On Extreme Points of a Class of Functions Related to a Convolution Conjecture

Richard Fournier[*] Stephan Ruscheweyh[†‡]

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[*]Centre de Recherches Mathématiques, Université de Montréal, C. P. 6128, succ. Centre-ville, Montréal, Québec H3C 3J7, Canada.fournier@DMS.UMontreal.CA

[†]Mathematisches Institut, Universität Würzburg, D–97074 Würzburg, Germany.ruscheweyh@mathematik.uni-wuerzburg.de

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Abstract

We study support points and extreme points of the class of functions analytic in the unit disc

$$\mathcal{D}' := \left\{F\colon F(0) = 1 \text{ and } \left|F'(z)\right| \leq \operatorname{Re} F(z) \text{ for } |z| < 1\right\}.$$

Introduction and Statement of the Results

Let $\mathcal{A}$ denote the set of analytic functions f in the unit disc $\mathbb{D} := \{z\colon |z| < 1\}$ of the complex plane, endowed with the topology of locally uniform convergence in $\mathbb{D}$. If $f(0) = 0$ or $f(0) = 1$, then we write $f \in \mathcal{A}_0$ or $f \in \mathcal{A}_1$, respectively. Let also

$$\mathcal{D}' := \{F \in \mathcal{A}_1\colon |F'(z)| \le \operatorname{Re} F(z) \text{ for } z \in \mathbb{D}\}.$$

In view of a recent convolution conjecture [3], [5], we became interested into the extreme points of $\mathcal{D}'$. The class $\mathcal{D}'$ is a convex, compact and rotationally invariant subset of $\mathcal{A}$; by the Krein-Milman Theorem all non-trivial linear extremal problems (and many non linear ones) admit a solution which is an extreme point of $\mathcal{D}'$. It is therefore of interest to study extreme points of $\mathcal{D}'$. The following result from [3] characterizes the extreme points of $\mathcal{D}'$ which are analytic in the closed unit disc.

Theorem A *Let G be analytic in $\overline{\mathbb{D}}$. Then G is an extreme point of $\mathcal{D}'$ if and only if*

$$|G'(z)| = \operatorname{Re} G(z), \quad |z| = 1. \tag{1}$$

>From the above result it is easily deduced that the functions

$$G_n(z) := \frac{1 + d_n z^n}{1 - d_n z^n}, \quad d_n = -n + \sqrt{1 + n^2},$$

and their rotations $G_n(e^{i\theta} z)$ are extreme points of $\mathcal{D}'$ and the question was raised in [3] whether these could be the only ones.

The first negative answer to this question came from the following result, which we state here (without proof), since it might have other consequences. Let

$$\mathcal{E}' := \overline{\operatorname{co}}\{G_n(e^{i\theta} z)\colon n \in \mathbb{N}, \theta \in \mathbb{R}\},$$

where $\overline{\operatorname{co}}\,\beta$ denotes the closed convex hull of a set $\beta \subset \mathcal{A}$.

Theorem 1 *For $G \in \mathcal{E}'$ the function G' extends continuously to $\overline{\mathbb{D}}$.*

It is easy to construct functions in $\mathcal{D}'$ which, by Theorem 1, do not belong to $\mathcal{E}'$: let, for instance,

$$G(z) := 1 + \frac{1}{2} \int_0^z B(t)\, dt,$$

where B is an infinite Blaschke product. Then

$$|G'(z)| = \frac{1}{2}|B(z)| \le \frac{1}{2} \le \operatorname{Re} G(z), \quad z \in \mathbb{D},$$

and thus $G \in \mathcal{D}'$. Clearly $G'(z) = \frac{1}{2}B(z)$ has no continuous extension to $\overline{\mathbb{D}}$. This shows that $\mathcal{E}' \underset{\neq}{\subset} \mathcal{D}'$ and therefore $\mathcal{D}'$ must have more extreme points than those described above.

We recall that a support point of $\mathcal{D}'$ is a function $G \in \mathcal{D}'$ such that

$$\max_{E \in \mathcal{D}'} \operatorname{Re} L(E) = \operatorname{Re} L(G)$$

for some continuous linear functional L over $\mathcal{A}$ whose real part is non-constant over $\mathcal{D}'$. It is not obvious what relations might exist between extreme points and support points of $\mathcal{D}'$; however, it is known [2] that a support point of $\mathcal{D}'$ analytic in $\overline{\mathbb{D}}$ is also an extreme point of $\mathcal{D}'$ and therefore satisfies equation (1); moreover [3] the functions G_n defined above are support points of $\mathcal{D}'$. We now have (the proof is omitted)

Theorem 2 *For integers $m > n \geq 1$, there exist complex numbers K_m, K_n such that*

$$\max_{G\in\mathcal{E}'} \operatorname{Re}\big(K_m G^{(m)}(0) + K_n G^{(n)}(0)\big) < \max_{G\in\mathcal{D}'} \operatorname{Re}\big(K_m G^{(m)}(0) + K_n G^{(n)}(0)\big). \quad (2)$$

It also follows from the Krein-Milman Theorem and Theorem 2 that $\mathcal{D}'$ has extreme points different from any rotation of the functions G_n.

The relatively simple form of the linear functional involved in (2) suggests indeed that $\mathcal{D}'$ has much more extreme points than $\mathcal{E}'$. This is confirmed by the following result [4]: given a finite Blaschke product B, there exists a solution G_B of (1), analytic in $\overline{\mathbb{D}}$ and such that B is the inner factor of G_B' (note that the functions $G_n(e^{i\theta}z)$ correspond to the Blaschke products $e^{i(n-1)\theta}z^{n-1}$).

As a partial extension of Theorem A we shall prove the following result. Here and in the sequel we shall use the notation

$$\gamma_f(z) := \big|f'(z)\big| - \operatorname{Re} f(z), \quad f \in \mathcal{A}.$$

Theorem 3 *Let $G \in \mathcal{D}'$ and assume that there exists a closed arc $I \subseteq \partial\mathbb{D}$ such that*

$$\lim_{\substack{z\to\zeta\\ z\in\mathbb{D}}} \gamma_G(z) = 0, \quad \textit{for all } \zeta \in I.$$

Then G is an extreme point of $\mathcal{D}'$.

Finally our next result will show that functions as in Theorem 3 really exist and are different from the functions $G_n(e^{i\theta}z)$ introduced above. We shall prove

Theorem 4 *Let $m \geq 1$ be a positive integer. There exist closed disjoint subarcs $I_j \subset \partial\mathbb{D}$, ($j = 1, \dots m$), and a function $G \in \mathcal{D}'$ such that*

$$\lim_{\substack{z\to\zeta\\ z\in\mathbb{D}}} \gamma_G(z) = 0 \iff \zeta \in \bigcup_{j=1}^{m} I_j.$$

Proof of Theorem 3

Let $G \in \mathcal{D}'$; then [3]

$$\sqrt{2} - 1 \leq \operatorname{Re} G(z) \leq \sqrt{2} + 1, \quad z \in \mathbb{D}. \quad (3)$$

Therefore G' is analytic and bounded on $\mathbb{D}$ and G admits a continuous extension to $\overline{\mathbb{D}}$. According to the hypothesis, there exists a closed subarc $I \subseteq \partial\mathbb{D}$. with

$$\lim_{\substack{z\to\zeta\\ z\in\mathbb{D}}} \big|G'(z)\big| - \operatorname{Re} G(z) = 0, \quad \zeta \in I, \quad (4)$$

and a convex domain Ω such that $\Omega \subseteq \mathbb{D}$, $\partial\Omega \cap \partial\mathbb{D} = I$ and, by (3) and (4), $\big|G'(z)\big|$ is continuous and does not vanish on $\overline{\Omega}$.

Let us assume that G is not an extreme point of $\mathcal{D}'$. It is well-known that there must exist $w \in \mathcal{A}_0$, $w \not\equiv 0$, with $G \pm w \in \mathcal{D}'$, i.e.,

$$\big|G'(z) \pm w'(z)\big| \leq \operatorname{Re}\big(G(z) \pm w(z)\big), \quad z \in \mathbb{D}. \quad (5)$$

We have for $z \in \mathbb{D}$,

$$\begin{aligned} \big|G'(z)\big| &\le \frac{1}{2}\big|G'(z)+w'(z)\big| + \frac{1}{2}\big|G'(z)-w'(z)\big| \\ &\le \frac{1}{2}\operatorname{Re}\big(G(z)+w(z)\big) + \frac{1}{2}\operatorname{Re}\big(G(z)-w(z)\big) \\ &= \operatorname{Re} G(z) \end{aligned} \tag{6}$$

and in particular for $z \in \Omega$,

$$1 \le \frac{1}{2}\left|1+\frac{w'(z)}{G'(z)}\right| + \frac{1}{2}\left|1-\frac{w'(z)}{G'(z)}\right| \le \frac{\operatorname{Re} G(z)}{|G'(z)|}.$$

It follows then from the hypothesis that

$$\lim_{\substack{z\to\zeta\in I\\ z\in\Omega}} \frac{1}{2}\left|1+\frac{w'(z)}{G'(z)}\right| + \frac{1}{2}\left|1-\frac{w'(z)}{G'(z)}\right| = 1. \tag{7}$$

We have for $z \in \mathbb{D}$,

$$\big|w'(z)\big| \le \frac{1}{2}\big|G'(z)+w'(z)\big| + \frac{1}{2}\big|G'(z)-w'(z)\big| \le \operatorname{Re} G(z) \le \sqrt{2}-1,$$

and the function $\frac{w'}{G'}$ is analytic and bounded over Ω; it therefore admits radial limits

$$a = a(\zeta) = \lim_{\substack{r\to 1\\ 0<r}} \frac{w'(r\zeta)}{G'(r\zeta)}, \text{ for almost all } \zeta \in I.$$

By (7) we have

$$\big|1+a(\zeta)\big| + \big|1-a(\zeta)\big| = 2$$

and clearly $-1 \le a(\zeta) \le 1$. It follows from a result of Caratheodory ([1], page 96) that for all $\zeta \in I$, the cluster set of $\frac{w'}{G'}$ at ζ is contained in $[-1,1]$.

Since $G \pm w \in \mathcal{D}'$, w is a continuous function over $\overline{\mathbb{D}}$ and by (6),

$$\lim_{\substack{z\to\zeta\\ z\in\Omega}} \big|G'(z) \pm w'(z)\big| = \operatorname{Re} G(\zeta) \pm \operatorname{Re} w(\zeta), \quad \zeta \in I,$$

and

$$\lim_{\substack{z\to\zeta\\ z\in\Omega}} \left|1 \pm \frac{w'(z)}{G'(z)}\right| = 1 \pm \frac{\operatorname{Re} w(\zeta)}{\operatorname{Re} G(\zeta)}, \quad \zeta \in I,$$

i.e., any cluster value $v = v(\zeta)$ of $\frac{w'}{G'}$ at $\zeta \in I$ belongs to $[-1,1]$ and satisfies

$$\big|1 \pm v(\zeta)\big| = 1 \pm \frac{\operatorname{Re} w(\zeta)}{\operatorname{Re} G(\zeta)}.$$

It follows that over I, $v(\zeta) \equiv \frac{\operatorname{Re} w(\zeta)}{\operatorname{Re} G(\zeta)}$, i.e., the function $\frac{w'}{G'}$ admits a real continuous extension to $\Omega \cup I$. By the reflection principle, $\frac{w'}{G'}$ is indeed analytic in a domain Ω^* containing the interior of I. The relation

$$\frac{w'(\zeta)}{G'(\zeta)} \equiv \frac{\operatorname{Re} w(\zeta)}{\operatorname{Re} G(\zeta)}, \quad \zeta \in I, \tag{8}$$

means that the function h defined as

$$h(z) := G(z)\frac{w'(z)}{G'(z)} - w(z)$$

is analytic over Ω and has its real part vanishing identically on I and also extends analytically to Ω^*. Since $h'(z) \equiv G(z)\big(\frac{w'(z)}{G'(z)}\big)'$, we either have $\frac{w'(z)}{G'(z)} \equiv 0$ or else G is holomorphic on a domain containing a subarc I_1 of I.

In the first case $w(z) \equiv A\big(G(z) - 1\big)$ with $-1 \le A \le 1$. By (5),

$$(1 \pm A)\big|G'(z)\big| \le (1 \pm A)\operatorname{Re} G(z) \mp A, \quad z \in \Omega$$

and as $z \to \zeta \in I$ we obtain $w \equiv 0$, a contradiction. In the second case, w is also holomorphic over I_1 and by (8)

$$\left(\frac{d}{d\theta}\operatorname{Re} w(e^{i\theta})\right)\Big/\operatorname{Re} w(e^{i\theta}) \equiv \left(\frac{d}{d\theta}\operatorname{Re} G(e^{i\theta})\right)\Big/\operatorname{Re} G(e^{i\theta}), \quad e^{i\theta} \in I_1.$$

As above it would then follow that $w \equiv 0$. This completes the proof of Theorem 3.

Proof of Theorem 4

Throughout our proof of Theorem 4, we shall consider a conformal map $b(z)$ of the unit disc such that $b(\mathbb{D})$ is a curvilinear polygon consisting of m disjoint closed arcs $I_j^* \subset \partial\mathbb{D}$ $(j = 1, 2, \ldots, m)$ together with m arcs of circles $I_j^{**} \subset \mathbb{D}$ $(j = 1, 2, \ldots, m)$: each arc of circle I_j^{**} connects one endpoint of I_j^* to another endpoint of I_{j+1}^*. Clearly we may assume that $0 \notin b(\overline{\mathbb{D}})$ and $b(\overline{\mathbb{D}})$ is a John domain; in particular,

$$b(z) := \sum_{n=0}^{\infty} b_n z^n \text{ with } \sum_{n=0}^{\infty} |b_n| < \infty. \tag{9}$$

Let $\mathcal{B} = \{w \in \mathcal{A} \mid \big|w(z)\big| \le |z|/2, z \in \mathbb{D}\}$. $\mathcal{B}$ is a compact and convex subset of the locally convex space $\mathcal{A}$. Let also $0 < r < 1$; we define an operator $T_r \colon \mathcal{B} \to \mathcal{A}$ by $T_r(w)(0) = 0$ and

$$T_r(w)'(z) = b(z)e^{\frac{1}{2\pi}\int_0^{2\pi}\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\ln\left(\frac{1-|w(re^{i\theta})|^2}{2}\right)d\theta}, \quad z \in \mathbb{D}.$$

Clearly T_r is continuous over $\mathcal{B}$ and since

$$\begin{aligned}\big|T_r(w)'(z)\big| &\le e^{\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right)\ln(1/2)d\theta} \\ &= 1/2, \quad w \in \mathcal{B},\ z \in \mathbb{D},\end{aligned}$$

we have $T_r(\mathcal{B}) \subset \mathcal{B}$. By the Schauder-Tychonoff fixed-point Theorem, there exists $w_r \in \mathcal{B}$ such that $T_r(w_r) = w_r$; i.e., we have for each $r \in (0, 1)$ a function $w_r \in \mathcal{B}$ with

$$w_r'(z) = b(z)e^{\frac{1}{2\pi}\int_0^{2\pi}\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\ln\left(\frac{1-|w_r(re^{i\theta})|^2}{2}\right)d\theta}. \tag{10}$$

We define $\widetilde{w}_r \in \mathcal{B}$ by $\widetilde{w}_r'(z) = w_r'(z)/b(z)$.

We have $\frac{3}{8} \le \big|\widetilde{w}_r'(z)\big| \le \frac{1}{2}$ if $z \in \mathbb{D}$ and as in ([4], Lemma 2.1 and Theorem 1.5), $\widetilde{w}_r$ is analytic in $\overline{\mathbb{D}}$ and convex univalent in $\mathbb{D}$. Given a sequence $0 < r_n < 1$, $\lim_{n\to\infty} r_n = 1$, there exists according

to a result of Pommerenke [6], a subsequence $\{\widetilde{w}_{r_{n_k}}\}_k$ uniformly convergent over $\overline{\mathbb{D}}$ to a function $\widetilde{w} \in \mathcal{B}$. Let us define

$$w(z) := b(z)\widetilde{w}(z) - \int_0^z b'(\zeta)\widetilde{w}(\zeta)\, d\zeta, \quad z \in \mathbb{D}.$$

Since

$$w_r(z) = b(z)\widetilde{w}_r(z) - \int_0^z b'(\zeta)\widetilde{w}_r(\zeta)\, d\zeta,$$

we have

$$\left|w(z) - w_{r_{n_k}}(z)\right| \leq \sup_{|\zeta|\leq 1}\left|\widetilde{w}(\zeta) - \widetilde{w}_{r_{n_k}}(\zeta)\right|\left(1 + \int_0^1 \left|b'(tz)\right| dt\right), \quad z \in \mathbb{D}$$

and by (9), (10), $w_{r_{n_k}} \to w$ uniformly over $\overline{\mathbb{D}}$. Taking limits in (10) we obtain

$$w'(z) = b(z)e^{\frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} \ln\left(\frac{1-|w(e^{i\theta})|^2}{2}\right) d\theta}, \quad z \in \mathbb{D}. \tag{11}$$

The function $G(z) := \frac{1+w(z)}{1-w(z)}$ belongs to $\mathcal{D}'$ and clearly for $\zeta \in \partial\mathbb{D}$,

$$\lim_{\substack{z\to\zeta \\ |z|<1}} \gamma_G(z) = \lim_{\substack{z\to\zeta \\ |z|<1}} \frac{2|w'(z)| - 1 + |w(z)|^2}{|1 - w(z)|^2}. \tag{12}$$

The right-hand side of (12) will vanish only if $\zeta \in \bigcup_{j=1}^m b^{-1}(I_j^*)$. This complete the proof of Theorem 4. The proof clearly shows that in fact there exist solutions w of (11) for $b(z) = b_0(z)B(z)$ where $b_0(\mathbb{D})$ is a John domain as above and $B(z)$ is a finite Blaschke product. In other words, there exist extreme points G of $\mathcal{D}'$ of the type described by Theorem 3 and with a finite number of preassigned zeros in $\mathbb{D}$ for G'. It seems rather difficult to obtain explicit formulae for these functions.

References

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