Series solution of an eddy current problem for a sphere with varying conductivity and permeability profiles

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Abstract

In the present paper two-parameter families of analytical solutions are found in the case of a single-turn coil symmetrically located above a two-layer sphere. Several cases are considered where the conductivity and relative magnetic permeability of the outer spherical layer are functions of the distance, ρ, from the sphere's center. The change in impedance of the coil is obtained in terms of a series containing Bessel functions. Computational impedance results are presented for different values of the parameters of the problem and the exact conductivity and permeability profiles are given diagrammatically for comparison's purpose with the impedance results.

Index Terms — eddy current; conductivity; permeability; nondestructive testing; change in impedance.

Résumé

On présente des solutions analytiques biparamétriques d'un problème d'une bobine à un seul tour en position symétrique au-dessus d'une sphère à deux couches. On considère plusieurs cas où la conductivité et la perméabilité magnétique relative de la couche extérieure est fonction de la distance ρ du centre de la sphère. On exprime le changement d'impédance de la bobine au moyen d'une série de fonctions de Bessel. On compare, sur figures, la conductivité exacte et les profiles de la perméabilité avec les résultats numériques sur l'impédance pour plusieurs valeurs des paramètres du problème.

1 INTRODUCTION

Industrial eddy current testing is widely used to control the size of products, to measure the thickness of metal coverings, and to determine the electrical conductivity of materials. Analytical solutions for eddy current testing problems in Cartesian, cylindrical or spherical geometry are known when the conductivity σ and the magnetic permeability μ of the medium are constant (see, for example, [1] and [2]).

Many industrial processes, such as surface hardening, decarburization, and spot welding, modify the electric and magnetic properties of materials. These modifications have to be taken into account when computing the change in impedance of an eddy current probe. One approach to the solution of eddy current problems with varying properties of the conducting medium is described in [3] and [4], where the medium is divided into a large number of subregions each of which having constant σ and μ.

In some cases, however, analytical solutions for eddy current testing problems can be found if the conductivity and/or magnetic permeability of the medium are functions of a single spatial coordinate. Examples of such solutions for a layered medium and for a cylindrical geometry are given in [5]–[7]. Of course, closed-form analytical solutions in terms of special functions can be found only for certain conductivity and/or magnetic permeability profiles. The purpose of constructing such solutions is twofold: first, the change in impedance of an eddy current probe can be found if the actual variations of σ and μ follow the formulae for which these solutions are constructed, and, second, analytical solutions can be used as benchmarks in more complicated cases where no analytical solution is available and solutions have to be constructed by numerical methods. At this stage, no experimental data nor independent numerical solutions of the problem in hand have been found to validate in some meaningful sense the analytical solution presented here.

The change in impedance of a single-turn coil due to the presence of a conducting two-layer sphere symmetrically situated with respect to the coil is computed in [8] where the conductivities of both layers are constant, but the relative magnetic permeability of the outer layer is a function of the distance ρ from the center of the sphere, namely, $\mu(\rho) = \rho^\alpha$, where α is an arbitrary real number.

In this paper, two-parameter families of analytical solutions are found for the case where a single-turn coil is symmetrically situated above a two-layer sphere with varying conductivity and relative magnetic permeability. Analytical solutions will be obtained for three types of conductivity and magnetic permeability profiles: $\sigma = \sigma_1(1 + \gamma/\rho^2)$, $\mu = \mu_1 = 1$; $\sigma = \sigma_1\rho^\alpha$, $\mu = \mu_1 = 1$; and $\sigma = \sigma_1\rho^\alpha$, $\mu = \mu_1\rho^\gamma$, where α and γ are arbitrary real numbers. The change in impedance in the coil is found in the form of an infinite series. Numerical impedance results are given for different values of the parameters of the problem and the exact conductivity and permeability profiles are given diagrammatically for comparison's purpose with the impedance results.

2 FORMULATION OF THE PROBLEM

Consider a two-layer sphere made of an inner ball of radius ρ_2 and an outer spherical shell of outer radius ρ_1. Consider also a coil of radius r_c located outside the sphere so that the axis of the coil passes through the center of the sphere (see Fig. 1).

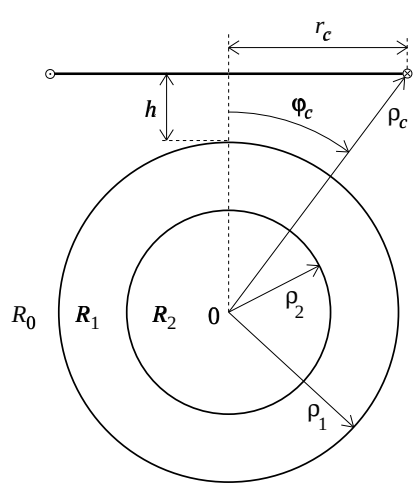

Figure 1: A single-turn coil above a two-layer sphere.

We introduce a spherical coordinate system (ρ, θ, φ) with center at O, where θ and φ stand for the longitude and latitude, respectively. We let the current in the coil be given by

$$\mathrm{I}^e = I_\theta^e \, e^{j\omega t} \mathrm{e}_\theta,$$

where $j = \sqrt{-1}$, ω is the frequency, I_θ^e is the current amplitude, and e_θ is a unit vector orthogonal to e_ρ and e_φ.

We consider the three regions R_0, R_1, and R_2 defined as follows:

(a) the empty space R_0: $\rho > \rho_1$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$, containing air;

(b) the outer spherical shell R_1: $\rho_2 < \rho < \rho_1$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$, which is a conducting medium with variable conductivity and relative magnetic permeability $\sigma_1(\rho)$ and $\mu_1(\rho)$;

(c) the inner ball R_2: $0 \le \rho < \rho_2$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$, which is a conducting medium with constant conductivity and relative magnetic permeability σ_2 and μ_2.

Introducing the vector potential A by the formula

$$\operatorname{curl} \mathrm{A} = \mathrm{B},$$

where B is the magnetic induction vector and taking into account the axial symmetry, we obtain that A has only one nonzero component, namely,

$$\mathrm{A} = (0, A(\rho, \varphi), 0)\, e^{j\omega t}.$$

Using Maxwell's equations and neglecting the displacement current we obtain the following equation for the function $A(\rho, \varphi)$:

$$\begin{aligned} &\frac{\partial^2 A}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial A}{\partial \rho} + \frac{\cot\varphi}{\rho^2}\frac{\partial A}{\partial \varphi} + \frac{1}{\rho^2}\frac{\partial^2 A}{\partial \varphi^2} - \frac{A}{\rho^2 \sin^2\varphi} \\ &\quad - \frac{1}{\mu(\rho)}\frac{d\mu}{d\rho}\left(\frac{A}{\rho} + \frac{\partial A}{\partial \rho}\right) - j\omega\sigma(\rho)\mu_0\mu(\rho)A = -\mu_0 I_\theta^e, \end{aligned} \tag{1}$$

where μ_0 is the magnetic constant.

3 MATHEMATICAL ANALYSIS

In general, analytical solutions to (1) can be found only for certain conductivity and relative magnetic permeability profiles $\sigma = \sigma(\rho)$ and $\mu = \mu(\rho)$, respectively. In the present paper, we consider three two-parameter families of conductivity and magnetic permeability profiles, namely,

$$\sigma(\rho) = \sigma_1(1 + \gamma/\rho^2), \qquad \mu(\rho) = \mu_1 = 1, \tag{2}$$

$$\sigma(\rho) = \sigma_1 \rho^\alpha, \qquad \mu(\rho) = \mu_1 = 1, \tag{3}$$

and

$$\sigma(\rho) = \sigma_1 \rho^\alpha, \qquad \mu(\rho) = \mu_1 \rho^\gamma, \tag{4}$$

where σ_1 and μ_1 are positive constants and α and γ are arbitrary real numbers. By choosing different values for γ and α (both positive or negative) formulae (2)–(4) can be used to model monotonically increasing or decreasing conductivity and relative magnetic permeability profiles.

Case 1. Consider the case where σ and μ are given by (2). Let $\rho_d = \rho/r_c$ be the dimensionless radial coordinate. In the sequel, the subscript d to ρ_d will be omitted and all the geometric quantities will be measured in units of r_c. We present the solution for Case 1 in greater detail while only the main formulae will be given for the remaining two cases.

Rewriting equation (1) for each of the subregions R_0, R_1 and R_2 we obtain

$$\begin{aligned} &\frac{\partial^2 A_0}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial A_0}{\partial \rho} + \frac{\cot\varphi}{\rho^2}\frac{\partial A_0}{\partial \varphi} + \frac{1}{\rho^2}\frac{\partial^2 A_0}{\partial \varphi^2} \\ &\quad - \frac{A_0}{\rho^2 \sin^2\varphi} = -\mu_0 I r_c^2 \delta(\rho - \rho_c)\delta(\varphi - \varphi_c), \quad \rho > \rho_1, \end{aligned} \tag{5}$$

$$\frac{\partial^2 A_1}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial A_1}{\partial \rho} + \frac{\cot\varphi}{\rho^2}\frac{\partial A_1}{\partial \varphi} + \frac{1}{\rho^2}\frac{\partial^2 A_1}{\partial \varphi^2} - \frac{A_1}{\rho^2 \sin^2\varphi} - j\beta_1^2 f(\rho) A_1 = 0, \quad \rho_2 < \rho < \rho_1, \tag{6}$$

$$\frac{\partial^2 A_2}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial A_2}{\partial \rho} + \frac{\cot\varphi}{\rho^2}\frac{\partial A_2}{\partial \varphi} + \frac{1}{\rho^2}\frac{\partial^2 A_2}{\partial \varphi^2} - \frac{A_2}{\rho^2 \sin^2\varphi} - j\beta_2^2 A_2 = 0, \quad 0 \le \rho < \rho_2, \tag{7}$$

where

$$\beta_1 = r_c\sqrt{\omega\sigma_1\mu_0\mu_1}, \quad \beta_2 = r_c\sqrt{\omega\sigma_2\mu_0\mu_2},$$

$A_i(\rho,\varphi)$ denotes the θ-component of the vector potential in region R_i, for $i = 0, 1, 2$, $\delta(x)$ is the Dirac delta function, and $f(\rho) = 1 + \gamma/\rho^2$. The boundary conditions are

$$A_0\Big|_{\rho=\rho_1} = A_1\Big|_{\rho=\rho_1}, \qquad \frac{\partial A_0}{\partial \rho}\Big|_{\rho=\rho_1} = \frac{\partial A_1}{\partial \rho}\Big|_{\rho=\rho_1}, \tag{8}$$

$$A_1\Big|_{\rho=\rho_2} = A_2\Big|_{\rho=\rho_2}, \qquad \frac{\partial A_1}{\partial \rho}\Big|_{\rho=\rho_2} = \frac{\partial A_2}{\partial \rho}\Big|_{\rho=\rho_2}. \tag{9}$$

Note that the second condition in (9) is written for the case $\mu_2 = 1$ since only this case is used in the computations.

To solve problem (5)–(9), we use the following integral transform

$$\widetilde{A}_i(\rho, n) = \frac{1}{D_n}\int_{-1}^{1} \widehat{A}_i(\rho,\xi) P_n^{(1)}(\xi)\, d\xi, \qquad i = 0, 1, 2, \tag{10}$$

where $\widehat{A}_i(\rho,\xi) = A_i(\rho, \cos\varphi)$, $P_n^{(1)}(\xi)$ is an associated Legendre function of the first kind, and

$$D_n := \int_{-1}^{1} \left[P_n^{(1)}(\xi)\right]^2 d\xi = \frac{2n(n+1)}{2n+1}.$$

Using the integral transform (10) we obtain

$$\frac{d^2\widetilde{A}_0}{d\rho^2} + \frac{2}{\rho}\frac{d\widetilde{A}_0}{d\rho} - \frac{n(n+1)}{\rho^2}\widetilde{A}_0 = -\mu_0 I r_c^2 \frac{2n+1}{2n(n+1)} P_n^{(1)}(\cos\varphi_c)\sin\varphi_c\, \delta(\rho - \rho_c), \tag{11}$$

$$\frac{d^2\widetilde{A}_1}{d\rho^2} + \frac{2}{\rho}\frac{d\widetilde{A}_1}{d\rho} - \frac{n(n+1)}{\rho^2}\widetilde{A}_1 - j\beta_1^2 f(\rho)\widetilde{A}_1 = 0, \tag{12}$$

$$\frac{d^2\widetilde{A}_2}{d\rho^2} + \frac{2}{\rho}\frac{d\widetilde{A}_2}{d\rho} - \frac{n(n+1)}{\rho^2}\widetilde{A}_2 - j\beta_2^2 \widetilde{A}_2 = 0, \tag{13}$$

with the boundary conditions

$$\widetilde{A}_0\Big|_{\rho=\rho_1} = \widetilde{A}_1\Big|_{\rho=\rho_1}, \qquad \frac{d\widetilde{A}_0}{d\rho}\Big|_{\rho=\rho_1} = \frac{d\widetilde{A}_1}{d\rho}\Big|_{\rho=\rho_1}, \tag{14}$$

$$\widetilde{A}_1\Big|_{\rho=\rho_2} = \widetilde{A}_2\Big|_{\rho=\rho_2}, \qquad \frac{d\widetilde{A}_1}{d\rho}\Big|_{\rho=\rho_2} = \frac{d\widetilde{A}_2}{d\rho}\Big|_{\rho=\rho_2}. \tag{15}$$

It is convenient to find solutions $\widetilde{A}_{01}$ and $\widetilde{A}_{02}$ to equation (11) in the two subregions $R_{01} = \{\rho_1 < \rho < \rho_c\}$ and $R_{02} = \{\rho > \rho_c\}$ of R_0, respectively. Thus, the bounded general solutions to (11) in R_{01} and R_{02} are

$$\widetilde{A}_{01}(\rho, n) = C_1\rho^n + C_2\rho^{-n-1}, \quad \rho_1 < \rho < \rho_c, \tag{16}$$

and

$$\widetilde{A}_{02}(\rho, n) = C_3\rho^{-n-1}, \quad \rho > \rho_c, \tag{17}$$

respectively. Since the vector potential is continuous at $\rho = \rho_c$, then

$$\widetilde{A}_{01}\Big|_{\rho=\rho_c} = \widetilde{A}_{02}\Big|_{\rho=\rho_c}. \tag{18}$$

Multiplying equation (11) by ρ^2, integrating from $\rho = \rho_c - \varepsilon$ to $\rho = \rho_c + \varepsilon$, and taking the limit as $\varepsilon \to +0$, we obtain the boundary condition

$$\frac{d\widetilde{A}_{02}}{d\rho}\Big|_{\rho=\rho_c} - \frac{d\widetilde{A}_{01}}{d\rho}\Big|_{\rho=\rho_c} = -\mu_0 I \tilde{\rho}_c^2 \frac{2n+1}{2n(n+1)} P_n^{(1)}(\cos\varphi_c)\sin\ \varphi_c. \tag{19}$$

Note that, in (19), $\tilde{\rho}_c$ is a variable with dimension. The general solution to (12) for the case $f(\rho) = 1 + \gamma/\rho^2$ is expressed in terms of Bessel functions (see [9], p. 146)

$$\widetilde{A}_1(\rho, n) = C_4 \rho^{-1/2} J_p(\beta\rho) + C_5 \rho^{-1/2} Y_p(\beta\rho), \tag{20}$$

where

$$p = \sqrt{j\beta_1^2\gamma + (n+1/2)^2}, \qquad \beta = \beta_1\sqrt{-j}.$$

The general solution to equation (13), which remains bounded as $\rho \to 0$, is given by

$$\widetilde{A}_2(\rho, n) = \frac{C_6}{\sqrt{\rho}} J_{n+1/2}(k\rho), \tag{21}$$

where $k = \beta_2\sqrt{-j}$. The arbitrary constants C_1 to C_6 in (16), (17), (20), and (21) are determined from the boundary conditions (14), (15), (18), and (19), that is,

$$\begin{aligned}
C_1 &= \frac{\mu_0 I \tilde{\rho}_c^2}{2n(n+1)} P_n^{(1)}(\cos\varphi_c)\sin\varphi_c\, \rho_c^{-n+1}, && (22)\\
C_2 &= C_1 \rho_1^{2n+1} c_{11}/c_{12}, && (23)\\
C_3 &= C_2 + C_1 \rho_c^{2n+1}, && (24)\\
C_4 &= a C_5, && (25)\\
C_5 &= \frac{C_1 \rho_1^{n+1/2} + C_2 \rho_1^{-n-1/2}}{a J_p(\beta\rho_1) + Y_p(\beta\rho_1)}, && (26)\\
C_6 &= \frac{C_4 J_p(\beta\rho_2) + C_5 Y_p(\beta\rho_2)}{J_{n+1/2}(k\rho_2)}, && (27)
\end{aligned}$$

where

$$a = \frac{a_{11}}{a_{12}},$$

$$a_{11} = kY_p(\beta\rho_2)J'_{n+1/2}(k\rho_2) - \beta Y'_p(\beta\rho_2)J_{n+1/2}(k\rho_2),$$

$$a_{12} = \beta J'_p(\beta\rho_2)J_{n+1/2}(k\rho_2) - kJ_p(\beta\rho_2)J'_{n+1/2}(k\rho_2),$$

and

$$\begin{aligned}
c_{11} &= (2n+1)aJ_p(\beta\rho_1) + (2n+1)Y_p(\beta\rho_1)\\
&\quad -2a\beta\rho_1 J'_p(\beta\rho_1) - 2\beta\rho_1 Y'_p(\beta\rho_1),\\
c_{12} &= (2n+1)aJ_p(\beta\rho_1) + (2n+1)Y_p(\beta\rho_1)\\
&\quad +2a\beta\rho_1 J'_p(\beta\rho_1) + 2\beta\rho_1 Y'_p(\beta\rho_1).
\end{aligned}$$

Inverting the integral transform (10) we obtain the solution to problem (5)–(9) in the form

$$A_i(\rho, \varphi) = \sum_{n=1}^{\infty} \widetilde{A}_i(\rho, n) P_n^{(1)}(\cos\varphi), \qquad i = 0, 1, 2. \tag{28}$$

The change in the coil impedance can be computed by the formula

$$Z = \frac{j\omega}{I} \oint_C \mathrm{A}_0^{\mathrm{ind}}(\rho, \varphi) \cdot d\mathrm{l}, \tag{29}$$

where C is the coil contour,

$$\mathrm{A}_0^{\mathrm{ind}}(\rho,\varphi) = A_0^{\mathrm{ind}}(\rho,\varphi)\mathrm{e}_\theta,$$

and $A_0^{\mathrm{ind}}(\rho,\varphi)$ is the induced part of the vector potential. The function $A_0^{\mathrm{ind}}(\rho,\varphi)$ is given by (28) with $i=0$ where $\widetilde{A}_0(\rho,n)$ is replaced with

$$\widetilde{A}_0^{\mathrm{ind}}(\rho,n) = C_2\rho^{-n-1}.$$

Substituting (28) into (29) we obtain the change in impedance in the form

$$Z = \pi\omega\mu_0\rho_1^2\tilde{\rho}_c^2\sin^2(\varphi_c)Z_0,$$

where, in the sequel, the results are computed in terms of the dimensionless variable

$$Z_0 = j\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(\frac{\rho_1}{\rho_c}\right)^{2n-1}\left[P_n^{(1)}(\cos\varphi_c)\right]^2\frac{c_{11}}{c_{12}}. \tag{30}$$

Case 2. Next, consider the case where σ and μ are given by (3). In this case the function $f(\rho)$ in (12) is

$$f(\rho) = \rho^\alpha$$

and the general solution to (12) can be written in the form (see [9], p. 146):

$$\widetilde{A}_1(\rho,n) = C_4\rho^{-1/2}J_p\left(\beta\rho^\nu\right) + C_5\rho^{-1/2}Y_p\left(\beta\rho^\nu\right), \tag{31}$$

where

$$\nu = \frac{\alpha}{2}+1, \quad p = \frac{n+1/2}{\nu}, \quad \beta = \frac{\beta_1\sqrt{-j}}{\nu}.$$

Determining the arbitrary constants in (16), (17), (21), and (31) from the boundary conditions (14), (15), (18), and (19) one can see that the structure of formulae (22)–(25) will be the same, but the coefficients c_{11}, c_{12} in (23) and a in (25) are replaced by

$$a = \frac{a_{11}}{a_{12}}, \tag{32}$$

where

$$\begin{aligned} a_{11} &= k\rho_2 J'_{n+1/2}(k\rho_2)Y_p\left(\beta\rho_2^\nu\right) - \beta\nu\rho_2^\nu Y'_p\left(\beta\rho_2^\nu\right)J_{n+1/2}(k\rho_2), \\ a_{12} &= \beta\nu\rho_2^\nu J'_p\left(\beta\rho_2^\nu\right)J_{n+1/2}(k\rho_2) - k\rho_2 J'_{n+1/2}(k\rho_2)J_p\left(\beta\rho_2^\nu\right), \end{aligned}$$

and

$$\begin{aligned} c_{11} &= (2n+1)aJ_p\left(\beta\rho_1^\nu\right) + (2n+1)Y_p\left(\beta\rho_1^\nu\right) \\ &\quad -2\beta\nu a\rho_1^\nu J'_p\left(\beta\rho_1^\nu\right) - 2\beta\nu\rho_1^\nu Y'_p\left(\beta\rho_1^\nu\right), \end{aligned} \tag{33}$$

$$\begin{aligned} c_{12} &= (2n+1)aJ_p\left(\beta\rho_1^\nu\right) + (2n+1)Y_p\left(\beta\rho_1^\nu\right) \\ &\quad +2\beta\nu a\rho_1^\nu J'_p\left(\beta\rho_1^\nu\right) + 2\beta\nu\rho_1^\nu Y'_p\left(\beta\rho_1^\nu\right). \end{aligned} \tag{34}$$

Similarly, the constants C_5 and C_6 in (26) and (27) are replaced by

$$C_5 = \frac{C_1\rho_1^{n+1/2} + C_2\rho_1^{-n-1/2}}{aJ_p\left(\beta\rho_1^\nu\right) + Y_p\left(\beta\rho_1^\nu\right)},$$

and

$$C_6 = \frac{C_4J_p\left(\beta\rho_2^\nu\right) + C_5Y_p\left(\beta\rho_2^\nu\right)}{J_{n+1/2}(k\rho_2)}.$$

The change in impedance of the coil is computed by means of (30) where a, c_{11}, and c_{12} are given by (32)–(34).

Case 3. Finally, consider the case where both σ and μ are functions of ρ given by (4). The procedure described above has to be modified. First of all, equation (6) and the corresponding equation (12) in the transformed space will have a slightly different form. More precisely, substituting (4) into (1) and using the integral transform (10) one has (instead of equation (12)) the following equation

$$\frac{d^2\widetilde{A}_1}{d\rho^2} + \frac{2-\gamma}{\rho}\frac{d\widetilde{A}_1}{d\rho} - \frac{n(n+1)}{\rho^2}\widetilde{A}_1 - \left(j\beta_1^2\rho^{\alpha+\gamma} + \frac{\gamma}{\rho^2}\right)\widetilde{A}_1 = 0. \tag{35}$$

Second, the boundary conditions (8)–(9) are replaced with

$$A_0\Big|_{\rho=\rho_1} = A_1\Big|_{\rho=\rho_1}, \quad \frac{\partial}{\partial\rho}(\rho A_0)\Big|_{\rho=\rho_1} = \frac{1}{\mu_{11}}\frac{\partial}{\partial\rho}(\rho A_1)\Big|_{\rho=\rho_1}, \tag{36}$$

$$A_1\Big|_{\rho=\rho_2} = A_2\Big|_{\rho=\rho_2}, \quad \frac{1}{\mu_{12}}\frac{\partial}{\partial\rho}(\rho A_1)\Big|_{\rho=\rho_2} = \frac{1}{\mu_2}\frac{\partial}{\partial\rho}(\rho A_2)\Big|_{\rho=\rho_2}, \tag{37}$$

where $\mu_{11} = \mu_1(\rho_1) = \rho_1^\gamma$ and $\mu_{12} = \mu_1(\rho_2) = \rho_2^\gamma$. As a consequence of these changes, the general solution to (35) can be written as follows

$$\widetilde{A}_1(\rho, n) = C_4\rho^b J_p(\beta\rho^s) + C_5\rho^b Y_p(\beta\rho^s), \tag{38}$$

where

$$b = \frac{\gamma-1}{2}, \qquad s = \frac{\alpha+\gamma}{2}+1, \qquad \beta = \frac{\beta_1\sqrt{-j}}{s},$$

and

$$p = \frac{\sqrt{(\gamma+1)^2+4n(n+1)}}{\alpha+\gamma+2}.$$

The arbitrary constants C_1 to C_6 in (16), (17), (21), and (38) can be found from the boundary conditions (36) and (37) transformed by (10), and conditions (18) and (19). As in Case 2, the structure of formulae (22)–(25) remains the same, but the coefficients c_{11}, c_{12} in (23) and a in (25) are replaced by

$$a = \frac{a_{11}}{a_{12}}, \tag{39}$$

where

$$\begin{aligned} a_{11} &= \mu_{12}Y_p(\beta\rho_2^s)\left[0.5J_{n+1/2}(k\rho_2) + k\rho_2 J'_{n+1/2}(k\rho_2)\right] \\ &\quad -\mu_2 J_{n+1/2}(k\rho_2)\left[(b+1)Y_p(\beta\rho_2^s) + \beta s\rho_2^s Y'_p(\beta\rho_2^s)\right], \\ a_{12} &= \mu_2 J_{n+1/2}(k\rho_2)\left[(b+1)J_p(\beta\rho_2^s) + \beta s\rho_2^s J'_p(\beta\rho_2^s)\right] \\ &\quad -\mu_{12}J_p(\beta\rho_2^s)\left[0.5J_{n+1/2}(k\rho_2) + k\rho_2 J'_{n+1/2}(k\rho_2)\right], \end{aligned}$$

and

$$\begin{aligned} c_{11} &= \mu_{11}(n+1)\left[aJ_p(\beta\rho_1^s) + Y_p(\beta\rho_1^s)\right] - a(b+1)J_p(\beta\rho_1^s) \\ &\quad -\beta s\rho_1^s a J'_p(\beta\rho_1^s) - (b+1)Y_p(\beta\rho_1^s) - \beta s\rho_1^s Y'_p(\beta\rho_1^s), \qquad (40) \\ c_{12} &= a(b+1)J_p(\beta\rho_1^s) + \beta s\rho_1^s a J'_p(\beta\rho_1^s) + (b+1)Y_p(\beta\rho_1^s) \\ &\quad +\beta s\rho_1^s Y'_p(\beta\rho_1^s) + \mu_{11}n\left[aJ_p(\beta\rho_1^s) + Y_p(\beta\rho_1^s)\right]. \qquad (41) \end{aligned}$$

The constants C_5 and C_6 in (26) and (27) are replaced by

$$C_5 = \frac{C_1\rho_1^{n-b} + C_2\rho_1^{-n-b-1}}{aJ_p(\beta\rho_1^s) + Y_p(\beta\rho_1^s)},$$

and

$$C_6 = \frac{C_4\rho_2^{b+1/2}J_p(\beta\rho_2^s) + C_5\rho_2^{b+1/2}Y_p(\beta\rho_2^s)}{J_{n+1/2}(k\rho_2)}.$$

The change in impedance of the coil is computed by formula (30) where a, c_{11}, and c_{12} are given by (39)–(41).

4 NUMERICAL RESULTS

Formula (30) was used to compute the change in impedance, $Z_0 = X + iY$, of the coil. Computations were done with Mathematica, version 3.0, because it evaluates Bessel functions of complex order. Since the problem contains many parameters, some of them were kept fixed in our computations, namely, the gap $h = 0.3$ between the sphere and the coil (see Fig. 1), $\beta_2 = 1$, and $\mu_2 = 1$.

The first set of computations is done for Case 1 with profile (2). The outer and inner radii are $\rho_1 = 0.9$ and $\rho_2 = 0.8$. The variation of X and Y as functions of β_1 is shown in Figs. 2 and 3.

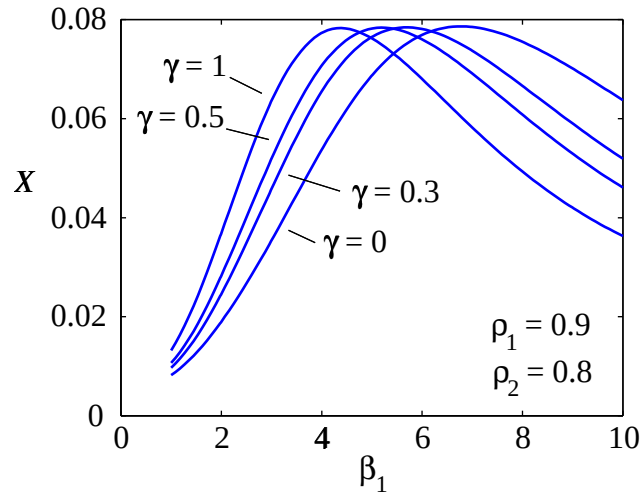

Figure 2: The real part, X, of the change in impedance as a function of β_1 for four values of γ with $\rho_1 = 0.9$, $\rho_2 = 0.8$.

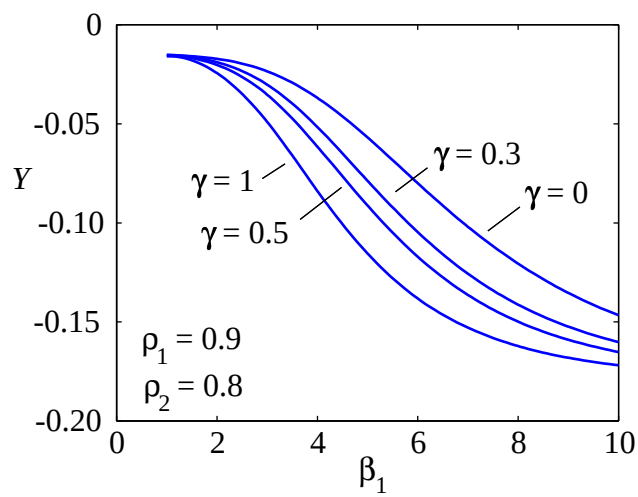

Figure 3: The imaginary part, Y, of the change in impedance as a function of β_1 for four values of γ with $\rho_1 = 0.9$, $\rho_2 = 0.8$.

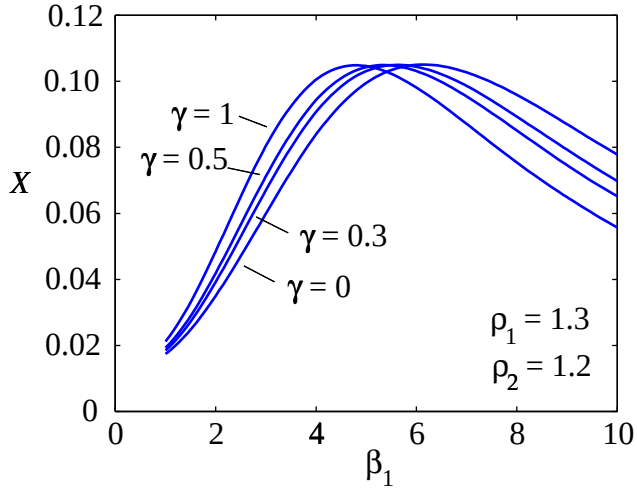

Figure 4: The real part, X, of the change in impedance as a function of β_1 for four values of γ with $\rho_1 = 1.3$, $\rho_2 = 1.2$.

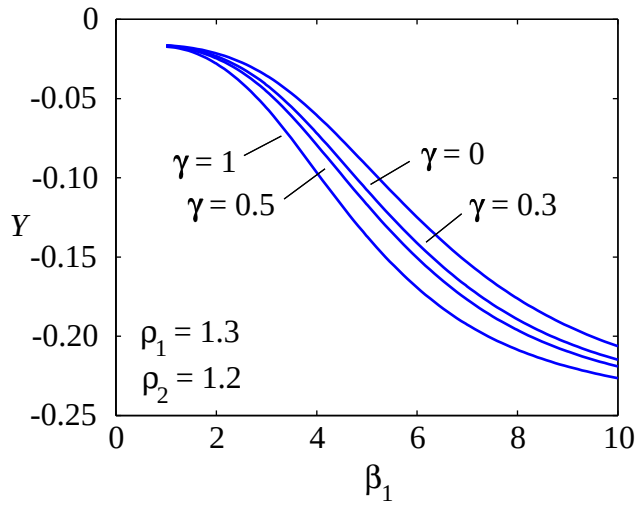

Figure 5: The imaginary part, Y, of the change in impedance as a function of β_1 for four values of γ with $\rho_1 = 1.3$, $\rho_2 = 1.2$.

The four curves corresponding to $\gamma = 0, 0.3, 0.5, 1$, respectively, describe the nonuniformity of the conductivity profile. The case $\gamma = 0$ corresponds to a two-layer sphere with constant conductivity in region R_1.

It is seen from Fig. 2 that the curves representing the real part of the change in impedance have maxima for each value of γ. However, the position of the maxima depends on β_1 and as γ increases, the maxima are shifted to the region of smaller β_1. Note also that the maximal values of X are approximately the same for all γ. This fact may be useful in controlling the properties of coverings with variable conductivity. On the other hand, $|Y|$ increases as both γ and β_1 increase, as can be seen from Fig. 3.

The curves in Figs. 4 and 5 for $\rho_1 = 1.3$ and $\rho_2 = 1.2$ are similar to those of Figs. 2 and 3 for $\rho_1 = 0.9$ and $\rho_2 = 0.8$. However, it is seen from Figs. 4 and 5 that the output signal is stronger for all values of γ in comparison with the signal shown in the previous two figures.

In Fig. 6, the exact conductivity profile $\sigma(\rho)/\sigma_1 = 1 + \gamma/\rho^2$ with $\gamma = 0, 0.3, 0.5, 1.0$ is plotted on the intervals $0.8 \le \rho \le 0.9$ and $1.2 \le \rho \le 1.3$ for comparison with Figs. 2–3 and Figs. 4–5, respectively.

The second set of computations is done for Case 2 with profile (3) and several values of α. The results presented in Figs. 7 and 8 indicate that the nonuniformity of the conductivity profile has a significant influence on the output signal of the eddy current probe. For example, for $8 \le \beta_1 \le 10$ the values of X and Y for $\alpha = 4$ and $\alpha = 0$ (region R_1 with constant conductivity) differ by as much as 50%.

The third set of computations is done for Case 3 with profile (4) and several values of α and γ. The change in

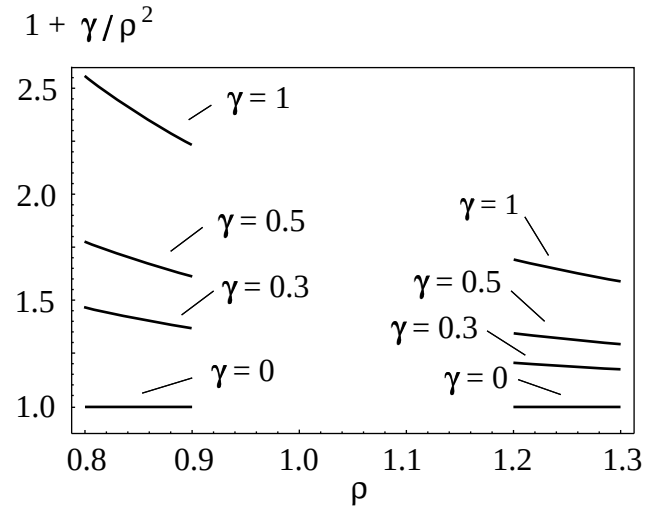

Figure 6: The exact conductivity profile $\sigma(\rho)/\sigma_1 = 1 + \gamma/\rho^2$ with $\gamma = 0, 0.3, 0.5, 1.0$ on the intervals $0.8 \leq \rho \leq 0.9$ and $1.2 \leq \rho \leq 1.3$ for comparison with Figs. 2–3 and Figs. 4–5, respectively.

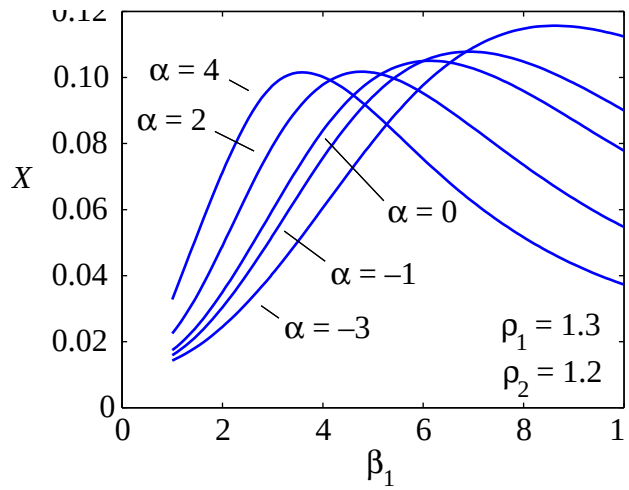

Figure 7: The real part, X, of the change in impedance as a function of β_1 for five values of α with $\rho_1 = 1.3$, $\rho_2 = 1.2$.

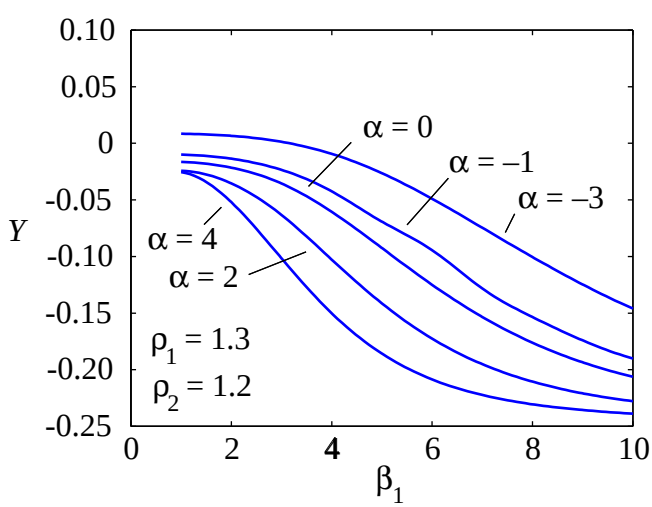

Figure 8: The imaginary part, Y, of the change in impedance as a function of β_1 for five values of α with $\rho_1 = 1.3$, $\rho_2 = 1.2$.

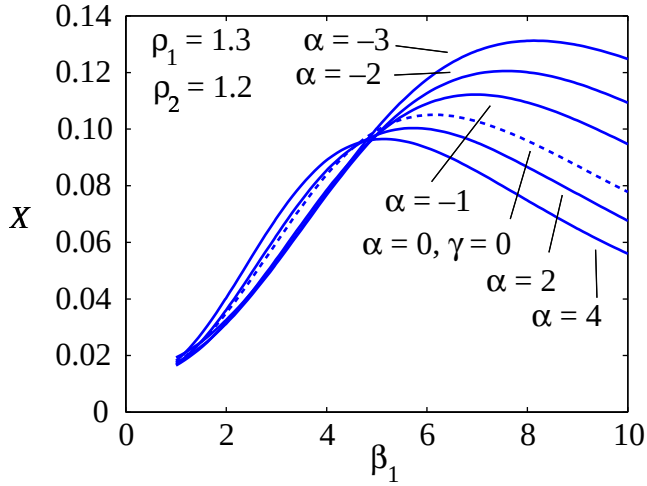

Figure 9: The real part, X, of the change in impedance as a function of β_1 for six values of α with $\rho_1 = 1.3$, $\rho_2 = 1.2$. The relative magnetic permeability and conductivity of the outer layer are $\mu_1 = 1/\rho^2$ and $\sigma_1 = \rho^\alpha$, respectively.

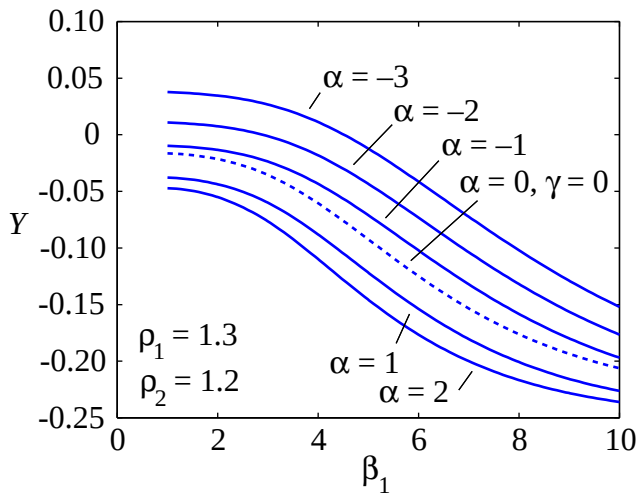

Figure 10: The imaginary part, Y, of the change in impedance as a function of β_1 for six values of α with $\rho_1 = 1.3$, $\rho_2 = 1.2$. The relative magnetic permeability and conductivity of the outer layer are $\mu_1 = 1/\rho^2$ and $\sigma_1 = \rho^\alpha$, respectively.

the real and imaginary parts in the coil impedance is shown in Figs. 9 and 10, respectively.

The outer and inner radii are $\rho_1 = 1.3$ and $\rho_2 = 1.2$. The conductivity and permeability are $\sigma_1 = \rho^\alpha$ and $\mu_1 = \rho^{-2}$, for different values of α. The dashed curves corresponding to constant properties of region R_1 ($\alpha = 0$, $\gamma = 0$) are shown for the sake of comparison with the case of nonconstant σ and μ. It is seen from Fig. 9 that the eddy current method is more selective if $\beta_1 > 5$ since the curves almost coincide if $1 < \beta_1 < 5$.

The variation of X and Y versus β_1 is shown in Figs. 11 and 12 for the case $\rho_1 = 1.3$, $\rho_2 = 1.2$, $\sigma_1 = \rho^{-2}$, and $\mu_1 = \rho^\gamma$ for different values of γ.

The dashed curves correspond to $\alpha = 0$ and $\gamma = 0$. Figure 12 shows that for $\gamma = 1$, -1, and -2 the values of Y for all β_1 in the interval $[1, 10]$ are almost the same so the three cases are almost indistinguishable.

In Fig. 13, the curve ρ^α is plotted with $\alpha = -3, -2, -1, 0, 1, 2$ representing (a) the exact conductivity profile $\sigma(\rho)/\sigma_1 = \rho^\alpha$ for Figs. 7–8, (b) the exact conductivity and permeability profiles $\sigma(\rho)/\sigma_1 = \rho^\alpha$ with $\alpha = -3, -1, 0, 1, 2$ and $\mu(\rho)/\mu_1 = \rho^{-2}$ for Figs. 9–10, and (c) the exact conductivity and permeability profiles $\sigma(\rho)/\sigma_1 = \rho^{-2}$ and $\mu(\rho)/\mu_1 = \rho^\gamma$ for Figs. 10–12.

The dashed curve corresponds to the exact conductivity and to the permeability with α or $\gamma = -2$. This figure is for comparison with the corresponding Figs. 7–12

Since the problem in hand contains several parameters, the use of mathematical modelling can help select, in the parameter space, the regions of high selectivity and high sensitivity of the eddy current method.

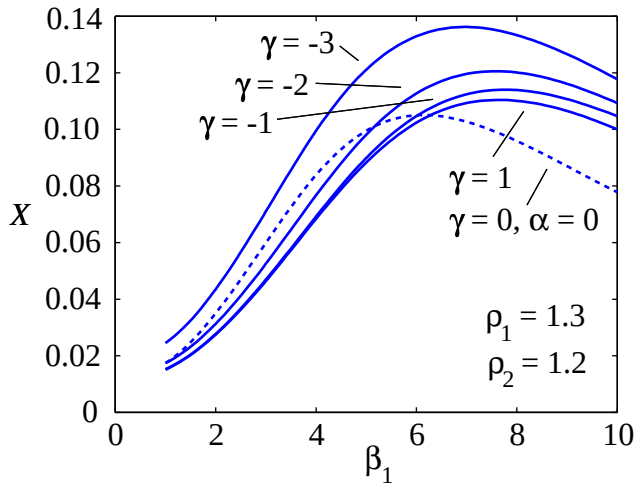

Figure 11: The real part of the change in impedance, X, as a function of β_1 for five values of γ with $\rho_1 = 1.3$, $\rho_2 = 1.2$. The relative magnetic permeability and conductivity of the outer layer are $\mu_1 = \rho^\gamma$ and $\sigma_1 = 1/\rho^2$, respectively.

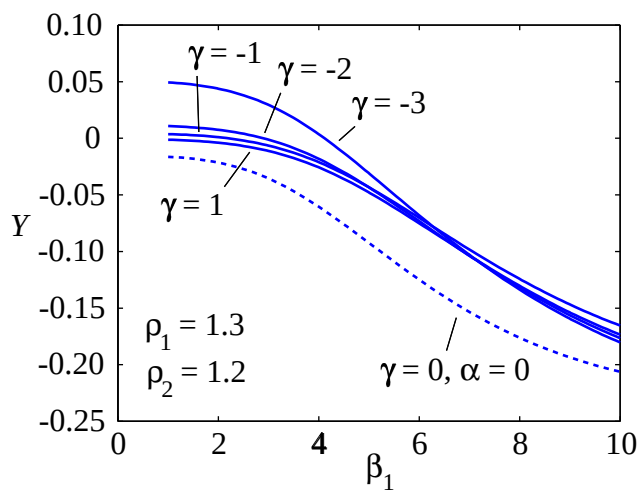

Figure 12: The imaginary part, Y, of the change in impedance as a function of β_1 for five values of γ with $\rho_1 = 1.3$, $\rho_2 = 1.2$. The relative magnetic permeability and the conductivity of the outer layer are $\mu_1 = \rho^\gamma$ and $\sigma_1 = 1/\rho^2$, respectively.

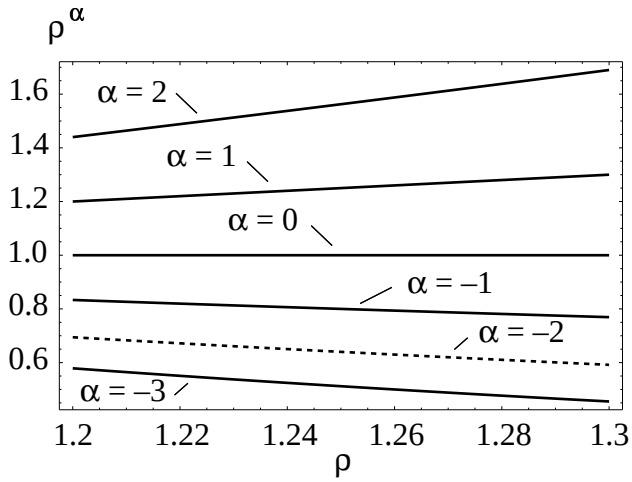

Figure 13: The curve ρ^α is plotted with $\alpha = -3, -2, -1, 0, 1, 2$ and represents the exact conductivity and permeability profiles corresponding to Figs. 7–12 (see text) for comparison with these figures. The dashed curve corresponds to the exact conductivity and permeability with $\gamma = -2$.

5 CONCLUSION

The change in impedance of a single-turn coil symmetrically situated above a two-layer sphere with nonconstant conductivity and relative magnetic permeability of the outer layer is obtained in the present paper. Analytical solutions are obtained for three types of conductivity and magnetic permeability profiles: $\sigma = \sigma_1(1 + \gamma/\rho^2)$, $\mu = \mu_1 = 1$; $\sigma = \sigma_1\rho^\alpha$, $\mu = \mu_1 = 1$; and $\sigma = \sigma_1\rho^\alpha$, $\mu = \mu_1\rho^\gamma$, where α and γ are arbitrary real numbers. Computational impedance results are presented for different values of the parameters of the problem and the exact conductivity and permeability profiles are given diagrammatically for comparison's purpose with the impedance results. The results of this paper can be used to estimate the nonconstant properties of spherical metallic products.

Acknowledgement

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