Graded contractions of Lie algebras and central extensions

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Abstract

We use the concept of *grading* of Lie algebras to investigate the appearance of central charges during the contraction process. As for the usual graded contractions, one finds simultaneously the central extensions of classes of algebras, rather than specific Lie algebras. To illustrate the method, we consider in detail two physical applications: the kinematical algebras of spacetime, and the graded contractions that occur in the general formalism of vector coherent state (VCS) representations of a Lie algebra.

Résumé

Nous utilisons l'idée de graduation d'une algèbre de Lie pour étudier comment apparaissent les charges centrales durant une contraction. Comme pour les contractions graduées habituelles, on trouve simultanément les extensions de classes d'algèbres plutôt que d'algèbres spécifiques. Afin d'illustrer notre méthode, nous considérons en détail deux applications physiques : les algèbres cinématiques et les contractions graduées qui apparaissent dans le formalisme des états cohérents vectoriels d'une algèbre de Lie.

1 Introduction

The objective of this paper is to generalize the method of graded contractions [1, 2] to include, using again the concept of Lie gradings [3], contractions with central charges. Although we will be primarily interested in contracting semisimple Lie algebras, our method can, in principle, be applied to non-semisimple Lie algebras, as well as Lie superalgebras and infinite dimensional Lie algebras (just like the usual graded contractions [1]).

Contractions are important in physics as they explain formally why some theories arise as a limit regime of more "complete" theories (see [4] and references therein). The paradigm is the passage from the Poincaré algebra to the Galilei algebra, in the limit where the speed of light approaches infinity [5]. Similarly, the de Sitter algebra can be contracted to the Poincaré algebra in the limit where the radius of the universe is large. Other examples include the so(3) algebra of rotations on a sphere, which contracts to translations for small angular displacements, and the dynamical algebra $\mathrm{sp}(2n,\mathrm{R})$ of the harmonic oscillators in n dimensions, which contracts to the $\mathrm{u}(n)\oplus\mathrm{hw}(2n)$ algebra (where $\mathrm{hw}(m)$ is the m'th Heisenberg-Weyl algebra) describing collective excitations at low energy: the $n = 3$ case is discussed in [6], whereas a realization applicable to specific problems in atomic physics was obtained in [7] for $n = 1$ and $n = 2$. From these examples, one can see that the existence of an "approximate", or "effective" theory is often the result of a contraction.

The importance of central charges in physics is likewise well appreciated [8]–[12]. Their appearance in a Lie algebra can be the counterpart of the presence of non-trivial phases in a projective representation of the corresponding group. We note, for instance, that the so-called "Schwinger term" (associated with anomalies) of conformal field theories can be associated with the central charge of the current algebra of the underlying theory [13, 14]. This term is crucial *e.g.* in the construction of Wess-Zumino-Witten models [15]. Mathematically, the general problem of finding the central charges "sits" halfway between the contraction procedure, where commutators that were originally nonvanishing are set to 0, and the opposite procedure of *deformation*, where initially vanishing commutators become nonzero. No method of "graded deformations" has been developed so far, and our scope is actually more modest.

Our first motivation is to understand in terms of gradings how the $\mathrm{Z}_2\times\mathrm{Z}_2$ contractions of the de Sitter algebras ($\mathrm{so}(3,2)$ and $\mathrm{so}(4,1)$) and of the Poincaré algebra, none of which admits a non-trivial central extension, can result in algebras such as the Galilei algebra, which admit non-trivial central extensions. Indeed, it was found in reference [16] that all physically possible kinematics, described by Bacry and Lévy-Leblond using very general assumptions, are $\mathrm{Z}_2\times\mathrm{Z}_2$ graded contractions of either de Sitter algebras [17]. However, only some of the contractions can be extended in a non-trivial manner by a central element. We will show how one can anticipate the appearance of central charges using only grading arguments.

Perhaps the most pleasant result of our method occurs for the (2+1) dimensional Galilei algebra. When considered on its own, this algebra admits three different central charges, one of which must eventually be eliminated by considering the transformations at group level. Within our formalism, which considers the Galilei algebra as a contraction of the de Sitter algebra, we find that this same charge cannot possibly occur at the algebra level. This illustrates that our results will, in general, differ from those obtained had we considered the problem of finding central charges *in vacuo*, *i.e.* without reference to another, uncontracted algebra. In fact, it is precisely the focus of this paper to determine how charges in an uncontracted algebra $\mathcal{L}$, such as de Sitter or Poincaré, can become (or remain) non-trivial in the contracted algebra $\mathcal{L}'$, such as the Galilei algebra.

The second motivation stems from the general method of vector coherent states (VCS) [18] for obtaining representations of a Lie algebra. The VCS construction is interesting because [19] its *modus operandi* consists in realizing a Lie algebra $\mathcal{L}$ in terms of elements in the enveloping algebra of another Lie algebra $\mathcal{L}'$. The so-called holomorphic VCS representations are constructed using an algebra $\mathcal{L}'$ which is a contraction of $\mathcal{L}$ containing a central charge, so that VCS are not accommodated by the current method of graded contractions [1, 2]. An example of this is the contraction $\mathrm{sp}(6,\mathrm{R}) \to \mathrm{hw}(6) \oplus \mathrm{u}(3)$, which has applications in nuclear collective motion (for a review, see [20]). The VCS construction of a Lie algebra $\mathcal{L}$ relies heavily on a decomposition which we will relate to a grading decomposition of $\mathcal{L}$. This grading is a Z_3 grading for the A_n, B_n and C_n families of algebras, and a $\mathrm{Z}_2 \otimes \mathrm{Z}_3$ grading for the D_n series. In the tradition of graded contractions, our equations do not depend on the particular algebra but only on the graded structure, so that we need only consider in detail the $\mathrm{su}(n) \to \mathrm{u}(n-1) \oplus \mathrm{hw}(n-1)$ and $\mathrm{so}(2n+1) \to \mathrm{u}(n) \oplus \mathrm{hw}(n) \oplus \mathrm{hw}(\frac{1}{2}n(n-1))$ contractions. The analysis for the $\mathrm{so}(2n) \to \mathrm{u}(n) \oplus \mathrm{hw}(\frac{1}{2}n(n-1))$ and $\mathrm{sp}(2n) \to \mathrm{u}(n) \oplus \mathrm{hw}(2n)$ contractions does not differ from the $\mathrm{su}(n)$ case, as neither does the analysis of the contraction of the real forms of these algebras.

Let us finally mention that the search for central charges within the framework of graded contractions has already been performed for some specific Lie algebras, and specific finest gradings by [21]. These authors have used a fixed finest grading in order to find all the graded contractions and then all their central extensions. They have then classified the extensions according to whether they can be obtained through a contraction or not. Our approach is different in the sense that we contract and look for the central charges simultaneously. Therefore, for the specific cases considered in [21], we only obtain a subset of their solutions. Also, our approach is different because our equations depend only on the chosen grading of interest and not on a particular algebra.

2 Graded contractions with central extensions

A *grading* of the Lie algebra $\mathcal{L}$ is a decomposition into subspaces labeled by μ:

$$\mathcal{L} = \bigoplus_{\mu \in \Gamma} \mathcal{L}_\mu, \tag{1}$$

where μ takes on values in some index set Γ, such that [3]

$$[\mathcal{L}_\mu, \mathcal{L}_\nu] \subseteq \mathcal{L}_{\mu+\nu}, \tag{2}$$

which means that if $x \in \mathcal{L}_\mu$ and $y \in \mathcal{L}_\nu$, then $[x, y]$ belongs to the subspace $\mathcal{L}_{\mu+\nu}$. The commutator of two elements $l_{(\mu,i)} \in \mathcal{L}_\mu$ (where μ is a grading index and i is a generator index) and $l_{(\nu,j)} \in \mathcal{L}_\nu$ is

$$[l_{(\mu,i)}, l_{(\nu,j)}] = \sum_k c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)}\; l_{(\mu+\nu,k)}, \tag{3}$$

where $c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)}$ are the structure constants of $\mathcal{L}$.

We now extend $\mathcal{L}$ to $\overline{\mathcal{L}}$ by adding the unit operator $\mathbb{1}$, so that the commutation relations for $\overline{\mathcal{L}}$ read

$$\begin{aligned} \left[l_{(\mu,i)}, l_{(\nu,j)}\right] &= \sum_k c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)}\; l_{(\mu+\nu,k)} + \beta_{(\mu,i),(\nu,j)}\, \mathbb{1}\;, \\ \left[l_{(\mu,i)}, \mathbb{1}\right] &= 0. \end{aligned} \tag{4}$$

The *central charges* (or central parameters) $\beta_{(\mu,i),(\nu,j)}$ play the role of structure constants for the unit operator.

The Jacobi identities force the structure constants to satisfy the quadratic conditions

$$\sum_l \left(c^{(\mu+\nu,l)}_{(\mu,i),(\nu,j)} c^{(\mu+\nu+\sigma,q)}_{(\mu+\nu,l),(\sigma,k)} + c^{(\nu+\sigma,l)}_{(\nu,j),(\sigma,k)} c^{(\mu+\nu+\sigma,q)}_{(\nu+\sigma,l),(\mu,i)} + c^{(\sigma+\mu,l)}_{(\sigma,k),(\mu,i)} c^{(\mu+\nu+\sigma,q)}_{(\sigma+\mu,l),(\nu,j)} \right) = 0, \tag{5}$$

and they constrain the β's to satisfy

$$\sum_l \left(c^{(\mu+\nu,l)}_{(\mu,i),(\nu,j)} \beta_{(\mu+\nu,l),(\sigma,k)} + c^{(\nu+\sigma,l)}_{(\nu,j),(\sigma,k)} \beta_{(\nu+\sigma,l),(\mu,i)} + c^{(\sigma+\mu,l)}_{(\sigma,k),(\mu,i)} \beta_{(\sigma+\mu,l),(\nu,j)} \right) = 0\,. \tag{6}$$

Recall further that the solutions to equations (6) are not unique. If one shifts the infinitesimal generators as

$$\tilde{l}_{(\mu,i)} \equiv l_{(\mu,i)} + \alpha_{(\mu,i)}\, \mathbb{1}\,, \tag{7}$$

and then compares the commutator (4) with its "shifted" version:

$$[\tilde{l}_{(\mu,i)}, \tilde{l}_{(\nu,j)}] = \sum_k \tilde{c}^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)}\, \tilde{l}_{(\mu+\nu,k)} + \tilde{\beta}_{(\mu,i),(\nu,j)}\, \mathbb{1}\,, \tag{8}$$

one can see that $\tilde{c}^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)} = c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)}$ and that if the set $\{\beta_{(\mu,i),(\nu,j)}\}$ is a solution of (6), then so is the set $\{\tilde{\beta}_{(\mu,i),(\nu,j)}\}$ defined by

$$\tilde{\beta}_{(\mu,i),(\nu,j)} = \beta_{(\mu,i),(\nu,j)} - \sum_k \alpha_{(\mu+\nu,k)}\, c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)}\,, \tag{9}$$

for any values of the "shift parameters" $\alpha_{(\mu,j)}$. Therefore, two sets of central parameters $\{\beta_{(\mu,i),(\nu,j)}\}$ and $\{\tilde{\beta}_{(\mu,i),(\nu,j)}\}$ which can be related through (9) are called *equivalent* and in particular, a parameter $\beta_{(\mu,i),(\nu,j)}$ is *trivial* if it is equivalent to zero (*i.e.* all the $\tilde{\beta}$'s in (9) are zero).

Note that the β's in equations (4) may or may not be trivial in $\overline{\mathcal{L}}$. But since we are seeking to find which charges in $\mathcal{L}$ become non-trivial in $\overline{\mathcal{L}}'$, we must, initially at least, explicitly keep even the trivial charges in $\overline{\mathcal{L}}$. Although tedious in general, the process of finding central charges in $\overline{\mathcal{L}}$ is straightforward when the initial algebra is semisimple, since they are all trivial. We will come back to this point later in this section.

A graded contraction with central extensions involves two types of parameters, both depending only on the grading indices. The parameters $\varepsilon_{\mu,\nu}$ control the contraction by scaling commutators of $\mathcal{L}$ into commutators of the contracted algebra $\mathcal{L}'$. The parameters $\eta_{\mu,\nu}$, which again depend only on grading indices, scale all the β's in a family of commutators, thereby controlling the possible appearance of central charges. Put altogether, the initial algebra is first extended to $\overline{\mathcal{L}}$, whose commutators are then deformed into those of the algebra $\overline{\mathcal{L}}_{\varepsilon,\eta}$. Finally, for specific limit values of ε and η, the family of algebras $\overline{\mathcal{L}}_{\varepsilon,\eta}$ results in the contracted (and extended) algebra $\overline{\mathcal{L}}'$.

The commutators in $\overline{\mathcal{L}}$ are redefined into those of $\overline{\mathcal{L}}_{\varepsilon,\eta}$ as follows:

$$\begin{aligned}
[\text{ł}_{(\mu,i)}, \text{ł}_{(\nu,j)}] \;\; \to [\text{ł}_{(\mu,i)}, \text{ł}_{(\nu,j)}]_{\varepsilon,\eta} &= \varepsilon_{\mu,\nu}[\text{ł}_{(\mu,i)}, \text{ł}_{(\nu,j)}] + \eta_{\mu,\nu}\beta_{(\mu,i),(\nu,j)}\, \mathbb{1}\,, \\
&= \varepsilon_{\mu,\nu}\left(\sum_k c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)} l_{(\mu+\nu,k)} \right) + \eta_{\mu,\nu}\beta_{(\mu,i),(\nu,j)}\, \mathbb{1}\,.
\end{aligned} \tag{10}$$

Note that the parameters $\varepsilon_{\mu,\nu}$ and $\eta_{\mu,\nu}$ are *symmetric* under permutation of μ and ν. In order for these modified commutators to define a new Lie algebra, the parameters are subject to constraints derived from the Jacobi identity:

$$\begin{aligned}
0 &= [[l_{(\mu,i)}, l_{(\nu,j)}]_{\varepsilon,\eta}, l_{(\sigma,k)}]_{\varepsilon,\eta} + \text{cycl.perm.}, \\
&= [\varepsilon_{\mu,\nu}[l_{(\mu,i)}, l_{(\nu,j)}] + \eta_{\mu,\nu}\beta_{(\mu,i),(\nu,j)}\,\mathbb{1}\,, l_{(\sigma,k)}]_{\varepsilon,\eta} + \text{cycl.perm.}, \\
&= \varepsilon_{\mu,\nu}[[l_{(\mu,i)}, l_{(\nu,j)}], l_{(\sigma,k)}]_{\varepsilon,\eta} + \text{cycl.perm.}, \\
&= \varepsilon_{\mu,\nu}\left(\varepsilon_{\mu+\nu,\sigma}[[l_{(\mu,i)}, l_{(\nu,j)}], l_{(\sigma,k)}] + \eta_{\mu+\nu,\sigma}\left(\sum_l c^{(\mu+\nu,l)}_{(\mu,i),(\nu,j)}\beta_{(\mu+\nu,l),(\sigma,k)}\right)\mathbb{1}\right) \\
&\qquad\qquad + \text{cycl.perm.}
\end{aligned} \tag{11}$$

Taking into account the fact that the commutators $[\cdot,\cdot]$ of the original algebra $\overline{\mathcal{L}}$ already satisfy the Jacobi identities (5), one finds the usual equations determining graded contractions [1]:

$$\varepsilon_{\mu,\nu}\varepsilon_{\mu+\nu,\sigma} = \varepsilon_{\nu,\sigma}\varepsilon_{\nu+\sigma,\mu}. \tag{12}$$

From equation (11) and (6), we find also

$$\varepsilon_{\mu,\nu}\eta_{\mu+\nu,\sigma} = \varepsilon_{\nu,\sigma}\eta_{\nu+\sigma,\mu} \tag{13}$$

as a set of solutions of (11). The equations (12) and (13) are the central result of this paper.

Before solving (13), one should note that they must be slightly modified in three special circumstances: (i) a subspace $\mathcal{L}_\mu$ is empty, (ii) $[\mathcal{L}_\mu, \mathcal{L}_\nu] = 0$ (so that $c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)} = 0$, for all i, j), and (iii) the charges $\beta_{(\mu,i),(\nu,j)}$ are forced to be 0 for all i, j. Under such circumstances, the relevant term in equation (11) does not contribute to the sum and any product containing $\varepsilon_{\mu,\nu}$ or $\eta_{\mu,\nu}$ must be taken out of the relations (12) and (13). The parameters $\varepsilon_{\mu,\nu}$ or $\eta_{\mu,\nu}$ are then referred to as *irrelevant*. In [1], a grading containing irrelevant parameter is referred to as being *non-generic*. A *generic* grading is such that no commutator in (2) vanishes identically.

In [1], it was shown that the nonzero ε's can often be renormalized to 1 for complex Lie algebras, and to 1 or -1 for real Lie algebras. This fact provides us with the possibility of relating different real forms through a graded contraction, and this notion has proven useful in the context of kinematical groups [16]. Thus, the rescaling of the structure constants of $\overline{\mathcal{L}}$ through $\overline{\mathcal{L}}_\mu \to \overline{\mathcal{L}}_\mu{}' = a_\mu \overline{\mathcal{L}}_\mu$ leads to a rescaling of the ε's and η's as

$$\varepsilon'_{\mu,\nu} \equiv \frac{a_\mu a_\nu}{a_{\mu+\nu}}\varepsilon_{\mu,\nu} \quad \text{and} \quad \eta'_{\mu,\nu} \equiv a_\mu a_\nu \eta_{\mu,\nu}. \tag{14}$$

Finally, once we have found those $\eta_{\mu,\nu}$'s that are not necessarily 0, one must remove the trivial parameters through a transformation (7). As for the contractions of Lie algebras, a *non-generic* grading will, in general, allow more non-trivial solutions for the η's, since (13) then contains fewer equations.

To summarize, the algorithm is as follows:

1. Choose a grading of some Lie algebra $\mathcal{L}$;

2. Extend $\mathcal{L}$ to $\overline{\mathcal{L}}$ but keep the same grading. (The unit, which commutes with everything, is added to the $\mathcal{L}_0$ subspace);

3. Contract $\overline{\mathcal{L}}$ to $\overline{\mathcal{L}}'$ by first removing the appropriate terms in (12) when the grading is non-generic, and then solving the equations for the ε's;

4. Given a set of solution ε's, solve for the η's using the linear equations (13), after removing therein the terms containing irrelevant η's;

5. The solutions η are then substituted in (10). The trivial charges are eliminated using, in equation (9), the deformed structure constants $\varepsilon_{\mu,\nu}\, c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)}$ and central parameters $\eta_{\mu,\nu}\, \beta_{(\mu,i),(\nu,j)}$ of the contracted algebra $\overline{\mathcal{L}}'$. Non-trivial charges necessarily appear when $\eta_{\mu,\nu} \neq 0$.

The step 2 is easy when $\mathcal{L}$ is a semisimple Lie algebra because then all the charges must be trivial. In other words, the central charge $\beta_{(\mu,i),(\nu,j)}$ can be written, using (7) and (9) with $\tilde{\beta} = 0$, as

$$\beta_{(\mu,i),(\nu,j)} = \sum_l c^{(\mu+\nu,l)}_{(\mu,i),(\nu,j)}\, \alpha_{(\mu+\nu,l)}\,, \tag{15}$$

where $\alpha_{(\mu+\nu,l)}$ are numbers chosen so that $\beta_{(\mu,i),(\nu,j)}$ is real.

In the literature (see, for instance, the reference [22]), one often finds equation (15) written in the form

$$\beta_{(\mu,i),(\nu,j)} = \Lambda([l_{(\mu,i)}, l_{(\nu,j)}])\,, \tag{16}$$

where Λ is a linear functional defined so that $\Lambda(l_{(\mu,i)}) = \alpha_{(\mu,i)}$. Please observe that this last equation involves a commutator in the original algebra $\mathcal{L}$. Using this, equation (10) takes the more "symmetric" form

$$[\text{ł}_{(\mu,i)}, \text{ł}_{(\nu,j)}]_{\varepsilon,\eta} = \varepsilon_{\mu,\nu}[\text{ł}_{(\mu,i)}, \text{ł}_{(\nu,j)}] + \eta_{\mu,\nu}\, \Lambda([\text{ł}_{(\mu,i)}, \text{ł}_{(\nu,j)}])\, \mathbb{1}\,.$$

Although $\eta_{\mu,\nu} = \varepsilon_{\mu,\nu}$ is a solution to equation (6), it is not the *only* solution: we have a non-trivial central extension when the scalings $\varepsilon_{\mu,\nu}$ and $\eta_{\mu,\nu}$ are not the same so that the central parameter does not "follow" the commutator.

Moreover, the equation (15) provides relations between charges. It follows from this equation that $\beta_{(\mu,i),(\nu,j)} = 0$ if $[l_{\mu,i}, l_{\nu,j}] = 0$ and that if two subspaces commute *i.e.* if $[\mathcal{L}_\mu, \mathcal{L}_\nu] = 0$, then the corresponding $\eta_{\mu,\nu}$ and $\varepsilon_{\mu,\nu}$ are irrelevant. Consider the following Z_3 grading of su(3):

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_+ + \mathcal{L}_-, \tag{17}$$

where

$$\begin{aligned} \mathcal{L}_0 &= \{E_{12}, E_{21}, E_{11} - E_{22}, 2E_{33} - E_{22} - E_{11}\}, \\ \mathcal{L}_+ &= \{E_{23}, E_{13}\}, \\ \mathcal{L}_- &= \{E_{32}, E_{31}\}. \end{aligned} \tag{18}$$

The commutation relations are given at the beginning of Section 4.1. Consider, for instance, the commutator $[E_{13}, E_{21}]$, in the notation of equation (4):

$$[E_{13}, E_{21}] = E_{23} \rightarrow [l_{(1,13)}, l_{(-1,21)}] = l_{(0,23)} + \beta_{(1,13),(-1,21)} \tag{19}$$

so that $\beta_{(1,13),(-1,21)} = \alpha_{(0,23)}$. Let $E_{11} - E_{22} = h_1$. We also have

$$[h_1, E_{23}] = -E_{23} \rightarrow [l_{(0,h_1)}, l_{(0,23)}] = -l_{(0,23)} + \beta_{(0,h_1),(0,23)} \tag{20}$$

and hence the relation

$$\beta_{(0,h_1),(0,23)} = -\alpha_{(0,23)} = -\beta_{(1,13),(-1,21)}\,, \tag{21}$$

between these central parameters.

This can be generalized. Recall, from the root diagram of a semisimple Lie algebra, that all the weight subspaces, with the exception of the zero weight subspace, are of dimension ≤ 1. Thus, if $[l_{(\mu,i)}, l_{(\nu,j)}] \neq 0$ and does not lie in the zero weight subspace, the sum in (15) contains exactly one term. Quite generally then, if two commutators are proportional to the same element not in the zero weight subspace, *e.g.* if $[l_{(\mu,i)}, l_{(\nu,j)}] = c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)} l_{(\mu+\nu,k)}$ and $[l_{(\mu',i')}, l_{(\nu',j')}] = c^{(\mu'+\nu',k)}_{(\mu',i),(\nu',j)} l_{(\mu'+\nu',k)}$ with $\mu+\nu = \mu'+\nu'$, then their respective charges $\beta_{(\mu,i),(\nu,j)}$ and $\beta_{(\mu',i'),(\nu',j')}$ are proportional to one another, as in equation (21). As all the applications discussed in this paper have as a starting point a semisimple Lie algebra, these relations will be extremely useful as they provide explicit relations between the charges in $\mathcal{L}'$ and $\mathcal{L}$.

Before turning our attention to physical applications, we conclude this section with the simplest example of a solution, a Z_2 grading, for which a general Lie algebra $\mathcal{L}$ decomposes into

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \tag{22}$$

with commutation relations expressed symbolically as

$$[\mathcal{L}_0, \mathcal{L}_0] = \mathcal{L}_0,\ [\mathcal{L}_0, \mathcal{L}_1] = \mathcal{L}_1,\ [\mathcal{L}_1, \mathcal{L}_1] = \mathcal{L}_0. \tag{23}$$

Then, the equations (12) give the following possible solutions [1] :

$$\begin{array}{l} \varepsilon^{I} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \quad \varepsilon^{II} = \begin{pmatrix} 0 & 0 \\ & 0 \end{pmatrix}, \quad \varepsilon^{III} = \begin{pmatrix} 1 & 1 \\ & 0 \end{pmatrix}, \\ \varepsilon^{IV} = \begin{pmatrix} 0 & 0 \\ & 1 \end{pmatrix}, \quad \varepsilon^{V} = \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}, \end{array} \tag{24}$$

where we use the form $\varepsilon = \begin{pmatrix} \varepsilon_{0,0} & \varepsilon_{0,1} \\ & \varepsilon_{1,1} \end{pmatrix}$. Equation (13) then reduces to

$$\begin{array}{rcl} \varepsilon_{0,0}\eta_{0,1} & = & \varepsilon_{0,1}\eta_{0,1}, \\ \varepsilon_{0,1}\eta_{1,1} & = & \varepsilon_{1,1}\eta_{0,0}, \end{array} \tag{25}$$

and, solving for the η's, we obtain

$$\begin{array}{l} \eta^{I} = \begin{pmatrix} a & b \\ & a \end{pmatrix}, \quad \eta^{II} = \begin{pmatrix} a & b \\ & c \end{pmatrix}, \quad \eta^{III} = \begin{pmatrix} a & b \\ & 0 \end{pmatrix}, \\ \eta^{IV} = \begin{pmatrix} 0 & a \\ & b \end{pmatrix}, \quad \eta^{V} = \begin{pmatrix} a & 0 \\ & b \end{pmatrix}, \end{array} \tag{26}$$

where a, b and c are free parameters. Obviously the net effect of these solutions depend on the Lie algebra considered, so that solutions which are inequivalent *a priori* might turn out to be equivalent in particular cases.

3 $\mathrm{Z}_2 \otimes \mathrm{Z}_2$ gradings and kinematical algebras

In this section, we determine the $\mathrm{Z}_2 \otimes \mathrm{Z}_2$ centrally extended contractions of the de Sitter algebras (in $(2+1)$ and $(3+1)$ dimensions) that lead to the kinematical algebras.

3.1 Generic $\mathrm{Z}_2 \otimes \mathrm{Z}_2$ graded contractions

A $\mathrm{Z}_2 \otimes \mathrm{Z}_2$ grading is a decomposition

$$\mathcal{L} = \mathcal{L}_{00} + \mathcal{L}_{01} + \mathcal{L}_{10} + \mathcal{L}_{11}. \tag{27}$$

Hereafter, the first Z_2 index gives the transformation properties of the generators under space inversion, whereas the second Z_2 refers to the time reversal.

Equation (12) is given in [1] for the generic $Z_2 \otimes Z_2$ grading, whereas (13) takes the form

$$\begin{array}{rcccl}
\varepsilon_{00,00}\eta_{00,k} & = & \varepsilon_{00,k}\eta_{00,k}, & & \\
\varepsilon_{00,k}\eta_{k,k} & = & \varepsilon_{k,k}\eta_{00,00}, & & \\
\varepsilon_{00,01}\eta_{01,10} & = & \varepsilon_{01,10}\eta_{00,11} & = & \varepsilon_{00,10}\eta_{01,10}, \\
\varepsilon_{00,10}\eta_{10,11} & = & \varepsilon_{10,11}\eta_{00,01} & = & \varepsilon_{00,11}\eta_{10,11}, \\
\varepsilon_{00,11}\eta_{01,11} & = & \varepsilon_{01,11}\eta_{00,10} & = & \varepsilon_{00,01}\eta_{01,11}, \\
\varepsilon_{01,10}\eta_{11,11} & = & \varepsilon_{10,11}\eta_{01,01} & = & \varepsilon_{01,11}\eta_{10,10}, \\
\varepsilon_{01,01}\eta_{00,10} & = & \varepsilon_{01,10}\eta_{01,11}, & & \\
\varepsilon_{01,01}\eta_{00,11} & = & \varepsilon_{01,11}\eta_{01,10}, & & \\
\varepsilon_{10,10}\eta_{00,01} & = & \varepsilon_{01,10}\eta_{10,11}, & & \\
\varepsilon_{10,10}\eta_{00,11} & = & \varepsilon_{10,11}\eta_{01,10}, & & \\
\varepsilon_{11,11}\eta_{00,01} & = & \varepsilon_{01,11}\eta_{10,11}, & & \\
\varepsilon_{11,11}\eta_{00,10} & = & \varepsilon_{10,11}\eta_{01,11}, & &
\end{array} \tag{28}$$

where $k = (01), (10), (11)$. This set of equations is maximal, in the sense that whenever one considers a non-generic grading, then one just has to remove the corresponding term from the equations above. Taking into account the symmetry of ε and η, we shall write the solutions in the form

$$\varepsilon = \begin{pmatrix} \varepsilon_{00,00} & \varepsilon_{00,01} & \varepsilon_{00,10} & \varepsilon_{00,11} \\ & \varepsilon_{01,01} & \varepsilon_{01,10} & \varepsilon_{01,11} \\ & & \varepsilon_{10,10} & \varepsilon_{10,11} \\ & & & \varepsilon_{11,11} \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_{00,00} & \eta_{00,01} & \eta_{00,10} & \eta_{00,11} \\ & \eta_{01,01} & \eta_{01,10} & \eta_{01,11} \\ & & \eta_{10,10} & \eta_{10,11} \\ & & & \eta_{11,11} \end{pmatrix}. \tag{29}$$

3.2 Kinematical algebras in $(2 + 1)$ dimensions

In $(2 + 1)$ dimensions, the de Sitter algebra so$(3, 1)$ is six dimensional, with commutation relations

$$\begin{array}{lcllcl}
[J, P_1] & = & P_2, & [J, P_2] & = & -P_1, \\
[J, K_1] & = & K_2, & [J, K_2] & = & -K_1, \\
[H, P_i] & = & -K_i, & [H, K_i] & = & -P_i, \\
[P_1, P_2] & = & J, & [K_1, K_2] & = & -J, \\
[P_i, K_j] & = & -\delta_{ij}H, & & &
\end{array} \tag{30}$$

where J is the angular momentum, H the energy, and P_i and K_i are the generators of translations and inertial transformations, respectively. Our $Z_2 \otimes Z_2$ grading (27) decomposes so$(3, 1)$ into the subspaces

$$\mathcal{L}_{00} = \{J\}, \ \mathcal{L}_{01} = \{H\}, \ \mathcal{L}_{10} = \{P_1, P_2\}, \ \mathcal{L}_{11} = \{K_1, K_2\}. \tag{31}$$

Then, with our definition (10), one obtains $\overline{\mathrm{so}(3,1)}_{\varepsilon,\eta}$, with the deformed commutation relations

$$\begin{array}{lcllcl}
[J, P_1]_{\varepsilon,\eta} & = & \varepsilon_{00,10}P_2 + \eta_{00,10}\alpha_{P_2}\,\mathbb{1}, & [J, P_2]_{\varepsilon,\eta} & = & -\varepsilon_{00,10}P_1 - \eta_{00,10}\alpha_{P_1}\,\mathbb{1}, \\
[J, K_1]_{\varepsilon,\eta} & = & \varepsilon_{00,11}K_2 + \eta_{00,11}\alpha_{K_2}\,\mathbb{1}, & [J, K_2]_{\varepsilon,\eta} & = & -\varepsilon_{00,11}K_1 - \eta_{00,11}\alpha_{K_1}\,\mathbb{1}, \\
[H, P_i]_{\varepsilon,\eta} & = & -\varepsilon_{01,10}K_i - \eta_{01,10}\alpha_{K_i}\,\mathbb{1}, & [H, K_i]_{\varepsilon,\eta} & = & -\varepsilon_{01,11}P_i - \eta_{01,11}\alpha_{P_i}\,\mathbb{1}, \\
[P_1, P_2]_{\varepsilon,\eta} & = & \varepsilon_{10,10}J + \eta_{10,10}\alpha_J\,\mathbb{1}, & [K_1, K_2]_{\varepsilon,\eta} & = & -\varepsilon_{11,11}J - \eta_{11,11}\alpha_J\,\mathbb{1}, \\
[P_i, K_j]_{\varepsilon,\eta} & = & -\varepsilon_{10,11}\delta_{i,j}H - \eta_{10,11}\delta_{i,j}\alpha_H\,\mathbb{1}, & & &
\end{array} \tag{32}$$

where we have used the notational shortcuts $\alpha_{P_2} = \alpha_{(10,2)}$ and so forth, to denote the shift parameters.

The parameters $\varepsilon_{00,00}, \varepsilon_{00,01}, \varepsilon_{01,01}$ and their corresponding η's are irrelevant, and the appropriate terms must be removed from (28). The possible kinematical algebra found in [17] all have $\varepsilon_{00,10}$ and $\varepsilon_{00,11}$ equal to 1. Hence, the equations (28) simplify to

$$\begin{array}{lcl} \eta_{01,10} & = & \varepsilon_{01,10}\eta_{00,11}, \\ \eta_{01,11} & = & \varepsilon_{01,11}\eta_{00,10}, \\ \varepsilon_{01,10}\eta_{11,11} & = & \varepsilon_{01,11}\eta_{10,10}, \\ \varepsilon_{10,10}\eta_{00,11} & = & \varepsilon_{10,11}\eta_{01,10}, \\ \varepsilon_{11,11}\eta_{00,10} & = & \varepsilon_{10,11}\eta_{01,11}. \end{array} \tag{33}$$

There is no restriction on the parameter $\eta_{10,11}$, which occurs in commutators of the type $[P_i, K_j]$.

The Galilei algebra is obtained from the de Sitter algebra by using the matrix $\varepsilon_{\mathrm{Gal}} = \begin{pmatrix} \emptyset & \emptyset & 1 & 1 \\ & \emptyset & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$. This, in turns, leads to the matrix $\eta_{\mathrm{Gal}} = \begin{pmatrix} \emptyset & \emptyset & a & b \\ & \emptyset & 0 & a \\ & & 0 & m \\ & & & k \end{pmatrix}$. (As in reference [1], $\emptyset$ denotes an entry associated to an irrelevant parameter.) The labeling of lines and column in each matrix is $(00), (01), (10), (11)$. In terms of explicit commutation relations, it transforms (32) into

$$\begin{array}{lcllcl} [J, P_1]_{\varepsilon,\eta} & = & P_2 + a\,\alpha_{P_2}, & [J, P_2]_{\varepsilon,\eta} & = & -P_1 - a\,\alpha_{P_1}, \\ [J, K_1]_{\varepsilon,\eta} & = & K_2 + b\,\alpha_{K_2}, & [J, K_2]_{\varepsilon,\eta} & = & -K_1 - b\,\alpha_{K_1}, \\ [H, P_i]_{\varepsilon,\eta} & = & 0, & [H, K_i]_{\varepsilon,\eta} & = & -P_i - a\,\alpha_{P_i}, \\ [P_1, P_2]_{\varepsilon,\eta} & = & 0, & [K_1, K_2]_{\varepsilon,\eta} & = & -k\,\alpha_J\,\mathbb{1}, \\ [P_i, K_j]_{\varepsilon,\eta} & = & -m\,\delta_{ij}\alpha_H\,\mathbb{1}. & & & \end{array} \tag{34}$$

The charges $a\,\alpha_{P_i}$ and $b\,\alpha_{K_i}$ are clearly equivalent to trivial charges as they can be eliminated by shifting the translation P_i and K_i generators using (7). This leaves the non-trivial charges of the extended Galilei algebra as $k\,\alpha_J$ and $m\,\delta_{ij}$. Thus, we finally have

$$\begin{array}{lcllcl} [J, P_1]_{\varepsilon,\eta} & = & P_2, & [J, P_2]_{\varepsilon,\eta} & = & -P_1, \\ [J, K_1]_{\varepsilon,\eta} & = & K_2, & [J, K_2]_{\varepsilon,\eta} & = & -K_1, \\ [H, P_i]_{\varepsilon,\eta} & = & 0, & [H, K_i]_{\varepsilon,\eta} & = & -P_i, \\ [P_1, P_2]_{\varepsilon,\eta} & = & 0, & [K_1, K_2]_{\varepsilon,\eta} & = & -k\,\alpha_J\,\mathbb{1}, \\ [P_i, K_j]_{\varepsilon,\eta} & = & -m\,\delta_{ij}\alpha_H\,\mathbb{1}. & & & \end{array} \tag{35}$$

This result is in accordance with the extended commutation relations found on page 240 of [22] (note that equation (3.29f) of this reference should read $[K_i, H] = P_i$), *except* in the following respect. In [22], the commutator $[J, H]$ in the Galilei algebra can be extended to $[J, H] = h\,\mathbb{1}$. However, it is shown that h must be 0 if a finite rotation by $2\pi + \theta$ is to coincide with a rotation by θ. When the Galilei algebra is considered as a contraction of the de Sitter algebra, however, the central parameter $\eta_{00,11}\,\beta_{(J,H)}$ vanishes immediately because $\beta_{(J,H)}$, the central parameter for the commutator $[J, H]$ in the de Sitter algebra, is found to be necessarily zero from equation (15).

3.3 Kinematical algebras in $(3+1)$ dimensions

The $\mathrm{Z}_2 \otimes \mathrm{Z}_2$ grading of the de Sitter algebras so$(4,1)$ and so$(3,2)$ is:

$$\begin{array}{rl} \mathcal{L}_{00} = \{\mathrm{J}\} : & 3 \text{ angular momentum operators}, \\ \mathcal{L}_{01} = \{H\} : & 1 \text{ energy operator}, \\ \mathcal{L}_{10} = \{\mathrm{P}\} : & 3 \text{ translation operators}, \\ \mathcal{L}_{11} = \{\mathrm{K}\} : & 3 \text{ inertial transformations}. \end{array} \tag{36}$$

Hereafter, we are interested in the graded contractions of the de Sitter algebras, with commutation relations given by

$$\begin{array}{rcl@{\qquad}rcl} [\mathrm{J},\mathrm{J}] &=& \mathrm{J}, & & & \\ [\mathrm{J},\mathrm{P}] &=& \mathrm{P}, & [\mathrm{J},\mathrm{K}] &=& \mathrm{K}, \\ [H,\mathrm{P}] &=& \pm\mathrm{K}, & [H,\mathrm{K}] &=& \mathrm{P}, \\ [\mathrm{P},\mathrm{P}] &=& \pm\mathrm{J}, & [\mathrm{P},\mathrm{K}] &=& H, \\ [\mathrm{K},\mathrm{K}] &=& -\mathrm{J}, & & & \end{array} \tag{37}$$

where the upper sign applies to so$(4,1)$, and the lower sign to so$(3,2)$. Following the notation of [17], we let $[\mathrm{A},\mathrm{B}] = \mathrm{C}$ denotes generically a commutator $[A_i, B_j] = \epsilon_{ijk} C_k$, $[H,\mathrm{A}] = \mathrm{B}$ stands for $[H, A_i] = B_i$, and $[\mathrm{A},\mathrm{B}] = H$ means $[A_i, B_j] = \delta_{ij} H$.

The modified commutators of $\overline{\mathrm{so}(4,1)}_{\varepsilon,\eta}$ and $\overline{\mathrm{so}(3,2)}_{\varepsilon,\eta}$ take the form [16, 17]

$$\begin{array}{rcl@{\qquad}rcl} [\mathrm{J},\mathrm{J}]_{\varepsilon,\eta} &=& \varepsilon_{00,00}\mathrm{J} + \eta_{00,00}\alpha_{\mathrm{J}}\,\mathbb{1}\,, & & & \\ [\mathrm{J},\mathrm{P}]_{\varepsilon,\eta} &=& \varepsilon_{00,10}\mathrm{P} + \eta_{00,10}\alpha_{\mathrm{P}}\,\mathbb{1}\,, & [\mathrm{J},\mathrm{K}]_{\varepsilon,\eta} &=& \varepsilon_{00,11}\mathrm{K} + \eta_{00,11}\alpha_{\mathrm{K}}\,\mathbb{1}\,, \\ [H,\mathrm{P}]_{\varepsilon,\eta} &=& \pm\varepsilon_{01,10}\mathrm{K} \pm \eta_{01,10}\alpha_{\mathrm{K}}\,\mathbb{1}\,, & [H,\mathrm{K}]_{\varepsilon,\eta} &=& \varepsilon_{01,11}\mathrm{P} + \eta_{01,11}\alpha_{\mathrm{P}}\,\mathbb{1}\,, \\ [\mathrm{P},\mathrm{P}]_{\varepsilon,\eta} &=& \pm\varepsilon_{10,10}\mathrm{J} \pm \eta_{10,10}\alpha_{\mathrm{J}}\,\mathbb{1}\,, & [\mathrm{P},\mathrm{K}]_{\varepsilon,\eta} &=& \varepsilon_{10,11}H + \eta_{10,11}\alpha_H\,\mathbb{1}\,, \\ [\mathrm{K},\mathrm{K}]_{\varepsilon,\eta} &=& -\varepsilon_{11,11}\mathrm{J} - \eta_{11,11}\alpha_{\mathrm{J}}\,\mathbb{1}\,. & & & \end{array} \tag{38}$$

The relations that determine the existence of central charges are obtained from (28) by removing the terms that contain $\varepsilon_{00,01}$, $\varepsilon_{01,01}$, $\eta_{00,01}$ and $\eta_{01,01}$, and by setting, as in $(2+1)$ dimensions, $\varepsilon_{00,00} = \varepsilon_{00,10} = \varepsilon_{00,11} = 1$:

$$\begin{array}{rcl@{\qquad}l} \eta_{10,10} &=& \varepsilon_{10,10}\eta_{00,00}, & \eta_{11,11} = \varepsilon_{11,11}\eta_{00,00}, \\ \eta_{01,10} &=& \varepsilon_{01,10}\eta_{00,11}, & \eta_{01,11} = \varepsilon_{01,11}\eta_{00,10}, \\ \varepsilon_{10,10}\eta_{00,11} &=& \varepsilon_{10,11}\eta_{01,10}, & \varepsilon_{11,11}\eta_{00,10} = \varepsilon_{10,11}\eta_{01,11}, \\ \varepsilon_{01,10}\eta_{11,11} &=& \varepsilon_{01,11}\eta_{10,10}. & \end{array} \tag{39}$$

The parameter $\eta_{10,11}$, which occurs in the extension of the commutator $[\mathrm{P},\mathrm{K}]$, is the only parameter on which there are no conditions, as expected (see, for instance, [17])!

The Poincaré algebra is obtained from the de Sitter algebras by using the matrix $\varepsilon_{\mathrm{Poin}} = \begin{pmatrix} 1 & \emptyset & 1 & 1 \\ & \emptyset & 0 & 1 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix}$, which in turns leads to $\eta_{\mathrm{Poin}} = \begin{pmatrix} a & \emptyset & b & c \\ & \emptyset & 0 & b \\ & & 0 & d \\ & & & a \end{pmatrix}$. Then equation (38) becomes

$$\begin{array}{rcl@{\qquad}rcl} [\mathrm{J},\mathrm{J}]_{\varepsilon,\eta} &=& \mathrm{J} + a\,\alpha_{\mathrm{J}}\,\mathbb{1}\,, & & & \\ [\mathrm{J},\mathrm{P}]_{\varepsilon,\eta} &=& \mathrm{P} + b\,\alpha_{\mathrm{P}}\,\mathbb{1}\,, & [\mathrm{J},\mathrm{K}]_{\varepsilon,\eta} &=& \mathrm{K} + c\,\alpha_{\mathrm{K}}\,\mathbb{1}\,, \\ [H,\mathrm{P}]_{\varepsilon,\eta} &=& 0, & [H,\mathrm{K}]_{\varepsilon,\eta} &=& \mathrm{P} + b\,\alpha_{\mathrm{P}}\,\mathbb{1}\,, \\ [\mathrm{P},\mathrm{P}]_{\varepsilon,\eta} &=& 0, & [\mathrm{P},\mathrm{K}]_{\varepsilon,\eta} &=& H + d\,\alpha_H\,\mathbb{1}\,, \\ [\mathrm{K},\mathrm{K}]_{\varepsilon,\eta} &=& -\mathrm{J} - a\,\alpha_{\mathrm{J}}\,\mathbb{1}\,. & & & \end{array} \tag{40}$$

As expected, all of the charges are trivial as they can be absorbed by shifting the generators as in (7).

The Galilei, Newton-Hooke and Static algebras are more interesting, since their central extensions are non-trivial. For instance, the contraction matrix that corresponds to the Galilei algebra is $\varepsilon_{\mathrm{Gal}} = \begin{pmatrix} 1 & \emptyset & 1 & 1 \\ & \emptyset & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$ and its η_{Gal} matrix is $\eta_{\mathrm{Gal}} = \begin{pmatrix} a & \emptyset & b & c \\ & \emptyset & 0 & b \\ & & 0 & d \\ & & & 0 \end{pmatrix}$. In this case, only the commutator $[\mathrm{P},\mathrm{K}]_{\varepsilon,\eta} = 0 + \, d\,\alpha_H \, \mathbb{1}$ can be extended in a non-trivial way. This result is in excellent agreement with [17], with the central charge being proportional to the mass operator.

Once can also verify that for the Newton-Hooke and Static algebras, defined by the matrices $\varepsilon_{\mathrm{NH}} = \begin{pmatrix} 1 & \emptyset & 1 & 1 \\ & \emptyset & 1 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$ and $\varepsilon_{\mathrm{Stat}} = \begin{pmatrix} 1 & \emptyset & 1 & 1 \\ & \emptyset & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$, the corresponding solutions, $\eta_{\mathrm{NH}} = \begin{pmatrix} a & \emptyset & b & c \\ & \emptyset & c & b \\ & & 0 & d \\ & & & 0 \end{pmatrix}$ and $\eta_{\mathrm{Stat}} = \begin{pmatrix} a & \emptyset & b & c \\ & \emptyset & 0 & 0 \\ & & 0 & d \\ & & & 0 \end{pmatrix}$ also reproduce the extended commutation relations given in [17].

4 Holomorphic vector coherent states representations and their contractions

The vector coherent states (VCS) theory [18] is a generalization of the usual (scalar) coherent state theory that provides a powerful and systematic method of expressing elements of a semisimple Lie algebra $\mathcal{L}$ as polynomials in the elements of another Lie algebra $\mathcal{L}'$ whose matrix elements are already known. As mentioned in the introduction, $\mathcal{L}'$ is a contraction of $\mathcal{L}$. In fact, in the "contraction limit" where one or many of the quantum numbers labeling a representation of $\mathcal{L}$ become asymptotically large, representations of $\mathcal{L}$ become indistinguishable from representations of $\mathcal{L}'$. Representations of all classical [19, 23, 24] Lie algebras and as well as exceptional ones have been constructed using VCS methods.

The VCS construction is based on a decomposition of $\mathcal{L}$ into

$$\mathcal{L} = \mathrm{n}_+ \oplus \mathcal{L}_0 \oplus \mathrm{n}_- \,, \tag{41}$$

where $\mathrm{n}_\pm$ span, respectively, nilpotent subalgebras of raising and lowering operators. This decomposition turns out to be a Z_3 grading. Matrix elements of $\mathcal{L}$ are computed using the $\mathcal{L} \supset \mathcal{L}_0$ subalgebra chain.

In this section, we consider first the $\mathrm{su}(n+1) \to \mathrm{u}(n) \oplus \mathrm{hw}(\frac{1}{2}n(n+1))$ contraction. The analysis for the $\mathrm{sp}(2n)$ and $\mathrm{so}(2n)$ algebras is identical to the $\mathrm{su}(n+1)$ algebra because for these three series the subalgebras $\mathrm{n}_\pm$ are abelian, *i.e.*

$$[\mathrm{n}_+, \mathrm{n}_+] = [\mathrm{n}_-, \mathrm{n}_-] = 0\,. \tag{42}$$

It is therefore possible to identify $\mathrm{n}_\pm$ with the graded subspaces $\mathcal{L}_\pm$, that occur in the decomposition (17). One should also note that the operators in $\mathrm{n}_\pm$ are the components of an irreducible tensor under the subalgebra $\mathcal{L}_0$. This is discussed in the Section 4.1.

The so$(2n+1)$ case is different because even if $\mathcal{L}_0$ remains u(n), the subalgebras $\mathrm{n}_\pm$ are now adjoint-nilpotent, *i.e.*

$$[\mathrm{n}_+, \mathrm{n}_+] \subseteq \mathrm{n}_+ \,, \qquad [\mathrm{n}_+, [\mathrm{n}_+, \mathrm{n}_+]] = 0 \,. \tag{43}$$

This requires a refinement of the Z_3 grading of the previous series. The subspaces $\mathrm{n}_\pm$ each contain two irreducible tensors of $\mathcal{L}_0$. The details for so$(2n+1)$ are found in Section 4.2.

4.1 The su$(n) \to$ u$(n-1) \oplus$ hw$(n-1)$ contractions

In this section, we concentrate on holomorphic representations of su(n) based on u$(n) \oplus$ hw$(\frac{1}{2}n(n+1))$. The treatment is identical for the so$(2n)$ and sp$(2n)$ algebras because the natural grading for the VCS construction is identical to that of su(n) in both cases.

Following [23], we give first a basis for the Lie algebra u(n) in terms of n^2 operators

$$\{E_{ij}\,,\ i,j = 1, \ldots, n\}\,, \tag{44}$$

with commutation relations

$$[E_{ij}, E_{kl}] = E_{il}\delta_{jk} - E_{kj}\delta_{il}\,. \tag{45}$$

A basis for su(n) is extracted by selecting the subset of generators

$$\begin{aligned}
E_{ij}\,, &\quad i > j = 1, \ldots, n\,, && \text{lowering operators}\,,\\
h_i = E_{ii} - E_{i+1,i+1}\,, &\quad i = 1, \ldots, n-2\,, && n-2 \text{ Cartan operators}\,,\\
W = (n-1)E_{nn} - \sum_{i=1}^{n-1} E_{ii}\,, &&& \text{last Cartan operator}\,,\\
E_{ij}\,, &\quad j > i = 1, \ldots, n\,, && \text{raising operators}\,.
\end{aligned} \tag{46}$$

We now decompose $\mathcal{L}$ in three nonempty subspaces as in (17). This a Z_3 grading where $\mathcal{L}_0$ consists of an su$(n-1)$ subalgebra with ladder operators $\{C_{ij} = E_{ij}\,, i \neq j = 1, \ldots, n-1,\}$, together with the Cartan subalgebra of the initial su(n), the first $n-2$ elements of which form the Cartan subalgebra of su$(n-1)$. The Cartan operator of u$(n-1)$ not in su$(n-1)$ is W. The $\mathcal{L}_\pm$ subspaces consist of the (commuting) raising and lowering operators $\{A_j = E_{jn}\,, j = 1, \ldots, n-1\}$ and $\{B_j = E_{nj}\,, j = 1, \ldots, n-1\}$, respectively.

We now consider the contraction of su(n) where only the commutation relations between elements of the subalgebra $\mathcal{L}_0 \sim$ u$(n-1)$ remain unchanged, whereas everything else is forced to commute. In terms of ε, this amounts to setting $\varepsilon_{01} = \varepsilon_{0,-1} = \varepsilon_{1,-1} = 0$ but keeping $\varepsilon_{00} = 1$. The parameters ε_{11} and $\varepsilon_{-1,-1}$ are irrelevant. One can check that these values are solutions of (12) once the irrelevant parameters have been removed. The corresponding ε matrix is given by

$$\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ & \emptyset & 0 \\ & & \emptyset \end{pmatrix}\,, \tag{47}$$

where the lines and columns are ordered according to the grading labels $0, 1, -1$.

To investigate the possible central charges associated with this contraction, we must solve for the η's. Noting that only ε_{00} is nonzero, we find that the only non-trivial relations between the ε and the η are obtained when only two in the triple of indices (μ, ν, σ) are 0. Choosing $\mu = \nu = 0$ therefore yields

$$\varepsilon_{00}\eta_{0\sigma} = \varepsilon_{0\sigma}\eta_{0\sigma} \to \eta_{0\sigma} = 0\,, \qquad \sigma \neq 0\,, \tag{48}$$

with no conditions on η_{00} or $\eta_{1,-1}$. The parameters η_{11} and $\eta_{-1,-1}$ are irrelevant since the commutators $[A_i, A_j] \sim [\mathcal{L}_1, \mathcal{L}_1]$ and $[B_i, B_j] \sim [\mathcal{L}_{-1}, \mathcal{L}_{-1}]$ are zero before the contraction. Thus we have the solution matrix

$$\eta = \begin{pmatrix} x & 0 & 0 \\ & \emptyset & y \\ & & \emptyset \end{pmatrix} , \tag{49}$$

where x and y are arbitrary parameters.

The commutators in $\overline{\mathcal{L}}'$ are now given by

$$\begin{array}{rcll} [A_j, B_i]_{\varepsilon,\eta} & = & y\,\beta_{(+,j),(-,i)}\,, & (i\,,j \neq n)\,, \\ [A_j, C_{ik}]_{\varepsilon,\eta} & = & 0\,, & (i\,,j\,,k \neq n)\,, \\ [B_i, C_{kj}]_{\varepsilon,\eta} & = & 0\,, & (i\,,j\,,k \neq n)\,, \\ [C_{jk}, C_{li}]_{\varepsilon,\eta} & = & \delta_{kl}C_{ji} - \delta_{ij}C_{lk} + x\,\beta_{(0,ji),(0,li)}\,, & (i\,,j\,,k\,,l \neq n)\,. \end{array} \tag{50}$$

First, we consider the $\mathrm{u}(n-1)$ commutators of the type $[C_{jk}, C_{li}]_{\varepsilon,\eta}$. Choose an $\mathrm{su}(n-1) \oplus \mathrm{u}(1)$ basis in this subspace, with W as the $\mathrm{u}(1)$ generator. If $[C_{jk}, C_{li}]_{\varepsilon,\eta}$ is a commutator of two $\mathrm{su}(2)$ elements, then the corresponding $\beta_{(0,jk),(li)}$ is either 0 or equivalent to 0 because $\mathrm{su}(2)$ is semisimple. If we have a commutator of the type $[W, C_{jk}]$, then the corresponding $\beta_{(0,jk),(li)}$ is necessarily 0 by equation (9) because W commutes with every element in $\mathrm{su}(n-1)$. Thus, since x is arbitrary, all the $\beta_{(0,jk),(0,li)}$'s are equivalent to 0 and we have

$$[C_{jk}, C_{li}]_{\varepsilon,\eta} = [C_{jk}, C_{li}]\ . \tag{51}$$

Consider now the commutators $[A_j, B_i]_{\varepsilon,\eta}$. Suppose first that $i \neq j$. If $\overline{\mathcal{L}}'$ is considered by itself we then have no reasons to eliminate the charge $y\,\beta_{(+,j),(-,i)}$. However, because we consider $\overline{\mathcal{L}}'$ as a contraction of $\mathcal{L}$, there are further constraints (of the type found in equation (21)) on $\beta_{(+,j),(-,i)}$; we will show that it is in fact equivalent to a trivial charge. Indeed, using (15), we have $\beta_{(+,j),(-,i)} = \alpha_{(0,ji)}$ of an $\mathrm{su}(n-1) \subset \mathcal{L}_0$ ladder operator, which in turn is equal to $\beta_{(0,h_k),(0,ji)}$, which has just been shown to be trivial. On the other hand, when $i = j$ we have

$$\beta_{(+,i),(-,i)} = \alpha_{(0,ii)} - \alpha_{(0,nn)}\,, \tag{52}$$

which can be expressed as a combination of shifts of Cartan operators from the semisimple algebra $\mathrm{su}(n-1) \subset \mathcal{L}_0$, all of which are equivalent to 0, plus a shift for the operator W, which is *not* in $\mathrm{su}(n-1)$. In fact, there is no commutator $[E_{ij}, E_{kl}]_{\varepsilon,\eta}$ that will yield something proportional to W since it is in the center of $\mathrm{u}(n-1)$: the structure constants $\varepsilon_{\mu,\nu} c^{(0,W)}_{(\mu,i),(\nu,j)}$ are all 0, so that the charge $\beta_{(+,i),(-,i)} \sim \alpha_{(0,WW)}$ cannot be eliminated by a transformation of the type found in (9) and is therefore non-trivial. Thus, we can write

$$[A_j, B_i]_{\varepsilon,\eta} = \delta_{ij}\, y\, \alpha_{(0,WW)}\,, \tag{53}$$

which are the commutation relations of $\mathrm{hw}(n-1)$. Thus, combining (51) and (53), we see that when $\mathrm{su}(n)$ is contracted using the contraction matrix of equation (47), the only possible central charge appears so as to contract $\mathrm{su}(n)$ into the direct sum $\mathrm{u}(n-1) \oplus \mathrm{hw}(n-1)$.

The same reasoning can be repeated for the analysis of the $\mathrm{sp}(2n) \to \mathrm{u}(n) \oplus \mathrm{hw}(n)$ and $\mathrm{so}(n) \to \mathrm{u}(n) \oplus \mathrm{hw}(n)$ contractions: since the graded structure of these is identical to that of $\mathrm{su}(n+1)$, and since the structure of the ε matrix governing the contraction is the same as in equation (49), the equations linking the η's and β's will be identical, as will be the final solutions.

4.2 The $\mathrm{so}(2n+1) \to \mathrm{u}(n) \oplus \mathrm{hw}(n) \oplus \mathrm{hw}(\frac{1}{2}n(n-1))$ contractions

The $\mathrm{so}(2n+1)$ algebra is naturally Z_2 graded, with $\mathcal{L}_0$ the $\mathrm{so}(2n)$ subalgebra spanned by generalized angular momentum operators $\{L_{ij}, i = 1, \ldots, 2n\}$, antisymmetric in ij, and $\mathcal{L}_1$ spanned by the extra generators

$$L_{2n+1,2i-1}\,, \qquad L_{2n+1,2i}\,, \qquad i = 1, \ldots, n\,,$$

which are again antisymmetric under exchange of indices. The Z_2 grading property (23) can be verified from the commutation relations

$$[L_{ij}, L_{kl}] = -i\left(\delta_{jk}L_{il} - \delta_{jl}L_{ik} + \delta_{il}L_{jk} - \delta_{ik}L_{jl}\right)\,. \tag{54}$$

This grading is well-known: whereas operators in the $\mathcal{L}_0$ subalgebra can be realized as fermion pair operators (*i.e.* bosons),

$$A_{ij} \sim b_i^\dagger b_j^\dagger\,, \qquad C_{ij} \sim \tfrac{1}{2}(b_i^\dagger b_j - b_j b_i^\dagger)\,, \qquad B_{ij} \sim A_{ij}^\dagger\,, \tag{55}$$

the operators in $\mathcal{L}_0$ are realized in terms of single fermions,

$$\mathcal{A}_i \sim \tfrac{1}{\sqrt{2}}\, b_i^\dagger\,, \qquad \mathcal{B}_i \sim A_i^\dagger\,, \tag{56}$$

with the usual fermionic anticommutation relations:

$$\{b_i, b_j\} = \{b_i^\dagger, b_j^\dagger\} = 0\,, \qquad \{b_i, b_j^\dagger\} = \delta_{ij}\,. \tag{57}$$

This Z_2 grading of $\mathrm{so}(2n+1)$ was explicitly exploited in [19] (see Eq. (3.31) therein) as a first step in the VCS construction. We need now to refine this grading so as to decompose further the $\mathcal{L}_0$ subspace, which contains an $\mathrm{so}(2n)$ subalgebra. The Z_3 grading appropriate for the analysis of $\mathrm{so}(2n)$ is similar to the one that we used in the $\mathrm{su}(n)$ example of the previous section. Thus, the grading underlying the analysis of $\mathrm{so}(2n)$ is a $\mathrm{Z}_2 \otimes \mathrm{Z}_3$ grading:

$$\mathcal{L} = \mathcal{L}_{00} \oplus \mathcal{L}_{01} \oplus \mathcal{L}_{0,-1} \oplus \mathcal{L}_{1,-1} \oplus \mathcal{L}_{11}\,, \tag{58}$$

with the subspace $\mathcal{L}_{10}$ empty.

These subspaces are spanned by

$$\begin{aligned}
\mathcal{L}_{00} &= \{C_{ij}, 1 \le i,j \le n\}\,, & C_{ij} &= \tfrac{1}{2}\Big(L_{2i-1,2j} + L_{2i,2j-1} + \mathrm{i}L_{2i,2j} + \mathrm{i}L_{2i-1,2j-1}\Big)\,,\\
\mathcal{L}_{01} &= \{A_{ij}, 1 \le i,j \le n\}\,, & A_{ij} &= \tfrac{1}{2}\Big(L_{2i-1,2j} + L_{2i,2j-1} + \mathrm{i}L_{2i,2j} - \mathrm{i}L_{2i-1,2j-1}\Big)\,,\\
\mathcal{L}_{0,-1} &= \{B_{ij}, 1 \le i,j \le n\}\,, & B_{ij} &= \tfrac{1}{2}\Big(L_{2i-1,2j} + L_{2i,2j-1} - \mathrm{i}L_{2i,2j} + \mathrm{i}L_{2i-1,2j-1}\Big)\,,\\
\mathcal{L}_{11} &= \{\mathcal{B}_i, 1 \le i \le n\}\,, & \mathcal{B}_i &= \tfrac{1}{\sqrt{2}}\left(L_{2n+1,2i} + \mathrm{i}L_{2n+1,2i-1}\right)\,,\\
\mathcal{L}_{1,-1} &= \{\mathcal{A}_i, 1 \le i \le n\}\,, & \mathcal{A}_i &= \tfrac{1}{\sqrt{2}}\left(L_{2n+1,2i} - \mathrm{i}L_{2n+1,2i-1}\right)\,.
\end{aligned} \tag{59}$$

This decomposition is presented explicitly for $\mathrm{so}(5)$ in figure 1. It is a non-generic grading, the parameters $\varepsilon_{k,(10)}, \varepsilon_{(01),(01)}, \varepsilon_{(0,-1),(0,-1)}, \varepsilon_{(01),(1,-1)}$ and $\varepsilon_{(0,-1),(11)}$ being irrelevant. The subspaces $\mathcal{L}_{01}$ and $\mathcal{L}_{1,-1}$ form a set of raising operators where

$$[\mathcal{L}_{1,-1}, \mathcal{L}_{1,-1}] \subset \mathcal{L}_{01}\,, \tag{60}$$

by virtue of the cyclicity modulo 2 and 3, respectively, of the addition of the grading indices in Z_2 and Z_3, and where $\mathcal{L}_{01}$ forms an abelian nilpotent subalgebra. The parallel observations hold for the set of lowering operators spanned by elements in the $\mathcal{L}_{0,-1}$ and $\mathcal{L}_{1,1}$ subspaces.

The subspace $\mathcal{L}_{00}$ spans a u(n) subalgebra of so($2n+1$). It contains a semisimple part, the su(n) subalgebra with ladder operators $\{C_{ij}, i \neq j = 1, \ldots, n\}$, together with the Cartan subalgebra of so($2n+1$), the first $n-1$ elements of which span the Cartan subalgebra of the aforementioned su(n) semisimple algebra. Again, the Cartan operator of $\mathcal{L}_{00}$ that is not in su(n) is the operator $W = \sum_{i=1}^{n} C_{ii}$ of equation (46).

The VCS contraction so$(2n+1) \to$ u$(n) \oplus$ hw$(n) \oplus$ hw$(\frac{1}{2}n(n-1))$ leaves only the commutators $[\mathcal{L}_{00}, \mathcal{L}_{00}]$ unchanged whereas all the other ones become 0. This can be realized in terms of a $Z_2 \otimes Z_3$ contraction in two steps, the first of which is to set any Z_2 commutators of the type $[\mathcal{L}_{0i}, \mathcal{L}_{1j}]$ and $[\mathcal{L}_{1i}, \mathcal{L}_{1j}]$ to zero while leaving $[\mathcal{L}_{0i}, \mathcal{L}_{0j}]$ unchanged. This corresponds to the contraction matrix ε^V of (24). The second step appropriately contracts the so($2n$) subalgebra of the Z_2 subspace labeled by 0. The contraction matrix was given in (47) of the previous section. The desired $Z_2 \otimes Z_3$ contraction is obtained by tensoring the Z_2 and Z_3 solutions and corresponds to the solution matrix

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ & \emptyset & 0 \\ & & \emptyset \end{pmatrix}. \tag{61}$$

The lines and columns are ordered as $(00), (01), (0,-1), (1-1), (1,1)$ in the solution matrix ε (Note that we have removed the fourth line and fourth column, corresponding to the subspace (10), as this subspace is empty in the grading decomposition). Using the properties of tensor product of matrices, one can verify that a solution to (13) is given by the tensor product of the appropriate η matrices:

$$\eta = \begin{pmatrix} a & 0 \\ & b \end{pmatrix} \otimes \begin{pmatrix} x & 0 & 0 \\ & \emptyset & y \\ & & \emptyset \end{pmatrix} = \begin{pmatrix} ax & 0 & 0 & 0 & 0 & 0 \\ & \emptyset & ay & 0 & \emptyset & 0 \\ & & \emptyset & 0 & 0 & \emptyset \\ & & & bx & 0 & 0 \\ & & & & \emptyset & by \\ & & & & & \emptyset \end{pmatrix}, \tag{62}$$

where a, b, x, y are arbitrary parameters. Again we must remove from this matrix the fourth line and fourth column corresponding to the subspace (10). Thus, we have

$$\eta_{(00),(00)} = ax\,, \qquad \eta_{(01),(0,-1)} = ay\,, \qquad \eta_{(1,1),(1,-1)} = yb\,. \tag{63}$$

The commutation relations now read

$$\begin{aligned} [A_{ij}, B_{kl}]_{\varepsilon,\eta} &= ay\,\beta_{(01,ij),(0-1,kl)}\,, \\ [\mathcal{A}_i, \mathcal{B}_j]_{\varepsilon,\eta} &= yb\,\beta_{(1-1,i),(11,j)}\,, \\ [C_{ij}, C_{kl}]_{\varepsilon,\eta} &= [C_{ij}, C_{kl}] + ax\,\beta_{(00,ij),(00,kl)}\,. \end{aligned} \tag{64}$$

The subspace $\mathcal{L}_{00}$ spans a u(n) subalgebra. We can then repeat the argument that led to (51) to show that all the charges $\beta_{(00,ij),(00,kl)}$ are equivalent to 0. Furthermore, by repeating the argument that lead to (53), we see that the only non-trivial charges occur when the commutators $[A_{ij}, B_{kl}]_{\varepsilon,\eta}$ and $[\mathcal{A}_i, \mathcal{A}_j]_{\varepsilon,\eta}$ are proportional to elements in the Cartan subalgebra, *i.e.*

$$[A_{ij}, B_{kl}]_{\varepsilon,\eta} = \delta_{ik}\delta_{jl}\,ay\,\beta_{(01,ij),(0-1,ij)}\,, \qquad [\mathcal{A}_i, \mathcal{B}_j]_{\varepsilon,\eta} = \delta_{ij}\,yb\,\beta_{(1-1,i),(11,i)}\,, \tag{65}$$

for otherwise the appropriate β's are proportional to shifts of generators in the semisimple subalgebra su$(n-1) \subset \mathcal{L}_0$, and therefore equivalent to 0.

Consider now

$$[A_{ij}, B_{ij}]_{\varepsilon,\eta} = ay\,\beta_{(01,ij),(0-1,ij)} = -\,ay\,(\alpha_{(00,ii)} + \alpha_{(00,jj)})\,. \tag{66}$$

This shift is a linear combination of a shift of Cartan operators in the semisimple part of $\mathcal{L}_0$ and a shift $\alpha_{(00,WW)}$ of the operator W not in this semisimple part. Eliminating the su(n) shifts as they are equivalent to 0, we are left, in general, with

$$[A_{ij}, B_{kl}]_{\varepsilon,\eta} = -\delta_{ik}\delta_{jl}\; ay\,\alpha_{(00,WW)}\,, \tag{67}$$

which are the commutation relations of the hw$((\frac{1}{2}n(n-1)))$ algebra.

Finally, consider

$$[\mathcal{A}_i, \mathcal{B}_i]_{\varepsilon,\eta} = yb\,\beta_{(1-1,i),(11,i)} = yb\,\alpha_{(00,ii)}\,. \tag{68}$$

Once again, $\alpha_{(00,ii)}$ is a linear combination of shifts from the Cartan subalgebra of the semisimple su(n) algebra and a shift from the trace W, which is not in the semisimple part of $\mathcal{L}_0$. After eliminating the former shifts as equivalent to 0, we find, in general,

$$[\mathcal{A}_i, \mathcal{B}_j]_{\varepsilon,\eta} = \delta_{ij}\,yb\,\alpha_{(00,WW)}\,, \tag{69}$$

which are the commutation relations for the algebra hw(n). Combining all these, we find $\overline{\mathcal{L}}' \sim$ u$(n) \oplus$ hw$(n) \oplus$ hw$(\frac{1}{2}n(n-1))$, as expected.

The solution η of equation (62) is *not* the most general solution to the equations (6) coupling η and ε (as was the case in [1] for the ε's). The most general solution would have $\eta_{(01),(11)}$ and $\eta_{(0,-1),(1,-1)}$ nonzero. Our interpretation of our solution is that it corresponds to a sequence of deformations: first a Z_2, then a Z_3 contraction.

5 Conclusion

In this paper, we have introduced a way to generalize the theory of graded contractions in order to include central charges, and therefore generate central extensions which have one more dimension than the original algebra. The method has been applied to two different physical settings: the kinematical algebras and the vector coherent states construction. In both cases, the grading decomposition of the original uncontracted algebra $\mathcal{L}$ reflects the tensorial nature of the subspaces decomposing $\mathcal{L}$. For the kinematical algebras, each $\mathrm{Z}_2 \otimes \mathrm{Z}_2$ subspace carries an irreducible representation of the group $\Pi \otimes \Theta$ of space and time inversion. For the VCS construction, each subspace carries an irreducible representation of the u$(n-1)$ subalgebra contained in the $\mathcal{L}_0$ or $\mathcal{L}_{00}$ subspace. It is therefore surprising that the location of the central charges in $\overline{\mathcal{L}}'$ can be inferred from the grading decomposition of $\mathcal{L}$.

Indeed, the non-trivial charges always occur, by construction, in the commutator of two commuting abelian subalgebras in $\overline{\mathcal{L}}'$. Furthermore, we have $[l_{\mu,i}, l_{\nu,j}]_{\varepsilon,\eta} = \eta_{\mu,\nu}\,\beta_{(\mu,i),(\nu,j)} \neq 0$ if and only if (i) $l_{\mu+\nu,k}$ commutes with every element in $\mathcal{L}_0$ or $\mathcal{L}_{00}$, and (ii) $[l_{\mu,i}, l_{\nu,j}] \neq 0$ in $\mathcal{L}$. Consider, for instance, the $(2+1)$ dimensional de Sitter algebra, with $\mathcal{L}_{00} = \{J\}$. This subspace trivially commutes with itself, and we can have $[K_1, K_2]_{\varepsilon,\eta} = -k\alpha_J\,\mathbb{1}$ in the Galilei algebra. In the $(3+1)$ dimensional case, however, $\mathcal{L}_{00} = \{\mathrm{J}\}$ no longer contains abelian generators, and it is impossible to extend $[\mathrm{K}, \mathrm{K}]_{\varepsilon,\eta}$ in the $(3+1)$ dimensional Galilei algebra. A similar line of reasoning can be applied to the VCS-type contractions: the only non-trivial central parameter occurs when a commutator in $\mathcal{L}$ is proportional to the operator W, which commutes with everything in the u$(n-1)$ subalgebra of either $\mathcal{L}_0$ or $\mathcal{L}_{00}$.

We believe that our formalism is obviously not limited to the examples presented in this paper. For instance, despite the fact that the interest in central extensions was originally related to the representations, we have not considered them at all. Also the method could be used to investigate the infinite dimensional Lie algebras and Sugawara construction, as it is done in [25] by using standard Wigner-Inönü contractions. An obvious continuation of our current work is to study the deformations with central extensions at the group level. Other possibilities include the extensions by spaces higher than one dimension.

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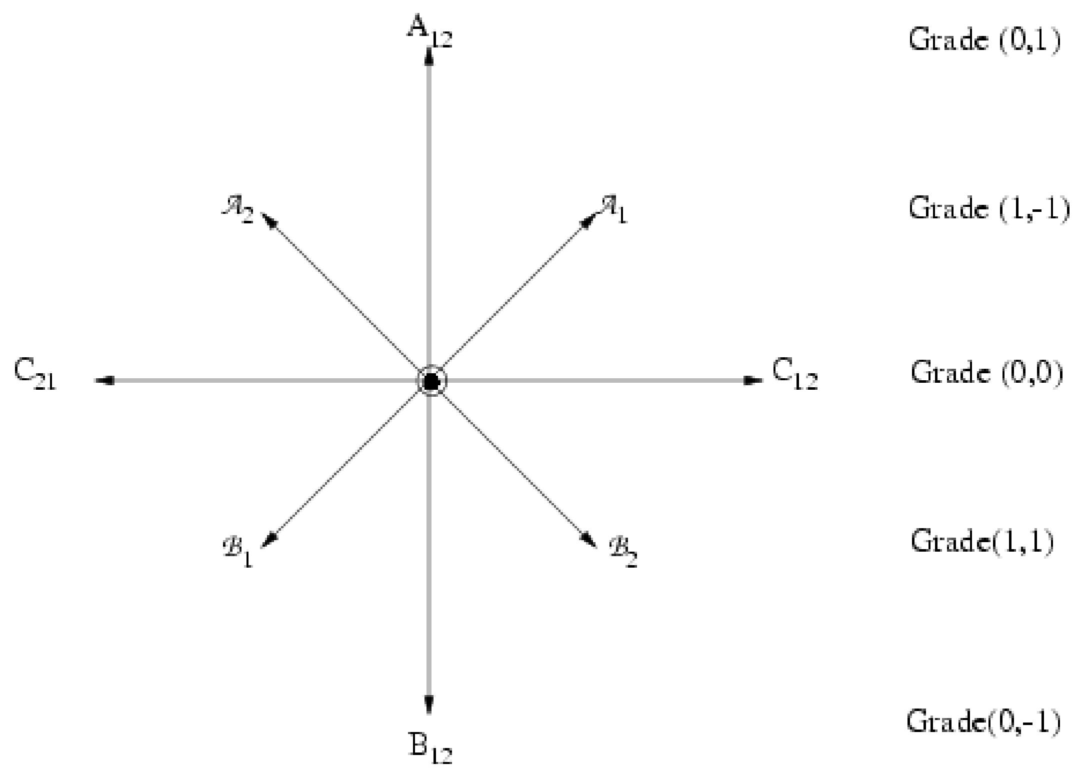

Figure 1: The root decomposition of so(5) along with the $Z_2 \otimes Z_3$ grading labels.