lambda) + N(\lambda)(1+S_\lambda(x))^{r(\lambda)}(1+O(m(\lambda))) - M(\lambda)x, \tag{75}$$

with $N(\lambda) = b_1(\lambda)[a(\lambda)m(\lambda)]^{r(\lambda)}$ and where the asymptotic property not only holds for the function but also for all its derivatives in x up to an a priori chosen order. Then

$$V'(x,\lambda) = N(\lambda)r(\lambda)(1+S_\lambda(x))^{r(\lambda)-1}S'_\lambda(x)(1+O(m(\lambda))) - M(\lambda). \tag{76}$$

This function has a finite number of small zeros if we can prove that the function $W(x) = (1+S_\lambda(x))^{r(\lambda)-1}S'_\lambda(x)$ has a nonzero derivative at $x=0$ for $\lambda=0$. Formally this is the case if we do not have

$$R(x) = R_{2,0}(x) = 1 + S(x) = (1+rx)^{1/r} \tag{77}$$

where $r = r(0)$. This will be verified if we prove that $(R(x))^r$ is not an affine map, i.e., has a nonvanishing higher derivative. This is done in Lemma 5.2.

Lemma 5.2. *For graphics (I^2_{14a}), (I^2_{15a}) inside quadratic systems the function $(R(x))^r$ is not an affine map and has a nonvanishing higher derivative.*

Proof. The proof is simple and relies on the results of [2], namely on the choice of adequate normalizing change of coordinates near the two saddle-nodes. Indeed, using the sectorial normalizing theorem it is possible to choose normalizing coordinates (x_i, y_i), $i = 2, 3$ which are analytic in the parabolic sectors and then to include these in a normalizing change of coordinates for the whole family (see results in Appendix). In these coordinates the map $R(x)$ is analytic. It is always possible to choose the section on which $R(x)$ is defined with end point on the invariant line which we suppose to be at $x = x_0$, and the image section having end point on the equator. To show that $(R(x))^r$ is not an affine map and has a nonvanishing higher derivative it suffices to prove it at one point. We prove it at a point x near x_0. Indeed near x_0 the map $R(x)$ is the composition of the regular transitions considered in Theorem 3.1 (we called them R and S and the r used here is the s of Theorem 3.1) with the Dulac map for a point with hyperbolicity ratio r. Since these transitions have a non-vanishing second derivative for $r \neq 1$ and since the saddle is non integrable for $r = 1$, then $(R(x))^r$ is not an affine map in the neighborhood of $x = x_0$. (The r here is the s in Theorem 5.1.) $\square$

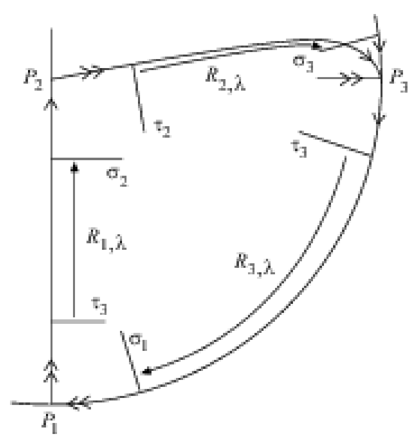

FIG. 6

Theorem 5.3. *The graphic (I^2_{15b}) (Figure 6) has cyclicity ≤ 2.*

Proof. We take sections as in Figure 6. We start exactly as in the proof of Theorem 5.1, the only difference being that the passage near P_2 is now center-unstable. We use the same displacement map $V(x,\lambda)$ from τ_2 to σ_2 given in (71) which we write $V(x,\lambda) = H(x,\lambda) - D_2^{-1}(x,\lambda)$. The map $D_2^{-1}(x,\lambda)$ is a solution of a Pfaff form $\omega(x,\lambda) = F(x,\lambda)dy - ydx$, with $F(x,\lambda)$ of the form $F(x,\lambda) = x^2(1+A(\lambda)x) + \epsilon(\lambda)$. This allows to use the method of fewnomials of Khovanskii: solutions of $V(x,\lambda) = 0$ are solutions of the system

$$\begin{cases} H(x,\lambda) = y \\ F(x,\lambda)dy - ydx. \end{cases} \tag{78}$$

The number of solutions of (78) is bounded by one plus the number of contact points of $\omega(x,\lambda)$ on $y = H(x,\lambda)$. These are given by

$$\begin{cases} y = b_0(\lambda) + N(\lambda)(1+S_\lambda(x))^{r(\lambda)}(1+O(m(\lambda))) \\ y = N(\lambda)r(\lambda)F(x,\lambda)(1+S_\lambda(x))^{r(\lambda)-1}S'_\lambda(x)(1+O(m(\lambda))) \end{cases} \tag{79}$$

and elimination of y yields

$$0 = b_0(\lambda) + N(\lambda)\left[(1+S_\lambda)^{r(\lambda)}(1+O(m(\lambda))) - r(\lambda)F(x,\lambda)S'_\lambda(x)(1+S_\lambda(x))^{r(\lambda)-1} + O(m(\lambda))\right], \tag{80}$$

which has at most one small zero since the first derivative does not vanish for small x and for λ such that $N(\lambda) \neq 0$. □

Appendix

A.1. Derivatives of regular transition maps.

Proposition A.1. *Let*

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \tag{A.1}$$

be a vector field. We consider the transition map $R(x)$ of (A.1) between two arcs without contact: $\Sigma = \{(x,y) = (x, f_1(x))\}$ and $\widetilde{\Sigma} = \{(x,y) = (x, f_2(x))\}$, in a region where $Q(x,y) \neq 0$. Let $x = x(x_0, y_0, y)$ be the solution with initial condition $x(x_0, y_0, y_0) = x_0$. Then

$$\frac{dR}{dx_0}(x_0) = \exp\left(\int_{f_1(x_0)}^{f_2(R(x_0))} \left(\frac{P'_xQ - PQ'_x}{Q^2}\right)\bigg|_{x=x(x_0,f_1(x_0),y)} dy\right) \frac{1-\left(\frac{P}{Q}\right)(x_0, f_1(x_0))f'_1(x_0)}{1-\left(\frac{P}{Q}\right)(x_0, f_2(R(x_0)))f'_2(R(x_0))}. \tag{A.2}$$

Formulas for the first and second derivatives are given in the particular case where $x_0 = 0$ and $P(0,y) \equiv 0$. Let $y_i = f_i(0)$.

$$R'(0) = \exp\left(\int_{y_1}^{y_2} \frac{P'_x}{Q}(0,y)dy\right). \tag{A.3}$$

$$\begin{aligned} R''(0) = R'(0)\Bigg[&2\left(f'_2(0)R'(0)\left(\frac{P_x}{Q}\right)(0,y_2) - f'_1(0)\left(\frac{P_x}{Q}\right)(0,y_1)\right) \\ &+ \int_{y_1}^{y_2}\left(\frac{P''_x}{Q}(0,y) - 2\frac{P'_xQ'_x}{Q^2}(0,y)\right)\exp\left(\int_{y_1}^{y}\frac{P'_x}{Q}(0,z)\,dz\right)dy\Bigg]. \end{aligned} \tag{A.4}$$

Proof. We transform (A.1) into the equivalent differential equation

$$\frac{dx}{dy} = \frac{P}{Q}. \tag{A.5}$$

The solution is $x = x(x_0, f_1(x_0), y)$ with initial condition $x(x_0, f_1(x_0), f_1(x_0)) = x_0$. We have that $R(x_0) = x(x_0, f_1(x_0), f_2(R(x_0)))$. Moreover

$$\frac{\partial}{\partial y}\frac{\partial x}{\partial x_0} = \frac{\partial}{\partial x_0}\frac{\partial x}{\partial y} = \frac{\partial}{\partial x_0}\frac{P(x(x_0,f_1(x_0),y),y)}{Q(x(x_0,f_1(x_0),y),y)} = \frac{P'_xQ - PQ'_x}{Q^2}\,\frac{\partial x}{\partial x_0}, \tag{A.6}$$

from which

$$\frac{\partial x}{\partial x_0} = \exp\left(\int_{f_1(x_0)}^{y}\frac{P'_xQ - PQ'_x}{Q^2}\,dy\right) \tag{A.7}$$

follows. Hence we can rewrite

$$\frac{dR}{dx_0}(x_0) = \exp\left(\int_{f_1(x_0)}^{f_2(R(x_0))} \left(\frac{P'_xQ - PQ'_x}{Q^2}\right)\bigg|_{x=x(x_0,f_1(x_0),y)} dy\right) \frac{1-\left(\frac{P}{Q}\right)(x_0, f_1(x_0))f'_1(x_0)}{1-\left(\frac{P}{Q}\right)(x_0, f_2(R(x_0)))f'_2(R(x_0))}. \tag{A.8}$$

The second derivative of R is most easily calculated from this formula. However the general formula is very long. In the particular case $x_0 = 0$ we get (A.3) and (A.4) for $R'(0)$ and $R''(0)$. □

Proposition A.2. *Let*

$$\mathcal{X}_\delta = P(x,y,\delta)\frac{\partial}{\partial x} + Q(x,y,\delta)\frac{\partial}{\partial y} \tag{A.1}$$

be a vector field depending on a small parameter δ. We consider the transition map $R(x,\delta)$ of (A.1) *between two arcs without contact: $\Sigma = \{(x,y) = (x, f_1(x))\}$ and $\widetilde{\Sigma} = \{(x,y) = (x, f_2(x))\}$, in a region where $Q(x,y,\delta) \neq 0$. Let $x = x(x_0, y_0, y, \delta)$ be the solution with initial condition $x(x_0, y_0, y_0, \delta) = x_0$. Let*

$$F(x_0,y,\delta) = \exp\left(\int_{f_1(x_0)}^{y} \left(\frac{P'_x Q - PQ'_x}{Q^2}\right)\Big|_{x=x(x_0,f_1(x_0),z,\delta)} dz\right). \tag{A.9}$$

Then

$$\frac{\partial R}{\partial \delta} = F(x_0, f_2(R(x_0,\delta)),\delta)\left[1 - \left(\frac{P}{Q}\right)(x_0, f_1(x_0), f_2(R(x_0,\delta)),\delta).f_2'(R(x_0,\delta))\right]^{-1}$$
$$\int_{f_1(x_0)}^{f_2(R(x_0,\delta))} \left(\frac{P'_\delta Q - PQ'_\delta}{Q^2}\right)\Big|_{x=x(x_0,f_1(x_0),y,\delta)} F^{-1}(x_0,y,\delta)\, dy. \tag{A.10}$$

Proof. The proof is very similar to that of the previous proposition. We start by deriving $R(x_0,\delta) = x(x_0, f_1(x_0), f_2(R(x_0,\delta)),\delta)$ with respect to δ and get

$$\frac{\partial R}{\partial \delta}(x_0,\delta)\left[1 - \left(\frac{P}{Q}\right)(x_0, f_1(x_0), f_2(R(x_0,\delta)),\delta).f_2'(R(x_0,\delta))\right] = \frac{\partial x}{\partial \delta}(x_0, f_1(x_0), f_2(R(x_0,\delta)),\delta). \tag{A.11}$$

We let $x = x(x_0, f_1(x_0), y, \delta)$ with initial condition $x(x_0, f_1(x_0), f_1(x_0), \delta) = x_0$ be the solution of (A.5) (which now depends on δ). Then

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial x}{\partial \delta}\right) &= \frac{\partial}{\partial \delta}\frac{P(x(x_0,f_1(x_0),y,\delta),y,\delta)}{Q(x(x_0,f_1(x_0),y,\delta),y,\delta)}\\ &= \left(\frac{P}{Q}\right)_x \frac{\partial x}{\partial \delta} + \left(\frac{P}{Q}\right)_\delta.\end{aligned} \tag{A.12}$$

The result follows by integration and evaluation at $y = f_2(R(x_0,\delta))$. □

A.2. Normalizing coordinates near a saddle-node.

We recall the results of [2].

An analytic planar saddle-node germ v of multiplicity 2 is formally orbitally equivalent by means of a transformation $(x,y) \mapsto (z,w)$ to a polynomial normal form

$$v_0 = z^2(1+az)^{-1}\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}; \tag{A.13}$$

the time orientation may be reversed. An unfolding of a germ v, depending on the multi-parameter λ is finitely smoothly orbitally equivalent to the local family

$$v_\lambda = \epsilon(\lambda)\frac{\partial}{\partial z} + z^2(1+a(\lambda)z)^{-1}\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}, \tag{A.14}$$

where $\epsilon(0) = 0$, [8].

Any complex saddle-node of multiplicity 2 is orbitally analytically equivalent to a germ

$$v = v_0 + z^2 R(z,w)\frac{\partial}{\partial w}. \tag{A.15}$$

To state the theorems of [2] we suppose that v has the form (A.15)

Theorem A.3. *Consider a real analytic germ of a saddle-node vector field on $(\mathbb{R}^2, 0)$ with one zero and one negative eigenvalue and with multiplicity 2. Then it is C^∞ orbitally equivalent to its normal form* (A.13) *by means of a change of coordinate $(Z,W) = (z,w) + o(|(z,w)|)$. The equivalence may be taken analytic outside the stable manifold.*

Theorem A.4. *For any C^∞ unfolding of a germ from Theorem A.3 there exists a finitely smooth orbital equivalence with the polynomial normal form* (A.14)*. For the critical parameter value this equivalence is analytic outside the stable manifold of the saddle-node germ.*

Acknowledgements

The authors are grateful to Mohamed El Morsalani, Yulij Ilyashenko, Robert Roussarie and Sergey Yakovenko for helpful discussions.

References

1. F. Dumortier, M. El Morsalani and C. Rousseau, *Hilbert's 16th problem for quadratic systems and cyclicity of elementary graphics*, Nonlinearity **9** (1996), 1209–1261.
2. F. Dumortier, Y. Ilyashenko and C. Rousseau, *Normal forms near a saddle-node and applications to finite cyclicity of graphics*, preprint, Centre de Recherches Mathématiques (2000).
3. F. Dumortier, R. Roussarie and C. Rousseau, *Hilbert's 16th problem for quadratic vector fields*, J. Differential Equations **110** (1994), 86–133.
4. A. Guzmán and C. Rousseau, *Genericity conditions for finite cyclicity of elementary graphics*, J. Differential Equations **155** (1999), 44–72.
5. D. Hilbert, *Mathematische Probleme (lecture), The second International Congress of Mathematicians Paris 1900*, Nachr. Ges. Wiss. Gottingen Math.-Phys. Kl.1900, 253–297; *Mathematical developments arising from Hilbert's problems Proceedings of Symposium in Pure Mathematics*, AMS, F. Browder editor **28** (1976), 50–51.
6. M. El Morsalani, *Bifurcations de polycycles infinis pour les champs de vecteurs polynomiaux*, Annales de la Faculté des Sciences de Toulouse **3** (1994), 387–410.
7. Yu Ilyashenko, *Nonlinear Stokes phenomena*, Advances in Soviet Mathematics, volume 14, American Mathematical Society, 1993, 1–55.
8. Yu Ilyashenko and S. Yakovenko, *Finitely-smooth normal forms of local families of diffeomorphisms and vector fields*, Russian Math. Surveys **46** (1991), 1–43.
9. Khovanskii, *Real analytic varieties with the property of finiteness and complex Abelian integrals*, Func. Anal. Appl. **18** (1984), 40–50.
10. A. Mourtada, *Cyclicité finie des polycycles hyperboliques des champs de vecteurs du plan: mise sous forme normale*, Springer Lecture Notes in Mathematics **1455** (1990), 272–314.
11. A. Mourtada, *Projections de sous-ensembles quasi réguliers d'Hilbert II*, preprint, Université de Bourgogne # 207 (1999).
12. R. Roussarie, *A note on finite cyclicity and Hilbert's 16th problem*, Springer Lecture Notes in Mathematics **1331** (1988), 161–168.
13. H. Zhu and C. Rousseau, *Finite cyclicity of graphics through a nilpotent singularity of elliptic or saddle type*, to appear in J. Differential Equations.
14. H. Zhu and C. Rousseau, *Graphics with a nilpotent singularity in quadratic systems and Hilbert's 16th problem*, in preparation (2000).
15. D. Schlomiuk, *Algebraic particular integrals, integrability and the problem of the center*, Trans. Amer. Math. Soc. **338** (1993), 799–841.