Approximation on closed sets by analytic or meromorphic solutions of elliptic equations and applications

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Abstract
Given a homogeneous elliptic partial differential operator $L$ with constant complex coefficients and a class of functions (jet-distributions) which are defined on a (relatively) closed subset of a domain $\Omega$ in $\mathbb{R}^n$ and which belong locally to a Banach space $V$, we consider the problem of approximating in the norm of $V$ the functions in this class by “analytic” and “meromorphic” solutions of the equation $Lu = 0$. We establish new Roth, Arakelyan (including tangential) and Carleman type theorems for a large class of Banach spaces $V$ and operators $L$. Important applications to boundary value problems of solutions of homogeneous elliptic partial differential equations are obtained, including the solution of a generalized Dirichlet problem.

Key words: Approximation on closed sets, elliptic operator, strongly elliptic operator, $L$-meromorphic and $L$-analytic functions, localization operator, Banach space of distributions, Dirichlet problem.

Résumé
Étant donné $L$ un opérateur aux dérivées partielles elliptique, homogène et à coefficients complexes constants, et une classe de fonctions (jet-distributions) définies sur un sous-ensemble (relativement) fermé d’un domaine $\Omega$ de $\mathbb{R}^n$ et appartenant localement à un espace de Banach $V$, nous considérons le problème d’approcher dans la norme de $V$ les fonctions de cette classe par des solutions “analytiques” ou “méro-"meromorphes” de l’équation $Lu = 0$. Nous obtenons de nouveaux théorèmes d’approximation de type Roth, Arakelyan (y compris tangentiel) et Carleman pour une grande classe d’espaces de Banach $V$ et opérateurs $L$. D’importantes applications aux problèmes du comportement à la frontière des solutions d’équations aux dérivées partielles elliptiques homogènes sont obtenues, y compris la solution d’un problème de Dirichlet généralisé.
1 Introduction.

Let \( L \) be a homogeneous elliptic partial differential operator with constant complex coefficients (such as powers of the Cauchy-Riemann operator \( \overline{\partial} \) or the Laplacean \( \Delta \)). In \([2]\), given a Banach space \((V, \| \|)\) of functions (distributions) on \( \mathbb{R}^n, n \geq 2\), we studied the problem of approximating, on a closed subset \( F \) of \( \mathbb{R}^n \), the solutions of the equation \( Lu = 0 \) by global (\( L \)-analytic or \( L \)-meromorphic) solutions of the equation. Approximation theorems of Runge-type and Arakelyan-type were obtained whenever the operator \( L \) and the Banach space \( V \) satisfied certain conditions.

In this paper, we first generalize the results of \([2]\) and \([11]\) to Banach spaces of functions (distributions) defined on any domain \( \Omega \) of \( \mathbb{R}^n (n \geq 2) \). As already mentioned in \([2]\), the only purpose of one of the important conditions on \( L \) and \( V \) (\([2]\), Condition (4))] was to obtain a “special maximum principle” ([2, Lemma 1]). Weakened assumptions of this Lemma have now become our new Condition 4 (see Section 2 below), and consequently our proof has been slightly modified (and improved).

For all operators \( L \) under consideration, our conditions are satisfied by a large class of classical (non-weighted) spaces. Using results on the solution of the Dirichlet problem for strongly elliptic equations in bounded smooth domains, we find (see Proposition 2 below) that in this case our conditions are also satisfied by a wide class of spaces, for which an application of our theorems gives important new examples in the theory of tangential approximation (see Theorem 4 (iii)).

Using Carleman-type approximation results (see Lemma 4 and Proposition 5), we obtain in Section 6 some very interesting examples of the possible boundary behaviour of solutions of homogeneous elliptic partial differential equations, analogous to those described in [5, Chapter IV, §5B] for functions holomorphic in a disc. First, given a domain \( \Omega \) satisfying some mild conditions, we construct an \( L \)-analytic function \( f \) in \( \Omega \) such that the limit of \( f \) and of all its derivatives along any path ending at the boundary of \( \Omega \) does not exist (Theorem 5). To our knowledge, only very special cases of this result were known for the \( \overline{\partial} \) equation in \( \mathbb{R}^2 \) and the Laplacean in \( \mathbb{R}^n, n \geq 2 \) (see [5, Chapter IV, §5], [6, §8]).

When the boundary of \( \Omega \) is sufficiently smooth, we are also able to solve (see Theorem 6) a “weakened” Dirichlet problem where the boundary values of an \( L \)-analytic function, together with the boundary values of a fixed number of its derivatives are prescribed (almost everywhere on \( \partial \Omega \)) as we approach the boundary in the normal direction.

2 Definitions and notations.

For the reader’s convenience, we summarize the definitions and main notations of \([2]\). Note that in \([2]\), these were given only for \( \mathbb{R}^n \), but here we extend them very naturally to general domains.

Let \( \Omega \) be any fixed domain in \( \mathbb{R}^n, n \geq 2 \). We let \( V = V(\Omega) \) stand for a Banach space, whose norm is denoted by \( \| \| \), which contains \( C_0^\infty(\Omega) \), the set of test functions in \( \Omega \) and is contained in \( (C_0^\infty(\Omega))^* \), the space of distributions on \( \Omega \). We make some additional assumptions on \( V \).

**Conditions 1 and 2:** We assume that \( V \) is a topological \( C_0^\infty(\Omega) \)-submodule of \( (C_0^\infty(\Omega))^* \), which means that for \( f \in V \) and \( \varphi \in C_0^\infty(\Omega) \), we have \( \varphi f \in V \) with

\[
\| \varphi f \| \leq C(\varphi)\| f \| \tag{1}
\]

and

\[
| < f, \varphi > | \leq C(\varphi)\| f \|. \tag{2}
\]
where \( < f, \varphi > \) denotes the action in \( \Omega \) of the distribution \( f \) on the test function \( \varphi \) and \( C(\varphi) \) is a constant independent of \( f \). We note that this implies that the imbeddings \( C^\infty_0(\Omega) \hookrightarrow V \) and \( V \hookrightarrow (C^\infty_0(\Omega))^* \) are continuous (see [2, section 2.1]).

Given a closed subset \( F \in \Omega \), let \( I(F) \) be the closure in \( V \) of (the family of) those \( f \in V \) whose support in \( \Omega \) in the sense of distributions (which will be denoted by \( \text{supp}(f) \)) is disjoint from \( F \), and let \( V(F) = V/I(F) \). The Banach space \( V(F) \), endowed with the quotient norm, should be viewed as the natural (Whitney type) version of \( V \) on \( F \) (see [14, Chapter 6]). We will write \( \| f \|_F \) for the norm of the equivalence class (jet) \( f(F) := f + I(F) \) in \( V(F) \) of the distribution \( f \in V \).

For any open set \( D \in \Omega \), let

\[
V_{\text{loc}}(D) = \{ f \in (C^\infty_0(D))^* \mid f \varphi \in V \text{ for each } \varphi \in C^\infty_0(D) \},
\]

where \( \varphi \) and \( f \varphi \) are extended to be identically zero in \( \Omega \setminus D \). We endow \( V_{\text{loc}}(D) \) with the projective limit topology of the spaces \( V(K) \) partially ordered by inclusion of the compact sets \( K \subset D \). For a closed set \( F \in \Omega \), define \( V_{\text{loc}}(F) = V_{\text{loc}}(\Omega)/J(F) \), where \( J(F) \) is the closure in \( V_{\text{loc}}(\Omega) \) of the family of those distributions in \( V_{\text{loc}}(\Omega) \) whose support is disjoint from \( F \). The topology on \( V_{\text{loc}}(F) \) will be the quotient topology. Note that for compact sets \( K \), the topological spaces \( V(K) \) and \( V_{\text{loc}}(K) \) are identical.

For \( f \in V_{\text{loc}}(\Omega) \), we put \( f_{(F),\text{loc}} := f + J(F) \). If \( D \) is a neighbourhood of \( F \) in \( \Omega \), then each \( h \in V_{\text{loc}}(D) \) naturally defines an element (jet) \( h_{(F),\text{loc}} \) in \( V_{\text{loc}}(F) \) by taking \( h_{(F),\text{loc}} \) to be the closure in \( V_{\text{loc}}(\Omega) \) of the set of \( f \in V_{\text{loc}}(\Omega) \) such that \( f = h \) (as distributions) in some neighbourhood (depending on \( f \)) of \( F \). In particular, this works for each \( h \in C^\infty(D) \subset V_{\text{loc}}(D) \). For \( f_{(F),\text{loc}} \in V_{\text{loc}}(F) \), we will write \( f_{(F),\text{loc}} \in V(F) \) (or more briefly \( f \in V(F) \)), if \( V \cap f_{(F),\text{loc}} \neq \emptyset \). We will then write \( \| f_{(F),\text{loc}} \|_F \), to mean \( \| g \|_F \), where \( g \in V \cap f_{(F),\text{loc}} \). Practically the same proof as in [2, section 2.1] shows that \( V \cap J(F) = I(F) \) holds for each closed set \( F \) in \( \Omega \), which means that \( \| f_{(F),\text{loc}} \|_F \) is well-defined.

For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \), with \( \alpha_j \in \mathbb{Z}_+(:= \{ 0, 1, 2, \ldots \}) \), we let \( |\alpha| = \alpha_1 + \cdots + \alpha_n \), \( \alpha! = \alpha_1! \cdots \alpha_n! \), \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( \partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n} \).

We denote by \( B(a, \delta) \) (respectively \( \overline{B}(a, \delta) \)) the open (respectively closed) ball with center \( a \in \mathbb{R}^n \) and radius \( \delta > 0 \). If \( B = B(a, \delta) \) and \( \theta > 0 \), then \( \theta B = B(a, \theta \delta) \) and \( \overline{\theta B} = \overline{B}(a, \theta \delta) \).

Throughout this paper we let \( L(\xi) = \sum_{|\alpha|=r} a_\alpha \xi^\alpha \), \( \xi \in \mathbb{R}^n \), be a fixed homogeneous polynomial of degree \( r \) (\( r \geq 1 \)) with complex constant coefficients and which satisfies the ellipticity condition \( L(\xi) \neq 0 \), \( \xi \neq 0 \). We associate to \( L \) the homogeneous elliptic operator of order \( r \)

\[
L = L(\partial) = \sum_{|\alpha|=r} a_\alpha \partial^\alpha.
\]

Let \( D \) be an open set in \( \mathbb{R}^n \) and denote by \( L(D) \) the set of distributions \( f \) in \( D \) such that \( Lf = 0 \) in \( D \) in the sense of distributions. It is well known [7, Theorem 4.4.1] that \( L(D) \hookrightarrow C^\infty(D) \). Therefore if \( D \subset \Omega \), then \( L(D) \subset V_{\text{loc}}(D) \), and if \( \{ f_m \} \) is a sequence in \( L(D) \) with \( f_m \to f \) in \( V_{\text{loc}}(D) \) as \( m \to \infty \), then \( f \in L(D) \), since convergence in \( V_{\text{loc}}(D) \) is stronger than convergence in the sense of distributions, which preserves \( L(D) \) [7, Theorem 4.4.2].

Functions from \( L(D) \) will be called \( L \)-analytic in \( D \). We shall also say that a distribution \( g \) in \( D \) is \( L \)-meromorphic in \( D \) if \( \text{supp}(Lg) \) is discrete in \( D \) and for each \( a \in \text{supp}(Lg) \) (\( a \in D \)) there exist \( h \), which is \( L \)-analytic in a neighbourhood of \( a \), \( k \in \mathbb{Z}_+ \) and \( \lambda_\alpha \in \mathbb{C} \), \( 0 \leq |\alpha| \leq k \), such that

\[
g(x) = h(x) + \sum_{|\alpha|\leq k} \lambda_\alpha \partial^\alpha \Phi(x - a)
\]
in some neighbourhood of \( a \), where \( \Phi \) is a special fundamental solution of \( L \) as described in [7, Theorem 7.1.20]. The points \( a \in \text{supp}(Lg) \) will be called the poles of \( g \).

We recall (see [3, p. 239] or [15, p. 163]) that there exists a \( k > 1 \) such that if \( T \) is a distribution with compact support contained in \( B(a, \delta) \) and \( f = \Phi \ast T \), then, for \( |x - a| > k\delta \), we have the Laurent-type expansion:

\[
f(x) = <T(y), \Phi(x - y)> = \sum_{|\alpha| \geq 0} c_{\alpha} \partial^\alpha \Phi(x - a),\]

where \( c_{\alpha} = (-1)^{\alpha}(\alpha!)^{-1} <T(y), (y - a)^{\alpha}> \). The series converges in \( C^\infty(\{ |x - a| > k\delta \}) \), which means that the series can be differentiated term by term and all such series converge uniformly on \( \{ |x - a| \geq k'\delta \} \), \( k' > k \).

Let \( \varphi \in C_0^\infty(\Omega) \). The Vitushkin localisation operator \( V_\varphi : (C_0^\infty(\Omega))^* \to (C_0^\infty(\Omega))^* \) associated to \( L \) and \( \varphi \) is defined as \( V_\varphi f = (\Phi \ast (\varphi Lf))|_\Omega \), where in the last equality \( \ast \) denotes the convolution operator in \( \mathbb{R}^n \).

**Condition 3:** We require that for each \( \varphi \in C_0^\infty(\Omega) \), the operator \( V_\varphi \) be invariant on \( V_{\text{loc}}(\Omega) \), i.e. \( V_\varphi \) must send continuously \( V_{\text{loc}}(\Omega) \) into \( V_{\text{loc}}(\Omega) \). This means that if \( K \) is a compact subset of \( \Omega \) and \( \text{supp}(\varphi) \subset K \), then for each \( f \in V_{\text{loc}}(\Omega) \) one has \( V_\varphi f \in V_{\text{loc}}(\Omega) \) and

\[
\|V_\varphi f\|_K \leq C\|f\|_K, \tag{4}
\]

where \( C \) is independent of \( f \).

We make one more assumption on \( V \) in relation with \( L \).

**Condition 4:** For each open ball \( B \) with \( 3\overline{B} \subset \Omega \), there exist \( d > 0 \) and \( C > 0 \) such that for each \( h \in C^\infty(\mathbb{R}^n) \) satisfying \( LH = 0 \) outside of \( B \) and \( h(x) = O(|x|^{-d}) \) as \( |x| \to \infty \), one can find \( v \in L(\Omega) \) with

\[
(h - v) \in V \quad \text{and} \quad \|h - v\| \leq C\|h\|_{3\overline{B}}. \tag{5}
\]

In this assumption, instead of the constant 3, one can take any fixed real number greater than 1.

## 3 Some remarks on Conditions 1 to 4

All Conditions 1 to 4 are satisfied by classical (non-weighted) spaces on any domain \( \Omega \) in \( \mathbb{R}^n \), for example \( BC^m(\Omega) \), \( BC^{m+\mu}(\Omega) \), \( VMO(\Omega) \) and the Sobolev spaces \( W^p_0(\Omega) \), \( 1 \leq p < \infty \). We shall give the definitions and prove this assertion only for the spaces \( V = BC^m(\Omega) \) and \( BC^{m+\mu}(\Omega) \).

For \( m \in \mathbb{Z}_+ \), let \( BC^m(\Omega) \) be the space of all \( m \)-times continuously differentiable functions \( f : \Omega \to \mathbb{C} \) with (finite) norm

\[
\|f\|_{m,\Omega} = \max_{|\alpha| \leq m} \sup_{x \in \Omega} |\partial^\alpha f(x)|.
\]

If \( m \in \mathbb{Z}_+ \) and \( 0 < \mu < 1 \), then

\[
BC^{m+\mu}(\Omega) = \{ f \in BC^m(\Omega) \mid \omega^m_\mu(f, \infty) < \infty \text{ and } \omega^m_\mu(f, \delta) \to 0 \text{ as } \delta \to 0 \},
\]
where \( \omega_m^\rho(f, \delta) = \sup_{x \neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^\rho} \), the supremum being taken over all multi-index \( \alpha \) such that \(|\alpha| = m \) and all \( x, y \in \Omega \) with \( 0 < |x - y| < \delta \). The norm in this space is defined as

\[
\|f\|_{m+\rho, \Omega} = \max\{\|f\|_{m, \Omega}, \omega_m^\rho(f, \infty)\}.
\]

We shall omit the index \( \Omega \) in the last norm whenever \( \Omega = \mathbb{R}^n \). Finally, for any \( \rho \geq 0 \), we set \( C^\rho(\Omega) = (BC^\rho(\Omega))_{loc} \).

**Proposition 1** Let \( \Omega \) be a domain in \( \mathbb{R}^n \), \( n \geq 2 \), and let \( \rho \geq 0 \). Then the pair \((L, V(\Omega))\) with \( V(\Omega) = BC^\rho(\Omega) \) satisfies Conditions 1, 2, 3 and satisfies Condition 4 with \( v = 0 \).

**Proof.** Conditions 1 and 2 are easily verified. Condition 3 is proved in [10, Corollary 5.6] in the case \( \Omega = \mathbb{R}^n \) for all spaces mentioned above, since \( C_0^\infty(\mathbb{R}^n) \) is locally dense in each of them. As Condition 3 is local, it holds for each pair \((L, V(\Omega))\) under consideration.

To obtain Condition 4 with \( v = 0 \), we can easily use [2, Lemma 1] (see also [11, Lemma 2]). In fact, by this Lemma, for each open ball \( B \) with \( 3B \subset \Omega \), we even can find \( d > 0 \) and \( C > 0 \) such that if \( h \) satisfies the hypotheses of Condition 4 with this \( d \), then

\[
\|h\|_\rho \leq C\|h\|_{\rho, 3B}.
\]

Since \( \|h\|_{\rho, \Omega} \leq \|h\|_\rho \), the proof is complete.

\[ \blacksquare \]

In [2, Corollary 1] (see also the brief discussion thereafter) and [11, Theorem 4] one sees how (whenever Conditions 1 to 3 are satisfied) Condition 4 can affect L-meromorphic and L-analytic approximation in the special case of weighted uniform holomorphic approximation \((n = 2, L = \overline{\partial})\).

We also wish to present here an example of a pair \((L, V)\) satisfying Conditions 1, 2 and 4 (with \( v = 0 \), but not 3. Hence, this example eludes our method. The example seems new even without considering Condition 4.

Take \( L = \overline{\partial}, \Omega = \mathbb{R}^2(= \mathbb{C}), B_1 = \{z \in \mathbb{C} \mid |z| < 1\} \) (the unit disk), and let

\[
V = BC^0(R^2) \cap BC^1(B_1) \quad \text{with norm } \|f\| = \max\{\|f\|_0, \|f\|_{1, B_1}\}.
\]

Conditions 1 and 2 are easily verified. Condition 4 (with \( v = 0, d = 1 \)) follows from the maximum principle and from trivial estimates of derivatives (outside \( 2B \)) of a function, holomorphic outside \( \overline{B} \) and vanishing at \( \infty \). Finally, fixing any \( \varphi \in C_0^\infty(3B_1) \) such that \( \varphi(z) = \overline{z} \) on \( 2B_1 \), one can check that there exists \( f \in BC^0(R^2), f = 0 \) in \( B_1 \), with \( \mathcal{V}_\varphi f|_{B_1} \) not in \( BC^1(B_1) \). In fact, in this case

\[
\mathcal{V}_\varphi f(w) = f(w)\varphi(w) - \frac{1}{\pi} \int \frac{f(z)\overline{\partial}\varphi(z)}{w - z} \; dx_1 dx_2 , \quad z = x_1 + ix_2 ,
\]

so that one needs only to study the behavior (in \( B_1 \)) of the function

\[
\int_{2B_1 \setminus B_1} \frac{f(z)}{(w - z)^2} \; dx_1 dx_2 .
\]

Easily, there is \( g \in C(R^2), g \geq 0, \text{supp}(g) \subset \{x_1 \geq 2|x_2|\} \cap B_1 \), such that

\[
\int \frac{g(z)}{|z|^2} \; dx_1 dx_2 = +\infty .
\]
It is enough to take \( f(z) = g(z - 1) \) and let \( w \in (0, 1) \) tend to 1. Indeed, set \( 1 - w = \delta \). Then, it is enough to show that

\[
\int \frac{g(z)}{(z + \delta)^2} dx_1 dx_2
\]

is unbounded as \( \delta \) tends to zero. In fact,

\[
\text{Re}\left(\frac{1}{(z + \delta)^2}\right) \geq \frac{1}{2|z + \delta|^2}
\]

on \( \text{supp}(g) \), and if the integrals

\[
\int \frac{g(z)}{|z + \delta|^2} dx_1 dx_2
\]

were uniformly bounded for \( \delta \in (0, 1) \), then by Fatou’s lemma, the integral with \( \delta = 0 \) would be convergent, which is not the case.

The following proposition provides us with another class of examples for which Condition 1 to 4 are satisfied. These in turn will allow us to obtain in Section 4 new results on “tangential” approximation. Given \( m \) and \( q \) in \( Z_+ \), with \( q \leq m \), and a bounded domain \( \Omega \), set

\[
BC^m_q(\Omega) = \{ f \in BC^m(\Omega) \mid \text{for each } \alpha, |\alpha| \leq q, \lim_{x \to \partial \Omega} \partial^\alpha f(x) = 0 \},
\]

which is a Banach space with the norm \( \|f\|_{m, \Omega} \).

**Proposition 2** Let \( L \) be a strongly elliptic operator of order \( r = 2\ell \), \( \ell \in \mathbb{Z}_+ \), \( \ell \geq 1 \) (see [1, p. 46]). Let \( m, q \in \mathbb{Z}_+ \), \( m \geq \ell - 1 \), \( q \leq \ell - 1 \). If \( \Omega \) is bounded and \( \partial \Omega \) is of class \( C^s \), \( s = \max\{2\ell, [n/2] + 1 + m\} \) (see [1, p. 128]), then the pair \( (L, V = BC^m_q(\Omega)) \) satisfies Conditions 1 to 4.

**Proof.** Since \( (BC^m_q(\Omega))_{\text{loc}} = C^m(\Omega) \), Conditions 1, 2 and 3 are satisfied. Let us prove Condition 4. Fix any ball \( B, 3\overline{B} \subset \Omega \), and take any \( h \in C^\infty(\mathbb{R}^n) \) with \( Lh = 0 \) outside \( B \). Now, results on solvability and regularity of the classical Dirichlet problem applied to the operator \( L \) (see [1, Theorem 8.2 and Lemma 7.7, Theorem 9.8 and Lemma 9.1, Theorem 3.9]) show that under the hypotheses of Proposition 2, there exists \( v_0 \in C^m(\overline{\Omega}) \cap L(\Omega) \) such that \( u_0 = h - v_0 \) satisfies \( \partial^\alpha u_0|_{\partial \Omega} = 0 \) for each \( \alpha, |\alpha| \leq \ell - 1 \) (so that \( h - v_0 \in V \)), and moreover

\[
\|u_0\| \equiv \|u_0\|_{m, \Omega} \leq C_1\|h\|_{s, \Omega},
\]

where \( C_1 \) is independent of \( h \). We observe that we have not used here the property \( Lh = 0 \) in \( \mathbb{R}^n \setminus B \). We also remark that our notations for \( m \) and \( \| \cdot \|_{m, \Omega} \) are different from those of [1], and that the last inequality follows from [1, (9.23)] since

\[
\|u_0\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} \leq C_2\|h\|_{W^2(\Omega)},
\]

by [1, Theorems 8.1 and 8.2].

By [11, Lemmas 1 and 3], we can choose \( d > 0 \) and \( C_3 > 0 \) (independently of \( h \)) such that if additionally \( h(x) = O(|x|^{-d}) \) as \( |x| \to \infty \), then (see also [2, Lemma 1])

\[
h = \Phi \ast Lh, \quad \text{and} \quad \|h\|_{m, \Omega} \leq \|h\|_{m, \mathbb{R}^n} \leq C_3\|h\|_{m, 3B}.
\]

Fix \( \chi \in C^\infty_0(\frac{3}{2}B) \), \( \chi = 1 \) on \( B \). Then for \( x \in \mathbb{R}^n \setminus 2\overline{B} \), we get

\[
h(x) = \int_B \Phi(x - y)Lh(y)\chi(y)dy = \int_B L(\chi(y)\Phi(x - y))h(y)dy
\]
and so since $\Omega$ is bounded,
\[ \|h\|_{s,\Omega \setminus 2B} \leq C_4 \|h\|_{0, i/2B} \leq C_4 \|h\|_{m, 3B}. \]
We can now find a function $h_1 \in C^\infty(\mathbb{R}^n)$, $h_1 = h$ on $\mathbb{R}^n \setminus 2B$ such that
\[ \|h_1\|_{s,\Omega} \leq C_5 \|h\|_{s,\Omega \setminus 2B} \leq C_6 \|h\|_{m, 3B}. \]
Let now $v_1$ and $u_1 = h_1 - v_1$ satisfy the same properties as the functions $v_0$ and $u_0$ above, but taken with $h_1$ instead of $h$. Then
\[ \|u_1\|_{m,\Omega} \leq C_2 \|h_1\|_{s,\Omega} \leq C_7 \|h\|_{m, 3B}. \]
The function $v = v_1$ is as desired. In fact, since $\partial^\alpha u_1 = 0$ on $\partial \Omega$ for $|\alpha| \leq \ell - 1$, then
\[ \partial^\alpha (h - v)|_{\partial \Omega} = \partial^\alpha (h_1 - v_1)|_{\partial \Omega} = 0 \]
for $|\alpha| \leq \ell - 1$, so that $h - v \in V(\Omega)$. Finally
\[ \|h - v\|_{m,\Omega} = \|h - h_1 + h_1 - v_1\|_{m,\Omega} \leq \|h\|_{m,\Omega} + \|h_1\|_{m,\Omega} + \|u_1\|_{m,\Omega} \leq C \|h\|_{m, 3B}, \]
since clearly
\[ \|h_1\|_{m,\Omega} \leq \|h_1\|_{s,\Omega} \leq C_6 \|h\|_{m, 3B}. \]
Note that the constants $C_2$ to $C_7$ and $C$ are independent of $h$. This ends the proof.

Let $\Omega$ be any domain in $\mathbb{R}^n$. Denote by $\Omega^* = \Omega \cup \{\ast\}$ the one point compactification of $\Omega$ and by $X^*$ the interior of a set $X$. For $i \geq 1$, let
\[ X_i = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \geq 1/i, \ |x| \leq i \}. \]
Then each $X_i$ is a compact subset of $\Omega$ such that both $\Omega^* \setminus X_i$ and $\Omega^* \setminus X_{i+1}^*$ are connected and such that $X_i \subset X_{i+1}^*$.

In the next sections, we shall need frequently the following easy consequence of a very general version of Runge’s theorem.

**Proposition 3** Assume $V = V(\Omega)$ satisfies Conditions 1 and 2. Then, given $i \geq 1$, $\varepsilon_i > 0$ and $f \in L(X_{i+1}^*)$, one can find $h_i \in L(\Omega)$ such that
\[ \|f - h_i\|_{X_i} \leq \varepsilon_i. \]

**Proof.** By the generalization of Runge’s theorem found in [7, Theorem 4.4.5], there exists a sequence $\{g_m\}_{m=1}^\infty \subset L(\Omega)$ such that $g_m \to f$ in $C^\infty(X_{i+1}^*)$ and hence $g_m \to f$ in $V(X_i)$ as $m \to \infty$, which gives the result if one takes $h_i = g_m$ for some $m$ sufficiently large.  

\[ \blacksquare \]
4 Approximation theorems.

As in [2, Section 3], a closed set \( F \) in \( \Omega \) will be called a Roth-Keldysh-Lavrent’ev set in \( \Omega \), or more simply an \( \Omega \)-RKL set, if \( \Omega^* \setminus F \) is connected and locally connected. In this section, we formulate our main approximation results. They extend the analogous ones of [2] from \( \mathbb{R}^n \) to general domains \( \Omega \). Using Proposition 2, concrete new applications to “tangential” approximation are also obtained (see Theorem 4 (iii)). Note that Carleman-type approximation results will also be presented in Section 6 with interesting applications to the boundary behaviour of \( L \)-analytic functions.

We first obtain sufficient conditions for approximation of Runge-type on closed sets.

**Theorem 1** Let \( \Omega \) be a domain in \( \mathbb{R}^n \), \( n \geq 2 \). Let \( (L, V(\Omega)) \) be a pair satisfying Conditions 1 to 4, \( F \) be a (relatively) closed subset of \( \Omega \), and \( f \) be \( L \)-analytic in some neighbourhood of \( F \) in \( \Omega \). Then, for each \( \varepsilon > 0 \), there exists an \( L \)-meromorphic function \( g \) on \( \Omega \) with poles off \( F \) such that \( (f(F),_{\text{loc}} g(F),_{\text{loc}}) \in V(F) \) and

\[
\|f - g\|_F < \varepsilon.
\]

Moreover, if \( F \) is an \( \Omega \)-RKL set, then \( g \) can be chosen in \( L(\Omega) \).

The next theorem deals with approximation of a single function and shows that the problem is essentially local.

**Theorem 2** Let \( \Omega \) be a domain in \( \mathbb{R}^n \) \((n \geq 2)\), \( (L, V(\Omega)) \) be a pair satisfying Conditions 1 to 4, \( F \) be a (relatively) closed subset of \( \Omega \), and \( f \in V_{\text{loc}}(\Omega) \). Then the following are equivalent:

(i) for each positive number \( \varepsilon \), there exists an \( L \)-meromorphic function \( g \) in \( \Omega \) with poles off \( F \) such that \( (f(F),_{\text{loc}} g(F),_{\text{loc}}) \in V(F) \) and \( \|f - g\|_F < \varepsilon \);

(ii) for each ball \( B, \overline{B} \subset \Omega \) and positive number \( \varepsilon \), there exists \( g \) such that \( Lg = 0 \) on some neighbourhood of \( F \cap \overline{B} \) and \( \|f - g\|_{F \cap \overline{B}} < \varepsilon \);

(iii) the previous property is satisfied by each ball from some locally finite family of balls \( \{B_j\} \) covering \( F \), where \( \overline{B_j} \subset \Omega \) for each \( j \).

For any subset \( X \) of \( \mathbb{R}^n \), we let \( L(X) \) stand for the collection of all functions \( f \) defined and \( L \)-analytic in some neighbourhood (depending on \( f \)) of \( X \). For a closed set \( F \) in \( \Omega \) we denote by \( M_{LV}(F) \) (respectively \( E_{LV}(F) \)) the space of all \( f(F),_{\text{loc}} \in V_{\text{loc}}(F) \) which satisfy the following property: for each \( \varepsilon > 0 \) there exists an \( L \)-meromorphic function \( g \) in \( \Omega \) with poles outside of \( F \) (respectively a function \( g \in L(\Omega) \)) such that \( f - g \in V(F) \) and \( \|f - g\|_F < \varepsilon \). We also introduce the space \( V_L(F) = V_{\text{loc}}(F) \cap L(\mathbb{C}) \). Whenever Conditions 1 to 4 hold, we have that by Theorem 1, \( M_{LV}(F) \) is the closure in \( V_{\text{loc}}(F) \) of the space \( \{h(F),_{\text{loc}} \in V_{\text{loc}}(F) \mid h \in L(F) \} \). Moreover, if \( F \) is an \( \Omega \)-RKL set, then \( M_{LV}(F) = E_{LV}(F) \).

We now study the necessity of being a \( \Omega \)-RKL set for approximation by \( L \)-analytic functions.

Let \( K \) be a compact set in \( \Omega \). Denote by \( \overline{K} \) the union of \( K \) and all the (connected) components of \( \Omega \setminus K \) which are pre-compact in \( \Omega \). Obviously, the property \( \overline{K} = K \) means precisely that \( \Omega^* \setminus K \) is connected, so that \( K \) is a \( \Omega \)-RKL set.

Define

\[
N(K) = N_{LV}(K) = \{a \in \overline{K} \setminus K \mid \Phi_a(K) \notin E_{LV}(K)\},
\]

where \( \Phi_a(x) = \Phi(x - a) \).

**Condition N.** We shall say that a pair \((L, V(\Omega))\) satisfies Condition N (“nonremovability of holes”) if \( N(K) \neq \emptyset \) for each compact set \( K \) with “holes”, i.e. such that \( K \neq \overline{K} \).
Remark 1 The same proof as in [2, Proposition 2] shows that \((L,V(\Omega))\) satisfies Condition \(N\) whenever all of the following conditions hold:

1. \((L,V(\Omega))\) satisfies Conditions 1 and 2;
2. \(n = 2\) or \(n \geq 3\) and \(L\) has the following symbol:
   \[L(\xi) = P_2(\xi)Q_{r-2}(\xi), \quad \xi \in \mathbb{R}^n,\]
   where \(P_2\) is some homogeneous (elliptic) polynomial of order two with real coefficients (so that \(P_2\) has constant sign in \(\mathbb{R}^n \setminus \{0\}\)), and \(Q_{r-2}\) is some homogeneous polynomial of order \(r - 2\);
3. \(\text{Ord}(V) \geq r - 1\).

For the definition of \(\text{Ord}(V)\) when \(\Omega = \mathbb{R}^n\), see [2, Section 4.3]. Replacing \(\mathbb{R}^n\) by \(\Omega\) everywhere in that definition, we get the corresponding definition of \(\text{Ord}(V(\Omega))\) for an arbitrary domain \(\Omega\).

One can also find in [2, Section 4.2] some informative examples concerning Condition \(N\).

Theorem 3 If \((L,V(\Omega))\) satisfies Conditions 1 to 4, then the following statements are equivalent:

(i) For each (relatively) closed set \(F \subset \Omega\) one has
   \[M_{LV}(F) = E_{LV}(F) \iff \{F\ \text{is a } \Omega\text{-RKL set}\};\]

(ii) For each compact set \(K \subset \Omega\),
   \[M_{LV}(K) = E_{LV}(K) \iff \{\Omega^* \setminus K \text{ is connected}\};\]

(iii) The pair \((L,V(\Omega))\) satisfies Condition \(N\).

Remark 2 Our proof of \((i) \implies (iii)\) in fact shows that if for some compact set \(K\) in \(\Omega\) there is a function \(f \in L(K)\) which is not in \(E_{LV}(K)\), then the same is true for some \(\Phi_a\), \(a \in \overline{K} \setminus K\).

From Theorems 2 and 3, it is not difficult to obtain the corresponding approximation (reduction) theorems for classes of functions (jets), analogous to that of [2, Proposition 1]. In this direction, we present only the following result which extends [2, Theorem 4]. Note that (iii) is a result on tangential approximation.

Theorem 4 Let \(L\) (of order \(r\)) satisfy property (2) of Remark 1, \(\Omega\) be an arbitrary domain in \(\mathbb{R}^n\) and \(F\) be a closed subset of \(\Omega\).

(i) For \(V = BC^p(\Omega)\), where \(p \in (r - 1, r)\) (see Section 3), the equality \(V_L(F) = M_{LV}(F)\) holds if and only if there exists a constant \(a \in (0, +\infty)\) such that for each ball \(B\) in \(\Omega\)
   \[M^{n-r+\rho}(B \setminus F^o) \leq AM^{n-r+\rho}(B \setminus F).\]

(ii) For \(V = BC^m(\Omega)\) \((m = r, r + 1, \ldots)\) or \(V = BC^p(\Omega)\) \((\rho > r, \rho \notin \mathbb{Z})\) the equality \(V_L(F) = M_{LV}(F)\) holds if and only if \(F^o\) is dense in \(F\).

(iii) Let \(L, \Omega\) and \(V = BC^m(\Omega)\) be as in Proposition 2, and additionally suppose that \(m \geq r\). Then the equality \(V_L(F) = M_{LV}(F)\) holds if and only if \(F^o\) is dense in \(F\).

(iv) For each space \(V(\Omega)\), which is mentioned in (i), (ii) or (iii), the equality \(V_L(F) = E_{LV}(F)\) holds if and only if \(V_L(F) = M_{LV}(F)\) and (at the same time) \(F\) is a \(\Omega\text{-RKL set}\).

Here \(M^{n-r+\rho}(\cdot)\) and \(M^{n-r+\rho}(\cdot)\) are the Hausdorff and lower Hausdorff contents of order \(n - r + \rho\) respectively (cf. [15]).
5 Proofs of Theorems 1, 2, 3 and 4.

Fix a pair \((L, V(\Omega))\) satisfying Conditions 1 to 4, and let \(k = k(L) > 1\) be the constant which appears in (3).

Lemma 1 Let \(B = B(a, \delta)\) be a ball in \(\Omega\) with \(6kB \subset \Omega\), and \(T\) be a distribution with \(\text{supp}(T) \subset B\). Set \(h = \Phi \ast T\) and let

\[
h_m = \sum_{0 \leq |\alpha| \leq m} c_\alpha \partial^\alpha \Phi(x - a)
\]

be the partial sums of the Laurent series expansion of \(h\) outside \(kB\) (see (3)). Then there exists \(M \in \mathbb{Z}_+\) such that for all \(m \geq M\), one can find \(v_m \in L(\Omega)\) such that \(h - h_m - v_m \in V(\Omega \setminus 2kB)\) and

\[
\|h - h_m - v_m\|_{\Omega \setminus 2kB} \to 0 \quad \text{as} \quad m \to \infty.
\]

Proof. First recall that \(h_m \to h\) in \(C^\infty(\Omega \setminus kB)\). Let \(\psi \in C^\infty(\mathbb{R}^n)\) such that

\[
\psi = \begin{cases} 
0 & \text{in a neighbourhood of} \ kB \\
1 & \text{in a neighbourhood of} \ R^n \setminus 2kB.
\end{cases}
\]

Take \(d\) from Condition 4 for the ball \(2kB\) and the pair \((L, V)\). Since we have that \(\psi h_m \to \psi h\) in \(C^\infty(\Omega)\), there exists \(M \in \mathbb{Z}_+\) such that for \(m \geq M\), one has

\[
h^*_m \equiv \psi(h - h_m) = O(|x|^{-d}) \quad \text{as} \quad |x| \to \infty.
\]

Using Condition 4 when \(m \geq M\), we can find \(v_m \in L(\Omega)\) such that \((h^*_m - v_m) \in V\) and

\[
\|h^*_m - v_m\| \leq C\|h^*_m\|_{6kB} \to 0 \quad \text{as} \quad m \to \infty.
\]

By definition, \((h - h_m - v_m) \in V(\Omega \setminus 2kB)\) and

\[
\|h - h_m - v_m\|_{\Omega \setminus 2kB} \leq \|h^*_m - v_m\| \to 0 \quad \text{as} \quad m \to \infty.
\]

The lemma is proved.

Proof of Theorem 1. The proof relies on a localization technique. Let \(f\) be a function \(L\)-analytic on some neighbourhood \(U\) of \(F\) in \(\Omega\) and \(U_1\) be a neighbourhood of \(F\), with \(\overline{U}_1 \subset U\). We extend \(f\) to a function (also denoted by \(f\)) in \(C^\infty(\Omega)\) so that \(f\) is still \(L\)-analytic in a neighbourhood of \(\overline{U}_1\). We can find a family of couples \(\{B(a_j, \delta_j), \varphi_j\}_{j=1}^\infty\) where the family of balls \(\{B_j = B(a_j, \delta_j)\}\) is locally finite in \(\Omega\), \(6kB_j \subset \Omega \setminus F\), each \(\varphi_j \in C^\infty_0(B_j)\), with \(0 \leq \varphi_j \leq 1\) and \(\sum_{j=1}^\infty \varphi_j = 1\) on some neighbourhood \(U_2\) of \(\Omega \setminus U_1\).

Let \(f_j = \nu_{\varphi_j} f = \Phi \ast (\varphi_j Lf)\). Each \(f_j\) is in \(C^\infty(\mathbb{R}^n)\). Let \(\{X_i\}, i \geq 1\), be the sequence of compact sets described before Proposition 3. Put \(J_i = \{j \mid B_j \cap X_{i+1} \neq \emptyset\}\). Note that \(L(f - \sum_{j \in J_1} f_j) = Lf - \sum_{j \in J_1} \varphi_j Lf = Lf(1 - \sum_{j \in J_1} \varphi_j) = 0\) (i.e. \(f - \sum_{j \in J_1} f_j\) is \(L\)-analytic) in \(X_2\). By Proposition 3, one can find \(P_1 \in L(\Omega)\) such that

\[
\|f - (\sum_{j \in J_1} f_j) - P_1\|_{X_1} < \frac{1}{2}.
\]
Now, since \( f - (\sum_{J_1} f_j) - P_1 - (\sum_{J_2\setminus J_1} f_j) \) is \( L \)-analytic in \( X_3^0 \), there exists \( P_2 \in L(\Omega) \) such that

\[
\| f - (\sum_{J_1} f_j) - P_1 - (\sum_{J_2\setminus J_1} f_j) - P_2 \|_{X_2} < \frac{1}{2^2}.
\]

Inductively, we can thus find \( P_i \in L(\Omega) \) such that

\[
\| f - (\sum_{J_1} f_j) - P_1 - (\sum_{J_2\setminus J_1} f_j) - P_2 - \cdots - (\sum_{J_i\setminus J_{i-1}} f_j) - P_i \|_{X_i} < \frac{1}{2^i}.
\]

so that, setting \( J_0 = \emptyset \), the equality

\[
f = \sum_{i=1}^{\infty} \left( \sum_{J_i\setminus J_{i-1}} f_j + P_i \right)
\]

holds in \( V_{loc}(\Omega) \).

Now, from (3), each \( f_j \) has a Laurent series expansion

\[
f_j(x) = \sum_{|\alpha|\geq 0} c_{\alpha}^j \partial^\alpha \Phi(x - a_j)
\]

valid outside \( kB_j \), and thus on a neighbourhood of \( F \). Using Lemma 1, given any \( \eta_j > 0 \), there exists \( m_j \in \mathbb{Z}_+ \) and \( v_j \in L(\Omega) \) such that if

\[
g_j(x) = \sum_{|\alpha|=0}^{m_j} c_{\alpha}^j \partial^\alpha \Phi(x - a_j),
\]

then \( (f_j - g_j - v_j) \in V(\Omega \setminus 2kB_j) \) and \( \| f_j - g_j - v_j \|_{\Omega \setminus 2kB_j} < \eta_j \).

Put \( F_1 = \Omega \setminus \bigcup_j (2kB_j) \); then \( F \subset F_1 \) and, for all \( j \), \( (f_j - g_j - v_j) \in V(F_1) \), \( \| f_j - g_j - v_j \|_{F_1} < \eta_j \). Fix \( \varepsilon > 0 \) and choose the sequence \( \{\eta_j\} \), \( \eta_j > 0 \), such that \( \sum_j \eta_j < \varepsilon \). Define

\[
g = \sum_{i=1}^{\infty} \left( \sum_{J_i\setminus J_{i-1}} (g_j + v_j) + P_i \right).
\]

Since for each \( m \geq 1 \) the series

\[
\sum_{i=m+1}^{\infty} \left( \sum_{J_i\setminus J_{i-1}} (g_j + v_j) + P_i \right)
\]

converges in \( V(X_m) \), \( g \) is \( L \)-meromorphic in \( \Omega \) with “poles” only (possibly) at \( a_j, j = 1, 2, \ldots \). Moreover \( g \in V_{loc}(F_1) \) and

\[
(f - g)(F_1, loc) = \sum_{i=1}^{\infty} \left( \sum_{J_i\setminus J_{i-1}} (f_j - g_j - v_j)(F_1, loc) \right).
\]

But then \( f - g \in V(F) \) and

\[
\| f - g \|_F < \varepsilon,
\]

10
since \((f - g)(F)_{\text{loc}}\) can be defined by the element

\[
\sum_{i=1}^{\infty} \left( \sum_{j_i \in \mathcal{J}} \Psi_j \right),
\]

where \(\Psi_j \in V\) are such that \((\Psi_j)_{(\Omega \setminus 2kB_j)} = (f_j - g_j - v_j)_{(\Omega \setminus 2kB_j)}\) and \(\|\Psi_j\| \leq \eta_j\). This proves the first part of Theorem 1.

Now assume that \(F\) is a RKL-set in \(\Omega\), i.e. \(\Omega^* \setminus F\) is connected and locally connected. It suffices to show that there exists a function \(h \in L(\Omega)\) such that

\[
\|g - h\|_F < \varepsilon.
\]

Let \(\{a_j\}_{j \geq 1}\) be the sequence of “poles” of \(g\) in \(\Omega\). Each \(a_j \in \Omega \setminus F\) and the sequence has no limit points in \(\Omega\). Since \(\Omega^* \setminus F\) is connected and locally connected at the “point” *, we can find paths \(\sigma_j\) from \(a_j\) to *, \(\sigma_j \subset \Omega \setminus F\), such that the family of curves \(\{\sigma_j\}\) is locally finite in \(\Omega\).

For a fixed \(j\), we can find sequences \(\{a_{jm}\}_{m=0}^{\infty} \subset \sigma_j\) and \(\{r_{jm}\}_{m=0}^{\infty} \subset (0, 1)\) such that \(a_{j0} = a_j, a_{jm} \to *\) as \(m \to \infty\), \(|a_{jm} - a_{jm+1}| < r_{jm} + 1\), \(B_{jm} = B(a_{jm}, r_{jm}) \subset \Omega \setminus F\). Additionally we can require that the family of balls \(\{B_{jm}\}\) is locally finite in \(\Omega\). If \(G_j = \bigcup_{m=0}^{\infty} B_{jm}\) then \(\overline{G}_j \cap F = \emptyset\) and \(\{G_j\}\) is also locally finite in \(\Omega\).

Set \(h_0 = g\). We construct a sequence of functions \(h_j\) such that \(h_j\) is \(L\)-meromorphic on \(\Omega\), \(h_j\) has the same poles (and singular parts) as \(h_{j-1}\) except at \(a_j\) where \(h_j\) is \(L\)-analytic, and such that

\[
\|h_{j-1} - h_j\|_{\Omega \setminus G_j} < \frac{\varepsilon}{2^j}.
\]

If such a sequence exists, then \(h = \lim_{j \to \infty} h_j\) is in \(L(\Omega)\). Indeed, by construction (since \(\{G_j\}\) is locally finite), we have \(G_j \to \{\ast\}\) as \(j \to \infty\), and thus \(\{h_j\}\) is a Cauchy sequence in \(V(X_i)\) for each \(i\). Moreover convergence in \(V_{\text{loc}}(\Omega)\) preserves \(L\)-analyticity. Finally we would have

\[
\|g - h\|_F < \varepsilon,
\]

as desired.

To construct the functions \(h_j\) (\(h_0 = g\)), assume that \(h_{\ell}\) has been constructed for \(\ell \leq j - 1\). Let \(s_0\) be the singular part of \(h_{j-1}\) at \(a_j = a_{j_0}\). By Lemma 1 (applied to \(h = s_0\) and \(a = a_{j_0}\)), we can find an \(L\)-meromorphic function \(s_1\) in \(\Omega\) whose only singularity is at \(a_{j_1}\) and such that

\[
\|s_0 - s_1\|_{\Omega \setminus B_{j_1}} < \left(\frac{1}{2}\right) \frac{\varepsilon}{2^j}.
\]

By induction, construct an \(L\)-meromorphic function \(s_m\) whose only singularity is at \(a_{j_m}\) and such that

\[
\|s_{m-1} - s_m\|_{\Omega \setminus B_{j_m}} < \left(\frac{1}{2^m}\right) \frac{\varepsilon}{2^j}.
\]

Finally set

\[
h_j = h_{j-1} + \sum_{m=1}^{\infty} (s_m - s_{m-1}).
\]

The function \(h_j\) has the desired properties. 

The proofs of Theorems 2, 3 and 4 are also very similar to the proofs of the corresponding Theorems in [2], and will not be reproduced here. We simply note that \( \mathbb{R}^n \) is to be replaced everywhere by \( \Omega, \{ \infty \} \) (of \( \mathbb{R}^n \)) by \( \{ \ast \} \), \( P_{LV} \) by \( E_{LV} \), “bounded” by “precompact in \( \Omega \)” and so on. Our Theorem 1 above replaces the corresponding Theorem 1 of [2]. Balls also sometimes need to be replaced by sets having appropriate properties. For example, in the proof of Theorem 1 above, balls \( B(0,i) \) were replaced by sets \( X_i \), where, for each \( i \), \( \Omega^* \setminus X_i \) was connected. In Theorem 3, the balls \( B(0,R), B(0,2R) \) and \( B(0,3R) \) need to be replaced respectively by \( \Omega \)-precompact domains \( U, U_1 \) and \( U_2 \) such that \( \partial U \) is smooth, \( \overline{U} \subset U_1 \subset U \subset U_2 \) and the union of all \( \Omega \)-precompact components of \( (\Omega \setminus F) \setminus \overline{U} \) is not precompact in \( \Omega \). The existence of such domains follows from the existence of an exhaustion of \( \Omega \) by smooth domains (which are precompact in \( \Omega \)) and the assumption made in the proof that \( \Omega^* \setminus F \) is not locally connected (see also [5, Chapter IV, §2 B]). Of course, the corresponding conditions on \( D_m \) and \( a_m \) need to be changed accordingly. Moreover, the following version of [2, Lemma 5] is needed in Theorem 3.

**Lemma 2** For each open sets \( U_1, U_2 \) such that \( \overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset \Omega \), there exists a positive constant \( A \) (depending only on the space \( V \) and the sets \( U_1 \) and \( U_2 \)) such that for any compact set \( K \) and for each \( f(K) \in V(K) \) one has

\[
\|f\|_K \leq A(\|f\|_{K \cap \overline{U}_2} + \|f\|_{K \setminus U_1}).
\]

We leave the details to the reader.

## 6 Boundary Behaviour of \( L \)-analytic functions

Let \( \mathcal{S}_r^n \) stand for the class of all homogeneous elliptic operators of order \( r \) in \( \mathbb{R}^n \) \((n \geq 2, r \geq 1)\) with constant complex coefficients (see Section 2 above).

In this section, given \( L \in \mathcal{S}_r^n \) and a domain \( \Omega \) satisfying some mild conditions, we will construct in \( \Omega \) solutions of the equation \( Lu = 0 \) having some prescribed boundary behaviour.

### 6.1 No limits at the boundary

Let \( \Omega \) be a domain in \( \mathbb{R}^n \), \( n \geq 2 \), \( \Omega \neq \mathbb{R}^n \), and let \( b \in \partial \Omega \). We shall say that a (continuous) path \( \gamma : [0,1] \rightarrow \mathbb{R}^n \) is admissible for \( \Omega \) with end point \( b \) if \( \gamma : [0,1] \rightarrow \Omega \) and \( \gamma(1) = b \). Given a continuous function \( f \) in \( \Omega \), denote by \( C_\gamma(f) \) the cluster set of \( f \) along \( \gamma \) at \( b \), that is:

\[
C_\gamma(f) = \{ w \in \mathbb{C}^* \mid \text{there exists a sequence } \{ t_n \} \subset [0,1) \text{ such that } t_n \rightarrow 1 \text{ and } f(\gamma(t_n)) \rightarrow w \text{ as } n \rightarrow \infty \}.
\]

**Theorem 5** Let \( L \in \mathcal{S}_r^n \), and let \( \Omega \subset \mathbb{R}^n \), \( \Omega \neq \mathbb{R}^n \), be a domain such that its boundary \( \partial \Omega \) has no (connected) components that consist of a single point. Then there exists \( g \in L(\Omega) \) with the property that for each \( b \in \partial \Omega \), for each admissible path \( \gamma \) for \( \Omega \) ending at \( b \) and for each \( \alpha \in \mathbb{Z}_+^n \), one has

\[
C_\gamma(\partial^\alpha g) = \mathbb{C}^*.
\]

The following proposition and remark show that, at least for \( L = \Delta \) in \( \mathbb{R}^n \) and \( L = \partial/\partial \bar{z} \) in \( \mathbb{R}^2 \), our theorem is close to being sharp.

**Proposition 4** If \( \Omega \) is a domain in \( \mathbb{R}^n \) such that \( \partial \Omega \) has an isolated point \( b \in \mathbb{R}^n \cup \{ \infty \} \), then for each function \( f \) harmonic in \( \Omega \) or (if \( n = 2 \)) for each function \( f \) holomorphic in \( \Omega \), there exists an admissible path \( \gamma \) for \( \Omega \) ending at \( b \) such that \( C_\gamma(f) \) is a single point in \( \mathbb{C}^* \).
Remark 3 It follows from Proposition 4 that for each \( \alpha \in \mathbb{Z}_+^n \) there exists an admisible path \( \gamma_\alpha \) for \( \Omega \) ending at \( b \) such that \( C_{\gamma_\alpha}(\partial^\alpha f) \) is just a single point in \( C^* \) since the point \( b \) is also an isolated singularity of the harmonic (or holomorphic) function \( \partial^\alpha f \).

Proof of Proposition 4. It is well known that if \( f \) is bounded at \( b \) (that is in some punctured neighbourhood of \( b \)), then \( f \) has a removable singularity at \( b \) and that consequently the proposition holds for every admissible path.

If \( f \) is unbounded at \( b \), then the result follows from a generalization of a theorem of Iversen due to B. Fuglede (see [4, Corollary 1]).

Lemma 3 Let \( L \in \mathfrak{N}^d \). For each \( \beta \in \mathbb{Z}_+^n \) there exists a homogeneous polynomial \( P_\beta \in L(\mathbb{R}^n) \) of degree \( |\beta| \) with \( \partial^\beta P_\beta \equiv 1 \).

Proof. The Lemma is obvious if \( |\beta| < r \). So let us assume that \( |\beta| \geq r \). We claim that \( \partial^\beta \Phi \not\equiv 0 \) on \( \mathbb{R}^n \setminus \{0\} \), where \( \Phi \) is a special fundamental solution for \( L \) as before (see Section 2).

Assuming the claim, fix a point \( a \neq 0 \) where \( \partial^\beta \Phi(a) \neq 0 \). By Taylor’s formula, we have

\[
\Phi(x) = \sum_{k=0}^{\infty} Q_k(x)
\]

where

\[
Q_k(x) = \sum_{|\alpha|=k} \frac{\partial^\alpha \Phi(a)}{\alpha!} (x-a)^\alpha
\]

belongs to \( L(\mathbb{R}^n) \) (see [2, Section 2.4]). It suffices to take

\[
P_\beta = \frac{Q_{|\beta|}}{\partial^\beta \Phi(a)}.
\]

To prove the claim, note that by [15, Lemma 1.1], one has in fact that

\[
\partial^\beta \Phi(x) = \sum_{|\alpha|=|\beta|-r} c_\alpha \partial^\alpha \delta(x) + K(x),
\]

where \( \delta(\cdot) \) is the Dirac delta function, \( c_\alpha \in \mathbb{C} \) and \( K \) is a Calderón-Zygmund \( (n+|\beta|-r) \)-dimensional kernel. Assuming that \( \partial^\beta \Phi(x) = 0 \) for all \( x \neq 0 \), then \( K(x) \equiv 0 \). Thus

\[
(-i)^r \xi^\beta \tilde{\Phi}(\xi) = \sum_{|\alpha|=|\beta|-r} c_\alpha \xi^\alpha,
\]

where \( \tilde{\Phi} \) denotes the Fourier transform of \( \Phi \). On the other hand, since \( L\Phi = \delta(\cdot) \), one has

\[
(-i)^r L(\xi) \tilde{\Phi}(\xi) \equiv 1.
\]

It follows that \( \xi^\beta = A(\xi)L(\xi) \), where \( A \) is a polynomial. Choose \( \eta = (\eta_1, \cdots, \eta_n) \) with \( \eta_j > 0 \), \( j = 1, \cdots, n \), and fix \( (\xi_2, \cdots, \xi_n) = (\eta_2, \cdots, \eta_n) \). We have, for all \( \xi_1 \) (after division by \( \eta_2^{\beta_2} \cdots \eta_n^{\beta_n} \)):

\[
\xi_1^{\beta_1} = A_1(\xi_1) L_1(\xi_1),
\]

where \( L_1(\xi_1) = L(\xi_1, \eta_2, \cdots, \eta_n) \) and \( A_1(\xi_1) \) are also polynomials. The polynomial \( L_1(\xi_1) \) has no zeros (on \( \mathbb{R} \)) and divides \( \xi_1^{\beta_1} \), so that it is constant. Similarly, we can show that \( L \) is constant on
each line through \( \eta \) which is parallel to a coordinate axis. Since this is true for each point \( \eta \) in the open cone \( \{ \eta \mid \eta_j > 0, j = 1, \ldots, n \} \), we conclude that the polynomial \( L(\xi) \) is constant in this cone and hence identically constant. Thus \( L = L(0) = 0 \), since \( L \) is homogeneous of order \( r \geq 1 \). This contradicts the ellipticity hypothesis, proves the claim and ends the proof of the lemma.

Proof of Theorem 5. Following the idea of the proof of [5, Chapter IV, §5, Theorem 4], we will construct a set of Carleman approximation which must be intersected infinitely often by every admissible path.

By Whitney’s approximation theorem [9, Theorem 1.6.5], we can find a real analytic function \( \Psi \) on \( \Omega \) such that for each \( x \in \Omega \) one has

\[
\frac{1}{2} \min \left( \text{dist}(x, \partial \Omega), \frac{1}{|x|} \right) \leq \Psi(x) \leq 2 \min \left( \text{dist}(x, \partial \Omega), \frac{1}{|x|} \right)
\]  

(6)  

From Sard’s theorem [9, Theorem 1.4.6], we can find a sequence \( \{\rho_j\}_{j=0}^{\infty}, \rho_j \nearrow 0 \) as \( j \to \infty \) such that the level sets \( R_j = \{ x \in \Omega \mid \Psi(x) = \rho_j \} \) do not contain any critical point of \( \Psi \), i.e. \( \nabla \Psi \neq 0 \) on \( R_j \) and \( R_j \) consists of only finitely many \( C^\infty \)-smooth (in fact real analytic) hypersurfaces. Let \( \Omega_j = \{ x \in \Omega \mid \Psi(x) > \rho_j \} \). We additionally require (as we can) that \( (\Omega_j)^\circ \subset \Omega_{j+1} \). We define \( E_j = \partial(\Omega_j) \) and note that \( E_j \) also consists of finitely many \( C^\infty \)-smooth closed hypersurfaces which we denote \( E_{j\nu}, 1 \leq \nu \leq k_j \).

For positive but small enough \( \delta_j \), the \( \delta_j \)-neighbourhood \( \Omega_j' \) of \( (\Omega_j)^\circ \) is \( C^\infty \)-smooth, \( (\Omega_j')^\circ = \overline{\Omega_j} \) and \( E'_j = \partial(\Omega_j') \) has the same number \( k_j \) of components \( E'_{j\nu} \) as \( E_j \). The sequence \( \{\delta_j\} \) is also chosen to satisfy \( \delta_j \nearrow 0 \) as \( j \to \infty \), \( \Omega_j' \subset \Omega_{j+1} \), \( \text{dist}(E'_j, E_{j+1}) \geq 2\delta_j \) and \( \delta_j < \min_\nu (\text{diam} E_{j\nu})/10 \). Choose \( a_{j\nu} \in E_{j\nu} \) and \( a'_{j\nu} \in E'_{j\nu} \) such that

\[
|a_{j\nu} - a'_{j\nu}| \geq \frac{\text{diam}(E_{j\nu})}{2}.
\]  

(7)  

Now let

\[
K_j = \overline{\Omega_j},
\]

\[
F_j = \bigcup_{\nu=1}^{k_j} \{(E_{j\nu} \setminus B(a_{j\nu}, \delta_j)) \cup (E'_{j\nu} \setminus B(a'_{j\nu}, \delta_j))\},
\]

and define

\[
F = \bigcup_{j=0}^{\infty} F_j.
\]

For each \( j \), we can find disjoint closed \( \eta_j \)-neighbourhoods \( G_j \) of \( F_j \) (with \( 0 < \eta_j < \delta_j/4 \)) such that \( G_{j+1} \cap K_j = \emptyset \) and \( \Omega^* \setminus (G_{j+1} \cup K_j) \) is connected.

Finally we define the function \( f \), \( L \)-analytic in some neighbourhood of the set \( G = \bigcup_{j=0}^{\infty} G_j \) as follows. For each \( \beta \in \mathbb{Z}_+^n \), we can find \( I_\beta \subset \mathbb{Z}_+ \) such that \( \cup_{\beta \in \mathbb{Z}_+^n} I_\beta = \mathbb{Z}_+ \), each \( I_\beta \) contains infinitely many elements and \( I_\beta \cap I_{\beta'} = \emptyset \) for \( \beta \neq \beta' \). Let \( \{\lambda_i^\beta\}_{i \in I_\beta} \) be a fixed sequence in \( C \) such that \( C^* \) is the set of its limit points. Now fix \( j \in \mathbb{Z}_+ \). Then \( j \) is in position \( i_j \) in \( I_\beta \) for some (unique) \( \beta \in \mathbb{Z}_+^n \). Let \( P_\beta \in L(\mathbb{R}^n) \) be a polynomial of degree \( |\beta| \) with \( \partial^\beta P_\beta \equiv 1 \) (see Lemma 3), and let \( U_j \) be pairwise disjoint (open) neighbourhoods of \( G_j \) such that \( \overline{U_{j+1}} \cap K_j = \emptyset \) for all \( j \). Then define \( f \) on \( U_j \) as

\[
f(x) = \lambda_{i_j}^\beta P_\beta(x).
\]

We will need the following “Carleman-type” approximation lemma.
Lemma 4 Let $f$ and $G$ be as above. Then for any sequence $\{\epsilon_j\}_{j=0}^\infty$, $\epsilon_j \searrow 0$ as $j \to \infty$, there exists $g \in L(\Omega)$ such that

$$\|f - g\|_{0, \mathcal{C}_1} \leq \epsilon_j$$

where $\| \cdot \|_{0, \mathcal{E}}$, as before, denotes the uniform norm on $\mathcal{E}$.

Assuming the lemma, fix a sequence $\{\tau_j\}_{j=0}^\infty$, $\tau_j \searrow 0$ as $j \to \infty$. Now choose a sequence $\{\epsilon_j\}$, $\epsilon_j \searrow 0$ as $j \to \infty$ such that if (8) is satisfied for a function $g \in L(\Omega)$, then

$$\|\partial^j g - \lambda_j^\beta\|_{0, F_j} < \epsilon_j, \quad j \in I_\beta.$$  (9)

This can be done by choosing $\epsilon_j$ small enough, since $\partial^j f = \lambda_j^\beta$ on $F_j$.

The function $g$ has the desired properties. Indeed, let $\gamma$ be an admissible path for $\Omega$ with end point $b \in \partial \Omega$. Then we claim that $[\gamma] = \gamma([0, 1])$ must intersect all $F_j$, except possibly finitely many of them. Combining the claim with (9) and the choice of $\{\lambda_j^\beta\}$ proves the theorem.

To prove the claim, assume that $[\gamma]$ does not intersect infinitely many $F_j$, say $\{F_{j_m}\}_{m=1}^\infty$ with $j_m \not\searrow \infty$ as $m \to \infty$. It then follows that there exists an $m_0$ such that for each $m > m_0$, one can find $\nu = \nu(j_m)$ such that $[\gamma]$ intersects $B(a_{j_m\nu}, \delta_j)$ and $B(a'_{j_m\nu}, \delta_j)$ and where each $E_{j_m\nu}$ is either the outer boundary (in $\mathbb{R}^n$) of $(\Omega_{j_m})^\circ$ or $E_{j_m\nu}$ surrounds the point $b$. Notice that by (7),

$$|a_{j_m\nu} - a'_{j_m\nu}| \geq \frac{\text{diam}(E_{j_m\nu})}{2} \geq 5\delta_j,$$

and thus, from the continuity of $\gamma$ at $b$, we must have that $\text{diam}(E_{j_m\nu}) \to 0$ as $j_m \to \infty$. But this is impossible. In fact, if $E_{j_m\nu}$ is the boundary of the unbounded component of $(\Omega_{j_m})^\circ$, then $\text{diam}(E_{j_m\nu}) = \text{diam}(\Omega_{j_m})$ which grows with $m$, so that all but a finite number of $E_{j_m\nu}$ must be “inner” components of the boundary of $(\Omega_{j_m})^\circ$ which surround the component of the boundary of $\Omega$ containing $b$. But our assumption on the boundary of $\Omega$ also makes this impossible. This proves the claim and completes the proof of Theorem 5.

\[ \square \]

Proof of Lemma 4. Lemma 4 is a consequence of a rather general theorem of A. Sinclair [13, Theorem 1], but we include the following relatively simple proof for the reader’s convenience.

Let $\{\epsilon'_k\}_{k=0}^\infty$ be the sequence of positive numbers satisfying $\epsilon_j = \sum_{k \geq j} \epsilon'_k$. Since $G_0$ is an $\Omega$-RKL set and $f \in L(U_0)$, then by Theorem 1, one can find $g_0 \in L(\Omega)$ with

$$\|f - g_0\|_{0, G_0} \leq \epsilon'_0.$$  

Let $U'_j$ be a neighborhood of $K_j$ such that $U'_j \cap U_{j+1} = \emptyset$. Define

$$f_1(x) = \left\{ \begin{array}{ll}
g_0(x), & x \in U'_0 \\
f(x), & x \in U_1. \end{array} \right.$$  

Since $K_0 \cup G_1$ is a RKL-set in $\Omega$ and $f_1 \in L(U'_0 \cup U_1)$, we can find $g_1 \in L(\Omega)$ such that

$$\|f_1 - g_1\|_{0, K_0 \cup G_1} \leq \epsilon'_1.$$  

Inductively, for $j \geq 1$, we define

$$f_{j+1}(x) = \left\{ \begin{array}{ll}
g_j(x), & x \in U'_j \\
f(x), & x \in U_{j+1}, \end{array} \right.$$
and choose \( g_{j+1} \in L(\Omega) \) such that
\[
\| f_{j+1} - g_{j+1} \|_{0, K_j \cup G_j + 1} \leq \varepsilon_j + 1.
\]

Since \( K_j \not\subset \Omega \), we have that
\[
g = \lim_{j \to \infty} g_j \quad (\in L(\Omega))
\]
satisfies the Lemma.

## 6.2 A Dirichlet problem

Our next example is in some sense in the opposite direction of the first one. Given a (smooth) domain \( \Omega \), we would like to prescribe (almost everywhere on \( \partial \Omega \)) the boundary values of an \( L \)-analytic function in \( \Omega \), together with the boundary values of a fixed number of its derivatives, as we approach the boundary of \( \Omega \) in the normal direction (a “weakened” Dirichlet problem).

We first prove an abstract Carleman-type approximation theorem when \( F \) is without interior.

**Proposition 5** Let \( L \in \mathbb{N}_+ \), \( \Omega \) be a domain in \( \mathbb{R}^n \) and let \( V = V(\Omega) \) be a Banach space such that the pair \( (L, V) \) satisfies Conditions 1 and 2. Let \( F \) be a closed subset of \( \Omega \) with \( F_0 = \emptyset \) and \( K_j \subset K_{j+1} \) and \( \bigcup_{j=0}^\infty K_j = \Omega \) which is “compatible” with \( F \) in the sense that for each \( j \geq 0 \), one has
\[
V_L(K_j \cup (K_{j+2} \cap F)) = E_L(V(K_j \cup (K_{j+2} \cap F)).
\] (10)

Then for each sequence \( \{\varepsilon_j\}_{j=0}^\infty \), \( \varepsilon_j \searrow 0 \) as \( j \to \infty \) and for each \( f \in V_{loc}(F) \), one can find \( g \in L(\Omega) \) such that, for all \( j \geq 0 \),
\[
\| f - g \|_{F \setminus K_j^e} < \varepsilon_j.
\]

**Proof.** Fix \( \{\delta_j\}_{j=0}^\infty \subset (0, \infty) \), with \( \sum_{j=0}^\infty \delta_j < \infty \). Let \( g_0 = f \). For each \( j \geq 1 \), we shall find \( g_j \in V_{loc}(\Omega) \cap L(K_j) \) such that
\[
\| g_{j-1} - g_j \|_{K_{j-1}} < \delta_{j-1},
\] (11)

and
\[
\| g_{j-1} - g_j \|_{F \setminus K_j^e} < \varepsilon_j \quad \text{for each } k \geq 0.
\] (12)

Letting \( g = \lim_{j \to \infty} g_j = g_0 + \sum_{j=1}^\infty (g_j - g_{j-1}) \) will give the result.

First, for each \( j \geq 1 \), fix \( \varphi_j \in C_0^\infty(K_{j+1}^e) \), \( 0 \leq \varphi_j \leq 1 \) and \( \varphi_j \equiv 1 \) on some neighbourhood of \( K_j \). We now proceed by induction on \( j \). By (10) with \( j = 0 \), we can find \( h_1 \in L(\Omega) \) such that
\[
\| g_0 - h_1 \|_{K_2 \cap F} < \mu_1,
\]
where \( \mu_1 \in (0, \infty) \) will be specified below. Let
\[
g_1 = h_1 \varphi_1 + g_0(1 - \varphi_1).
\]

Then \( g_1 \in V_{loc}(\Omega) \cap L(K_1) \), and it follow from Condition 1 that
\[
\| g_0 - g_1 \|_F = \| (g_0 - h_1) \varphi_1 \|_F \leq C(\varphi_1) \| g_0 - h_1 \|_{K_2 \cap F} < C(\varphi_1) \mu_1
\]
and
\[\|g_0 - g_1\|_{F \setminus K_2^2} = 0.\]
Consequently, (11) and (12) hold for \( j = 1 \) if \( C(\varphi_1)\mu_1 \leq \varepsilon_1/2 \). Note that (11) is an empty condition at this stage since \( K_0 \) is the empty set.

Suppose now that we have found \( g_0, \ldots, g_J \) such that (11) and (12) hold for \( 1 \leq j \leq J \). By (10) with \( j = J \), one can find \( h_{J+1} \in L(\Omega) \) such that
\[
\|g_J - h_{J+1}\|_{K_J \cup (K_{J+2} \cap F)} < \mu_{J+1},
\]
where \( \mu_{J+1} \) is a small positive constant to be chosen later. Let
\[
g_{J+1} = h_{J+1}\varphi_{J+1} + g_J(1 - \varphi_{J+1}).
\]
Then
\[
\|g_J - g_{J+1}\|_{K_J} = \|(g_J - h_{J+1})\varphi_{J+1}\|_{K_J} = \|g_J - h_{J+1}\|_{K_J} < \mu_{J+1},
\]
which gives (11) (with \( j = J + 1 \)) whenever \( \mu_{J+1} \leq \delta_J \). Since \( \|g_J - g_{J+1}\|_{F \setminus K_{J+2}} = 0 \), it is enough, in order to get (12), to require that
\[
\|g_J - g_{J+1}\|_F < \frac{\varepsilon_{J+1}}{2^{J+1}}.
\]
But this follows from (13) and Condition 1 if \( \mu_{J+1} \) is small enough. Indeed,
\[
\|g_J - g_{J+1}\|_F = \|(g_J - h_{J+1})\varphi_{J+1}\|_F \leq C(\varphi_{J+1})\|g_J - h_{J+1}\|_{F \setminus K_{J+2}} < C(\varphi_{J+1})\mu_{J+1},
\]
and thus it suffices to take \( \mu_{J+1} = \min(\delta_J, \varepsilon_{J+1}/(2^{J+1}C(\varphi_{J+1}))) \). This completes the proof. \( \blacksquare \)

We shall also need the following lemma.

**Lemma 5** For \( 0 < d < 1 \), denote by \( Q_d' = [-d, d]_{y_1} \times [-d, d]_{y_2} \times \cdots \times [-d, d]_{y_{n-1}} \) the \( n - 1 \) dimensional closed cube centered at zero in \( \mathbb{R}^{n-1} \) and let \( Q_d = Q_d' \times [0, 2d]_{y_n} \). Let \( s \in \mathbb{Z}_+ \) be fixed. Given \( h_0, \ldots, h_s \in C(Q_d') \), there exists a function \( H \in C^\infty(Q_d \setminus (Q_d' \times \{0\})) \cap C(Q_d) \) such that, if \( y' = (y_1, y_2, \ldots, y_{n-1}) \), then
\[
\frac{\partial^k H}{\partial y^k_n}(y', y_n) \rightarrow h_k(y')
\]
uniformly on \( Q_d' \) as \( y_n \rightarrow 0, 0 \leq k \leq s \).

**Remark 4** We first note that (14) and the mean-value theorem implies that the one-sided derivatives at zero exist and
\[
\frac{\partial^k H}{\partial y^k_n}(y', 0+) = h_k(y').
\]

**Remark 5** The Lemma is easily proved if we assume that \( h_0, h_1, \ldots, h_s \in C^\infty(Q_d') \) since in this case it suffices to take
\[
H(y', y_n) = \sum_{k=0}^{s} \frac{y_n^k}{k!} h_k(y').
\]
The proof of the general case is an adaptation of this idea using approximation and a partition of unity.
Proof of Lemma 5. Let \( \{ \varphi_j \} \), \( j = 2, 3, \ldots \), \( \varphi_j \in C^\infty(\mathbb{R}) \) such that \( \text{supp}(\varphi_j) \subset (\frac{1}{j+1}, \frac{1}{j-1}) \), \( 0 \leq \varphi_j \leq 1 \), and \( \sum_{j=2}^\infty \varphi_j \equiv 1 \) on \((0, 1/2)\). Let \( \| \varphi_j^{(k)} \|_0 =: \lambda_{kj} \) and \( M := \max_{0 \leq k \leq s} \| h_k \|_{0,Q''_j} \). Let \( \{ \varepsilon_j \}_{j=2}^\infty \subset (0, 1) \) be a sequence of decreasing numbers tending to zero. By the Weierstrass approximation theorem in several variables, for each \( k \) and \( j \), \( 0 \leq k < s \) and \( j = 2, 3, \ldots \), we can find \( h_{kj} \in C^\infty(Q') \) (in fact polynomials) such that

\[
\| h_{kj} - h_k \|_{0,Q''_j} < \varepsilon_j.
\]

We claim that the function

\[
H(y', y_n) = \sum_{k=0}^s \sum_{j=2}^\infty \frac{y_n^k}{k!} h_{kj}(y') \varphi_j(y_n), \quad \text{when } y_n > 0,
\]

\[
H(y', 0) = h_0(y')
\]

has the desired properties whenever the sequence \( \{ \varepsilon_j \} \) is chosen to satisfy \( \sum_{j \geq 2} \varepsilon_j \lambda_{kj} < \infty \), for each \( k \), \( 0 \leq k < s \). Indeed let us assume that \( 0 < y_n < \frac{1}{j_0+1} < 1/2 \). Then

\[
|H(y', y_n) - h_0(y')| = \left| \sum_{j=2}^\infty (h_{0j}(y') - h_0(y')) \varphi_j(y_n) + \sum_{k=1}^s \sum_{j=2}^\infty \frac{y_n^k}{k!} h_{kj}(y') \varphi_j(y_n) \right|
\]

\[
\leq 2\varepsilon_{j_0} + (M + 1) \sum_{k=1}^s \frac{y_n^k}{k!},
\]

and thus \( |H(y', y_n) - h_0(y')| \to 0 \) uniformly as \( y_n \to 0 \). Similarly, since \( \sum_{j \geq 2} \varphi'_j(y_n) = 0 \), \( 0 < y_n < 1/2 \), we have

\[
\left| \frac{\partial H}{\partial y_n}(y', y_n) - h_1(y') \right|
\]

\[
= \left| \sum_{k=1}^s \sum_{j=2}^\infty \frac{y_n^{k-1}}{(k-1)!} h_{kj}(y') \varphi_j(y_n) + \sum_{k=0}^s \sum_{j=2}^\infty \frac{y_n^k}{k!} h_{kj}(y') \varphi_j(y_n) - \sum_{j=2}^\infty h_1(y') \varphi_j(y_n) \right|
\]

\[
\leq \left| \sum_{j=2}^\infty (h_{1j}(y') - h_1(y')) \varphi_j(y_n) \right| + \sum_{k=2}^s \sum_{j=0}^\infty \frac{y_n^{k-1}}{(k-1)!} h_{kj}(y') \varphi_j(y_n)
\]

\[
+ \sum_{k=0}^s \sum_{j=0}^{j_0} \frac{y_n^k}{k!} (h_{kj}(y') - h_k(y')) \varphi_j(y_n)
\]

\[
\leq 2\varepsilon_{j_0} + (M + 1) \sum_{k=2}^s \frac{y_n^{k-1}}{(k-1)!} + \sum_{k=0}^s \sum_{j=0}^{j_0} \frac{y_n^k}{(k-1)!} \varepsilon_j \lambda_{1j},
\]

assuming that \( 0 < y_n < \frac{1}{j_0+1} \). Thus \( \left| \frac{\partial H}{\partial y_n}(y', y_n) - h_1(y') \right| \to 0 \) uniformly as \( y_n \to 0 \). The proof of the other cases is very similar.

\[\]

Theorem 6 Let \( L \in S^r_n \) and let \( \Omega \) be a domain of class \( C^{r+1} \) in \( \mathbb{R}^n \). Let \( h_k, k = 0, 1, \ldots, r-1, \) be \( \sigma \)-measurable functions which are finite \( \sigma \)-almost everywhere, where \( \sigma \) is the \( n-1 \) dimensional Lebesgue measure on \( \partial \Omega \). Then there exists \( h \in L(\Omega) \) such that, for \( k = 0, \ldots, r-1, \) and for \( \sigma \)-almost all \( x \in \partial \Omega \), the limit of \( \frac{\partial h_k}{\partial x^j}(y) \) is equal to \( h_k(x) \), where the derivatives are taken in the direction of the outer normal at \( x \), and \( y \in \Omega \) tends to \( x \in \partial \Omega \) along that normal direction.
Proof. We will begin the proof by constructing a special family of $C^r$-diffeomorphisms from $n$-dimensional closed cubes into $\Omega$. We will use the notations introduced in Lemma 5. Fix a point $b$ on the boundary of $\Omega$ and choose an (orthonormal) coordinate system $y = (y_1, \cdots, y_n)$ such that $y(b) = 0$ and for some $\delta > 0$ there is $\psi \in C^{r+1}(Q'_\delta)$ with $\psi(0') = 0$, $\frac{\partial \psi}{\partial y_k} \big|_{0'} = 0$ ($k = 1, 2, \ldots, n - 1$) such that

$$\{ y \mid y = (y', y_n) \in \partial \Omega, y' \in Q'_\delta, |y_n| < 2\delta \} = \{ y \mid y_n = \psi(y'), y' \in Q'_\delta \}.$$ 

Moreover we suppose that

$$\{ y \mid \psi(y') < y_n < 2\delta, y' \in Q'_\delta \} \subset \Omega.$$

Let us define $\Psi : Q'_\delta \times \mathbb{R} \rightarrow \mathbb{R}^n$ by:

$$\Psi(y', y_n) = (y', \psi(y')) - y_n \tilde{n}_\delta.$$

Here $\tilde{n}_\delta$ denotes the outer normal (unit) vector to $\partial \Omega$ at the point $\tilde{y} = (y', \psi(y'))$. The Jacobian of $\Psi$ at the origin is the identity. By the inverse mapping theorem, there exists $\epsilon > 0$ and $d > 0$ such that $\Psi$ is a $C^r$-diffeomorphism of $Q_d$ on $\Psi(Q_d)$ and such that $\Psi(Q_d) \subset \Omega$.

Using the fact that $\partial \Omega$ is compact, we now choose a finite family of maps $\Psi_\nu$ and closed cubes $Q_{\nu} := Q_{d_\nu} = Q'_{d_\nu} \times [0, 2d_\nu] =: Q'_{d_\nu} \times [0, 2d_\nu]$ such that $\Psi_\nu|_{Q_{\nu}}$ is a $C^r$-diffeomorphism, $\Psi_\nu(Q_{\nu} \times \{0\}) \subset \partial \Omega, \Psi_\nu(Q_{\nu} \setminus (Q'_{d_\nu} \times \{0\})) \subset \Omega$ and $\partial \Omega \subset \cup_{\nu} \Psi_\nu(U_{(\nu)} \times \{0\})$, where $U_{(\nu)} := (-d_\nu, d_\nu)_{y_1} \times \cdots \times (-d_\nu, d_\nu)_{y_{n-1}}$.

Let $h_0, \ldots, h_{r-1}$ be any $r \sigma$-measurable functions defined and $\sigma$-finite almost everywhere on $\partial \Omega$. We can construct a family $\{ E_m \}_{m=1}^\infty$, $E_m \subset \partial \Omega$ with the following properties:

a) The sets $E_m$, $m = 1, 2, \ldots$, are compact, pairwise disjoint, nowhere dense subsets of $\partial \Omega$ with $\sigma(E_m) \neq 0$.

b) For each $k \in \{0, 1, \ldots, r-1\}$ and $m \in \{1, 2, \ldots\}$, we have $h_k \in C(E_m)$.

c) $\sigma(\partial \Omega \setminus (\cup_mE_m)) = 0$.

d) For each $m \in \{1, 2, \ldots\}$, there exists $\nu_m$ such that $\Psi_\nu^{-1}(E_m) \subset (U_{(\nu_m)} \times 0)$ where $\Psi_\nu$ belongs to the finite family of diffeomorphisms chosen above.

e) For some fixed $\mu \in (0, 1)$ and for each $m \in \{1, 2, \ldots\}$ there is a $c > 0$ such that for any $x \in E_m$ and $\epsilon < d_{\nu_m}$ one has

$$M^{n-2+\mu}(\{B(x, \epsilon) \cap \Psi_\nu(E_m \setminus \{0\})\} \setminus E_m) \geq c\epsilon^{n-2+\mu},$$

where $M^\lambda$ denotes the $\lambda$-dimensional Hausdorff content.

For example, the first three properties are obtained using Lusin’s theorem [12, Theorem 2.24], and the fourth follows easily. In order to have additionally property (e), we use the following Lemma, taking products of the set $E$ from this Lemma with $n-2$-dimensional closed cubes which gives an $n-1$-dimensional analog of the Lemma, that is (15).

**Lemma 6** For each $\mu \in (0, 1)$ and $\eta > 0$, there exist a compact set $E \subset [0, 1]$ and a constant $c > 0$ (independent of $\eta$) such that $M^1(E) > 1 - \eta$ and for each $t \in \mathbb{R}$ and each $\epsilon > 0$, one has

$$M^\mu(\{ \tau \mid |\tau - t| < \epsilon \} \setminus E) \geq c\epsilon^\mu.$$
We extend \( h \). If \( V \) is a Cantor-type set \( K \subset [0, 1] \) with \( M^1(K) = 0 \) and \( M^\mu(K) > 0 \). For \( m \in \mathbb{Z}_+ \) and \( j \in \{0, \ldots, 2^m - 1\} \), define \( K^\prime_m = \{\tau + j2^{-m} \mid \tau \in K\} \). Since \( M^1(K^\prime_m) = 0 \), there are open sets \( U^j_m \) containing \( K^\prime_m \) with \( M^1(U^j_m) < \eta 2^{-m-1} \). It suffices to take (as can be easily checked)

\[
E = [0, 1] \setminus \bigcup_{m=0}^{\infty} \bigcup_{j=0}^{2^m-1} U^j_m.
\]

We now return to the proof of Theorem 6. Given \( \{E_m\} \) as above, define

\[
F_1 = \Psi_{\nu_1}((\Psi_{\nu_1}^{-1}(E_1) \times (0, \delta_1]),
\]

where \( 0 < \delta_1 \leq d_{\nu_1} \), and for \( m \geq 2 \),

\[
F_m = \Psi_{\nu_m}((\Psi_{\nu_m}^{-1}(E_m) \times (0, \delta_m]),
\]

where \( 0 < \delta_m \leq \min\{d_{\nu_m}, \delta_{m-1}/2\} \) is so small that \( F_m \) is disjoint from \( F_1 \cup \cdots \cup F_{m-1} \) and \( \{F_m\} \) is a locally finite family in \( \Omega \).

Let \( F = \bigcup_{m=1}^{\infty} F_m \). We note that \( F \) is a (relatively) closed \( \Omega \)-RKL set with no interior. Let \( G_m = \Psi_{\nu_m}^{-1}(E_m) \) and \( h_{k,m}^* (y') = h_k(\Psi_{\nu_m}(y', 0)) \) and note that \( h_{k,m}^* \) is (defined and) continuous on \( G_m \). We extend \( h_{k,m}^* \) continuously to all of \( Q_{(\nu_m)} \) and still denote this extension by \( h_{k,m}^* \). Using Lemma 5 with \( s = r - 1 \), for each \( m \geq 1 \), there exist functions \( H_m^* \in C^r(Q_{(\nu_m)} \setminus \{Q_{(\nu_m)} \times \{0\}\}) \cap C(Q_{(\nu_m)}) \) such that for each \( k, 0 \leq k \leq r - 1 \),

\[
\frac{\partial^k H_m^*(y', y_n)}{\partial y_n^k} \longrightarrow h_{k,m}^*(y')
\]

uniformly on \( Q_{(\nu_m)} \) as \( y_n \to 0^+ \). Define \( H_m \) in \( C^r(\Psi_{\nu_m}(Q_{(\nu_m)})) \) by \( H_m(x) = H_m^*(\Psi_{\nu_m}^{-1}(x)) \).

From our construction, it follows that one can choose (open) neighbourhoods \( \Omega_m \) of \( F_m \) such that the sets \( \Omega_m \) are still pairwise disjoint and \( \Omega_m \subset \Psi_{\nu_m}(Q_{(\nu_m)}) \). Define

\[
f|_{\Omega_m} = H_m|_{\Omega_m}.
\]

If \( V \) is the space \( BC^{r-1+\mu}(\Omega) \) then \( f \in V_{loc}(F) \) (note that \( f \) can be extended from (possibly smaller) neighbourhoods \( \Omega_m \) of \( F_m \) to a function in \( V_{loc}(\Omega) \).

It follows also from our construction of \( F \) (recalling (15)) that there exists an exhaustion of \( \Omega \) by compact sets \( K_j \) such that

1) each \( Y_j = K_j \cup (K_{j+2} \cap F) \) is an \( \Omega \)-RKL set;

2) for each \( Y_j \), there exists a constant \( c_j = c(Y_j) > 1 \) such that for all balls \( B(x, \varepsilon) \subset \Omega \) we have

\[
c_j M^{n-1+\mu}(B(x, \varepsilon) \setminus Y_j) \geq \varepsilon^{n-1+\mu} \geq M^n_{*}(B(x, \varepsilon) \setminus Y_j^o).
\]

It then follows from Theorem 4 ((i) and (iv)) that \( V_L(Y_j) = M_{LV}(Y_j) = E_{LV}(Y_j) \). Thus by Proposition 5, one can find \( h \in L(\Omega) \) such that

\[
\|f - h\|_{F\setminus K_j} < \frac{1}{j}.
\]

The function \( h \) has the desired properties.

It can be proved that Theorem 6 remains true if we require only \( C^r \)-smoothness of \( \partial \Omega \).
7 Remerciements

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References


