

# Spectral Transformations of the Laurent Biorthogonal Polynomials

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### **Abstract**

We study the simplest closure conditions of chains of spectral transformations of the Laurent biorthogonal polynomials (LBP). It is shown that the 1-periodic  $q$ -closure condition leads to the  $q$ -Appel LBP which are a special case of the LBP introduced by Pastro.

**Keywords:** Laurent orthogonal polynomials,  $q$ -Appel polynomials, spectral transformations.

### **Résumé**

Nous considérons les conditions les plus simples sous lesquelles se terminent les chaînes de transformations spectrales des polynômes de Laurent biorthogonaux. Nous montrons que la condition de  $q$ -terminaison de période 1 conduit aux polynômes de Laurent de type  $q$ -Appel qui sont une classe particulière des polynômes de Laurent introduits par Pastro.



# 1. Introduction.

The Laurent biorthogonal polynomials LBP  $P_n(z)$  appeared in problems connected with the two-points Padé approximations (see, e.g. [6]).

We shall recall their definition and general properties (see, e.g. [6], [4], [5]).

Let  $\mathcal{L}$  be some linear functional defined on all possible monomials  $z^n$  by the moments

$$c_n = \mathcal{L}\{z^n\}, \quad n = 0, \pm 1, \pm 2 \dots \quad (1.1)$$

In general the moments  $c_n$  are arbitrary complex numbers. The functional  $\mathcal{L}$  is thus defined on the space of Laurent polynomials  $\mathcal{P}(z) = \sum_{n=-N_1}^{N_2} a_n z^n$  where  $a_n$  are arbitrary complex numbers and  $N_{1,2}$  arbitrary integers:

$$\mathcal{L}\{\mathcal{P}(z)\} = \sum_{n=-N_1}^{N_2} a_n c_n.$$

The monic LBP  $P_n(z)$  are defined by the determinant [4]

$$P_n(z) = (\Delta_n)^{-1} \begin{vmatrix} c_0 & c_1 & \dots & c_n \\ c_{-1} & c_0 & \dots & c_{n-1} \\ \dots & \dots & \dots & \dots \\ c_{1-n} & c_{2-n} & \dots & c_1 \\ 1 & z & \dots & z^n \end{vmatrix},$$

where  $\Delta_n$  is defined as

$$\Delta_n = \begin{vmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_{-1} & c_0 & \dots & c_{n-2} \\ \dots & \dots & \dots & \dots \\ c_{1-n} & c_{2-n} & \dots & c_0 \end{vmatrix}.$$

It is obvious from the definition (1.2) that the polynomials  $P_n(z)$  satisfy the orthogonality property

$$\mathcal{L}\{P_n(z)z^{-k}\} = h_n \delta_{kn}, \quad 0 \leq k \leq n, \quad (1.2)$$

where the normalization constants  $h_n$  are

$$h_0 = c_0, \quad h_n = \Delta_{n+1}/\Delta_n. \quad (1.3)$$

This orthogonality property can be rewritten as the biorthogonal relation [9], [4],

$$\mathcal{L}\{P_n(z)Q_m(1/z)\} = h_n \delta_{nm}, \quad (1.4)$$

where the polynomials  $Q_n(z)$  are defined by the formula

$$Q_n(z) = (\Delta_n)^{-1} \begin{vmatrix} c_0 & c_{-1} & \dots & c_{-n} \\ c_1 & c_0 & \dots & c_{1-n} \\ \dots & \dots & \dots & \dots \\ c_{n-1} & c_{n-2} & \dots & c_{-1} \\ 1 & z & \dots & z^n \end{vmatrix}. \quad (1.5)$$

We note that the polynomials  $Q_n(z)$  are again LBP with moments  $c_n^{\{Q\}} = c_{-n}$ .

In what follows we will assume that

$$\Delta_n \neq 0, \quad n = 1, 2, \dots \quad (1.6)$$

and that

$$\Delta_n^{(1)} \neq 0, \quad n = 1, 2, \dots \quad (1.7)$$

where by  $\Delta_n^{(j)}$  we denote the determinants

$$\Delta_0^{(j)} = 1; \quad \Delta_n^{(j)} = \begin{vmatrix} c_j & c_{j+1} & \dots & c_{n+j-1} \\ c_{j-1} & c_j & \dots & c_{n+j-2} \\ \dots & \dots & \dots & \dots \\ c_{1+j-n} & c_{2+j-n} & \dots & c_j \end{vmatrix}. \quad (1.8)$$

If the conditions (1.6) and (1.7) are fulfilled, the polynomials  $P_n(z)$  satisfy the recurrence relation (see, e.g. [3])

$$P_{n+1}(z) + (d_n - z)P_n(z) = zb_n P_{n-1}(z), \quad n \geq 1 \quad (1.9)$$

where the recurrence coefficients are

$$d_n = -\frac{P_{n+1}(0)}{P_n(0)} = h_n^{-1} \frac{T_{n+1}}{T_n} \neq 0, \quad n = 0, 1, \dots \quad (1.10)$$

$$b_n = d_n \frac{h_n}{h_{n-1}} \neq 0, \quad n = 1, 2, \dots \quad (1.11)$$

with  $T_n = \Delta_n^{(1)}$ .

There is a one-to-one correspondence between the moments  $c_n$  and the recurrence coefficients  $b_n, d_n$  (provided restrictions  $b_n d_n \neq 0$  are fulfilled).

The recurrence relation (1.9) will be our starting point.

In addition to the polynomials  $P_n(z)$  it is useful to introduce the  $j$ -associated polynomials  $P_n^{(j)}(z)$  that satisfy the recurrence relation

$$P_{n+1}^{(j)}(z) + d_{n+j} P_n^{(j)}(z) = z(P_n^{(j)}(z) + b_{n+j} P_{n-1}^{(j)}(z)) \quad (1.12)$$

with  $P_0^{(j)} = 1$ ,  $P_1^{(j)}(z) = z - d_j$ . Note that  $P_n(z)$  and  $P_{n-j}^{(j)}(z)$  are two independent solutions of the difference equation

$$\phi_{n+1} + d_n \phi_n = z(\phi_n + b_n \phi_{n-1}). \quad (1.13)$$

There is an obvious scaling property of the LBP.

Assume that the polynomials  $P_n(z)$  are LBP with  $c_n$  as moments and  $d_n, b_n$  as recurrence coefficients. The polynomials  $\tilde{P}_n(z) = \kappa^{-n} P_n(\kappa z)$  are also LBP with the scaled recurrence parameters  $\tilde{b}_n = b_n/\kappa$ ,  $\tilde{d}_n = d_n/\kappa$ , and moments  $\tilde{c}_n = \kappa^{-n} c_n$ . The proof of this statement is obvious. This property means that simultaneously multiplying the recurrence coefficients by the same constant merely leads to a rescaling of the argument  $z$  of the polynomials  $P_n(z)$ .

We say that the LBP are regular if  $b_n d_n \neq 0$ . It can be shown that for regular LBP the following Lemma holds (see, e.g. [14])

**Lemma 1** *Let  $P_n(z)$  be a set of regular LBP. Assume that these  $P_n(z)$  satisfy the relation*

$$S_1(z; n) P_n(z) + S_2(z; n) P_{n-1}(z) = 0, \quad n = 1, 2, \dots, \quad (1.14)$$

where  $S_{1,2}(z; n)$  are polynomials in  $z$  whose coefficients depend on  $n$  but whose degrees are fixed and independent of  $n$ . Then the polynomials  $S_{1,2}$  should identically vanish

$$S_1(z; n) = S_2(z; n) \equiv 0. \quad (1.15)$$

It is interesting to note that the LBP were identified recently in the physical context of the relativistic Toda chain [7].

The main purpose of this paper is to study chains of spectral transformations of LBP and their closure properties. For the ordinary orthogonal polynomials this was developed in [11] and [15]. We show that the simplest quasi-periodic closure leads to q-Appel LBP.

## 2. Spectral transformations of LBP.

In this section we describe the two simplest spectral transformations of LBP, namely the Christoffel (CT) and the Geronimus (GT) transformations which are the exact analogs of the spectral transformations of ordinary orthogonal polynomials bearing the same name (see, e.g. [11], [15]). In what follows we will assume the standard normalization condition  $c_0 = 1$ .

By CT we mean the following transformation

$$\tilde{P}_n(z) = \frac{P_{n+1}(z) - U_n P_n(z)}{z - \mu}, \quad (2.1)$$

where  $\mu$  is an arbitrary parameter and

$$U_n = \frac{P_{n+1}(\mu)}{P_n(\mu)}.$$

It is easily verified that the polynomials  $\tilde{P}_n(z)$  constructed according to (2.1) are again monic LBP with moments

$$\tilde{c}_n = (c_1 - \mu)^{-1}(c_{n+1} - \mu c_n), \quad n = 0, \pm 1, \pm 2, \dots \quad (2.2)$$

In terms of the moment functional  $\mathcal{L}$ , the relation (2.2) can be rewritten in the form

$$\tilde{\mathcal{L}} = (c_1 - \mu)^{-1}(z - \mu)\mathcal{L}. \quad (2.3)$$

The formulas (2.1) and (2.3) coincide with those defining the Christoffel transformation of the ordinary orthogonal polynomials [1], [13].

If the polynomials  $P_n(z)$  satisfy the recurrence relation (1.9), the polynomials  $\tilde{P}_n(z)$  satisfy the recurrence relation

$$\tilde{P}_{n+1}(z) + \tilde{d}_n \tilde{P}_n(z) = z(\tilde{P}_n(z) + \tilde{b}_n \tilde{P}_{n-1}(z)), \quad (2.4)$$

where

$$\tilde{b}_n = b_n \frac{b_{n+1} + U_n}{b_n + U_{n-1}}, \quad (2.5)$$

$$\tilde{d}_n = d_n \frac{d_{n+1} + U_{n+1}}{d_n + U_n}. \quad (2.6)$$

The reciprocal of the CT of LBP is the Geronimus transformation (GT) of LBP defined as

$$\tilde{P}_n(z) = V_n P_n(z) + z(1 - V_n) P_{n-1}(z), \quad (2.7)$$

with some coefficients. Compatibility of relation (2.7) with three-term recurrence relations (1.9) of the polynomials  $P_n(z)$  and  $\tilde{P}_n(z)$  leads to the following restriction

$$V_0 = 1, \quad V_n = \frac{\mu}{\mu - \phi_n/\phi_{n-1}}, \quad n = 1, 2, \dots \quad (2.8)$$

Here  $\mu$  is an arbitrary parameter and  $\phi_n$  is an arbitrary solution of the second-order difference equation

$$\phi_{n+1} + d_n \phi_n = \mu (\phi_n + b_n \phi_{n-1}). \quad (2.9)$$

The polynomials  $\tilde{P}_n(z)$  satisfy the recurrence relation (2.4) with recurrence coefficients

$$\tilde{b}_n = b_{n-1} \frac{1 - V_n}{1 - V_{n-1}}, \quad n = 2, 3, \dots; \quad (2.10)$$

$$\tilde{d}_n = d_n V_{n+1}/V_n. \quad (2.11)$$

One can easily show that the CT and the GT are reciprocal.

Indeed, we have the following:

**Proposition 1** *Assume that the LBP  $R_n(z)$  have the recurrence coefficients  $d_n, b_n$  and are related with the LBP  $P_n(z)$  by the formula*

$$R_n(z) = V_n P_n(z) + z(1 - V_n) P_{n-1}(z), \quad (2.12)$$

where  $V_n$  are some coefficients. Then

$$P_n(z) = \frac{R_{n+1}(z) - U_n R_n(z)}{z - \mu}, \quad (2.13)$$

where

$$\mu = V_{n+1} \left( d_n + \frac{V_n b_n}{1 - v_{n_1}} \right) = \text{const}, \quad (2.14)$$

$$U_n = \frac{b_n V_{n+1}}{1 - V_{n+1}} = \frac{R_{n+1}(\mu)}{R_n(\mu)}. \quad (2.15)$$

*Proof.* From relation (2.12) and the same relation with  $n \rightarrow n + 1$ , we express  $R_{n+1}(z)$  in terms of  $P_n(z)$  and  $P_{n-1}(z)$  using (2.13) with  $\mu$  and  $U_n$  given by the first equation of (2.14) and (2.15). The second equation of (2.12) (i.e.  $\mu = \text{const}$ ) follows from (2.8) and (2.9). The second equation of (2.15) follows from the fact that the lhs of (2.13) cannot have a pole at  $z = \mu$ .

We thus proved that the polynomials  $P_n(z)$  are expressed in terms of  $R_n(z)$  through the CT (2.13).

Note that the GT contains (with respect to the CT) an additional parameter because the function  $\phi_n = P_n(\mu) + \chi P_{n-1}^{(1)}(\mu)$  is a linear combination of two independent solutions of the recurrence relation (2.9).

When  $\mu \rightarrow 0$  then both the CT and the GT become extremely simple. In this case for the CT we have (note that  $P_n(0) \neq 0$  due to the 1-regularity)

$$\tilde{P}_n(z) = z^{-1}(P_{n+1} + d_n P_n(z)) = P_n(z) + b_n P_{n-1}(z). \quad (2.16)$$

The moments are then merely shifted

$$\tilde{c}_n = \frac{c_{n+1}}{c_1}, \quad n = 0, \pm 1, \pm 2, \dots \quad (2.17)$$

and the recurrence coefficients transformationed according to

$$\tilde{b}_n = b_n \frac{b_{n+1} - d_n}{b_n - d_{n-1}}, \quad n = 1, 2, \dots \quad (2.18)$$

$$\tilde{d}_0 = d_0 - b_1, \quad \tilde{d}_n = d_{n-1} \frac{b_{n+1} - d_n}{b_n - d_{n-1}}, \quad n = 1, 2, \dots, \quad (2.19)$$

For the GT, when  $\mu \rightarrow 0$  we have

$$\tilde{P}_n(z) = \frac{P_{n+1}(z) - zP_n(z)}{b_n - d_n} = \frac{zb_n P_{n-1}(z) - d_n P_n(z)}{b_n - d_n}; \quad (2.20)$$

$$\tilde{c}_n = \frac{c_{n-1}}{c_{-1}}, \quad (2.21)$$

$$\tilde{b}_1 = -\frac{b_1 d_0}{b_1 - d_1}, \quad \tilde{b}_n = b_n \frac{b_{n-1} - d_{n-1}}{b_n - d_n}, \quad n = 2, 3, \dots, \quad (2.22)$$

$$\tilde{d}_0 = -\frac{d_0 d_1}{b_1 - d_1}, \quad \tilde{d}_n = d_{n+1} \frac{b_n - d_n}{b_{n+1} - d_{n+1}}, \quad n = 1, 2, \dots \quad (2.23)$$

In what follows we will denote (for brevity) by  $\mathcal{C}(\mu)\{P_n(z)\}$  and  $\mathcal{G}(\mu, \chi)\{P_n(z)\}$  the result of the Christoffel and Geronimus transformations of the LBP  $P_n(z)$  (i.e. (2.1) and (2.7)). For  $\mu = 0$  we will write  $\mathcal{C}(0)\{P_n(z)\}$  and  $\mathcal{G}(0)\{P_n(z)\}$  with formulas (2.16) and (2.20) in mind. Note that for  $\mu = 0$  the Geronimus transformation does not depend on an auxiliary parameter.

### 3. Involutions of the LBP

Apart from the spectral transformations (i.e. Christoffel and Geronimus transformations) there are two fundamental transformations of the LBP. These transformations are *involutions*, i.e. they yield the identity transformation when applied twice.

The first of these involutions is the transformation of the LBP  $P_n(z)$  into their biorthogonal partners  $Q_n(z)$  introduced in (1.5). In terms of  $P_n(z)$  the polynomials  $Q_n(z)$  are written as

$$Q_n(z) = \frac{z^n P_{n+1}(1/z) - z^{n-1} P_n(1/z)}{P_{n+1}(0)}, \quad (3.1)$$

where  $P_{n+1}(0) = (-1)^{n+1} d_0 d_1 \dots d_n$ . It is verified that  $Q_n(z)$  are monic LBT satisfying recurrence relation (1.9) with recurrence parameters

$$\tilde{b}_n = \frac{b_n (d_{n+1} - b_{n+1})}{d_n d_{n+1} (d_n - b_n)}, \quad (3.2)$$

$$\tilde{d}_n = \frac{d_{n+1} - b_{n+1}}{d_{n+1} (d_n - b_n)}. \quad (3.3)$$

The moments  $\tilde{c}_n$  corresponding to the polynomials  $Q_n(z)$  are

$$\tilde{c}_n = c_{-n}. \quad (3.4)$$

Introduce the operator  $\mathcal{Q}$  (acting on the space of polynomials) such that  $\mathcal{Q}\{P_n(z)\} = Q_n(z)$ . It is then obvious from (3.4) that  $\mathcal{Q}^2 = I$ , where  $I$  is identity operator:  $I\{P_n(z)\} = P_n(z)$ . Thus the transformation  $\mathcal{Q}: P_n(z) \rightarrow Q_n(z)$  is indeed an involution.

The second involution is the transformation of the polynomials  $P_n(z)$  into the polynomials  $T_n(z)$  with the argument inverted

$$T_n(z) = \frac{z^n P_n(1/z)}{P_n(0)}. \quad (3.5)$$

It is verified that  $T_n(z)$  are monic LBP having the recurrence coefficients

$$\tilde{b}_n = \frac{b_n}{d_n d_{n+1}}, \quad (3.6)$$

$$\tilde{d}_n = \frac{1}{d_n}. \quad (3.7)$$

The moments  $\tilde{c}_n$  corresponding to the polynomials  $T_n(z)$  are

$$\tilde{c}_n = \frac{c_{1-n}}{c_1}. \quad (3.8)$$

Introduce the operator  $\mathcal{T}$  such that  $\mathcal{T}\{P_n(z)\} = T_n(z)$ . Again it is clear from (3.8) (or from (3.6) and (3.7)) that  $\mathcal{T}^2 = I$  and hence the transformation  $\mathcal{T}$  is an involution.

Consider composition  $\mathcal{T}\mathcal{Q}$  of these involutions. (This means that we first perform the transformation  $\mathcal{Q}$  and then the transformation  $\mathcal{T}$ ). For the corresponding moments we have

$$\mathcal{T}\mathcal{Q}\{c_n\} = \frac{c_{n+1}}{c_1}. \quad (3.9)$$

This means that  $\mathcal{T}\mathcal{Q} = \mathcal{C}(0)$ . The special Christoffel transformation  $\mathcal{C}(0)$  has been defined at the end of section 2.

Analogously,  $\mathcal{Q}\mathcal{T} = \mathcal{G}(0)$ , where  $\mathcal{G}(0)$  is the special Geronimus transformation also defined in section 2.

The composition of these two involutions leads therefore to special spectral transformations.

It is interesting to note that these two involutions and their compositions are connected with the discrete-time dynamics of the relativistic Toda chain (for details see [7]).

## 4. Algebraic interpretation of the spectral transformations

In this section we consider the algebraic interpretation of the Christoffel transformation of the LBP. (The Geronimus transformation can also be interpreted in the same manner but we do not do this here).

Let  $|n\rangle$ ,  $n = 0, 1, 2, \dots$  be some basis. Consider four 2-diagonal operators

$$A|n\rangle = |n-1\rangle + d_n|n\rangle, \quad n = 1, 2, \dots, \quad A|0\rangle = d_0|0\rangle; \quad (4.1)$$

$$B|n\rangle = |n\rangle + b_{n+1}|n+1\rangle, \quad n = 0, 1, \dots; \quad (4.2)$$

$$J|n\rangle = |n-1\rangle + \xi_n|n\rangle, \quad n = 1, 2, \dots, \quad J|0\rangle = \xi_0|0\rangle; \quad (4.3)$$

$$K|n\rangle = |n-1\rangle + \eta_n|n\rangle, \quad n = 1, 2, \dots, \quad K|0\rangle = \eta_0|0\rangle, \quad (4.4)$$

where  $\xi_n, \eta_n$  are some coefficients to be determined.

Consider the generalized eigenvalue problem (GEVP)

$$A|\psi\rangle = zB|\psi\rangle, \quad (4.5)$$

where  $z$  is the (generalized) eigenvalue and  $|\psi\rangle$  is the (generalized) eigenstate. Expand the eigenstate  $|\psi\rangle$  over the basis  $|n\rangle$ :

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n(z)|n\rangle, \quad (4.6)$$

where  $C_n(z)$  are some coefficients depending on  $z$ . We can normalize these coefficients by choosing  $C_0 = 1$ . It is then easily seen that  $C_n(z) = P_n(z)$  with  $P_n(z)$  the LBP satisfying the recurrence relation (1.9). Thus, the LBP  $P_n(z)$  can be interpreted as objects arising from the generalized eigenvalue problem (4.5).

Assume moreover that the following operator relations

$$\tilde{A} J = K A, \quad (4.7)$$

$$\tilde{B} J = K B \quad (4.8)$$

take place where the operators  $\tilde{A}$  and  $\tilde{B}$  have the same form as (4.1) and (4.2) with  $d_n, b_n$  replaced by  $\tilde{d}_n, \tilde{b}_n$ . We then have the following:

**Proposition 2** *If  $|\psi\rangle$  is an eigenstate of the GEVP (4.5) with eigenvalue  $z$  then the state*

$$|\tilde{\psi}\rangle = F(z)J|\psi\rangle \quad (4.9)$$

(where  $F(z)$  is an arbitrary function of  $z$ ) is the eigenstate of another GEVP

$$\tilde{A}|\tilde{\psi}\rangle = z\tilde{B}|\tilde{\psi}\rangle \quad (4.10)$$

with the same eigenvalue  $z$ .

The proof of this proposition is obvious.

We have correspondingly for the coefficients, the transformation law

$$\tilde{C}_n(z) = F(z) (C_{n+1}(z) - \xi_n C_n(z)). \quad (4.11)$$

If we demand that both  $C_n(z) = P_n(z)$  and  $\tilde{C}_n(z) = \tilde{P}_n(z)$  are monic LBP, the conditions

$$\xi_n = -\frac{P_{n+1}(\mu)}{P_n(\mu)}, \quad (4.12)$$

$$\eta_n = \xi_{n-1} \frac{\xi_n - b_{n+1}}{\xi_{n-1} - b_n}, \quad (4.13)$$

$$F(z) = 1/(z - \mu, ) \quad (4.14)$$

where  $\mu$  is an arbitrary parameter are seen to hold. The transformation law (4.11) then becomes simply the Christoffel transformation (2.1) with  $U_n = \xi_n$ . We thus obtain an algebraic (operator) interpretation of the CT in terms of the relations (4.7), (4.8).

If  $Q$  is the operator such that  $Q|n\rangle = q^n|n\rangle$ , it is verified that

$$qQAQ^{-1}|n\rangle = |n-1\rangle + qd_n|n\rangle; \quad (4.15)$$

$$QBQ^{-1}|n\rangle = |n\rangle + qb_{n+1}|n+1\rangle \quad (4.16)$$

We thus see that the action of the operators

$$A(q) = qQAQ^{-1}, \quad B(q) = QBQ^{-1} \quad (4.17)$$

has the same structure as that of the operators  $A, B$  but with the coefficients  $b_n, d_n$  replaced with  $qb_n, qd_n$ .

## 5. Self-similar chains of spectral transformations

>From a given polynomials  $P_n(z)$  we can construct a chain of new polynomials

$$P_n(z; \mu_1, \mu_2, \dots, \mu_N) = \mathcal{C}(\mu_N) \mathcal{C}(\mu_{N-1}) \dots \mathcal{C}(\mu_1) \{P_n(z)\} \quad (5.1)$$

by applying successively  $N$  CT at the points  $\mu_1, \mu_2, \dots, \mu_N$ .

Our closure condition is that the polynomials  $P_n(z; \mu_1, \mu_2, \dots, \mu_N)$  coincide with the initial LBP up to a rescaling :

$$P_n(z; \mu_1, \mu_2, \dots, \mu_N) = q^n P_n(z/q) \quad (5.2)$$

with  $q$  some parameter.

Note that the analogous closure conditions for the ordinary orthogonal polynomials were introduced and analyzed in [11]. In this case, the closure condition (5.2) leads to a special class of semi-classical polynomials (on special exponential grids) introduced by Magnus [8].

In terms of the recurrence coefficients  $b_n, d_n$ , the closure condition (5.2) reads

$$b_n^{(N)} = q b_n, \quad d_n^{(N)} = q d_n, \quad (5.3)$$

where  $b_n^{(N)}$  stands for the coefficients obtained from  $b_n$  by the application of  $N$  CT at the points  $\mu_1, \mu_2, \dots, \mu_N$ . For arbitrary  $N$  the relations (5.3) are very complicated non-linear difference equations. However in particular cases, these equations can be solved in terms of elementary functions.

In operator form, the closure condition (5.2) can be written as

$$A(q) J(N) = K(N) A, \quad B(q) J(N) = K(N) B, \quad (5.4)$$

where the operators  $A(q), B(q)$  are as defined in (4.17) and the operators  $K(N), J(N)$  specified by the following procedure. Assume that we start from the polynomials  $P_n(z)$ . At the first step we get the polynomials  $P_n(z; \mu_1)$  by applying a CT at the point  $\mu_1$ . We then have the operator relations  $A_1 J_1 = K_1 A$ ,  $B_1 J_1 = K_1 B$  where the operators  $A_1, B_1$  correspond to the eigenproblem for the LBP  $P_n(z; \mu_1)$ . At the second step, we choose the point  $\mu_2$  and get the relation  $A_2 J_2 = K_2 A_1$ ,  $B_2 J_2 = K_2 B_1$  where  $A_2, B_2$  correspond to the polynomials  $P_n(z; \mu_1, \mu_2)$ . Clearly we have  $A_2 J_2 J_1 = K_2 K_1 A$ ,  $B_2 J_2, J_1 = K_2 K_1 B$ . Repeating this procedure we get the relation (5.4) where

$$J(N) = J_N J_{N-1} \dots J_1, \quad K(N) = K_N K_{N-1}, \dots K_1. \quad (5.5)$$

## 6. The simplest $N = 1$ closure

In this section we consider the simplest q-closure with  $N = 1$ , i.e. this means that

$$\mathcal{C}(\mu) \{P_n(z)\} = q^n P_n(z/q), \quad (6.1)$$

where  $\mu$  is an arbitrary parameter.

>From the formulas (2.5) and (2.6) we get the relations

$$\frac{b_{n+1} + U_n}{b_n + U_{n-1}} = q, \quad (6.2)$$

$$\frac{d_{n+1} + U_{n+1}}{d_n + U_n} = q, \quad (6.3)$$

where the coefficients  $U_n = P_{n+1}(\mu)/P_n(\mu)$  obey the relation

$$U_n + d_n = \mu(1 + b_n/U_{n-1}), \quad n = 1, 2, \dots \quad (6.4)$$

as follows from the recurrence relation (1.9). Moreover, the initial condition yields

$$U_0 = \mu - d_0. \quad (6.5)$$

It is easily found that the most general solution of relations (6.2), (6.3) (compatible with initial condition (6.5)) is

$$U_n \equiv -\beta, \quad (6.6)$$

$$b_n = \beta(1 - q^n), \quad (6.7)$$

$$d_n = \mu q^n + \beta, \quad (6.8)$$

where  $\beta$  is an arbitrary nonzero constant.

The polynomials  $P_n(z)$  thus obtained have their recurrence coefficients  $b_n, d_n$  expressed in terms of elementary functions (6.7), (6.8) and depending on two parameters  $\beta$  and  $\mu$ . In fact, only one of these parameters (say,  $\mu$ ) is essential because the parameter  $\beta$  can be set equal to 1 with the help of a scaling transformation.

It can be verified that these polynomials can be expressed in terms of basic hypergeometric function

$$P_n(z) = (-\beta)^n (-\mu/\beta; q)_n {}_2\phi_1 \left( \begin{matrix} q^{-n}, 0 \\ \frac{-\beta q^{1-n}}{\mu} - \frac{zq}{\mu} \end{matrix} \right), \quad (6.9)$$

where  $(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1})$  is the  $q$ -shifted factorial. (The notation for the basic hypergeometric functions is the standardized one of ref [2]).

In order to find the moments corresponding to these polynomials we exploit the formula (2.2) for the transformation of the moments  $c_n$  with respect to the CT. Taking into account that the scaling transformation  $P_n(z) \rightarrow q^n P_n(z/q)$  leads to the transformation  $c_n \rightarrow q^n c_n$ , we see that the closure condition (6.1) is equivalent to the equation

$$c_{n+1} = (\beta q^n + \mu) c_n, \quad n = 0, \pm 1, \pm 2, \dots, \quad (6.10)$$

whence

$$c_n = \mu^n (-\beta/\mu; q)_n, \quad (6.11)$$

where for negative  $n$  the  $q$ -shifted factorials are defined [2] as  $(a; q)_{-n} = 1/(aq^{-n}; q)_n$ .

In the limiting case  $\mu = 0$ , we have for the coefficients  $b_n = \beta(1 - q^n)$ ,  $d_n = \beta$ , and the explicit expression for the polynomials reads

$$P_n(z) = (-\beta)^n \sum_{k=0}^n \frac{(q^{-n})_k}{(q)_k} q^{-k(k-1)/2} (zq^n/\beta)^k. \quad (6.12)$$

The corresponding moments are

$$c_n = \beta^n q^{n(n-1)/2}. \quad (6.13)$$

It is interesting to note that for  $\beta = q^{1/2}$  we get a *symmetric* moment problem:  $c_n = c_{-n} = q^{n^2}$ . When  $0 < q < 1$ , the corresponding polynomials coincide with those found by Szegő [12]. The Szegő polynomials are orthogonal on the unit circle.

## 7. q-Appel LBP

The LBP satisfying the property

$$\mathcal{D}_q\{P_{n+1}(z)\} = [n+1] P_n(z) \quad (7.1)$$

will be called q-Appel LBP. The symbol  $\mathcal{D}_q$  used above denotes the q-derivative operator defined by

$$\mathcal{D}_q\psi(z) = \frac{\psi(qz) - \psi(z)}{z(q-1)} \quad (7.2)$$

while  $[n] = (q^n - 1)/(q - 1)$  is the so-called q-number. Let us show that condition (7.1) is equivalent to closure condition (6.1).

Denote  $R_n(z) = q^{-n} P_n(qz)$ , the q-Appel condition (7.1) is then rewritten as

$$R_n(z) = V_n P_n(z) + z(1 - V_n) P_{n-1}(z) \quad (7.3)$$

with  $V_n = q^{-n}$ . In view of proposition 1, (7.3) leads to the reciprocal relation

$$R_n(z) = \frac{P_{n+1}(z) - U_n P_n(z)}{z - \mu} \quad (7.4)$$

with some constant  $\mu$ . This last relation (7.4) coincides with closure condition (6.1), the q-Appel LBP are thus given by (6.9).

It is interesting to consider the limit  $q \rightarrow 1$  of the q-Appel polynomials.

Using the recurrence relation (1.9) we see that for  $\mu \rightarrow \infty$  the Christoffel transformation becomes (to within terms  $O(1/\mu)$ )

$$\mathcal{C}(\mu)\{P_n(z)\} = P_n(z) + \frac{b_n(zP_{n-1}(z) - P_n(z))}{\mu} + O(1/\mu^2). \quad (7.5)$$

Thus for  $\mu \rightarrow \infty$  the Christoffel transformation differs from the identity transformation (i.e.  $P_n(z) \rightarrow P_n(z)$ ) only by terms of the order of  $O(1/\mu)$ .

On the other hand for  $q = 1 + \varepsilon + O(\varepsilon^2)$  we have

$$q^n P_n(z/q) = P_n(z) + \varepsilon(nP_n(z) - zP'_n(z)) + O(\varepsilon^2). \quad (7.6)$$

Putting

$$\varepsilon = -1/\mu, \quad (7.7)$$

we get from (6.1), in the limit  $\varepsilon \rightarrow 0$ , the following condition

$$nP_n(z) - zP'_n(z) = b_n(zP_{n-1}(z) - P_n(z)). \quad (7.8)$$

Equating, in (7.8), the coefficients in front of  $z^{n-1}$ , we find that  $b_n = -n$  and hence (7.8) becomes

$$P'_n(z) = nP_{n-1}(z). \quad (7.9)$$

Condition (7.9) defines the Appel polynomials. They have the recurrence coefficients (see, e.g. [14])  $b_n = \beta n$ ,  $d_n = \beta(n + \gamma)$  where  $\beta$ ,  $\gamma$  are arbitrary constants. In our case  $\beta = -1$  according to the choice (7.7). (Obviously the constant  $\beta$  plays the role of the scaling parameter). The explicit expression for these polynomials is

$$P_n(z) = (\gamma)_n {}_1F_1 \left( \begin{matrix} -n \\ 1 - n - \gamma \end{matrix}; z \right), \quad (7.10)$$

where  $(a)_n$  is the Pochhammer symbol and  ${}_1F_1$  the ordinary hypergeometric function. The Appel property (7.9) is an elementary consequence of the relation

$$\frac{d}{dz} {}_1F_1 \left( \begin{matrix} a \\ b \end{matrix}; z \right) = (a/b) {}_1F_1 \left( \begin{matrix} a+1 \\ b+1 \end{matrix}; z \right), \quad (7.11)$$

where  $a, b$  are arbitrary parameters.

## 8. Operator algebra corresponding to the q-Appel LBP

In this section we consider the operator algebra corresponding to closure condition (5.2). Here the operators  $J, K$  have the simple form

$$J|n\rangle = |n-1\rangle + \beta|n\rangle, \quad K|n\rangle = |n-1\rangle + q\beta|n\rangle, \quad (8.1)$$

and the relation between these operators becomes

$$K = qQJQ^{-1}. \quad (8.2)$$

We can thus eliminate the operator  $K$  using this relation. We then get from (5.4) the following closure conditions for  $N = 1$

$$AQ^{-1}J = JQ^{-1}A, \quad (8.3)$$

$$BQ^{-1}J = qJQ^{-1}B. \quad (8.4)$$

Moreover, the definition (8.1) entails an additional condition for the operator  $Q$  and  $J$

$$Q^{-1}J - qJQ^{-1} = \beta(1-q)Q^{-1}. \quad (8.5)$$

We thus have operator algebra generated by three relations (8.3), (8.4) and (8.5) for 4 operators  $A, B, J, Q$ . Note that this algebra seems to be more complicated than the simple q-oscillator algebra obtained in the case of the ordinary OP [10], [11] for  $N = 1$  closure.

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