Modular Invariants and Generalized Halphen Systems

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CRM-1597

December 1998

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Talk presented by J. Harnad at the SIDE III international meeting, Sabaudia, May, 1998. Research supported in part by the Natural Sciences and Engineering Research Council of Canada and the Fonds FCAR du Québec

Abstract

Generalized Halphen systems are solved in terms of functions that uniformize genus zero Riemann surfaces, with automorphism groups that are commensurable with the modular group. Rational maps relating these functions imply subgroup relations between their automorphism groups and symmetrization relations between the associated differential systems.

1. Introduction. Halphen Systems and Modular Invariants.

1a. Darboux-Halphen Equations.

The Darboux–Halphen differential system:

$$\begin{aligned} w_1' &= w_1(w_2+w_3) - w_2 w_3 \\ w_2' &= w_2(w_3+w_1) - w_3 w_1 \\ w_3' &= w_3(w_1+w_2) - w_1 w_2 \end{aligned} \tag{1.1}$$

originally appeared in the work of Darboux [Da] on orthogonal coordinate systems. It was subsequently solved by Halphen [Ha], who related it to the hypergeometric equation of Legendre type

$$\lambda(1-\lambda)\frac{d^2y}{d\lambda^2} + (1-2\lambda)\frac{dy}{d\lambda} - \frac{1}{4}y = 0, \tag{1.2}$$

and also generalized it to a 3–parameter family of systems admitting a similar relation to the general hypergeometric equation [Ha, Br]. The general solution to (1.1) may be expressed [Ha] in terms of the *elliptic modular function*; that is, the square of the elliptic modulus, viewed as a function of the ratio τ of the periods of the Jacobi elliptic functions. More recently, the system (1.1) has found applications in mathematical physics in relation to magnetic monopole dynamics [AH], self–dual Einstein equations [GP, Hi, To] and topological field theory [Du].

To relate the Darboux–Halphen system to the hypergeometric equation (1.2), we first form the ratio of two linearly independent solutions of the latter

$$\tau(\lambda) := \frac{y_1}{y_2}, \tag{1.3}$$

and note that the inverse function $\lambda(\tau)$ satisfies the Schwarzian equation [GS, H]

$$\{\lambda,\tau\} + \frac{\lambda^2-\lambda+1}{2\lambda^2(1-\lambda)^2}\lambda'^2 = 0, \tag{1.4}$$

where the Schwarzian derivative is defined as

$$\{f,\tau\} := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2, \qquad (f' := \frac{df}{d\tau}). \tag{1.5}$$

In view of the invariance properties of the Schwarzian derivative, the general solution of (1.4) is obtained by composing a particular one with the Möbius (linear fractional) transformations

$$\tau \to \frac{a\tau+b}{c\tau+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{C}). \tag{1.6}$$

The solutions to the Darboux-Halphen system are then given [Br, Ha] by setting

$$w_1 := \frac{1}{2}\frac{d}{d\tau}\ln\frac{\lambda'}{\lambda}, \qquad w_2 := \frac{1}{2}\frac{d}{d\tau}\ln\frac{\lambda'}{(\lambda-1)}, \qquad w_3 := \frac{1}{2}\frac{d}{d\tau}\ln\frac{\lambda'}{\lambda(\lambda-1)}, \tag{1.7}$$

where $\lambda(\tau)$ is a solution of (1.4). A particular solution is provided [H, GS] by the ellliptic modular function

$$\lambda(\tau) = k^2(\tau), \tag{1.8}$$

(with τ is interpreted as the ratio of two elliptic periods), whose automorphism group is the principal congruence subgroup $\Gamma(2) \subset \Gamma$ of the full modular group $\Gamma := PSL(2, \mathbf{Z})$.

An explicit representation of $\lambda(\tau)$ may be given in terms of null ϑ-functions [WW]

$$\lambda(\tau) = \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)} = 1 - \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}. \tag{1.9}$$

Substituting this in (1.7) and using the differential identities satisfied by the null theta functions leads to the explicit formulae

$$w_1 = 2\frac{d}{d\tau}\ln\vartheta_4, \qquad w_2 = 2\frac{d}{d\tau}\ln\vartheta_2, \qquad w_3 = 2\frac{d}{d\tau}\ln\vartheta_3. \tag{1.10}$$

1b. Symmetrization under $\mathbf{S}_3 = \Gamma/\Gamma(2)$***. The Chazy Equation.***

The modular transformations

$$\tau \mapsto \ \tau + 1, \ -\frac{1}{\tau} \tag{1.11}$$

do not leave λ invariant, but generate the *group of anharmonic ratios* [H]

$$\lambda \mapsto \lambda, \ \frac{1}{\lambda}, \ 1-\lambda, \ \frac{1}{1-\lambda}, \ \frac{\lambda}{\lambda-1}, \ \frac{\lambda-1}{\lambda}. \tag{1.12}$$

The symmetric invariant for this group is given by Klein's J–function

$$J = \frac{4(\lambda^2-\lambda+1)^3}{27\lambda^2(\lambda-1)^2} = \frac{(\vartheta_2^8+\vartheta_3^8+\vartheta_4^8)^3}{54\vartheta_2^8\vartheta_3^8\vartheta_4^8}, \tag{1.13}$$

which has the full modular group Γ as automorphism group, and satisfies the Schwarzian equation

$$\{J,\tau\} + \frac{36J^2-41J+32}{72J^2(J-1)^2}J'^2 = 0. \tag{1.14}$$

In a similar manner to the above, this may be associated with the hypergeometric equation

$$J(1-J)\frac{d^2y}{dJ^2} + \left(\frac{2}{3} - \frac{7}{6}J\right)\frac{dy}{dJ} - \frac{1}{144}y = 0. \tag{1.15}$$

In terms of the Halphen variables (1.7), it follows that the corresponding elementary symmetric polynomials

$$\sigma_1 := w_1 + w_2 + w_3, \qquad \sigma_2 := w_1w_2 + w_2w_3 + w_3w_1, \qquad \sigma_3 := w_1w_2w_3, \tag{1.16}$$

satisfy the symmetrized system

$$\sigma_1' = \sigma_2 \qquad \sigma_2' = 6\sigma_3 \qquad \sigma_3' = 4\sigma_1\sigma_3 - \sigma_2^2, \tag{1.17}$$

which reduces to the Chazy equation [Ch]

$$W''' = 2WW'' - 3W'^2 \tag{1.18}$$

for

$$W := 2\sigma_1 = \frac{1}{2}\frac{d}{d\tau}\ln\frac{J'^6}{J^4(J-1)^3}. \tag{1.19}$$

More generally, it is easy to see that, forming the ratio, as in (1.3), of a pair (y_1, y_2) of linearly independent solutions of the general hypergeometric equation

$$f(1-f)\frac{d^2y}{df^2} + (c-(a+b+1)f)\frac{dy}{df} - aby = 0, \tag{1.20}$$

and *assuming* that the inverse function $f = f(\tau)$ is well–defined, this similarly provides solutions to the Schwarzian equation

$$\{f,\tau\} + \frac{1}{2}\left(\frac{1-\lambda^2}{f^2} + \frac{1-\mu^2}{(f-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{f(f-1)}\right)f'^2 = 0, \tag{1.21}$$

where (λ, μ, ν) are the relative Frobenius exponents for (1.20) at $(0, 1, \infty)$, given in terms of the parameters (a, b, c) by

$$\lambda := 1-c, \qquad \mu := c-a-b, \qquad \nu := b-a. \tag{1.22}$$

Introducing the general *Halphen-Brioschi variables* [Br]

$$W_1 := \frac{1}{2}\frac{d}{d\tau}\ln\frac{f'}{f}, \quad W_2 := \frac{1}{2}\frac{d}{d\tau}\ln\frac{f'}{(f-1)}, \quad W_3 := \frac{1}{2}\frac{d}{d\tau}\ln\frac{f'}{f(f-1)}, \tag{1.23}$$

these are similarly seen to satisfy the *general Halphen system*:

$$\begin{aligned} W_1' =& W_1(W_2+W_3) - W_2W_3 + X(\lambda,\mu,\nu) \\ W_2' =& W_2(W_3+W_1) - W_3W_1 + X(\lambda,\mu,\nu) \\ W_3' =& W_3(W_1+W_2) - W_1W_2 + X(\lambda,\mu,\nu), \end{aligned} \tag{1.24}$$

where

$$\begin{aligned} X(\lambda,\mu,\nu) :=& \mu^2W_1^2 + \lambda^2W_2^2 + \nu^2W_3^2 + (\nu^2-\lambda^2-\mu^2)W_1W_2 \\ &+ (\lambda^2-\mu^2-\nu^2)W_3W_1 + (\mu^2-\lambda^2-\nu^2)W_2W_3. \end{aligned} \tag{1.25}$$

This procedure, although formally identical to the two cases treated above, can really only be viewed as providing global solutions to such generalized systems if certain additional conditions, relating to functional inversion and modularity of the resulting functions, are satisfied. We therefore end this introductory section by posing the following questions:

1. When is the functional inversion $\tau(f) \to f(\tau)$ well-defined?
2. Are there other cases in which the resulting function $f(\tau)$ is a modular function, with known properties, analogous to λ and J?
3. Are there any further generalizations of the above systems admitting solutions in terms of modular functions?

These questions will be addressed in the following sections.

2. Modular solutions of general Halphen systems.

2a. Replicable functions, Hauptmoduls and Schwarzian equations.

A sufficient condition for the existence of a well-defined inverse function $f(\tau)$ is that the (projectivized) monodromy group of the hypergeometric equation in question be a Fuchsian group of the first type. Essentially, this means that the possibly infinite multivaluedness of the functions $\tau(f)$ defined by the ratio of two solutions can be characterized by subdividing its image in the τ–plane into fundamental domains (each having a finite number of sides), which are permuted amongst themselves by the action of the monodromy group, and into which each branch of the function is mapped in a single–valued way. This characterization may be applied not only to hypergeometric equations, but more generally, to Fuchsian differential equations having an arbitrary number of regular singular points, the number coinciding generally with the number of vertices in a fundamental domain. In the case of hypergeometric equations, there are three such singular points, at $(0, 1, \infty)$, and the domains are necessarily triangular.

In this section, a class of modular functions $f(\tau)$ is presented that provide instances of globally defined inverses of the multivalued functions $\tau(f)$ given by ratios of solutions of the hypergeometric equation (1.20) for certain particular parameter values (a, b, c). By "globally", we here understand an open domain of definition with natural boundary, in the interior of which the functions are holomorphic, and beyond which they do not admit analytic continuation. As in the case of the modular functions λ and J, this domain in the first instance consists of the upper half of the complex τ plane, with the real axis as boundary, but upon application of the Möbius transformations (1.6), it may become the interior of any disc. In subsequent sections, examples of functions within the same general class will be given that provide solutions to further generalizations of Halphen systems.

The particular class of functions to be considered here are the *replicable functions* [CN, FMN], which arose originally in the context of "modular moonshine". These functions provide generalizations of the J–function and are similarly defined in the upper half–plane by a normalized q–series ([CN, FMN]

$$F(q) = \frac{1}{q} + \sum_{n=1}^{\infty} a_n q^n, \qquad q := e^{2i\pi\tau} \tag{2.1}$$

satisfying suitable invariance properties under generalized Hecke operators. To relate these series to the functions obtained from solutions of the hypergeometric equations, an affine transformation must be applied

$$F(\tau) = af(\tau) + b, \tag{2.2}$$

with the constants (a, b) chosen so that the values of f at the vertices of the fundamental domains are $(0, 1, \infty)$. The main properties of these functions that are of importance in what follows are:

1. They are *uniformizing functions* for genus zero Riemann surfaces $\mathfrak{H}/\mathfrak{G}_f$ formed by quotienting the upper half–plane $\mathfrak{H}$ by the automorphism group $\mathfrak{G}_f$ of the function. Such functions are referred to as *Hauptmoduls*.
2. The automorphism group $\mathfrak{G}_f$ is a subgroup of $PGL(2, \mathbf{Q})$ that is *commensurable* with the modular group Γ; that is, the intersection $\mathfrak{G}_f \cap \Gamma$ is of finite index in both.

3. In view of the form of the q–series (2.1), each such function has a *cusp* at $\tau = i\infty$.
4. The automorphism group $\mathfrak{G}_f$ in each case contains a subgroup of type

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}),\ c \equiv 0 \bmod N \right\}, \tag{2.3}$$

   with the smallest such N referred to as the *level* of the function (or group).
5. Finally, for the cases to be considered here, the coefficients a_n are all integers (although this is not part of the properties shared by all replicable functions).

A complete list of such replicable functions, together with many of their properties, is provided in refs. [CN, FMN].

The fact that each such function is the generator of the field of meromorphic functions on a genus zero Riemann surface implies that they all satisfy a Schwarzian equation of the same general form as (1.21)

$$\{f,\tau\} + 2R(f)f'^2 = 0, \tag{2.4}$$

where $R(f)$ is *some* rational function. In the case where the fundamental domains are triangular, with vertices mapping to $(0,1,\infty)$ in the f–plane, $R(f)$ will have the form appearing in eq. (1.21) and the functions may be associated to solutions of a corresponding hypergeometric equation. The angles at the vertices of the fundamental domain are determined in terms of the hypergeometric parameters as $(\lambda\pi, \mu\pi, \nu\pi)$. In general, $R(f)$ could have any number of poles at arbitrary locations, but in the case of the replicable functions arising in [CN, FMN], this number never exceeds 25. The remainder of this section concerns only the triangular case; in the following sections, cases with higher numbers of vertices and the corresponding generalized Halphen systems will be discussed.

2b. Triangular replicable functions.

Table 1 below, which is taken from ref. [HM], lists all cases, up to equivalence under Möbius transformations (1.6) and affine transformations of f, of replicable triangular functions with integer a_n's. These are the modular functions whose automorphism groups are the arithmetic triangular groups of noncompact type classified in [Ta].

The first column identifies the groups according to the notation of [CN, FMN], with the integer indicating the level. The second and third columns give the associated hypergeometric parameters and angles at the vertices $(0,1,\infty)$, respectively. The fourth column gives the generators $(\rho_0\ ,\ \rho_1)$ of the automorphism group fixing a pair of finite vertices mapping to the points 0 and 1 in the f–plane. The third generator, stabilizing the point $i\infty$, is

$$\rho_\infty\ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.5}$$

and the three generators together satisfy the consistency relation

$$\rho_\infty \rho_1 \rho_0 = \mathbf{I}. \tag{2.6}$$

The fifth column in the table indicates the relation between the function F normalized as in (2.1) and f normalized as in the hypergeometric equation, so that the vertices are

mapped to $(0, 1, \infty)$. The last column gives explicit expressions for the modular function f in terms of null theta functions or the Dedekind eta function $\eta(\tau)$. In this table, the case $1A$ is just the J function, which therefore determines solutions to the symmetrized Halphen system (1.17), while the case 4C is essentially the function λ (more precisely, composed with the transformation $\tau \to 2\tau$, $\lambda \to 1/\lambda$ as indicated in the sixth column), and hence determines solutions to the original Darboux–Halphen system (1.1). The other cases provide seven further examples of globally defined modular solutions to the general Halphen system (1.24).

Table 1. Triangular Replicable Functions

Name	(a,b,c)	(λ,μ,ν)	ρ_0	ρ_1	F	$f(\tau)$
$1A$ $\sim\Gamma$	$(\frac{1}{12},\frac{1}{12},\frac{2}{3})$	$(\frac{1}{3},\frac{1}{2},0)$	$\begin{pmatrix}0&-1\\1&-1\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$1728f-744$	$J=\frac{(\vartheta_2^8+\vartheta_3^8+\vartheta_4^8)^3}{54\vartheta_2^8\vartheta_3^8\vartheta_4^8}$
$2A$	$(\frac{1}{8},\frac{1}{8},\frac{3}{4})$	$(\frac{1}{4},\frac{1}{2},0)$	$\begin{pmatrix}0&-1\\2&-2\end{pmatrix}$	$\begin{pmatrix}0&-1\\2&0\end{pmatrix}$	$256f-104$	$\frac{\left(\vartheta_3^4+\vartheta_4^4\right)^4}{16\vartheta_2^8\vartheta_3^4\vartheta_4^4}$
$3A$	$(\frac{1}{6},\frac{1}{6},\frac{5}{6})$	$(\frac{1}{6},\frac{1}{2},0)$	$\begin{pmatrix}0&-1\\3&-3\end{pmatrix}$	$\begin{pmatrix}0&-1\\3&0\end{pmatrix}$	$108f-42$	$\frac{\left(\eta^{12}(\tau)+27\eta^{12}(3\tau)\right)^2}{108\eta^{12}(\tau)\eta^{12}(3\tau)}$
$2B$ $\sim\Gamma_0(2)$	$(\frac{1}{4},\frac{1}{4},\frac{1}{2})$	$(\frac{1}{2},0,0)$	$\begin{pmatrix}1&-1\\2&-1\end{pmatrix}$	$\begin{pmatrix}-1&0\\2&-1\end{pmatrix}$	$64f-40$	$1+\frac{1}{64}\left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{24}$ $=\frac{\left(\vartheta_3^4(\tau)+\vartheta_4^4(\tau)\right)^2}{\vartheta_2^8(\tau)}$
$3B$ $\sim\Gamma_0(3)$	$(\frac{1}{3},\frac{1}{3},\frac{2}{3})$	$(\frac{1}{3},0,0)$	$\begin{pmatrix}1&-1\\3&-2\end{pmatrix}$	$\begin{pmatrix}-1&0\\3&-1\end{pmatrix}$	$27f-15$	$1+\frac{1}{27}\left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^{12}$
$4C^*$ $\sim\Gamma_0(4)$	$(\frac{1}{2},\frac{1}{2},1)$	$(0,0,0)$	$\begin{pmatrix}1&-1\\4&-3\end{pmatrix}$	$\begin{pmatrix}-1&0\\4&-1\end{pmatrix}$	$16f-8$	$\frac{1}{\lambda(2\tau)}=\frac{\vartheta_3^4(2\tau)}{\vartheta_2^4(2\tau)}$ $=1+\frac{1}{16}\left(\frac{\eta(\tau)}{\eta(4\tau)}\right)^8$
$2a$	$(\frac{1}{6},\frac{1}{6},\frac{2}{3})$	$(\frac{1}{3},\frac{1}{3},0)$	$\begin{pmatrix}2&-3\\4&-4\end{pmatrix}$	$\begin{pmatrix}0&-1\\4&-2\end{pmatrix}$	$24\sqrt{3}i(2f-1)$	$\frac{\sqrt{3}i\left(e^{\pi i/3}\vartheta_3^4(2\tau)-\vartheta_2^4(2\tau)\right)^3}{9\vartheta_2^4(2\tau)\vartheta_3^4(2\tau)\vartheta_4^4(2\tau)}$
$4a$	$(\frac{1}{4},\frac{1}{4},\frac{3}{4})$	$(\frac{1}{4},\frac{1}{4},0)$	$\begin{pmatrix}4&-5\\8&-8\end{pmatrix}$	$\begin{pmatrix}0&-1\\8&-4\end{pmatrix}$	$-16i(2f-1)$	$-\frac{i\left(\vartheta_3^2(2\tau)+i\vartheta_4^2(2\tau)\right)^4}{8\vartheta_2^4(2\tau)\vartheta_3^2(2\tau)\vartheta_4^2(2\tau)}$
$6a$	$(\frac{1}{3},\frac{1}{3},\frac{5}{6})$	$(\frac{1}{6},\frac{1}{6},0)$	$\begin{pmatrix}6&-7\\12&-12\end{pmatrix}$	$\begin{pmatrix}0&-1\\12&-6\end{pmatrix}$	$6\sqrt{3}i(2f-1)$	$-\frac{\sqrt{3}i\left(\eta^6(2\tau)+3\sqrt{3}i\eta^6(6\tau)\right)^2}{36\eta^6(2\tau)\eta^6(6\tau)}$

Although these cases are distinct, it is important to note that, just as the functions λ and J are related by the rational map defined in (1.13), so all these replicable functions are linked by algebraic relations, some of which may similarly be expressed in terms of explicit rational maps. This implies that the general Halphen–Brioschi variables for the various cases are also algebraically related, just as in the Halphen-Chazy case, where the ones for the J case are essentially the elementary symmetric invariants of those for the λ case. A further illustrative

example of such algebraic relations is given in the next subsection; the full set of rational maps relating various triangular cases are given in [HM], to which the reader is referred for details.

2c. Rational transformations.

If $f(\tau)$ satisfies a Schwarzian equation of the form (1.21) (where $R(f)$ is any rational function) and there is a map

$$f = Q(g), \tag{2.7}$$

relating it to another function $g(\tau)$, where $Q(g)$ satisfies the Schwarzian equation

$$\{Q, g\} + 2R(Q(g))Q'^2 = 2\tilde{R}(g), \tag{2.8}$$

for some function $\tilde{R}(g)$, then g satisfies the transformed Schwarzian equation

$$\{g, \tau\} + 2\tilde{R}(g)g'^2 = 0. \tag{2.9}$$

Also, if $y(f)$ satisfies the second order linear equation

$$\frac{d^2y}{df^2} + R(f)y = 0, \tag{2.10}$$

then $\tilde{y}(g) := (Q')^{-\frac{1}{2}}y(Q(g))$ satisfies the transformed equation:

$$\frac{d^2\tilde{y}}{dg^2} + \tilde{R}(g)\tilde{y} = 0. \tag{2.11}$$

Similar transformations may be applied to second order equations in which first order derivative terms are also present. For the case of hypergeometric equations, such transformations, with $Q(g)$ rational functions of degree up to four, were studied by Goursat [Go] in his thesis. They imply corresponding *symmetrizations* of the associated general Halphen systems, such as the ones relating (1.1) and (1.17). These are discussed in detail in ref. [HM]; here, we present just one illustrative example taken from [HM].

Example 2.1. $3B \mapsto 1A$.

In this case the relevant identity relating the associated hypergeometric functions is

$$F\left(\frac{1}{12}, \frac{1}{12}; \frac{2}{3}; \frac{x(x+8)^3}{64(x-1)^3}\right) = (1-x)^{\frac{1}{4}} F\left(\frac{1}{3}, \frac{1}{3}; \frac{2}{3}; x\right). \tag{2.12}$$

The corresponding rational map $Q(g)$ is therefore defined by

$$f = \frac{g(g+8)^3}{64(g-1)^3}, \tag{2.13}$$

where f denotes the Hauptmodul of case $1A$ (i.e., the modular function J), and g is the Hauptmodul of case $3B$. The relationship between the corresponding automorphism groups is given by a symmetrization quotient which may also be expressed as a quotient of finite groups:

$$\mathfrak{G}_f/\mathfrak{G}_g = \Gamma/\Gamma_0(3) = \mathbf{S}_{1A}^{9B}/\mathbf{S}_{3B}^{9B} = \mathbf{A}_4/\mathbf{Z}_3 \tag{2.14}$$

(The subgroup $\mathfrak{G}_g$ for this case is not normal in $\mathfrak{G}_f$, so the corresponding field extension is non-Galois. The expressions $\mathbf{S}_{1A}^{9B}$, $\mathbf{S}_{3B}^{9B}$ denote the finite groups obtained by quotienting the respective automorphism group of $1A$ and $3B$ by the largest subgroup that is normal in both, which in this case also corresponds to a replicable function, denoted by $9B$, appearing in Table 2 below.) We may associate the following polynomial invariants to this symmetrization:

$$\begin{aligned}\Sigma_1 :=& 3w_1 + 2w_2 + w_3,\\ \Sigma_2 :=& (w_1 - w_3)(9w_1 - 8w_2 - w_3),\\ \Sigma_3 :=& (w_1 - w_3)(27w_1^2 - 36w_1w_2 + 8w_2^2 - 18w_1w_3 + 2w_2w_3 - w_3^2),\end{aligned} \tag{2.15}$$

where (w_1, w_2, w_3) are the Halphen–Brioschi variables for the case $3B$, in terms of which the corresponding variables for the case $1A$, denoted (W_1, W_2, W_3) are determined by

$$\begin{aligned}3W_1 + 2W_2 + W_3 =& \quad \Sigma_1\\ W_1 - W_3 =& -\frac{\Sigma_2^2}{\Sigma_3}\\ W_2 - W_3 =& -\frac{\Sigma_3}{\Sigma_2}.\end{aligned} \tag{2.16}$$

3. Generalized Halphen Systems.

3a. Fuchsian equations, monodromy and automorphism groups.

Up to projective equivalence (i.e., multiplication of solutions by a common function that does not alter the location of the singular points), second order Fuchsian equations may be expressed in the form (2.10), where $R(f)$ is a rational function of the form

$$R(f) = \frac{N(f)}{(D(f))^2}, \qquad D(f) = \prod_{i=1}^{n}(f - a_i), \tag{3.1}$$

and the numerator is a polynomial $N(f)$ of degree $\leq 2n - 2$. The ratio of two linearly independent solutions

$$\tau(f) := \frac{y_1}{y_2} \tag{3.2}$$

then satisfies the Schwarzian differential equation [GS, H]

$$\{\tau, f\} = 2R(f), \tag{3.3}$$

and the inverse function $f = f(\tau)$ (if well defined) satisfies eq. (2.4) with $R(f)$ given by (3.1). The image of the monodromy representation for eq. (2.10) determines a subgroup $\mathfrak{G}_f \subset PGL(2, \mathbf{C})$ that acts on the ratio τ of solutions by linear fractional transformations (1.6) leaving the inverse function $f(\tau)$ invariant.

3b. Constrained Dynamical Systems and Flows on the $SL(2,\mathbf{C})$ Manifold.

Following Ohyama [Oh], we may associate to any such Fuchsian system a dynamical system in $n+1$ variables, subject to $n-2$ independent quadratic constraints. Define the following $n+1$ variables, which serve to generalize the Halphen–Brioschi variables,

$$X_0 := \frac{1}{2}\frac{d}{d\tau}\ln f', \quad X_i := \frac{1}{2}\frac{d}{d\tau}\ln\frac{f'}{(f-a_i)^2}, \quad i=1,\dots n. \tag{3.4}$$

Equivalently, we may use the linear combinations

$$u := X_0 = \frac{1}{2}\frac{f''}{f'}, \quad v_i := \frac{1}{2}(X_0 - X_i) = \frac{1}{2}\frac{f'}{f-a_i}. \tag{3.5}$$

These satisfy the set of quadratic constraints

$$(a_i - a_j)v_iv_j + (a_j - a_k)v_jv_k + (a_k - a_i)v_kv_i = 0, \qquad 1 \le i,j,k \le n \tag{3.6}$$

(of which $n-2$ are independent), and the differential equations:

$$v_i' = -2v_i^2 + 2uv_i, \qquad i=1,\dots n, \tag{3.7a}$$

$$u' = u^2 - \sum_{i,j=1}^{n} r_{ij}v_iv_j, \tag{3.7b}$$

where the coefficients r_{ij} defining the quadratic form

$$r(v) := \sum_{i,j=1}^{n} r_{ij}v_iv_j$$

appearing in (3.7b) are obtained by expressing $R(f)$ in the form

$$R(f) = \frac{1}{4}\sum_{i,j=1}^{n}\frac{r_{ij}}{(f-a_i)(f-a_j)}. \tag{3.8}$$

(This leaves a residual ambiguity in their definition, which just consists of adding any linear combination of the vanishing quadratic forms appearing in (3.6). The definition may be made unique by fixing an ordering for the singular points, and requiring that all coefficients r_{ij} vanish, except when $i=j, j\pm 1$.) The quadratic form $r(v)$ encodes all the relevant information about the monodromy of the associated Fuchsian system and, in the cases where the associated group is modular, about the geometry of the fundamental domains. In particular, the angles at the vertices are $\{\alpha_i\pi\}_{i=1,n}$ where

$$r_{ii} = 1 - \alpha_i^2. \tag{3.9}$$

Equivalently, these systems may be viewed as unconstrained dynamical systems on the $SL(2,\mathbf{C})$ group manifold. To see this, let

$$g(\tau) := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbf{C}) \tag{3.10}$$

denote an integral curve in $SL(2,\mathbf{C})$ for the equation

$$g' = \begin{pmatrix} 0 & \gamma \\ -1 & 0 \end{pmatrix} g, \tag{3.11}$$

where

$$\gamma := -\frac{1}{4C^2} \sum_{i,j=1}^{n} \frac{r_{ij}}{(Ca_i + D)(Ca_j + D)} = -\frac{1}{C^2} R\left(-\frac{D}{C}\right). \tag{3.12}$$

Defining $f(\tau)$ as

$$f := -\frac{D}{C}, \tag{3.13}$$

it follows that this satisfies (2.4), and that

$$u := \frac{A}{C}, \qquad v_{a_i} := \frac{1}{2C(Ca_i + D)}, \qquad i = 1, \dots n, \tag{3.14}$$

satisfy the system (3.7a), (3.7b) and the constraints (3.6). Ohyama's variables (3.4) are recovered by applying g as a linear fractional transformation to $\{\infty, a_1, \dots a_n\}$.

$$X_0 = \frac{A}{C}, \qquad X_i = \frac{Aa_i + B}{Ca_i + D}, \qquad i = 1, \dots n. \tag{3.15}$$

3c. Examples with Four Vertices.

Table 2 below, which is also taken from [HM], lists all cases of replicable functions having integer q–series coefficients, whose fundamental domains have four vertices, and which are related by a rational map of degree ≤ 4 to one of the triangular cases. These functions, composed with the Möbius transformation (1.6), therefore provide the general solutions to generalized Halphen systems of the type (3.6)–(3.7b), with $n = 3$.

Table 2. Four Vertex Replicable Functions

(admitting a rational map of degree ≤ 4 to a triangular replicable function)

Name	(a_1, a_2, a_3)	ρ_1 ρ_2 ρ_3	$\sum_{i,j=1}^{3} r_{ij} v_i v_j$	F	$f(\tau)$
$6C$	$(-3, 0, 1)$	$\begin{pmatrix} 3 & -2 \\ 6 & -3 \end{pmatrix}$ $\begin{pmatrix} 3 & -1 \\ 12 & -3 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix}$	$\frac{3}{4}v_1^2 + \frac{3}{4}v_2^2 + v_3^2$ $-\frac{1}{2}v_2 v_3 - v_1 v_3$	$4f + 2$	$1 + \frac{1}{4}\frac{\eta^6(\tau)\eta^6(3\tau)}{\eta^6(2\tau)\eta^6(6\tau)}$
$6D$	$(\beta, \bar{\beta}, 1)$ $\beta := -\frac{3}{4} + \sqrt{2}i$	$\begin{pmatrix} 4 & -3 \\ 6 & -4 \end{pmatrix}$ $\begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix}$	$\frac{3}{4}v_1^2 + \frac{3}{4}v_2^2 + v_3^2$ $+\frac{131}{162}v_1 v_2$ $-\frac{28-16\sqrt{2}i}{81}v_1 v_3$ $-\frac{28+16\sqrt{2}i}{81}v_2 v_3$	$4f$	$1 + \frac{1}{4}\frac{\eta^4(\tau)\eta^4(2\tau)}{\eta^4(3\tau)\eta^4(6\tau)}$
$6E$ $\sim \Gamma_0(6)$	$(-\frac{1}{8}, 0, 1)$	$\begin{pmatrix} 5 & -3 \\ 12 & -7 \end{pmatrix}$ $\begin{pmatrix} 5 & -2 \\ 18 & -7 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix}$	$v_1^2 + v_2^2 + v_3^2$ $-\frac{10}{9}v_2 v_3 - \frac{8}{9}v_1 v_3$	$8f - 3$	$1 + \frac{1}{8}\frac{\eta^5(\tau)\eta(3\tau)}{\eta(2\tau)\eta^5(6\tau)}$
$6c$	$(-1, 1, 0)$	$\begin{pmatrix} 8 & -7 \\ 12 & -10 \end{pmatrix}$ $\begin{pmatrix} 2 & -1 \\ 12 & -4 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 12 & -1 \end{pmatrix}$	$\frac{8}{9}v_1^2 + \frac{8}{9}v_2^2 + v_3^2$ $+\frac{16}{9}v_1 v_2$	$i3\sqrt{3}f$	$-\frac{i}{3\sqrt{3}}\frac{\eta^6(2\tau)}{\eta^6(6\tau)}$
$8E$ $\sim \Gamma_0(8)$	$(-1, 0, 1)$	$\begin{pmatrix} 3 & -2 \\ 8 & -5 \end{pmatrix}$ $\begin{pmatrix} 3 & -1 \\ 16 & -5 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 8 & -1 \end{pmatrix}$	$v_1^2 + v_2^2 + v_3^2$ $-2v_1 v_3$	$4f$	$1 + \frac{1}{4}\frac{\eta^4(\tau)\eta^2(4\tau)}{\eta^2(2\tau)\eta^4(8\tau)}$ $= \frac{\vartheta_3^2(2\tau)+\vartheta_4^2(2\tau)}{\vartheta_3^2(2\tau)-\vartheta_4^2(2\tau)}$
$9B$ $\sim \Gamma_0(9)$	$(\omega, \bar{\omega}, 1)$ $\omega := e^{\frac{2\pi i}{3}}$	$\begin{pmatrix} 5 & -4 \\ 9 & -7 \end{pmatrix}$ $\begin{pmatrix} 2 & -1 \\ 9 & -4 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 9 & -1 \end{pmatrix}$	$v_1^2 + v_2^2 + v_3^2$ $-v_1 v_2$ $-(1-\omega)v_1 v_3$ $-(1-\bar{\omega})v_2 v_3$	$3f$	$1 + \frac{1}{3}\frac{\eta^3(\tau)}{\eta^3(9\tau)}$

The first column again gives the label of the associated automorphism group $\mathfrak{G}_f$ for each function $f(\tau)$ in the notation of [CN, FMN], while the second specifies the location of the finite singular points (a_1, a_2, a_3) of the associated Fuchsian equation (2.10). The third column gives the automorphism group generators stabilizing three finite vertices mapping to (a_1, a_2, a_3), and the fourth gives the quadratic form $r(v)$ appearing in eq. (3.7b). The fifth column lists the affine relations connecting the normalized q–series of the form (2.1) with the functions $f(\tau)$ mapping the vertices to the singular points (a_1, a_2, a_3) of (2.10), and the last column gives explicit expressions for $f(\tau)$ in terms of the Dedekind η–function.

There are also algebraic relations connecting these dynamical systems with triangular cases, following from the existence of rational maps satisfying (2.7)–(2.11). The following

example is an illustration of such rational maps and the implied relations between the generalized Halphen–Brioschi variables. (Further details and a list of the other cases may be found in [HM].)

Example 3.1. $8E \mapsto 4C$.

The rational map (2.7) relating these two cases is given by

$$f = \frac{(g+1)^2}{4g}, \tag{3.16}$$

where f denotes the Hauptmodul for the case $4C$ and g the one for $8E$. The automorphism group for $4C$ is $\Gamma_0(4)$, and that for $8E$ is $\Gamma_0(8)$, a normal subgroup, and therefore in this case the function field generated by g is a Galois extension of the one generated by f. The map is of degree two and the quotient group characterizing the symmetrization is

$$\mathbf{S}_{4C}^{8E} = \Gamma_0(4)/\Gamma_0(8) = \mathbf{Z}_2, \tag{3.17}$$

whose action is generated by

$$\tau \mapsto \frac{-\tau}{4\tau - 1}, \qquad g \mapsto \frac{1}{g}. \tag{3.18}$$

The effect of this on the generalized Halphen–Brioschi variables is

$$(u, v_{-1}, v_0, v_1) \mapsto (u - 2v_0, v_1 - v_0, -v_0, v_1 - v_0), \tag{3.19}$$

and the associated polynomial invariants are

$$\Sigma_1 := u + v_{-1} - v_0 - v_1, \quad \Sigma_1' := 4v_1 - 2v_0, \quad \Sigma_2 := 4v_0^2. \tag{3.20}$$

The Halphen–Brioschi variables for the $4C$ case are determined in terms of these by

$$W_1 = \Sigma_1, \quad W_1 - W_3 = \frac{\Sigma_2}{\Sigma_1'}, \quad W_2 - W_3 = \Sigma_1'. \tag{3.21}$$

3d. Two examples with 26 ***vertices.***

In the list of replicable functions with integer q–series coefficients, [CN, FMN], there are three cases, denoted $72e$, $96a$ and $144^{\sim}e$, for which the fundamental domains have 26 vertices, the maximal number occurring. We give here the corresponding generalized Halphen–Brioschi variables, and the quadratic forms determining the associated constrained dynamical system for the cases $72e$ and $96a$. (The case $144^{\sim}e$ is related to $72e$ by multiplication of f by $e^{\frac{i\pi}{12}}$.)

Example 3.2. $72e$. The modular function for this case may be expressed as follows in terms of the Dedekind η–function

$$f = \frac{\eta(24\tau)\eta(36\tau)}{\eta(12\tau)\eta(72\tau)}. \tag{3.22}$$

The fundamental domain has 25 finite vertices, and the rational function $R(f)$ to whose poles these are mapped is

$$R(f) = \frac{1}{4f^2}\left(1 + \frac{2^7 3^3 f^{12}(f^{12}+1)^2}{(f^{24} - 34f^{12} + 1)^2}\right). \tag{3.23}$$

The poles are located at the origin, and at the vertices of two regular dodecagons centered at the origin, at radial distances $(\sqrt{2}\pm 1)^{\frac{1}{3}}$, forming angles that are multiples of $\pi/12$ with the axes:

$$a_0 := 0, \quad a_m := e^{\frac{(m-1)\pi i}{6}}(\sqrt{2}-1)^{\frac{1}{3}}, \quad a_{12+m} := e^{\frac{(m-1)\pi i}{6}}(\sqrt{2}+1)^{\frac{1}{3}}, \qquad m = 1,\dots 12. \tag{3.24}$$

The quadratic form $r(v)$ defining the dynamical system in this case is

$$\sum_{i,j=1}^{n} r_{ij}v_iv_j = v_0^2 + \frac{3}{4}\sum_{m=1}^{24} v_m^2 - \frac{3}{8}\sum_{m=1}^{11}(1-e^{\frac{m\pi i}{6}})\big((2-\sqrt{2})v_mv_{m+1} + (2+\sqrt{2})v_{12+m}v_{13+m}\big). \tag{3.25}$$

Example 3.3. $96a$. For this case, the modular function may be expressed as

$$f = \frac{\eta^2(48\tau)}{\eta(24\tau)\eta(96\tau)}, \tag{3.26}$$

and the rational function $R(f)$ is

$$R(f) = \frac{1}{4f^2}\left(1 + \frac{2^{10}3^3f^{24}}{(f^{24}-2^6)^2}\right). \tag{3.27}$$

The poles are therefore located at the origin, and at the vertices of a regular 24–gon centered at the origin, at radial distance $2^{\frac{1}{4}}$, forming angles that are multiples of $\pi/24$ with the axes:

$$a_0 := 0, \quad a_m := e^{\frac{(m-1)\pi i}{12}}2^{\frac{1}{4}}, \qquad m = 1,\dots 24. \tag{3.28}$$

The quadratic form $r(v)$ defining the dynamical system for this case is

$$\sum_{i,j=1}^{n} r_{ij}v_iv_j = v_0^2 + \frac{3}{4}\sum_{m=1}^{24} v_m^2 - \frac{3}{4}\sum_{m=1}^{23}(1-e^{\frac{m\pi i}{12}})v_mv_{m+1}. \tag{3.29}$$

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