# Some Remarks on Jack's Lemma*

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*Dedicated to Prof. Petru Mocanu on the occasion of his 70th birthday.

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## Introduction and Statement of the Results

Let $r > 0$ and $D_r$ denote the disc $\{z \mid |z| < r\}$. We write $D_1 = D$, the open unit disc and $\overline{D}_r$ the closure of $D_r$. For a subset $X$ of the complex plane let $\mathcal{H}(X)$ be the set of functions analytic in some domain containing $X$; let also $\mathcal{B} = \{f \in \mathcal{H}(D) \mid |f(z)| \le |z|, z \in D\}$ and $S$ the set of univalent functions in $\mathcal{H}(D)$ normalized as usual.

Jack's famous lemma may be formulated as follows: *let $n \ge 1$ and $f \in \mathcal{H}(D)$ with a $n$-fold zero at the origin and $|f(z)| = \max_{|z|=r} |f(z)| > 0$ for some $0 < r < 1$. Then $rf'(r)/f(r) \ge n$, with equality if and only if $f(z) \equiv Kz^n$, $K \in \mathbb{C}$.* The result appeared in a paper by Jack [2] who attributed the proof to Clunie. It has been told that the result is contained implicitely in the work of Valiron and for sure is equivalent to a result of Loewner ([5], p. 373). A weaker statement appears in a paper of Hayman [1] and Ruscheweyh [6] has shown how the lemma follows from Julia's theorem. Miller and Mocanu [3] gave another proof (with the additional statement $\mathrm{Re}\big(1 + rf''(r)/f'(r)\big) \ge rf'(r)/f(r)\big)$ and made extensive use of it in their work on differential subordinations; their recent book [4] contains an encyclopedic set of references related to this topic. The case of equality in the lemma does not seem to be universally known and we shall supply a proof of it at the end of this work.

Our aim in this note is threefold. First we shall study how Jack's lemma can be used as to obtain information concerning points where an analytic function attains its minimum modulus. Next we shall introduce a sharpened version of the lemma for univalent functions and finally see what information can be obtained by taking into account two Taylor coefficients of functions. We shall prove

**Theorem 1.** *Let $n \ge 1$, $0 < r$ and $f(z) := \sum_{k=n}^{\infty} a_k z^k \in \mathcal{H}(\overline{D}_r)$. Let also $r\alpha_j$ $(0 < |\alpha_j| < 1,\ j = 1, 2, \ldots, N)$ be the zeros of $f$, including multiplicities, in the pointed disc $\overline{D}_r \setminus \{0\}$ and $|f(r)| = \min_{|z|=r} |f(z)|$. Then*

$$r\frac{f'(r)}{f(r)} \le n + \sum_{j=1}^{N} \frac{1 + |\alpha_j|}{1 - |\alpha_j|}.$$

**Theorem 2.** *Let $f(z) := \sum_{n=1}^{\infty} a_n z^n \in \mathcal{H}(D_r \cup \{r\})$ and univalent in $D_r$. Let also $|f(r)| = \max_{|z|<r} |f(z)|$; then*

$$\sqrt{\frac{|f(r)|}{r|a_1|}} \le r\frac{f'(r)}{f(r)}.$$

**Theorem 3.** *Let $n \ge 2$ and $f(z) := a_1 z + a_n z^n + \sum_{k=n+1}^{\infty} a_k z^k \in \mathcal{H}(\overline{D}_r)$ with $0 < |f(r)| = \max_{|z|\le r} |f(z)|$. Then*

$$r\frac{f'(r)}{f(r)} \ge \frac{(n-1)|f(r)| - (n-3)|a_1|r}{|f(r)| + |a_1|r} + \frac{2\big(|f(r)| - |a_1|r\big)^2}{|f(r)|^2 + |a_n|r^n|f(r)| - |a_1|^2 r^2}$$

Some applications of our results to the class $S$ will also be given.

## Proof of Theorem 1

Consider the Blaschke product $b(z) = \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}$ and the function $F(z) = z^{n+1}\frac{b(z/r)}{f(z)} \in \mathcal{H}(\overline{D}_r)$. Clearly $F$ has a zero of order at least 1 at the origin and $|F(r)| = \max_{|z|\le r} |F(z)|$. By Jack's lemma,

$$r\frac{F'(r)}{F(r)} = n + 1 + \frac{b'(1)}{b(1)} - r\frac{f'(r)}{f(r)} \ge 1$$

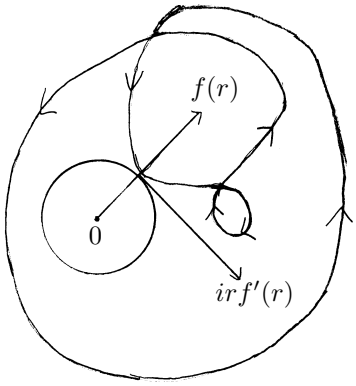
and since for $|u| = 1$,

$$u\frac{B'(u)}{B(u)} = \sum_{j=1}^{N} \frac{1 - |\alpha_j|^2}{|1 - \bar{\alpha}_j u|^2} \le \sum_{j=1}^{N} \frac{1 + |\alpha_j|}{1 - |\alpha_j|}$$

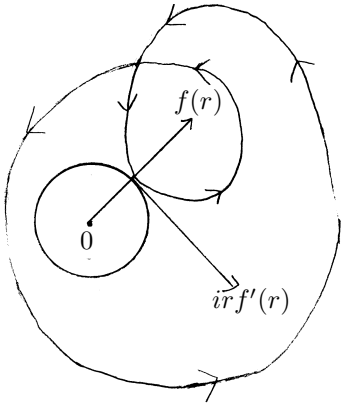
the result follows. The given estimate is best possible as seen from any function of the type $f_n(z) = z^n B(z/r)$ where $B$ is a finite Blaschke product with strictly positive zeros. It is interesting to remark that sometimes the quotient $rf'(r)/f(r)$ may be negative; for example the functions $g_n(z) = ze^{-z^n}$ are such that

$$\min_{|z|=r>0} |g_n(z)| = g_n(r)$$

and for example $r_n\frac{g'(r_n)}{g_n(r_n)} = 1 - n(1 - 1/n)^n \to -\infty$ if $r_n = 1 - 1/n$.




case 1




case 2

The result of Theorem 1 can be improved to

$$\frac{1-r}{1+r} \le r\frac{f'(r)}{f(r)} \le 1 \tag{1}$$

if the function $f$ is assumed to be univalent in $D$ and $0 < r < 1$. The right-hand side estimate follows from Theorem 1 upon noticing that for a univalent $f$, we must have $n = 1$ and the Blaschke product $b(z)$ must reduce to a constant. We notice that for a univalent $f$, $rf'(r)/f(r)$ is positive. This is best explained by a picture: if $rf'(r)/f(r) < 0$, at least one of the following "minimal" outcomes shall hold (remember that the analytic curve $f(\partial D_r)$ must wind about the origin):

In case 1 $f$ is not univalent but still locally univalent. In case 2 $f$ is not locally univalent. If therefore $rf'(r)/f(r) > 0$, we obtain by a well-known distortion theorem.

$$r\frac{f'(r)}{f(r)} = \left| r\frac{f'(r)}{f(r)} \right| \ge \frac{1-r}{1+r}$$

where equality shall hold for the function $-k(-z)$ where $k(z) = z/(1-z)^2$. The right-hand side of (1) is also sharp as seen from the identity function.

## Proof of Theorem 2

Our proof is based on the following

**Lemma 1.** *Let $w(z) = \sum_{k=1}^{\infty} a_k z^k \in \mathcal{B} \cap \mathcal{H}(\{1\})$. Assume that $w$ is univalent and $w(1) = 1$. Then*

$$w'(1) \ge |a_1|^{-1/2}.$$

Let us consider, for fixed $x$ with $|x| = 1$, the function $v(z) := \frac{k(xw(z))}{xa_1}$. Clearly $v \in S$ and by the growth theorem,

$$|v(z)| = \frac{\left|k\big(xw(z)\big)\right|}{|a_1|} \le k(|z|), \quad z \in D. \tag{2}$$

Since $x$ is arbitrary, we obtain from (2)

$$|w(z)| \le k^{-1}\big(|a_1|k(|z|)\big), \quad z \in D.$$

It follows from Loewner's result quoted above that $w'(1)$ is a positive number and by (2)

$$\begin{aligned} w'(1) = |w'(1)| &= \lim_{z\to 1}\left|\frac{w(z)-w(1)}{z-1}\right| \\ &\ge \lim_{\substack{z\to 1 \\ 0<z<1}} \frac{1-|w(z)|}{1-z} \\ &\ge \lim_{\substack{z\to 1 \\ 0<z<1}} \frac{1-k^{-1}\big(|a_1|k(z)\big)}{1-z}. \end{aligned} \tag{3}$$

The function $k^{-1}\big(|a_1|k(z)\big)$ is well-defined and belongs to $\mathcal{H}(D)$ (remember that $k(D)=\mathbb{C}\setminus(-\infty,-\frac{1}{4}]$ is a starlike domain with respect to the origin); indeed the function is also analytic at $z=1$. This can be seen by computing $k^{-1}(u)=\frac{2u}{1+2u+\sqrt{1+4u}}$, $u\in k(D)$ (here $\sqrt{1}=1$), and

$$k^{-1}\big(|a_1|k(z)\big)=\frac{2|a_1|z}{(1-z)^2+2|a_1|z+(1-z)\sqrt{(1-z)^2+4|a_1|}},z\in D.$$

Further $k^{-1}\big(|a_1|k(1)\big)=1$ and

$$\frac{1-k^{-1}\big(|a_1|k(z)\big)}{1-z}=\frac{1-z+\sqrt{(1-z)^2+4|a_1|z}}{(1-z)^2+2|a_1|z+(1-z)\sqrt{(1-z)^2+4|a_1|z}}. \tag{4}$$

A combination of (3) and (4) yields $w'(1)\geq|a_1|^{-1/2}$.

The proof of the Theorem is now rather straightforward; we write $w(z):=\frac{f(rz)}{f(r)}=\frac{a_1r}{f(r)}z+\cdots$. Lemma 1 applies for this $w$ and we obtain

$$r\frac{f'(r)}{f(r)}=w'(1)\geq\left(\frac{|a_1|r}{|f(r)|}\right)^{-1/2}.$$

This completes the proof. The estimate given is sharp: let us define, given $r>0$ and $t\in(0,1]$,

$$F(z)=Kk^{-1}\big(tk(z/r)\big),\quad z\in D_r$$

where $K$ is a fixed complex number. Clearly $F\in\mathcal{H}(D_r\cup\{r\})$ and fulfils the hypothesis of Theorem 2. A simple computation yields $F(r)=K$, $F'(r)=\frac{K}{r\sqrt{t}}$, $F'(0)=\frac{Kt}{r}$ and $r\frac{F'(r)}{F(r)}=\sqrt{\frac{|F(r)|}{r|F'(0)|}}$. It may also be of some interest to remark that Theorem 2 does not hold for at least one (non-univalent!) locally univalent function: let $0<r$, $p<\pi/r$ and $G(z):=e^{pz}-1$. Then $G$ is entire, locally univalent but non-univalent in $D_r$; we have $G(r)=\max_{|z|\leq r}|G(z)|$ and

$$r\frac{G'(r)}{G(r)}=\frac{rp}{1-e^{-rp}}<\sqrt{\frac{e^{rp}-1}{rp}}=\sqrt{\frac{|G(r)|}{r|G'(0)|}}$$

if $p$ is large enough.

We finally notice that for functions $f\in S$ as in Theorem 2 and $0<r<1$,

$$\sqrt{\frac{|f(r)|}{r}}\leq r\frac{f'(r)}{f(r)}\leq\frac{1+r}{1-r}$$

which is easily seen to be a sharp estimate.

## Proof of Theorem 3

We shall need

**Lemma 2.** *Let $t\in\overline{D}$, $n\geq2$ and $w(z)=a_1z+t(1-|a_1|^2)z^n+\sum_{k=n+1}^{\infty}a_kz^k\in\mathcal{B}\cap H(\{1\})$. Then, if $w(1)=1$,*

$$w'(1)\geq\frac{n+1+(n-1)|t|-(n-3)|t||a_1|-(n-1)|a_1|}{(1+|a_1|)(1+|t|)}.$$

The proof of the lemma is a standard application of Schwarz lemma. For $w$ as above,

$$\frac{1}{z^{n-1}}\frac{w(z)/z-a_1}{1-\bar{a}_1w(z)/z}=\frac{\rho(z)+t}{1+\bar{t}\rho(z)},\quad\rho\in\mathcal{B}$$

and

$$|w(z)|\leq|z|\frac{p_n(|z|)}{|z|^np_n(1/|z|)},\quad z\in D,$$

where the polynomial $p_n(u) := u^n + |t|u^{n-1} + |t||a_1|u + |a_1|$ has all its zeros in $\overline{D}$. As before, $w'(1)$ is real and positive and

$$\begin{aligned} w'(1) &\geq \lim_{\substack{z\to 1\\ 0<z<1}} \frac{1-|w(z)|}{1-|z|} \\ &\geq \lim_{\substack{z\to 1\\ 0<z<1}} \frac{1-p_n(z)/z^{n-1}p_n(1/z)}{1-z} \\ &= 2\frac{p'(1)}{p(1)} - (n-1) \\ &= \frac{n+1+(n-1)|t| - (n-3)|t||a_1| - (n-1)|a_1|}{(1+|a_1|)(1+|t|)}. \end{aligned} \tag{5}$$

The latter estimate is best possible since equality shall hold for the Blaschke products $w_n(z) = z\frac{p_n(z)}{z^n p_n(1/z)}$, where $0 < t < 1$, $0 < a_1 < 1$. It is remarkable that the sequence $\{w_n\}$ converges locally uniformly in $D$ to a multiple of the identity function while the right-hand side of (5) tends to infinity with $n$.

Theorem 3 follows easily from Lemma 2 by defining $w(z) := \frac{f(rz)}{f(r)} = \frac{a_1 r}{f(r)}z + \frac{a_n r^n}{f(r)}z^n + \cdots$ with $t = \frac{a_n r^n \overline{f(r)}}{|f(r)|^2 - |a_1|^2 r^2}$. We leave the details to the reader. A minimization of (5) with respect to $|t|$ (the minimum occurs when $|t| = 1$) yields the better looking estimate

$$r\frac{f'(r)}{f(r)} \geq \frac{n|f(r)| - (n-2)|a_1|r}{|f(r)| + |a_1|r}$$

which is sharp and almost as good as the Miller-Mocanu estimate in [3]. Finally when $n = 2$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is univalent and as in Theorem 3 our method shall give

$$r\frac{f'(r)}{f(r)} \geq \sqrt{\frac{|f(r)|^2 - |a_0|^2}{r|f(r)||a_1|}}\frac{|f(r) - a_0|^2}{|f(r)|^2 - |a_0|^2},$$

a slightly better estimate than in [3].

We shall prove here our claim concerning the case of equality in Jack's lemma.

Let first $w \in \mathcal{H}(\mathbb{D} \cup \{1\})$, $w(z) = a_n z^n + \cdots$, $w(D) \subseteq D$ and $w(1) = 1$. It follows that [5] $w'(1) \geq n$. We shall prove that $w'(1) = n$ if and only if $w(z) \equiv z^n$. Let $w(z) = z^n w_1(z)$ where $w_1(0) = a_n$, $w_1 \in H(D \cup \{1\})$, $w_1(D) \subseteq D$ and $w_1(1) = 1$; if $w_1$ is non-constant then $|a_n| < 1$ and we write $w_2(z) = \frac{w_1(z) - a_n}{1 - \bar{a}_n w_1(z)}\frac{1-\bar{a}_n}{1-a_n}$. Again by Loewner's result (applied to $w_2$), we get $w_1'(1) \geq \frac{|1-a_n|^2}{1-|a_n|^2} > 0$ and since $w'(z) = nz^{n-1}w_1(z) + z^n w_1'(z)$, we have

$$n = w'(1) = nw_1(1) + w_1'(1) \geq n + \frac{|1-a_n|^2}{1-|a_n|^2} > 0,$$

a contradiction $\cdots$. In other words $w'(1) = n \Leftrightarrow w(z) \equiv z^n$.

The case of equality for Jack's lemma follows at once: if $f \in \mathcal{H}(\overline{D}_r)$ with $f(z) = \alpha_n z^n + \cdots$ and $f(r) = \max_{|z|=r}|f(z)| > 0$, then, with $w(z) = \frac{f(rz)}{f(r)}$ we have

$$r\frac{f'(r)}{f(r)} = w'(1) = n \iff f(z) \equiv Kz^n, \quad K \in \mathbb{C}.$$

# References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