Bifurcartions of limit cycles from hamiltonian isochrones : a Darbouxlinearization approach

B. Toni

CRM-2629
September 1999

Depto de Matemáticas, Facultad de Ciencias, Universidad Autónoma del Estado de MorelosAv. Universidad 1001, Col. Chamilpa Cuernavaca 62210, Morelos, Mexico.toni@servm.fc.uaem.mx.

Centre de recherches mathématiques, Université de Montréal, C.P. 6128, suuc. Centre-ville, Montréal (Québec) H3C 3J7,. Canada.

Abstract

In this paper, we study the local bifurcations of limit cycles from isochrones. These isochrones have been determined using the method of Darboux linearization which provides a rational linearizing change of coordinates for the examples we analyze. By means of this linearizing transformation, the perturbation of a non-linear isochrone is actually reduced to that of the linear one, simplifying the analysis and avoiding the complexity of the Abelian integrals appearing in other approaches. As an application, we show that no more than two families of limit cycles can bifurcate from cubic Hamiltonian nonlinear isochrones. From the linear isochrone the maximum number is one. We also give the upper bound from an arbitrarily $n-$degree polynomial autonomous perturbation of the linear isochrone. Moreover there are small perturbations leading to the maximum number in each case.

Key words: Limit cycles, Isochrones, Perturbations.

Mathematics Subject Classifications: 34C15, 34C25, 58F14, 58F21, 58F30.

Résumé

Dans ce papier nous étudions les bifurcations locales de cycles limites de systèmes isochrones déterminés par la méthode de Darboux linéarisation. Cette méthode fournit en effet une transformation linéarisante birationnelle pour les exemples que nous analysons. Ainsi la perturbation du système non linéaire isochrone se ramène a celle du système linéaire isochrone et permet d'éviter la complexité due aux intégrales abéliennes qui apparaissent dans l'approche classique.

Nous montrons qu'au plus deux familles de cycles limites peuvent bifurquer du système hamiltonien cubique non linéaire isochrone dans une perturbation du premier ordre. Dans le cas du système linéaire isochrone au plus une famille de cycles limites bifurque dans une perturbation du premier ordre. Nous donnons aussi la borne supérieure des cycles limites dans une perturbation polynomiale de degré arbitraire n. En outre il existe des petites perturbations ayant exactement le maximum de cycles limites dans chaque cas.

1. Introduction

We consider a one parameter family $(\mathcal{X}_\lambda)$ of plane vector fields in the form

$$(\mathcal{X}_\lambda) \qquad \dot{u} = F(u) + \lambda G(u), \quad u \in \mathbb{R}^2$$

depending analytically on a small real parameter λ. In fact $(\mathcal{X}_\lambda)$ is an autonomous perturbation of a system $(\mathcal{X}_0)$. For a closed orbit (Γ_0) of the unperturbed system $(\mathcal{X}_0)$ an interesting question is the creation of limit cycles from (Γ_0) on passing from $\lambda = 0$ to close nonzero values of λ. Recall a limit cycle is a periodic cycle isolated in the set of all cycles of the vector field and whose germ of the return map is not the identity. This characteristic has motivated a great deal of research toward the determination of the maximum number $M(n)$ and relative positions of these limit cycles, particularly in terms of the degree of a given polynomial vector field. It is the famous Hilbert's $16th$ problem, from which were derived the other celebrated Poincaré finiteness conjecture (1908), namely, *A polynomial vector field* $(\mathcal{X})$ *on* $\mathbb{R}^2$ *has a finite number of limit cycles.* The conjecture was proved independently by Ecalle [E] and Ilyashenko [Il]. Many results for questions related to the Hilbert's problem are of the type $M(n) \geq C_n$. Indeed, by means of Hopf bifurcations from a weak focus or a centre and perturbation methods involving Abelian integrals, one may construct a polynomial vector field of degree n with at least C_n limit cycles.

To be specific, given (Γ_0) a periodic path of the unperturbed system $(\mathcal{X}_0)$, we investigate the number and relative positions of limit cycles of $(\mathcal{X}_\lambda)$ located in a sufficiently small neighborhood of $(-_0)$ for sufficiently small λ, i.e., limit cycles emerging from $(\mathcal{X}_0)$ on passing from $\lambda = 0$ to close values of λ. It is now classic that this question is equivalent to the analysis of the sufficiently small real roots of the displacement function $D(r, \lambda)$ for r in an open set of the Poincaré section to (Γ_0) and for sufficiently small values of λ. Such analysis is twofold. First, one can deal with all the possible modified systems sufficiently close to $(\mathcal{X}_0)$, i.e., in a sufficiently small neighborhood of the point $(\mathcal{X}_0)$ in the space (Ξ) of all dynamic systems. One result, now classic as well, is the following: *If* (Γ_0) *is a* $k-$*tuple limit cycle of* $(\mathcal{X}_0)$, *the maximum number of limit cycles created from* Γ_0 *on passing to other nearby systems, is* k. Note also that a simple limit cycle is a structurally stable path, that is, any perturbation will have precisely one limit cycle in a sufficiently small neighborhood of such a limit cycle.

A more restricted second possibility, indeed our line of study, is to confine the perturbed system to $(\mathcal{X}_\lambda)$ corresponding to small values of λ and no longer consider any possible system sufficiently close to $(\mathcal{X}_0)$; geometrically, we are no longer dealing with the entire neighborhood $\aleph(\mathcal{X}_0)$ of $(\mathcal{X}_0)$ in the space (Ξ) but only with some curve $(\mathcal{C}_\lambda)$ in this neighborhood, which passes through the point $(\mathcal{X}_0)$. We then analyze bifurcations of limit cycles in this neighborhood for motion along $(\mathcal{C}_\lambda)$ in (Ξ).

However our work is more akin to bifurcation methods involving Abelian integrals. Assume the system $(\mathcal{X}_\lambda)$ is an autonomous perturbation (p, q) of a polynomial system of degree n which we denote $(\mathcal{P}_\lambda)$. Moreover the unperturbed system

$$(\mathcal{P}_0) \qquad \dot{u} = F(u),$$

with components (P, Q) has a period annulus $\mathbb{A}$ with a Poincaré section Σ. We define the displacement function

$$D(r, \lambda) := R(r, \lambda) - r, \tag{1-1}$$

where $R(r, \lambda)$ is the Poincaré return map with the distance coordinate r on Σ. The zeros of $D(r, \lambda)$ correspond to periodic orbits of $(\mathcal{P}_\lambda)$ intersecting Σ. Clearly $D(r, 0) \equiv 0$. Upon investigating the number and position of the periodic orbits in the period annulus from which bifurcates a family of limit cycles, we reduce the analysis to that of finding the roots of a suitable bifurcation function derived from the displacement function. For instance, for a Hamiltonian unperturbed system with Hamiltonian $H(x, y)$, the displacement function may be expressed as

$$\begin{aligned} D(r, \lambda) =& R(r, \lambda) - r = \int_{\gamma_\lambda(r)} \dot{H} dt \\ =& \lambda M(r) + o(\lambda), \end{aligned} \tag{1-2}$$

with $\gamma_\lambda(r)$ the trajectory of the perturbed system $(\mathcal{P}_\lambda)$ starting at r on Σ, and

$$M(r) = \int_{\gamma(r)} (pdy - qdx) = \iint_{\Omega(r)} \operatorname{div}(p, q) dxdy \tag{1-3}$$

where $\gamma(r)$ is the boundary of simply connected region $\Omega(r)$ filled with periodic orbits of the Hamiltonian system with a positive orientation. $M(r)$ is an Abelian integral for H polynomial. From some techniques based on the Implicit function theorem, $M(r) = 0$ is a necessary condition for closed orbits to emerge from $\gamma(r)$ after perturbations. A

sufficient one is given by $M'(r) \neq 0$. More precisely, if $M(r_0) = 0$ and $M'(r_0) \neq 0$, then there exists $\lambda(r_0) > 0$ and a C^1-function $\omega(\lambda)$ defined for $|\lambda| < \lambda(r_0)$, with $\omega(0) = r_0$ and such that the vector field $(\mathcal{P}_\lambda)$ has a periodic orbit intersecting the section Σ at the point $r = \omega(\lambda)$. See [P] for details.

For $r = r_0$ a homoclinic loop, the Abelian integral $M(r)$ has the convergent asymptotic expansion

$$\begin{aligned} M(r) =& b_0 + a_1(r - r_0)log|r - r_0| + b_1(r - r_0) + \\ & a_2(r - r_0)^2 log|r - r_0| + b_2(r - r_0)^2 + \dots . \end{aligned} \tag{1-4}$$

If $a_k \neq 0$ (resp. $b_k \neq 0$) is the first nonzero coefficient in the expansion, then there are, for small $\lambda \neq 0$, at most $(2k - 1)$ (resp. $2k$) limit cycles, counting multiplicities and they tend to $\gamma(r_0)$ as $\lambda \longrightarrow 0$. ([R])

For perturbations of integrable polynomial systems, in general, a Hamiltonian system in the annulus region is derived by finding an integrating factor $K(x, y)$ and dividing $(\mathcal{P}_0)$ by K. It is the subject of Arnold's Hilbert's weakened 16th problem ([Ar]): *Investigate the number of zeros of the integral*

$$I_a(r) = \oint \frac{pdx + qdy}{K} dt \tag{1-5}$$

along the contours $H = r$ of a polynomial system with integrating factor K and where $p(x, y)$, $q(x, y)$ are polynomials of degree n, components of $G(u)$ in $(\mathcal{P}_\lambda)$.

Our particular interest is when the unperturbed system possesses an isochronous period annulus, i.e., all the cycles have the same constant period. We then analyze which one of the cycles and how many cycles survive after perturbation by giving birth to a continuous family of limit cycles of the perturbed system. In section two below we fully decribe our approach, *the Isochrone Reduction*, to study the bifurcations of limit cycles in an autonomous perturbation of a polynomial isochronous system. Section three is entirely devoted to the application of this method to the Hamiltonian case. The computational portions can be carried out by hands and a double checking may be done with the widely available computer algebra system Mathematica or Maple.

2. The Isochrone Reduction Approach

Given a periodic orbit Γ in the period annulus $\mathbb{A}$ and $\lambda_0 > 0$, the coordinates can be arranged so that the Poincaré section is an interval Σ of the positive x-axis transverse to $\mathbb{A}$. The displacement function D is defined globally on Σ. We next derive our bifurcation function B. Let consider an element r_0 such that

$$D_\lambda(r_0, 0) = 0, \quad \text{but } D_{r\lambda}(r_0, 0) \neq 0$$

i.e., r_0 is a simple zero of D_λ; then, by the Implicit function theorem, there exits a smooth function $r = \omega(\lambda)$ defined in some neighborhood of $\lambda = 0$, such that $\omega(0) = r_0$ and $D(\omega(\lambda), \lambda) \equiv 0$. This curve $\omega(\lambda)$ corresponds to a family of limit cycles emerging from the periodic trajectory of the unperturbed system which meets Σ at r_0. A first level of difficulties resides in the calculations and analysis of the partial derivatives of $D(r, \lambda)$. Of course for $D_\lambda(r, 0) \equiv 0$, or if one of its zeros is not simple, then higher order derivatives must be computed or else one must find an alternative approach. Actually, for a periodic orbit in a period annulus, we have

$$D_\lambda(r, 0) = 0 \quad \text{for all values of } r,$$

and we cannot apply the Implicit function theorem. From the Taylor series of D

$$\begin{aligned} D(r, \lambda) =& \lambda D_\lambda(r, 0) + O(\lambda^2) \\ =& \lambda(D_\lambda(r, 0) + O(\lambda)) \end{aligned} \tag{2-1}$$

we define a reduced displacement function by

$$B(r, \lambda) := D_\lambda(r, \lambda), \tag{2-2}$$

for small real values of λ. Clearly if $B(\omega(\lambda), \lambda) \equiv 0$ then $D(\omega(\lambda), \lambda)) \equiv 0$, the Implicit Function theorem does apply to B. In other words, a simple zero of B corresponds to the appearance of a family $r = \omega(\lambda)$ of periodic orbits. Such a zero of B is called *a branch point of periodic orbits* for system $(\mathcal{P}_\lambda)$.

Definition 2.1. A local family of periodic trajectories bifurcates from Γ if there is a number $\lambda_0 > 0$ and a continuous family of closed plane curves (Γ_λ) defined for $|\lambda| < \lambda_0$ such that $\Gamma_0 = \Gamma$ and Γ_λ is a periodic trajectory of $(\mathcal{P}_\lambda)$. Such a Γ is sometimes said to *survive* or to *persist* after perturbation.

Noting that the function B is analytic for analytic systems, a generalization along with a straightforward application of Weierstrass preparation theorem yields the following.

Lemma 2.2. *Assume that*

$$B(r,0) \equiv \partial_\lambda B(r,0) \equiv \partial_\lambda^2 B(r,0) \equiv \cdots \equiv \partial_\lambda^{k-1} B(r,0) \equiv 0. \tag{2-3}$$

If r_0 is a simple root of $\partial_\lambda^k B(r,0) = 0$, i.e.,

$$\partial_\lambda^k B(r_0,0) = 0 \quad \text{with } \partial_r \partial_\lambda^k B(r_0,0) \neq 0, \tag{2-4}$$

then there is a number $\lambda_1 > 0$ and a unique smooth function $\omega(\lambda)$, $|\lambda| < \lambda_1$, such that $\omega(0) = r_0$ and $B(\omega(\lambda),\lambda) \equiv 0$. Moreover, for $0 < |\lambda| < \lambda_1$, the point $r = \omega(\lambda)$ is a simple root of the equation $B(r,\lambda) = 0$. On the other hand, if r_0 is a root of multiplicity l, i.e.,

$$\partial_\lambda^k B(r_0,0) = \partial_r \partial_\lambda^k B(r_0,0) = \cdots = \partial_r^{l-1} \partial_\lambda^k B(r_0,0) = 0, \quad \partial_r^l \partial_\lambda^k B(r_0,0) \neq 0, \tag{2-5}$$

then there are at most l distinct smooth functions $\omega_i(\lambda)$ such that $\omega_i(0) = r_0$ and $B(\omega_i(\lambda),\lambda) \equiv 0$, for $i = 1,\ldots,l$.

An immediate consequence follows.

Bifurcation Theorem. *If the reduced displacement $B(r,\lambda)$ is such*

$$B(r,0) \equiv \partial_\lambda B(r,0) \equiv \partial_\lambda^2 B(r,0) \equiv \cdots \equiv \partial_\lambda^{k-1} B(r,0) \equiv 0. \tag{2-6}$$

and $\partial_\lambda^k B(r,0) = 0$ has N simple zeros on Σ, then there are exactly N branch points of periodic orbits of $(\mathcal{P}_\lambda)$ (corresponding to the zeros of B); while if $\partial_\lambda^k B(r,0) = 0$ has N zeros, counting multiplicities, then system $(\mathcal{P}_\lambda)$ has at most N branch points of periodic trajectories.

As may be observed, application of this theorem requires knowledge of the partial derivatives of the displacement function. In this regard, an important practical tool is given by Andronov et al., reformulated and reproved in [CJ2]. Among others, its application requires the period function $T(r,\lambda)$ seen in the more general sense as the minimum positive time taken for the trajectory starting at (r,λ) to first return to the Poincaré section Σ. It is therefore natural to first investigate bifurcations from isochrones. Chicone and Jacobs show, in [CJ2], that at most three limit cycles bifurcate from quadratic isochrones. Their approach did not take sufficiently advantage of the isochronal assumption as we will indicate below after a few recalls. We also give a practical expression of the bifurcation function in the isochrone case.

As stated before, actually we deal with systems whose unperturbed system is assumed to be isochronous, i.e., in the associated period annulus, all the integral curves have the same constant period. The bulk of our approach is this isochronal assumption, which is equivalent to the existence of an analytic change of coordinates in a neighborhood of the origin transforming the system into one which is linear. It is well known that only non-degenerate center can be isochronous. Indeed upon blowing up a degenerate center is replaced by a polycycle with transition time at the corners going to infinity. And also the basin of isochronous center can contain no finite singular points since otherwise the period would tend to infinity. In appropriate coordinates and upon rescaling of time, a polynomial system of degree n with a non-degenerate center at the origin is given by

$$\dot{u} = Au + f(u), \tag{2-7}$$

where $u = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, A the rotation matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $f(u) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$ with

$$f_1(x,y) = \sum_{i+j=2}^{n} a_{ij} x^i y^j, \quad f_2(x,y) = \sum_{i+j=2}^{n} b_{ij} x^i y^j. \tag{2-8}$$

It is also well known (details in [MRT] for instance) that the origin of (2-7) is isochronous if and only if there exists an analytic change of coordinates

$$(\mathcal{T}_l) \qquad \begin{aligned} X &= x + o(|(x,y)|) \\ Y &= y + o(|(x,y)|) \end{aligned}$$

reducing the system to the linear isochrone $(\mathcal{L}_i)$

$$\dot{X} = -Y$$
$$\dot{Y} = X$$

Our previous work in [T1] has motivated research toward understanding the mechanism that makes a center isochronous as well as the role played by isochronous systems in the whole family of parametrized systems. This question has been fully addressed in [MRT], where a systematic treatment of isochronicity has been given for several systems by constructing a linearizing change of coordinates. All the examples investigated have a rational first integral along with an algebraic linearizing transformation, hinting the use of arguments from algebraic geometry for further research like in [MJR]. That is, the linearizing coordinates are solutions of a system of polynomial equations in the form

$$(\bar{\mathcal{T}}_l) \qquad \begin{aligned} \bar{X}(x,y,X) &= \sum_{i=0}^{k} X_i(x,y)X^i = 0 \\ \bar{Y}(x,y,X) &= \sum_{i=0}^{l} Y_i(x,y)Y^i = 0, \end{aligned}$$

for some k, $l \in \mathbb{N}$, where $X_i(x,y)$ and $Y_i(x,y)$ are polynomials and $X_k(x,y)$ and $Y_l(x,y)$ are nonzero.

Moreover, for all the isochronous system investigated, the linearizing function in $(\mathcal{T}_l)$ is a Darboux function, i.e., in the form

$$Z(z,\bar{z}) = \Pi_{j=0}^{k} F_j^{\alpha_j}(z,\bar{z}), \quad \alpha_j \in \mathbb{C} \tag{2-9}$$

with either $F_j \in \mathbb{C}[z,\bar{z}]$ or $F_j = exp(G_j)$, with $G_j \in \mathbb{C}(z,\bar{z})$, for each $j = 0,\ldots,k$. The system is then said to be Darboux linearizable. A necessary condition has been shown to be the existence of two irreductible (complex conjugate) algebraic curves passing through the origin.

Once we know the linearizing transformation in $(\mathcal{T}_l)$, we may reduce the autonomous perturbation of the non-linear isochrone to that of a linear one; we then derive a simple expression of the bifurcation function B. Or else, the integral curves $\gamma(t)$ in the period annulus have to be determined first for every non-linear isochrone and apply the Andronov integration representation of the displacement function. This was the approach in [CJ2] for quadratic isochrones. It has the inconvenient of involving complex elliptic integrals. One may also take advantage of the effective criterion of Darboux linearizability, provided in [MJR], to test the applicability of our approach.

Practically speaking, we consider the perturbed system $(\mathcal{P}_\lambda)$

$$(\mathcal{P}_\lambda) \qquad \dot{u} = F(u) + \lambda G(u),$$

with $F(u) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$ and $G(u) = \begin{pmatrix} p(x,y) \\ q(x,y) \end{pmatrix}$ where the unperturbed polynomial system of degree n

$$(\mathcal{P}_0) \qquad \dot{u} = F(u),$$

has an isochronous center at the origin associated with a period annulus $\mathbb{A}$. It is thus amenable to the form

$$(\tilde{\mathcal{P}}_\lambda) \qquad \dot{u} = Au + \bar{F}(u),$$

with A as above and $\bar{F}$ without linear terms. Through the linearizing transformation $(\mathcal{T}_l)$, the perturbed system $(\mathcal{P}_\lambda)$ is therefore simplified to the weakly linear system

$$(\bar{\mathcal{P}}_\lambda) \qquad \dot{U} = AU + \lambda\bar{G}(U).$$

Recall the primary goal is the determination of the branch points of periodic orbits of $(\mathcal{P}_\lambda)$. We first reduce the appropriate displacement function to a bifurcation function so that to apply the Implicit Function theorem and its related corollaries. Then the perturbed system is reduced via the linearizing transformation. We next identify the bifurcation function in terms of this reduced perturbed system; the branch points are its simple zeros. We give below the expression of this bifurcation function.

Theoreom 2.3. *Consider a weakly linear system in the form*

$$(\mathcal{P}_\lambda) \qquad \dot{v} = Av + \lambda h(v), \quad v \in \mathbb{R}^2,$$

where $v = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $h(v) = \begin{pmatrix} h_1(x,y) \\ h_2(x,y). \end{pmatrix}$ *Assume the unperturbed system has a period annulus whose periodic orbits are parametrized by* r.

A branch point of periodic orbits of $(\mathcal{P}_\lambda)$ *is a simple zero of the function*

$$(\mathcal{F}_b) \qquad B(r) := \int_0^{2\pi} (h_1(r\cos t, r\sin t)\cos t + h_2(r\cos t, r\sin t)\sin t)\, dt,$$

where r *is taken in an interval of* $(0,\infty)$.

Proof. Given $D(r,\lambda)$ the associated displacement function, defined globally on the Poincaré section Σ, the bifurcation function has been defined as $B(r,\lambda) := D_\lambda(r,0)$ for small values of λ.

Using a periodic orbit Γ with integral curve $\gamma_r(t) := (x(t,r,\lambda), y(t,r,\lambda))$ starting at $(r,0)$ we get

$$D_\lambda(r,0) = \dot{x}(T(r,0),r,0) \times T_\lambda(r,0) + x_\lambda(T(r,0),r,0). \tag{2-10}$$

At $r=0$ we have $\dot{x}(T(r,0),r,0) = -y(0,r,0) = 0$. Thus we get

$$D_\lambda(r,0) = x_\lambda(T(r,0),r,0). \tag{2-11}$$

Looking for $x_\lambda(T(r,0),r,0)$ amounts to integrate the variational equation

$$\begin{aligned} \dot{x}_\lambda &= -y_\lambda + h_1(x,y) \\ \dot{y}_\lambda &= x_\lambda + h_2(x,y), \\ x_\lambda(0,r,0) &= y_\lambda(0,r,0) = 0. \end{aligned} \tag{2-12}$$

In matrix form it is expressed as

$$\begin{aligned} \dot{W} &= AW + H(t) \\ W(0) &= 0, \end{aligned} \tag{2-13}$$

where A is as given above and

$$H(t) = \begin{pmatrix} h_1(x(t,r,\lambda), y(t,r,\lambda)) \\ h_2(x(t,r,\lambda), y(t,r,\lambda)) \end{pmatrix} \tag{2-14}$$

By the method of variation of constants

$$\begin{aligned} W(T(r,0)) &= (x_\lambda(T(r,0),r,0), y_\lambda(T(r,0),r,0)) \\ &= \Phi(T(r,0)) \int_0^{T(r,0)} \Phi^{-1}(s) H(\gamma_r(s)) ds, \end{aligned} \tag{2-15}$$

where $\Phi(t)$ denotes the principal fundamental matrix solution of $\dot{W} = AW$ at $t=0$. We have

$$\Phi(t) = e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \tag{2-16}$$

and $H(\gamma_r(t) = \begin{pmatrix} h_1(r\cos t, r\sin t) \\ h_2(r\cos t, r\sin t) \end{pmatrix}$. Hence, for $T(r,0) = 2\pi$, it follows

$$\begin{pmatrix} x_\lambda(2\pi,r,0) \\ y_\lambda(2\pi,r,0) \end{pmatrix} = \begin{pmatrix} \int_0^{2\pi} (\cos s \times h_1(r\cos s, r\sin s) + \sin s \times h_2(r\cos s, r\sin s))\, ds \\ \int_0^{2\pi} (-\sin s \times h_1(r\cos s, r\sin s) + \cos s \times h_2(r\cos s, r\sin s))\, ds \end{pmatrix} \tag{2-17}$$

Thus we obtain

$$\begin{aligned} B(r,\lambda) = D_\lambda(r,0) &= x_\lambda(2\pi,r,0) \\ &= \int_0^{2\pi} (\cos s \times h_1(r\cos s, r\sin s) + \sin s \times h_2(r\cos s, r\sin s))\, ds \end{aligned} \tag{2-18}$$

□

The stability of the perturbed periodic orbit Γ_λ for small λ passing through the $(r,0)$ is determined by the size of $R_r(r,\lambda) \geq 0$. That is, if $R_r(r,\lambda) < 1$ then Γ_λ is asymptotically stable, and it is asymptotically unstable for $R_r(r,\lambda) > 1$. Indeed we have

$$R(r,\lambda) = R(r,0) + \lambda R_\lambda(r,0) + O(\lambda^2). \tag{2-19}$$

And, as a result,

$$\begin{aligned} R_r(r,\lambda) - 1 =& \lambda\left(D_{r\lambda}(r,0) + O(\lambda)\right) \\ =& \lambda B_r(r,0), \end{aligned} \tag{2-20}$$

for λ sufficiently small. Therefore the derivative of the return map at r has no eigenvalue with modulus 1, (i.e., the periodic orbit at r is *hyperbolic*) if and only $B_r(r,0) \neq 0$. In summary we have the following corollary.

Stability Theorem.

(1) *A hyperbolic periodic orbit of system* $(\mathcal{P}_0)$ *persits for autonomous perturbations.*
(2) *If* r_0 *is a simple zero of* B *then there is a continuous family* (Γ_λ) *of periodic trajectories with* $\Gamma_0 = \Gamma$ *which are hyperbolic limit cycles for sufficiently small* $\lambda \neq 0$. *Moreover, we have*

$$\begin{cases} \lambda B_r(r_0,0) < 0, & \textit{then } \Gamma_\lambda \textit{ is attracting, and} \\ \lambda B_r(r_0,0) > 0, & \Gamma_\lambda \textit{ is repelling.} \end{cases} \tag{2-21}$$

Here we mention in passing that, in general, the analytic associated period function has the following series expansion

$$T(x,\lambda) = 2\pi + \sum_{k=1}^{\infty} p_{2k}(\lambda) x^{2k}, \tag{2-22}$$

for $|x|$ and $|\lambda - \lambda_*|$ sufficiently small, and $p_{2k} \in \mathbb{R}[\lambda]$, the Noetherian ring of polynomial in the variables λ. For an isochronous centre all the period coefficients p_{2k}, $k \geq 1$, will vanish. An interesting bifurcation problem is that of the local bifurcations of critical periods in the neighborhood of the center. This question has been addressed and completely solved for various cubic systems in [RT1], [RT2], [T2]. The maximum number has been shown to be three and moreover in each of these cases, we have constructed perturbations leading to the maximum number.

Since for several systems (see e.g. [MRT]) the isochronous strata are known with an algebraic linearizing transformation explicitly given, so providing the basis of this method, there is ample opportunity to apply our approach. For the sake of illustration, we choose to address the bifurcations of limit cycles from polynomial Hamiltonian isochronous systems.

3. First Order Bifurcations from Hamiltonian Isochrones

It is well known that all invariant algebraic curves of a Hamiltonian system with a Hamiltonian H are given by $H - h = 0$ or by a factor of such a curve, with a cofactor equal to zero; only factors of $H - h$ (if they exist) can have nonzero cofactor. Note also that all hyperbolic singular points at infinity of a Hamiltonian system are of index one. By [MRT], an isochronous Hamiltonian system necessarily has a singular point at infinity, degenerate by an index argument. Vorobev in [V] gave a class of Hamiltonian isochrone systems obtained from the linear isochrone $\dot{X} = -Y, \quad \dot{Y} = X$ by means of the following canonical change of coordinates:

$$\begin{aligned} x =& X + a_2 f(a_1 X + a_2 Y) \\ y =& Y - a_1 f(a_1 X + a_2 Y). \end{aligned} \tag{3-1}$$

It is worth mentioning also the case of conservative second order scalar differential equations of the form

$$\ddot{u} + G(u,\lambda) = 0, \tag{3-2}$$

with the potential energy given by $V(u,\lambda) := \int_0^u G(s,\lambda)ds$ and Hamiltonian $H(u,\dot{u}) = \frac{(\dot{u})^2}{2} + V(u,\lambda)$. For $V(u,\lambda)$ analytic and $V'(0) = 0$ and $V"(0) = 1$, system (3-2) has a center at the origin and there are periodic orbits with energies $E = V(u)$ up to the first positive critical value of V. In [CJ1], it has been shown that for

$$V(u,\lambda) = \frac{(\dot{u})^2}{2} + \sum_{i=1}^{n-2} \lambda_{i+2} u^{i+2},$$

the origin is an isochronous center if and only if $\lambda = (\lambda_3, \ldots, \lambda_n) = 0$. Moreover for $G(u,\lambda) = u + \bar{G}(u,\lambda)$ where $\bar{G}(u,\lambda)$ is a nonzero polynomial with $\bar{G}(0,\lambda) = \bar{G}'(0,\lambda) = 0$, this system cannot be linearized by a smooth coordinate transformation.

3.1 Cubic Hamiltonian Isochrones.

Assuming the degenerate singularity on the $y-$axis without loss of generality, a cubic Hamiltonian system may be written as

$$(\mathcal{H}_3)\qquad \begin{aligned}\dot{x} =& -y - a_1x^2 - 2a_2xy - 3a_3y^2 - a_4x^3 - 2a_5x^2y\\ \dot{y} =& x + 3a_6x^2 + 2a_1xy + a_2y^2 + 4a_7x^3 + 3a_4x^2y + 2b_5xy^2,\end{aligned}$$

with Hamiltonian function

$$H(x,y) = \frac{x^2+y^2}{2} + a_6x^3 + a_1x^2y + a_2xy^2 + a_3y^3 + a_7x^4 + a_4x^3y + a_5x^2y^2. \tag{3-3}$$

A necessary condition for Darboux linearizability is that the curve $H(x,y) = 0$ factors in two complex conjugate algebraic curves. From this fact and after showing there are no cubic Darboux linearizable Hamiltonian systems with two invariant lines through a center, Rousseau et al have established the following characterization in [MMR].

Theorem 3.1. *The Hamiltonian cubic system $(\mathcal{H}_3)$ is Darboux linearizable if and only if it is of the form*

$$(\mathcal{H}_i)\qquad \begin{aligned}\dot{x} =& -y - Cx^2\\ \dot{y} =& x + 2Cxy + 2C^2x^3,\end{aligned}$$

This system is linearizable through the canonical change of coordinates

$$(\mathcal{D}_l)\qquad (u,v) = (x, y + Cx^2).$$

Remarks. This Darboux linearizable system is also obtained as one of Vorobev Hamiltonian isochrone systems by taking f as a quadratic polynomial in the coordinates change (3-1). Note also that the domain of the isochronous center is the whole plane.

3.2 First Order Perturbations.

We now consider a cubic autonomous perturbation $(\mathcal{H}_\lambda)$ of system $(\mathcal{H}_i)$

$$(\mathcal{H}_\lambda)\qquad \begin{aligned}\dot{x} =& -y - Cx^2 + \lambda p(x,y)\\ \dot{y} =& x + 2Cxy + 2C^2x^3 + \lambda q(x,y),\end{aligned}$$

where, along with small values of the parameter $\lambda \in \mathbb{R}$, we take

$$\begin{aligned}p(x,y) =& \sum_{i=1}^{3}\sum_{k=0}^{i} a_{i-k,k} \times x^{i-k}y^k\\ q(x,y) =& \sum_{i=1}^{3}\sum_{k=0}^{i} b_{i-k,k} \times x^{i-k}y^k.\end{aligned} \tag{3-4}$$

That is, the unperturbed system $(\mathcal{H}_0)$ is the Hamiltonian isochrone $(\mathcal{H}_i)$. In view of the approach fully described in the previous section, we use the Darboux change of coordinates $(\mathcal{D}_l)$ to transform a perturbation of the nonlinear isochrone into a perturbation of the linear one and then apply theorem (2-3) to identify the bifurcation function corresponding to $(\mathcal{H}_\lambda)$. We want to determine the number and position of the local families of limit cycles which emerge from periodic trajectories of $(\mathcal{H}_0)$ as the small parameter λ changes to nonzero values. As previously stated in introduction, for small λ, an interval J of the positive x-axis with left hand point at the origin is a Poincaré section Σ for the flow of $(\mathcal{H}_\lambda)$. Take r as the distance coordinate along the x-axis and the reduced displacement function $B(r,\lambda)$ defined and analytic on an open neighborhood of $\Sigma \times \{0\}$. Recall from theorem (2-3) that a family (Γ_λ) of limit cycles emerges from a periodic trajectory Γ of the isochrone $(\mathcal{H}_0)$ through the point r_0 if this point is a simple zero of B, i.e., a branch point of periodic orbits of $(\mathcal{H}_\lambda)$.

From the linearizing change of coordinates $(\mathcal{D}_l)$ rewritten as

$$\begin{aligned}u(x,y) =& f^*(x,y) = x\\ v(x,y) =& g^*(x,y) = y + Cx^2\end{aligned} \tag{3-5}$$

we derived the inverse transformation $\mathcal{D}_l^{-1}$

$$\begin{aligned} x(u,v) =& f(u,v) = u \\ y(u,v) =& g(u,v) = v - Cu^2. \end{aligned} \tag{3-6}$$

System $(\mathcal{H}_\lambda)$ is then transformed via $(\mathcal{D}_l)$ into a system $(\bar{\mathcal{H}}_\lambda)$ with the differential form

$$\begin{aligned} \dot{u} =& -v + \lambda \bar{p}(u,v) \\ \dot{v} =& u + \lambda \bar{q}(u,v) \end{aligned} \tag{3-7}$$

where we have

$$\begin{aligned} \bar{p}(u,v) &= \left(f_x^*(x,y)p(x,y) + f_y^*(x,y)q(x,y)\right)|_{(u,v)} \\ \bar{q}(u,v) &= \left(g_x^*(x,y)p(x,y) + g_y^*(x,y)q(x,y)\right)|_{(u,v)}, \end{aligned} \tag{3-8}$$

with

$$\begin{aligned} f_x^*|_{(u,v)} =& \frac{\partial f^*}{\partial x}|_{(u,v)} = 1 \quad \text{and } f_y^*|_{(u,v)} = \frac{\partial f^*}{\partial y}|_{(u,v)} = 0 \\ g_x^*|_{(u,v)} =& \frac{\partial g^*}{\partial x}|_{(u,v)} = 2Cu \quad \text{and } g_y^*|_{(u,v)} = \frac{\partial g^*}{\partial y}|_{(u,v)} = 1. \end{aligned} \tag{3-9}$$

Therefore it follows from (3-4) the polynomials $\bar{p}$ and $\bar{q}$ are expressed as

$$\begin{aligned} \bar{p}(u,v) &= \sum_{i=1}^{3}\sum_{k=0}^{i} a_{i-k,k}\,(u)^{i-k}\left(v - Cu^2\right)^k \\ \bar{q}(u,v) &= \sum_{i=1}^{3}\sum_{k=0}^{i} (u)^{i-k}\left(v - Cu^2\right)^k (\tilde{a}_{i-k,k}u + b_{i-k,k}), \end{aligned} \tag{3-10}$$

where $\tilde{a}_{i-k,k} = 2Ca_{i-k,k}$. Then using theorem (2-3), we look for the simple zeros of

$$B(r) = \int_0^{2\pi} (\bar{p}(r\cos t, r\sin t)\cos t + \bar{q}(r\cos t, r\sin t)\sin t)dt. \tag{3-11}$$

And this analysis is done for cubic autonomous perturbations of both the linear isochrone and then the non-linear one.

3.2.1 The Linear Isochrone.

After perturbation of the linear isochrone, we obtain

$$\begin{aligned} B(r) &= \int_0^{2\pi} (p(r\cos t, r\sin t)\cos t + q(r\cos t, r\sin t)\sin t)dt \\ &= \int_0^{2\pi} \left\{ \sum_{i=1}^{3}\sum_{k=0}^{i} \left((a_{i-k,k}\cos t + b_{i-k,k}\sin t)x^{i-k}y^k\right)|_{(r\cos t, r\sin t)} \right\} dt \\ &= \sum_{i=1}^{3} r^i \sum_{k=0}^{i} \left(\int_0^{2\pi} (a_{i-k,k}\cos t + b_{i-k,k}\sin t)\cos^{i-k} t \times \sin^k t dt \right). \end{aligned} \tag{3-12}$$

We then prove the following theorem

Theorem 3.2. *From a periodic trajectory Γ_0 in the period annulus $\mathbb{A}$ of the linear isochrone, at most one continuous family of limit cycles bifurcate from Γ_0 in the direction of the cubic autonomous perturbation (p,q). This maximum number one is attained if and only if the coefficients satisfy the condition $c_0 \times c_2 < 0$.*

Precisely, when this condition is met, this family emerges from the real simple roots of the quadratic

$$\delta(r) := c_0 + c_2 r^2.$$

where c_0 and c_2 are given below. Moreover the limit cycles are attracting (resp. repelling) for $\lambda \times c_0 > 0$ (resp. $\lambda \times c_0 < 0$) and $|\lambda|$ sufficiently small.

Proof. Computation of the expression in (3-12) amounts to that of the following quite easy integrals

$$\begin{aligned} T_{a_{ik}} =& a_{i-k,k} \int_0^{2\pi} \cos^{i-k+1} t \times \sin^k t dt \\ T_{b_{ik}} =& b_{i-k,k} \int_0^{2\pi} \cos^{i-k} t \times \sin^{k+1} t dt, \end{aligned} \tag{3-13}$$

for

$$(i,k) \in \{(1,0);(1,1);(2,0);(2,1);(2,2);(3,0);(3,1);(3,2);(3,3)\}. \tag{3-14}$$

These integrals are easily carried out by hands and can be double checked using Mathematica or MapleV. And, as a result

$$\begin{aligned} B(r) =& \left(r\pi(a_{10}+b_{01}) + r^3 \frac{\pi}{4}(3a_{30}+a_{12}+b_{21}+3b_{03}) \right) \\ =& (r\pi) \left((a_{10}+b_{01}) + \frac{r^2}{4}(3a_{30}+a_{12}+b_{21}+3b_{03}) \right). \end{aligned} \tag{3-15}$$

Thus, the roots of the quadratic

$$\delta(r) := a_{10}+b_{01} + \frac{r^2}{4}(3a_{30}+a_{12}+b_{21}+3b_{03}) \tag{3-16}$$

give the continuous families of limit cycles that bifurcate from the period annulus at the origin of the linear isochrone. Therefore, letting

$$\begin{aligned} c_2 :=& \frac{1}{4}(3a_{30}+a_{12}+b_{21}+3b_{03}) \\ c_0 :=& (a_{10}+b_{01}), \end{aligned} \tag{3-17}$$

if $c_0 = 0$, and $c_2 \neq 0$ then the origin is the only root for the polynomial $\delta(r)$. But $c_0 \neq 0$ implies

$$r^2 = -\frac{c_2}{c_0}. \tag{3-18}$$

Therefore condition $c_0 \times c_2 < 0$ gives exactly two real roots of opposite signs that must be simple. And the result follows from the Bifurcation theorem. Finally note that, at the root $r_0^2 = -\frac{c_2}{c_0}$, we have $\lambda \times B_\lambda(r_0) = -2\pi(\lambda c_0)$. Hence the claim. □

Comments. It is worth noting that for any two distinct periodic trajectories of the period annulus of the linear isochrone, there is no choice of perturbation so that simultaneously one continuous family of limit cycles is made to emerge from each of them, due to the uniqueness of the positive simple zero.

Generalization. We generalize the above result by considering a perturbation of degree n of the linear isochrone and prove the following theorem.

Theorem 3.3. *From the linear isochrone, to first order, no more than $\frac{n-1}{2}$, (resp. $\frac{n-2}{2}$) for n odd (resp. n even) continuous families of limit cycles can bifurcate in the direction of any autonomous polynomial perturbation of degree n. And we can construct small perturbations with the maximum number of limit cycles.*

Proof. In fact we take p and q polynomials of degre n in the expression (3-12) of the bifurcation function B and obtain

$$B(r) = \sum_{i=1}^{n} r^i \sum_{k=0}^{i} \left(\int_0^{2\pi} (a_{i-k,k}\cos t + b_{i-k,k}\sin t)\cos^{i-k} t \times \sin^k t dt \right). \tag{3-19}$$

This can be simplified using the well known rules

$$\begin{cases} \int_0^{2\pi} \cos^n t dt = \int_0^{2\pi} \sin^n t dt = 0, & \text{for } n \text{ odd} \\ \int_0^{2\pi} \cos^m t \times \sin^n t dt = 0, & \text{for } m \text{ or } n \text{ odd.} \end{cases} \tag{3-20}$$

As a result

$$B(r) = r \sum_{s=1, s odd}^{N} r^{s-1} c_s, \tag{3-21}$$

where

$$\begin{cases} N = n = 2l+1, & \text{for } n \text{ odd} \\ N = n-1 = 2l-1, & \text{for } n \text{ even} \end{cases} \tag{3-22}$$

and c_s is the nonzero constant

$$c_s = (a_{s0} + b_{0s}) + \sum_{k=1, k odd}^{s-2} (b_{s-k,k} + a_{s-k-1,k+1}) \int_0^{2\pi} \cos^{s-k} t \times \sin^{k+1} t dt. \tag{3-23}$$

Therefore the upper bound of the number of simple zeros of $B(r)$ is $M(n) = \frac{n-1}{2}$ for n odd and $\frac{n-2}{2}$ for n even. Perturbations with the maximum number are constructed as in the case below. $\square$

Remark. Note that, to first order, no limit cycles can emerge from periodic trajectories of the linear isochrone after a quadratic autonomous perturbation; this confirms a result in [CJ2].

3.2.2 The Non-Linear Isochrone.

We use the reduced perturbation $(\bar{\mathcal{H}}_\lambda)$ of the nonlinear isochrone and the corresponding expression of $B(r)$ in (3-11). That is

$$B(r) = \int_0^{2\pi} (\bar{p}(r\cos t, r\sin t)\cos t + \bar{q}(r\cos t, r\sin t)\sin t)\, dt, \tag{3-24}$$

with

$$\begin{aligned} \bar{p}(r\cos t, r\sin t) =& p(u,v)|_{(r\cos t, r\sin t)} \\ \bar{q}(r\cos t, r\sin t) =& ((2Cu \times p(u,v)) + q(u,v))\,|_{(r\cos t, r\sin t)}. \end{aligned} \tag{3-25}$$

To help debug the computations $B(r)$ is best displayed as follows.

$$B(r) = I_1 + I_2 + I_3, \tag{3-26}$$

where

$$\begin{aligned} I_1 =& \sum_{i=1}^{3} r^i \sum_{k=0}^{i} a_{i-k,k} \int_0^{2\pi} \cos^{i-k+1} t \times (\sin t - Cr\cos^2 t)^k dt \\ I_2 =& 2C \sum_{i=1}^{3} r^{i+1} \sum_{k=0}^{i} a_{i-k,k} \int_0^{2\pi} \cos^{i-k+1} t \times \sin t \times (\sin t - Cr\cos^2 t)^k dt \\ I_3 =& \sum_{i=1}^{3} r^i \sum_{k=0}^{i} b_{i-k,k} \int_0^{2\pi} \cos^{i-k} t \times \sin t \times (\sin t - Cr\cos^2 t)^k dt \end{aligned} \tag{3-27}$$

The calculations of the terms $I_s, \quad s = 1, 2, 3$ are lenghty but easily carried out. And the bifurcation function $B(r)$ is identified as

$$B(r) = c_1 r + c_3 r^3 + c_5 r^5 = r(c_1 + c_3 r^2 + c_5 r^4), \tag{3-28}$$

with

$$\begin{aligned} c_1 =& \pi(a_{10} + b_{01}) \\ c_3 =& \frac{\pi}{4}(a_{12} + 3(a_{30} + b_{03}) + b_{21} - (a_{11} + 2b_{02})) \\ c_5 =& \frac{\pi}{8}(a_{12} + 3b_{03})C^2 \end{aligned} \tag{3-29}$$

Therefore the branch points of periodic orbits are given by the simple zeros $r > 0$ of the quartic equation

$$\Delta(r) := c_1 + c_3 r^2 + c_5 r^4 = 0. \tag{3-30}$$

Note that $C = 0$ gives the previous nonlinear case with c_5 becoming zero. The upper bound $M(4)$ of positive zeros of equation (3-30) can be derived by various methods. Descartes rule of sign is one these. It allows the following result: denote ν the number of sign changes in the sequence of coefficients of $\Delta(r)$ and N_+ the number of positive zeros, we have

$$N_+ - \nu = 2k, \quad k \in \mathbb{N}. \tag{3-31}$$

Therefore

$$\begin{cases} c_1 \times c_3 < 0 \quad \text{and } c_3 \times c_5 < 0, & \text{we get } N_+ = 2 \text{ or } 0. \quad (1) \\ c_1 \times c_3 < 0 \quad \text{and } c_3 \times c_5 > 0, & \text{gives } N_+ = 1 \text{ or } 0. \quad (2) \\ c_1, \quad c_3, \quad c_5 \quad \text{of same sign}, & \text{there is no positive zeros} \quad (3) \end{cases} \tag{3-32}$$

The upper bound is clearly two. The determination of the exact number is done via the following lemma; its application yields a practical construction of small perturbations with an indicated number of families of limit cycles.

Lemma 3.4. *Let $s(x)$ be a real polynomial, $s \neq 0$, and let $s_0(x), s_1(x), \ldots, s_m(x)$ be the sequence of polynomials generated by the Euclidean algorithm started with $s_0 := s(x)$; $s_1 := s'(x)$. Then for any real interval $[\alpha, \beta]$ such that $s(\alpha) \times s(\beta) \neq 0$, $s(x)$ has exactly $\nu(\alpha) - \nu(\beta)$ distinct zeros in $[\alpha, \beta]$ where $\nu(x)$ denotes the number of changes of sign in the numerical sequence $(s_0(x), s_1(x), \ldots, s_m(x))$. Moreover all zeros of $s(x)$ in $[\alpha, \beta]$ are simple if and only if s_m has no zeros in $[\alpha, \beta]$.*

For a detailed proof, see [H, Theorem 6.3d]. Assume $C \neq 0$ and $c_5 \neq 0$ for a more general treatment. $c_5 = 0$ is similar to the previous section. We then write $\Delta(r)$ as

$$\Delta(r) = r^4 + \alpha_2 r^2 + \alpha_0, \tag{3-33}$$

with $\alpha_0 := \frac{c_1}{c_5}$; $\quad \alpha_2 := \frac{c_3}{c_5}$. We derive the following Euclidean sequence (up to constant factors):

$$\begin{cases} s_0(x) = \Delta(r), & \text{and } s_1(x) = \Delta'(r) \\ s_2(x) = -\frac{\alpha}{2} r^2 - \alpha_0, & \text{and } s_3(x) = \frac{-2\alpha_2^2 + 8\alpha_0}{\alpha_2} r \\ s_4(x) = \alpha_0. \end{cases} \tag{3-34}$$

We further assume $\alpha_0 \neq 0$ and $\alpha_2 \neq 0$, i.e., c_1 and c_3 nonzero. At $x = 0$ we obtain the sequence $(\alpha_0, 0, -\alpha_0, 0, \alpha_0)$; hence $\nu(0) = 2$. At ∞, where the leading terms dominate, we get $(1, 4, -\frac{\alpha_2}{2}, K_2, \alpha_0)$. As a result, to make $N_+ = 2$, (resp. 1) we must have $\nu(\infty) = 0$ (resp. 1). It amounts to taking all the terms $-\frac{\alpha_2}{2}$, K_2, and α_0 positive. Then it suffices to realize $c_1 \times c_5 > 0$, $c_3 \times c_5 < 0$ and $4c_1 \times c_5 < c_3^2$. And respectively $c_1 \times c_3 < 0$ and $c_3 \times c_5 < 0$. Moreover for $\alpha_0 \neq 0$, $s_4(x)$ is constant; therefore all zeros made to appear by the previous construction are simple.

Similar constructions may be done with same results using a simplified expression of $\Delta(r)$ by letting $z := r^2$. Thus we have established the following theorem.

Theorem 3.5.

From Γ_0, a periodic trajectory in the period annulus $\mathbb{A}$ of the non-linear isochrone $(\mathcal{H}_i)$, at most two continuous families of limit cycles bifurcate in the direction of the cubic perturbation (p, q). Moreover there are autonomous perturbations with exactly two families of limit cycles. These families emerge from the real positive simple roots of the quartic

$$\Delta(r) := c_1 + c_3 r^2 + c_5 r^4 = 0,$$

precisely when the coefficients $c_i, i = 1, 3, 5$ are chosen as above.

3.3 Concluding Theorem.

The previous analysis is summarized as follows:

Theorem 3.6. *If the Hamiltonian system $(\mathcal{H})$ has an isochronous center at the origin with period annulus $\mathbb{A}$, then there are at most two (resp. one) periodic trajectories in the nonlinear (resp. linear) annulus $\mathbb{A}$ at which a continuous family of limit cycles emerges in the perturbed $(\mathcal{H}_\lambda)$. Moreover, in both cases, there are small perturbations $(\mathcal{H}_\lambda)$ such that respectively two and one periodic trajectories can be made to appear from a corresponding number of arbitrarily prescribed periodic trajectories within the period annulus $\mathbb{A}$ of the isochronous center.*

In the nonlinear case one may construct a perturbation giving birth to one continuous family of limit cycles simultaneously from two distinct periodic orbits of the period annulus.

Final Words

We wish to emphasize here that the approach described above might, the *Isochrone Reduction*, be applied with much success to various isochronous strata (quadratic, symmetric cubic, $\dot{z} = iz + P(z)$) and in the more general case of Darboux linearizable systems. To our knowledge it is indeed the first application of Darboux linearization of isochronous centers to the study of bifurcations of limit cycles.

Acknowledgements

We are very grateful to Prof. C. Rousseau for fruitful discussions since this method was initiated. We would like to thank very much the CRM, and in particular Louis Pelletier and Josée Laferrière for the kind support.

REFERENCES

[An] Andronov, A.A., *Theory of bifurcations of dynamical systems on a plane*, Wiley, New York, 1973.

[Ar] Arnold, V.I., *Geometric Methods in the theory of ordinary differential equations*, Springer-Verlag, New-York, 1983.

[Ba] Bautin, N.N., *On the number of limit cycles which appear with the variation of the coefficients from an equilibrium point of focus or center type*, AMS Translation Series I **5** (1952), 396-413.

[CJ1] Chicone, C. and Jacobs, M, *Bifurcation of critical periods*, Trans.Amer.Math.Soc. **312** (1989), 433-486.

[CJ2] Chicone, C. and Jacobs, M, *Bifurcation of Limit Cycles from Quadratic Isochrones*, J. of Differential Equations **91** (1991), 268-327.

[CD] Christopher, C.J. and Devlin, J., *Isochronous Centres in Planar Polynomial Systems*, Preprint, University College of Wales (1994).

[D] Darboux, G, *Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré (Mélanges)*, Bull. des Sc.Math. (1878), 60-96; 123-144; 151-200.

[E] Ecalle, J., *Finitude des cycles limites et accelo-sommation de l'application de retour*, Springer Lectures Notes in Mathematics **1455** (1990), 74-159.

[FP] Françoise, J.P. and Pugh, C.C., *Keeping track of limit cycles*, J. Differential Equations **65** (1986), 139-157.

[H] Henrici, P., *Applied and Computational Complex Analysis*, vol. 1, John Wiley and Sons, 1974.

[Il] Il'yashenko, Y., *Finiteness theorems for limit cycles*, Russian.Math.Surveys **40** (1990), 143-200.

[MJR] Mardešić, P., Moser-Jauslin, L. and Rousseau, C., *Darboux Linearization and Isochronous centers with rational first integral*, Preprint (1995).

[MRT] Mardešić, P., Rousseau, C. and Toni, B., *Linearization of Isochronous centers*, J. Differential Equations **121** (1995), 67-108.

[P] Pontrjagin, L.S., *Uber Autoschwingungunssysteme die den Hamiltonschen nahe liegen*, Physikal Zeitschr.d.Sowjetanion **6** (1934), 25-28.

[R] Rousseau, C., *Bifurcation Methods in Polynomial Systems*, Preprint, Université de Montréal (1994).

[R] Roussarie, R., *Dynamical Systems, Lecture Notes in Maths, Springer-Verlag, New-York*, vol. 1331, 1990, pp. 161...168.

[RT1] Rousseau, C. and Toni, B., *Local bifurcation of critical periods in cubic symmetric vector fields*, Can.Math.Bull. **36(4)** (1993), 473...484.

[RT2] Rousseau, C. and Toni, B., *Local bifurcations of Critical Periods in the reduced Kukles Systems*, Can.Jour.Math. **49(2)** (1997), 338-358.

[T1] Toni, B., *Bifurcations de périodes critiques locales*, Ph.D Thesis, Université de Montréal (1994).

[T2] Toni, B., *Critical Periods for Cubic Vector fields in Kapteyn Normal Form*, Quaestiones Mathematicae **22** (1999), 43-61.