

Solutions of Nonlinear Differential and Difference Equations with Superposition Formulas

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Abstract

Matrix Riccati equations and other nonlinear ordinary differential equations with superposition formulas are, in the case of constant coefficients, shown to have the same exact solutions as their group theoretical discretizations.

Résumé

Nous démontrons que les équations de Riccati matricielles, ainsi que d'autres équations différentielles ordinaires avec formules de superposition ont, dans le cas de coefficients constants, les mêmes solutions exactes que leurs discrétisations associées.

1 Introduction

Several recent articles have been devoted to a study of nonlinear ordinary differential equations (ODEs) with superposition formulas [1, 2, 3, 4, 5, 6]. The entire approach is based on a theorem due to Sophus Lie [7] that states the following.

Theorem 1 (S. Lie) *The general solution $\vec{y}(t; c_1, \dots, c_n)$ of a system of n ordinary differential equations*

$$\dot{\vec{y}} = \vec{\eta}(\vec{y}, t), \quad \vec{y} \in \mathbb{C}^n \text{ (or } \mathbb{R}^n) \quad (1.1)$$

can be written as a function of m particular solutions and n arbitrary constants

$$\vec{y} = \vec{F}(\vec{y}_1(t), \dots, \vec{y}_m(t); c_1, \dots, c_n) \quad (1.2)$$

if and only if:

1. *The system (1.1) has the form*

$$\dot{\vec{y}}(t) = \sum_{k=1}^r Z_k(t) \vec{\eta}_k(\vec{y}, t), \quad 1 \leq r < \infty \quad (1.3)$$

2. *The vector fields*

$$\hat{X}_k = \sum_{\mu=1}^n \eta_k^\mu(t) \partial_{y_\mu}, \quad (1.4)$$

generate a Lie algebra L of finite dimension $\dim L = r < \infty$. The number of equations n , number of fundamental solutions m and the dimension of the Lie algebra r satisfy

$$nm \geq r. \quad (1.5)$$

A geometric approach to such equations has been developed [5]. The variables y_1, \dots, y_n are considered to be coordinates on a homogeneous space $M \sim G/G_0$ with the group G acting transitively on M . The Lie algebra L of vector fields (1.4) is the Lie algebra corresponding to the group G . The subgroup G_0 is the isotropy group of the origin and its Lie algebra L_0 is the subalgebra of L consisting of vector fields that vanish at the origin. Thus, it is possible to associate a system of ODEs satisfying Lie's criterion with every group-subgroup pair (G_0, G) , or with every Lie algebra pair (L_0, L) , $L_0 \subset L$.

We call the system (1.3) decomposable if a subsystem of equations can be split off that involves only a subset of variables, namely, y_1, \dots, y_k , $1 \leq k < n$, that itself satisfies Lie's theorem. This occurs if the action of G on M is not primitive, i.e., if a G -invariant foliation of M exists [5]. Indecomposable systems of ODE's with superposition formulas are obtained if the pair of Lie algebras (L_0, L) forms a transitive primitive Lie algebra [8, 9, 10, 11, 12]. This requires that L_0 be a maximal subalgebra of L , not containing any ideal of L . Transitive primitive Lie algebras over \mathbb{C} and \mathbb{R} have been classified [8, 9, 10, 11, 12]. Let us now consider a particularly interesting class of transitive primitive Lie algebras, namely one where L is a simple classical Lie algebra over \mathbb{C} or \mathbb{R} and L_0 is a maximal parabolic subalgebra. The space $M \sim G/G_0$ is then a Grassmannian. Using affine coordinates on this space we obtain vector fields with polynomial coefficients of order at most four [5]. The corresponding ODEs (1.3) have the same polynomial nonlinearities.

These equations have the following properties:

1. By construction, they allow a nonlinear superposition formula.
2. They are linearizable by introducing homogeneous coordinates on the Grassmannian, i.e., by an embedding into a larger system of linear equations.
3. They have the Painlevé property [13], i.e., their singularity structure is particularly simple: the only movable singularities of their solutions are poles.

A discretization of the nonlinear ODEs with superposition formulas has been proposed in a recent article [14]. It preserves all of these properties, in particular the Painlevé property, manifesting itself as singularity confinement [15].

The discretization was performed for the case when L was any classical complex Lie algebra and L_0 any of its maximal parabolic subalgebras. The discretization made use of the linearization of the equations (1.3). It can, however, be presented schematically without using the corresponding linear system.

Indeed, let (1.4) represent the vector fields of Lie's theorem. Eq. (1.3) can be rewritten as

$$\dot{\vec{y}} = \widehat{X}(t)\vec{y} \quad (1.6)$$

$$\widehat{X}(t) = \sum_{k=1}^r Z_k(t) \sum_{\mu=1}^n \eta_k^\mu(t) \partial_{y_\mu}, \quad (1.7)$$

As t varies, the vector field $\widehat{X}(t)$ follows some path in the Lie algebra L . The vector fields (1.4) can be integrated to give the corresponding local group action

$$\vec{y} = g\vec{u}, \quad (1.8)$$

where \vec{u} is some chosen point on M . The form of action (1.8) is independent of the equation (1.6), i.e. of the functions $Z_k(t)$. It involves $r = \dim L = \dim G$ group parameters. The general solution of eq. (1.6) can be represented in the form (1.8), where $g = g(t)$ is a function of t . As t varies, $g(t)$ will trace a path in the group manifold, corresponding to the path that $\widehat{X}(t)$ follows in the Lie algebra. Reconstructing this path is equivalent to solving the equation (1.6).

The difference equations with all the desired properties are obtained in the form (1.8) with g a function of time (i.e., each of the r group parameters in g depends on time). The equations are

$$\bar{y} = gy, \quad y \equiv y(t), \quad \bar{y} \equiv y(t + \sigma), \quad (1.9)$$

where σ is the spacing of the lattice. The group element $g = g(t)$ is to be evaluated at the same point as $y(t)$ on the right hand side. For $\sigma \rightarrow 0$ we have $g(t) = I + \sigma\widehat{X}(t)$ and the difference equation (1.9) goes into the differential equation (1.6).

The purpose of this article is to explore some properties of the difference equations (1.9) and in particular to show that in the case of constant coefficients in the differential equations (1.6) the solutions of the continuous and discrete equations coincide exactly.

A general theorem on solutions of continuous and discrete matrix Riccati equation is formulated and proven in Section 2. Section 3 is devoted to examples.

2 Continuous and Discrete Matrix Riccati Equations

Let us consider the Grassmannian $M \sim G/G_0$ where G is $SL(N, \mathbb{C})$ and G_0 is a maximal parabolic subgroup. We have

$$N = p + q, \quad G \sim \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}, \quad G_0 \sim \begin{pmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{pmatrix}, \quad (2.1)$$

where $G_{11} \in \mathbb{C}^{p \times p}$, $G_{12} \in \mathbb{C}^{p \times q}$, $G_{21} \in \mathbb{C}^{q \times p}$, $G_{22} \in \mathbb{C}^{q \times q}$ and $M = pq$, $\det G = 1$. Affine coordinates on this space (in the neighborhood of the origin) are given by the matrix elements of a matrix $W \in \mathbb{C}^{p \times q}$. The nonlinear ODEs with superposition formulas in this case are the Matrix Riccati Equations [3, 4, 5, 6, 16]

$$\dot{W} = A + BW + WC + WDW, \quad (2.2)$$

where

$$\xi(t) = \begin{pmatrix} B(t) & A(t) \\ -D(t) & -C(t) \end{pmatrix}, \quad \text{Tr } B = \text{Tr } C, \quad (2.3)$$

For t fixed, $\xi(t)$ is an element of the Lie algebra $\mathfrak{sl}(N, \mathbb{C})$. The division into blocks is the same in eq. (2.3) as in (2.1). The matrices A, B, C, D are known functions of t . As t varies the matrix $\xi(t)$ follows some path lying in $\mathfrak{sl}(N, \mathbb{C})$.

The Matrix Riccati Equation (2.2) is linearized by introducing homogeneous coordinates, i.e. the matrix elements of two matrices X and Y . We put

$$W = XY^{-1}, \quad X \in \mathbb{C}^{p \times q}, \quad Y \in \mathbb{C}^{q \times q}. \quad (2.4)$$

The origin on M corresponds to $X = O, Y = I$. When the matrices X, Y satisfy the linear system

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} B(t) & A(t) \\ -D(t) & -C(t) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad (2.5)$$

the matrix $W = XY^{-1}$ satisfies eq. (2.2).

The ordinary Riccati equation for $w(t)$ a scalar function, namely

$$\dot{w} = a + (b + c)w + dw^2, \quad (2.6)$$

is obtained for $p = q = 1$. For $p \geq 2, q = 1$ eq. (2.2) is called the *projective Riccati* equation, since it corresponds to the projective action of $SL(p + 1, \mathbb{C})$ on the space \mathbb{C}^p .

The discretization of eq. (2.5) is [14]

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \begin{pmatrix} B(t) & A(t) \\ -D(t) & -C(t) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad (2.7)$$

where $\bar{X} = X(t + \sigma), \bar{Y} = Y(t + \sigma)$ and the discrete matrix Riccati equation is obtained via the projection (2.4), i.e.,

$$\bar{W} = (G_{11}W + G_{12})(G_{21}W + G_{22})^{-1}. \quad (2.8)$$

Now let us restrict ourselves to the case of the Matrix Riccati Equations (2.2) with constant coefficients. In this case the matrix ξ of eq. (2.3) can be viewed as a one-dimensional subalgebra of $\mathfrak{sl}(N, \mathbb{C})$ and the corresponding one-dimensional subgroup of $SL(N, \mathbb{C})$ is obtained by exponentiating ξ

$$G = e^{\epsilon\xi} = I + \epsilon\xi + \frac{1}{2!}\epsilon^2\xi^2 + \dots \quad (2.9)$$

This matrix G is to be inserted into eq. (2.7) and eq. (2.8), first identifying the group parameter ϵ with the lattice spacing σ , for example, putting $\epsilon = \sigma$. Alternatively the form of the transformation G can be obtained by integrating the vector fields on the right hand side of eq. (2.2). To do this we rewrite eq. (2.2) in components as

$$\dot{W}_{\alpha a} = A_{\alpha a} + B_{\alpha\beta}W_{\beta a} + W_{\alpha b}C_{ba} + W_{\alpha b}D_{b\beta}W_{\beta a}, \quad (2.10)$$

where $1 \leq \alpha \leq p, 1 \leq a \leq q$ (repeated Greek labels are summed from 1 to p , Latin ones from 1 to q). Eq. (2.8) is obtained directly by integrating

$$\frac{d\bar{W}_{\alpha a}}{d\epsilon} = A_{\alpha a} + B_{\alpha\beta}\bar{W}_{\beta a} + \bar{W}_{\alpha b}C_{ba} + \bar{W}_{\alpha b}D_{b\beta}\bar{W}_{\beta a}, \quad (2.11)$$

where $\bar{W}_{\alpha a}|_{\epsilon=0} = W_{\alpha a}$.

We now come to our main result which we formulate as a theorem.

Theorem 2 *The general solution of the continuous matrix Riccati equation (2.2) with constant coefficients coincides with the general solution of the discrete matrix Riccati equation (2.8), with coefficients $G_{\mu\nu}$ satisfying eq. (2.9).*

Proof. Let us rewrite the discrete linear system (2.7) as

$$e^{\sigma d/dt}Z = e^{\sigma\xi}Z, \quad Z = \begin{pmatrix} X \\ Y \end{pmatrix} \quad (2.12)$$

and expand both sides in powers of σ :

$$1 + \sigma \frac{d}{dt} + \frac{\sigma^2}{2} \left(\frac{d}{dt} \right)^2 + \dots + \frac{\sigma^n}{n!} \left(\frac{d}{dt} \right)^n + \dots = 1 + \sigma\xi + \frac{\sigma^2}{2}\xi^2 + \dots + \frac{\sigma^n}{n!}\xi^n + \dots$$

Comparing coefficients of the powers of σ we have

$$\frac{d^n}{dt^n}Z = \xi^n Z. \quad (2.13)$$

For $n = 0$ this is satisfied trivially, for $n = 1$ we obtain the continuous linear system (2.5)

$$\dot{Z} = \xi Z. \quad (2.14)$$

Let $Z(t)$ be a solution of eq. (2.14) and hence W a solution of the continuous matrix Riccati equation (2.2). In order for Z to be a solution of the discrete system (2.7), and hence W a solution of the discrete matrix Riccati equation (2.8), Z must also satisfy eq. (2.13) for all $n \geq 2$. For $n = 2$ we have

$$\frac{d^2 Z}{dt^2} = \frac{d}{dt}(\xi Z) = \frac{d\xi}{dt}Z + \xi \frac{dZ}{dt} = \xi^2 Z, \quad (2.15)$$

since $d\xi/dt = 0$ and $\dot{Z} = \xi Z$. For $n \geq 3$ eq. (2.13) follows by induction, hence every solution of eq. (2.14) with constant ξ satisfies the difference equation (2.7). \square

The general solution of eq. (2.5) depends on p, q significant free parameters, the initial conditions. The same is true for the difference equation (2.7). Hence, the general solution of the differential equation (2.5) provides the general solution of the difference equation (2.7) and vice versa.

3 Examples

3.1 The Scalar Riccati Equation

Let us first consider a scalar Riccati equation with real constant coefficients:

$$\dot{u} = a + bu + cu^2, \quad c \neq 0, \quad (3.1)$$

and introduce $\Delta \equiv b^2 - 4ac$. We simplify this equation by using an appropriate linear transformation $u = \alpha y + \beta$. Depending on the sign of Δ we obtain three different standard forms of (3.1).

3.1.1 $\Delta > 0$

We choose α and β so as to obtain

$$\dot{y} = \omega(y^2 - 1), \quad \omega = \text{const.} \quad (3.2)$$

Integrating the vector field $\hat{X} = (y^2 + 1)d/dy$ we obtain the *discrete* Riccati equation

$$\bar{y} = \frac{y \cosh \sigma\omega - \sinh \sigma\omega}{-y \sinh \sigma\omega + \cosh \sigma\omega}, \quad (3.3)$$

where $y = y(t)$, $\bar{y} = y(t + \sigma)$. It is easy to verify by direct calculation that the general solution of both equations (3.2), (3.3) is

$$y = -\coth \omega(t - t_0), \quad (3.4)$$

where t_0 is an arbitrary constant.

3.1.2 $\Delta < 0$

An appropriate choice of α and β yields

$$\dot{y} = \omega(y^2 + 1), \quad \omega = \text{const.}, \quad (3.5)$$

Integration of the vector field $\hat{X} = (y^2 - 1)d/dy$ provides us with the discrete Riccati equation

$$\bar{y} = \frac{y \cos \sigma\omega + \sin \sigma\omega}{-y \sin \sigma\omega + \cos \sigma\omega}. \quad (3.6)$$

The general solution of these equations is

$$y = \tan \omega(t - t_0), \quad (3.7)$$

where t_0 is an arbitrary constant.

3.1.3 $\Delta = 0$

In this case the Riccati equation (3.1) can be reduced to

$$\dot{y} = y^2, \quad (3.8)$$

and its discrete form is

$$\bar{y} = \frac{y}{1 - \sigma y}. \quad (3.9)$$

The general solution of both equations (3.8) and (3.9) is

$$y = -\frac{1}{t - t_0}, \quad (3.10)$$

where t_0 is an arbitrary constant.

3.2 Coupled Riccati Equations Based on a Nonprimitive Group Action

Most of the earlier articles on nonlinear ODEs with superposition formulas concerned indecomposable systems of equations. A study of imprimitive actions and hence decomposable systems was initiated quite recently [17].

Here we shall give an example of such a system and its discretization. Consider the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, realized as a subalgebra of $\text{diff}(2, \mathbb{R})$, by the vector fields.

$$X_1 = \partial_x, \quad X_2 = x\partial_x + y\partial_y, \quad X_3 = x^2\partial_x + (2xy + ky^2)\partial_y, \quad (3.11)$$

when $k = 0$, or $k = 1$.

The corresponding ODEs are

$$\dot{x} = a + bx + cx^2, \quad \dot{y} = by + c(2xy + ky^2), \quad (3.12)$$

Let us now restrict to the case when a, b, c are constants and moreover assume that $b^2 - 4ac < 0$, $c \neq 0$. We can then, by means of a linear transformation of x take eq. (3.12) into the standard form

$$\dot{x} = \omega(x^2 + 1), \quad \dot{y} = \omega(2xy + ky^2), \quad \omega = \text{const.} \quad (3.13)$$

To obtain a discretization that preserves the underlying group structure, we integrate the vector field

$$\widehat{X} = \omega[(1 + x^2)\partial_x + (2xy + ky^2)\partial_y], \quad (3.14)$$

We have

$$\frac{d\bar{x}}{d\sigma} = \omega(1 + \bar{x}^2), \quad \frac{d\bar{y}}{d\sigma} = \omega(2\bar{x}\bar{y} + k\bar{y}^2), \quad (3.15)$$

where $\bar{x}|_{\sigma=0} = x$, $\bar{y}|_{\sigma=0} = y$. The result is

$$\begin{aligned} \bar{x} &= \frac{x \cos \sigma\omega + \sin \sigma\omega}{-x \sin \sigma\omega + \cos \sigma\omega}, \\ \bar{y} &= \frac{y(x^2 + 1)}{D}, \end{aligned} \quad (3.16)$$

where $D = (x^2 + 1 + kxy)(-x \sin \sigma\omega + \cos \sigma\omega)^2 - ky(x \cos \sigma\omega + \sin \sigma\omega)(-x \sin \sigma\omega + \cos \sigma\omega)$. Thus, eq. (3.16) is a difference equation that has eq. (3.13) as its continuous limit (for $\sigma \rightarrow 0$).

Eq. (3.13) is easy to solve and we obtain

$$\begin{aligned} x &= \tan \omega(t - t_0), \\ y &= \frac{1}{\alpha \cos^2 \omega(t - t_0) - k[\sin 2\omega(t - t_0)]/2}, \end{aligned} \quad (3.17)$$

where α and t_0 are integration constants. Expression (3.17) is also the general solution of the difference equation (3.16).

3.3 Projective Riccati Equations

Let us now consider the matrix Riccati equation (2.2) for $q = 1$ which is called the *projective* Riccati equations.

We rewrite the system as

$$\dot{y} = a + By + y^T Cy \quad (3.18)$$

where $a, y \in K^p$ are column vectors, $B, C \in K^{p \times p}$ are matrices and T denotes transposition. We consider the case when all coefficients are constants.

We simplify eq. (3.18) by a linear transformation with constant coefficients

$$y = Au + \rho \quad (3.19)$$

Choosing A and ρ appropriately we can, for instance, transform C and B in such a way that

$$C_i = \delta_{i1}, \quad B_{j1} = 0, \quad j = 1, 2, \dots, p \quad (3.20)$$

with some further freedom remaining.

Now let us consider the simplest case of $p = 2$. Changing notations we write the transformed projective Riccati equation as

$$\dot{x} = 1 + y + x^2, \quad \dot{y} = a + by + xy, \quad (3.21)$$

where a, b are constants.

In order to discretize the equations we must either exponentiate a matrix as in eq. (2.9) or integrate the vector field

$$\widehat{X} = (1 + y + x^2)\partial_x + (a + by + xy)\partial_y, \quad (3.22)$$

We choose the second procedure and put

$$\frac{d\bar{x}}{d\sigma} = (1 + \bar{y} + \bar{x}^2), \quad \frac{d\bar{y}}{d\sigma} = (a + b\bar{y} + \bar{x}\bar{y}), \quad (3.23)$$

where $\bar{x}|_{\sigma=0} = x, \bar{y}|_{\sigma=0} = y$.

Eliminating \bar{y} from eq. (3.23) we have

$$\ddot{\bar{x}} = 3\bar{x}\dot{\bar{x}} + b\dot{\bar{x}} - \bar{x}^3 - b\bar{x}^2 - \bar{x} + a - b \quad (3.24)$$

where dots denote differentiation with respect to σ .

Eq. (3.24) has the Painlevé property and can be transformed to one of the 50 standard forms [18], studied by Painlevé and Gambier. More specifically it is equivalent to the linearizable equation No. VI in the list reproduced by Ince [18] (see p. 334). Indeed, we put

$$\bar{x}(\sigma) = \alpha(\sigma)w(\tau) + \beta, \quad \tau = \tau(\sigma), \quad (3.25)$$

while the constant β must satisfy:

$$\beta^3 + b\beta^2 + \beta + b - a = 0$$

The functions $\alpha(\sigma)$ and $\tau(\sigma)$ must satisfy linear equations with constant coefficients:

$$\ddot{\alpha} - (b + 3\beta)\dot{\alpha} + (3\beta^2 + 2b\beta + 1)\alpha = 0 \quad (3.26)$$

$$\dot{\tau} = -\alpha. \quad (3.27)$$

The function $w(\tau)$ then satisfies

$$w_{\tau\tau} = -3ww_{\tau} - w^3 + q(\tau)(w_{\tau} + w^2) \quad (3.28)$$

$$q(\tau) = \frac{3\dot{\alpha} - 3\alpha\beta - b\alpha}{\alpha^2}. \quad (3.29)$$

Equation (3.28) is linearized by the Cole-Hopf transformation

$$w = \frac{f_{\tau}}{f}, \quad f_{\tau\tau\tau} = q(\tau)f_{\tau\tau}. \quad (3.30)$$

Eq. (3.30) can be solved in quadratures, however, we shall only consider the simplest case, when $q(\tau) = 0$. This implies a constraint between the constants a, b , namely

$$q(\tau) = 0, \quad a = \frac{2b(b^2 + 9)}{27} \quad (3.31)$$

Equation (3.30) in this case is solved trivially. Returning to the original variables \bar{x} and \bar{y} we have

$$\begin{aligned} x(\sigma) &= -k \frac{2Pe^{k\sigma} + Q}{Pe^{k\sigma} + Q + Re^{-k\sigma}} + \beta \\ y(\sigma) &= k \frac{4(\beta - k)Pe^{k\sigma} + (2\beta - k)Q}{Pe^{k\sigma} + Q + Re^{-k\sigma}} - \beta^2 - 1. \end{aligned} \quad (3.32)$$

Here P, Q, R are integration constants and we have put

$$\begin{aligned} k = \beta + \frac{b}{3} \neq 0, \quad \beta &= \frac{-b + \sqrt{3(b^2 - 3)}}{3}, \quad b^2 \neq 3, \\ \tau = e^{k\sigma}, \quad \alpha &= -ke^{k\sigma} \end{aligned} \quad (3.33)$$

We eliminate the constants P, Q, R by imposing initial conditions as in eq. (3.23). Finally, we obtain difference equations

$$\bar{x} = \frac{g_{11}x + g_{12}y + g_{13}}{g_{31}x + g_{32}y + g_{33}}, \quad \bar{y} = \frac{g_{21}x + g_{22}y + g_{23}}{g_{31}x + g_{32}y + g_{33}}, \quad (3.34)$$

where

$$\begin{aligned} g_{11} &= (\beta - 2k)(2\beta - k)e^{k\sigma} - 4(\beta - k)^2 + \beta(2\beta - 3k)e^{-k\sigma}, \\ g_{12} &= (\beta - 2k)e^{k\sigma} + 2(k - \beta) + \beta e^{-k\sigma}, \\ g_{13} &= (\beta - 2k)(-\beta^2 + k\beta + 1)e^{k\sigma} - 2(\beta - k) + \beta e^{-k\sigma} \\ g_{21} &= (4k\beta - 4k^2 - \beta^2 - 1)(2\beta - k)e^{k\sigma} - 4(2\beta k - k^2 - \beta^2 - 1)(\beta - k) \\ &\quad - (\beta^2 + 1)(2\beta - 3k)e^{-k\sigma}, \\ g_{22} &= (4k\beta - 4k^2 - \beta^2 - 1)e^{k\sigma} - 2(2\beta k - k^2 - \beta^2 - 1) - (\beta^2 + 1)e^{-k\sigma}, \\ g_{23} &= (4k\beta - 4k^2 - \beta^2 - 1)(-\beta^2 + \beta k + 1)e^{k\sigma} - 2(2\beta k - k^2 - \beta^2 - 1)(2\beta k + 1) \\ &\quad + (\beta^2 + 1)(\beta^2 + 2k^2 - 3\beta k + 1)e^{-k\sigma}, \\ g_{31} &= (2\beta - k)e^{k\sigma} - 4(\beta - k) + (2\beta - 3k)e^{-k\sigma}, \\ g_{32} &= e^{k\sigma} - 2 + e^{-k\sigma}, \\ g_{33} &= (1 + \beta k - \beta^2)e^{k\sigma} + 2(\beta^2 - 2\beta k + 1) + (-\beta^2 - 2k^2 + 3\beta k + 1)e^{-k\sigma}. \end{aligned} \quad (3.35)$$

As in the previous examples the general solutions of the projective Riccati equations (3.21), and their discretization (3.34), coincide.

4 Conclusion

The results of this article can be summed up as follows.

1. We have shown that matrix Riccati equations with constant coefficients and their group theoretical discretizations have the same solutions.
2. We have shown how other types of equations with superposition formulas can be discretized via an integration of the underlying vector fields. For constant coefficients the continuous and discrete equations again have the same solutions.

An open question, presently under study, is whether the results of this article can be used to obtain an effective perturbative approach to solving differential and difference equations that allow superposition formulas, but do not have constant coefficients.

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