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Abstract

We introduce the Schlesinger transformations of the Gambier equation. The latter can be written, in both the continuous and discrete cases, as a system of two coupled Riccati equations in cascade involving an integer parameter n . In the continuous case the parameter appears explicitly in the equation while in the discrete case it corresponds to the number of steps for singularity confinement. Two Schlesinger transformations are obtained relating the solutions for some value n to that corresponding to either $n + 1$ or $n + 2$.

Résumé

Nous introduisons les transformations de Schlesinger pour l'équation de Gambier. Cette dernière peut s'écrire, dans les deux cas discrets et continus, comme un système couplé de deux équations de Riccati en cascade comprenant un paramètre entier n . Dans le cas continu, le paramètre apparaît explicitement dans l'équation alors que dans le cas discret il correspond au nombre de pas pour le confinement de la singularité. Deux transformations de Schlesinger sont obtenues reliant les solutions pour une certaine valeur n à celles correspondant à $n + 1$ ou $n + 2$.

1. INTRODUCTION

The existence of Schlesinger transformations is one of the very special properties of Painlevé equations [1]. These transformations are a particular kind of auto-Bäcklund transformations [2]. The latter relate a solution of a given equation to a solution of the same equation but corresponding to a different set of parameters. Schlesinger transformations do just that but the changes of parameters correspond to integer or half-integer shifts in the monodromy exponents. Both continuous and discrete (whether difference or multiplicative) Painlevé equations have been shown to possess Schlesinger transformations [3]. For the discrete case (and in particular for q -Painlevé's) the relation of Schlesinger transformations to monodromy exponents is not quite clear and their derivation requires both experience and intuition, in particular in the choice of the proper parameters. With this minor *caveat* we hasten to say that we do possess a systematic approach to the construction of Schlesinger transformations. It is based on the bilinear formalism [4] which can be used to construct Miura transformations [5], the iteration of which can lead to auto-Bäcklund's.

The logical conclusion of the above introduction is that Schlesinger transformations should exist only for Painlevé equations. This is almost true with one exception. As we shall report in this paper there exists one equation which, without being a Painlevé, does possess Schlesinger's. This equation is known under the name of Gambier (who first derived it): it is the most general second-order ODE of linearisable type [6]. What makes possible the existence of Schlesinger transformations for this equation is the fact that its general expression involves an arbitrary integer n . It turns out that we can relate the solution of the equation for some value of n to that corresponding to $n + 2$. These transformations are, for the linearisable case, the analogues of the Schlesinger's. Moreover, as we shall show, the same procedure can be followed in the discrete case i.e. for the Gambier mapping [7,8].

In section 2 we shall review some basic facts about the Gambier equation. The Schlesinger transformations will be given in section 3 while section 4 is devoted to the study of the discrete case.

2. THE GAMBIER EQUATION

The Gambier equation is given as a system of two Riccati equations in cascade. This means that we start with a first Riccati for some variable y

$$y' = -y^2 + by + c \quad (2.1)$$

and then couple its solution to a second Riccati by making the coefficients of the latter depend explicitly on y :

$$x' = ax^2 + nxy + \sigma. \quad (2.2)$$

The precise form of the coupling introduced in (2.2) is due to integrability requirements. In fact, the application of singularity analysis shows that the Gambier system cannot be integrable unless the coefficient of the xy term in (2.2) is an integer n . This is not the only integrability requirement. Depending on the value of n one can find constraints on the a, b, c, σ (where the latter is traditionally taken to be constant 1 or 0) which are necessary for integrability.

The common lore [9] is that out of the functions a, b, c two are free. This turns out not to be the case. The reason for this is that the system (2.1-2) is not exactly canonical i.e. we have not used all possible transformations in order to reduce its form. We introduce a change of independent variable from t to T through $dt = gdT$ where g is given by $\frac{1}{g} \frac{dg}{dt} = b \frac{n}{2-n}$, a gauge through $x = gX$ and also $Y = gy - \frac{1}{n} \frac{dg}{dt}$. The net result is that system (2.1-2) reduces to one where $b = 0$ while σ remains equal to 0 or 1. It is clear from the equations above that n must be different from 2. On the other hand when $n = 2$ the integrability condition, if $\sigma = 1$, is precisely $b = 0$. So we can always take $b = 0$. (As a matter of fact in the case $\sigma = 0$ an additional gauge freedom allows us to take both b and c to zero for all n , even for $n = 2$). Thus the Gambier system can be written in full generality

$$y' = -y^2 + c \quad (2.3a)$$

$$x' = ax^2 + nxy + \sigma. \quad (2.3b)$$

One further remark is in order here. The system (2.3) retains its form under the transformation $x \rightarrow 1/x$. In this case $n \rightarrow -n$ and σ and $-a$ are exchanged. Thus in some cases it will be interesting to consider a Gambier system where σ is not constant but rather a function of t . Still, it is possible to show that we can always reduce this case to one where $\sigma = 1$, while preserving the form of (2.3a) i.e. $b = 0$. To this end we introduce the change of variables $dt = h dT$, $x = gX$ and $Y = hy - \frac{1}{2} \frac{dh}{dt}$ with $h = \sigma^{2/(n-2)}$, $g = \sigma^{n/(n-2)}$. With these transformations system (2.3) reduces to one with $\sigma = 1$ and $b = 0$. (In the special case $n = 2$, with $b = 0$ integrability implies $\sigma = \text{constant}$, whereupon its value can always be reduced to 1).

3. SCHLESINGER TRANSFORMATIONS FOR THE GAMBIER EQUATIONS

The theory of auto-Bäcklund transformations of Painlevé equations is well established. As was shown in [2] the general form of auto-Bäcklund transformations for most Painlevé equations is of the form:

$$\tilde{x} = \frac{\alpha x' + \beta x^2 + \gamma x + \delta}{\epsilon x' + \zeta x^2 + \eta x + \theta}. \quad (3.1)$$

In the case of the Gambier equation considered as a coupled system of two Riccati's it is more convenient to look for an auto-Bäcklund of the form:

$$\tilde{x} = \frac{\alpha xy + \beta x + \gamma y + \delta}{(\zeta y + \eta)(\theta x + \kappa)}. \quad (3.2)$$

with a factorized denominator, with hindsight from the discrete case. We require that the equation satisfied by \tilde{x} do not comprise terms nonlinear in y . We examine first the case $\zeta \neq 0$ and reach easily the conclusion that there exists no solution. So we take $\zeta = 0$, $\eta = 1$ which implies that α and γ do not both vanish (otherwise (3.2) would have been independent of y). We find in this case $\alpha = 0$ and thus the general form of the auto-Bäcklund can be written as:

$$\tilde{x} = \frac{\beta x + \gamma y + \delta}{\theta x + \kappa}. \quad (3.3)$$

>From (3.3) we can obtain the two possible forms of the Gambier system auto-Bäcklund:

$$\tilde{x} = \beta x + \gamma y + \delta \quad (3.4)$$

$$\tilde{x} = \frac{\beta x + \gamma y + \delta}{x + \kappa}. \quad (3.5)$$

As we shall see in what follows both forms lead to Schlesinger transformations.

Let us first work with form (3.4). Our approach is straightforward. We assume (3.4) and require that \tilde{x} satisfy an equation of the form (2.3b) while y is always the same solution of (2.3a). The calculation is easily performed leading to:

$$\tilde{x} = \gamma y + \frac{a\gamma}{n+1}x + \frac{\gamma'}{n}, \quad (3.6)$$

where γ satisfies:

$$\frac{\gamma'}{\gamma} = \frac{n}{n+2} \frac{a'}{a}. \quad (3.7)$$

Here we have assumed $a \neq 0$ otherwise \tilde{x} does not depend on x and (3.6) does not define a Schlesinger. The parameters of the equation satisfied by \tilde{x} are given (in obvious notations) by:

$$\tilde{n} + n + 2 = 0 \quad (3.8a)$$

$$\tilde{a} = \frac{n+1}{\gamma} \quad (3.8b)$$

and

$$\tilde{\sigma} = \gamma \left(c + \frac{a\sigma}{n+1} + \frac{1}{n+2} \frac{a''}{a} - \frac{n+3}{(n+2)^2} \frac{a'^2}{a^2} \right). \quad (3.8c)$$

Thus (3.6) is indeed a Schlesinger transformation since it takes us from a Gambier system with parameter n to one with parameter $\tilde{n} = -n - 2$. It suffices now to invert \tilde{x} in order to obtain an equation with parameter $N = n + 2$. Expressions (3.6) and (3.8) can be written in a more symmetric way:

$$\tilde{a}\tilde{x} - ax = (n+1)\left(y - \frac{a'}{\tilde{n}a}\right) \quad (3.9)$$

and

$$\begin{aligned} \tilde{n} + 1 &= -(n+1) \\ \tilde{n} \frac{\tilde{a}'}{\tilde{a}} &= n \frac{a'}{a} \\ \tilde{a}\tilde{\sigma} - a\sigma &= (n+1) \left(c - \frac{1}{\tilde{n}} \left(\frac{a'}{a} \right)' + \frac{1}{\tilde{n}^2} \frac{a'^2}{a^2} \right). \end{aligned} \quad (3.10)$$

The inverse transformation can be easily obtained if we introduce $\tilde{\gamma}$ such that $a\tilde{\gamma} = -(n+1) = -\tilde{a}\tilde{\gamma}$. We thus find

$$x = y\tilde{\gamma} + \frac{\tilde{a}\tilde{\gamma}}{\tilde{n}+1}\tilde{x} + \frac{\tilde{\gamma}'}{\tilde{n}} \quad (3.11)$$

and the relations (3.10) are still valid.

Iterating the Schlesinger transformations one can construct the integrable Gambier systems for higher n 's and obtain by construction the functions which appear in them. However it may happen that when we implement the Schlesinger we find $\tilde{\sigma} = 0$. If we invert x we get a system with $N = -\tilde{n} = n + 2$ but $A = 0$ for which one cannot iterate the Schlesinger further.

Let us give example of the application of this Schlesinger transformation. Let us start from $n = 0$, in which case we find $\tilde{n} = -2$ and, after inversion, $N = 2$. For $n = 0$ we start from $a = -1$ and $\sigma = 0$ or 1 (always possible through the appropriate changes of variable). This leads to $\tilde{a} = -1$, $\tilde{\sigma} = -c + \sigma$ and the Schlesinger reads: $\tilde{x} = -y + x$. Next we invert \tilde{x} and have $X = 1/(x - y)$. We find thus that the Schlesinger takes us from

$$\begin{aligned} y' &= -y^2 + c \\ x' &= -x^2 + \sigma \end{aligned} \quad (3.12)$$

to the system

$$\begin{aligned} y' &= -y^2 + c \\ X' &= AX^2 + 2Xy + \end{aligned} \quad (3.13)$$

with $A = c - \sigma$, $= 1$. In the particular case $n = 2$, a change of variables exists which allows us to put $A = -1$ (unless $A = 0$), without introducing b in the equation for y , while keeping $= 1$ and changing only the value of c . Thus the generic case of the Gambier equation for $n = 2$ can be written with $A = -1$. Eliminating y between the two equations we find:

$$x'' = \frac{x'^2}{2x} - 2xx' - \frac{x^3}{2} - \frac{1}{2x} + (2c + 1)x. \quad (3.14)$$

This is the generic form of the $n = 2$ Gambier equation and it contains just one free function. The nongeneric cases corresponding to $A = 0$ and $\sigma = 0$ or 1 can be constructed in an analogous way.

We now turn to the second Schlesinger transformation corresponding to the form (3.5). As we shall show, a Schlesinger transformation of this form does indeed exist and corresponds to changes in n with $n = 1$. Let us start from the basic equations (2.3). Next we ask that \tilde{x} defined by (3.5) indeed satisfy a system like (2.3). We find thus that $\kappa = -x_0$ and γ must be given by:

$$\frac{\gamma'}{\gamma} = y_0 + \frac{2ax_0}{n+1} \quad (3.15)$$

where y_0 is a solution of the Riccati (2.3a) and x_0 a solution of (2.3b), obtained with y replaced by y_0 . We introduce the quantities $\tilde{x}_0 = \frac{ax_0}{n+1}$, $\tilde{a} = -\frac{nx_0}{\gamma}$. In this case (3.15) becomes:

$$\frac{\gamma'}{\gamma} = y_0 + \frac{2\tilde{a}\tilde{x}_0}{\tilde{n}+1} = y_0 + \frac{2x_0\tilde{x}_0}{\gamma}, \quad (3.16)$$

where

$$\tilde{n} + n + 1 = 0. \quad (3.17)$$

We have thus, starting from a generic solution x, y of (2.3) for some n , the Schlesinger:

$$\tilde{x} = \tilde{x}_0 + \frac{\gamma(y - y_0)}{x - x_0} \quad (3.18)$$

where \tilde{x} is indeed a solution of (2.3) for $\tilde{n} = -n - 1$ for the same y

$$\tilde{x}' = -\tilde{a}\tilde{x}^2 + \tilde{n}\tilde{x}y + \tilde{\sigma} \quad (3.19)$$

where \tilde{a} has been defined as $-\frac{nx_0}{\gamma}$ and

$$\tilde{\sigma} = \frac{\gamma}{n+1} \left(a' + a^2 x_0 \frac{n+2}{n+1} + ay_0(n+2) \right). \quad (3.20)$$

Note that \tilde{x}_0 is a solution of the same equation with y replaced by y_0 . As in the previous case if we invert \tilde{x} we obtain an equation corresponding to $N = n + 1$.

As an application of the $n = 1$ Schlesinger we are going to construct the $n = 1$ equation starting from the $n = 0$ case, i.e. system (2.3) with $n = 0$. From (3.18), the Schlesinger reads

$$\tilde{x} = \gamma \left(a + \frac{y - y_0}{x - x_0} \right) \quad (3.21)$$

where γ satisfies the differential equation

$$\frac{\gamma'}{\gamma} = y_0 + 2ax_0 \quad (3.22)$$

and where y_0 is a particular of (2.3b) and x_0 is a solution of (2.3a) with $n = 0$. Then \tilde{x} satisfies a Gambier equation in which $n = -1$, $\tilde{\sigma} = \gamma(a' + 2a^2x_0 + 2ay_0)$ and $\tilde{a} = 0$. To obtain the $n = 1$ Gambier equation, we define $X = 1/\tilde{x}$ and we arrive at the following system

$$\begin{aligned} y' &= -y^2 + c \\ X' &= AX^2 + Xy \end{aligned} \quad (3.23)$$

where $A = -\tilde{\sigma}$. The system (3.23) is the generic $n = 1$ case since, in this case, the condition for (2.3) to be integrable is $\sigma = 0$.

It is worth pointing out here that the Schlesinger transformation corresponding to $n = 2$ was known to Gambier himself. As a matter of fact when faced with the problem of determining the functions appearing in his equation so as to satisfy the integrability requirement, Gambier proposed a recursive method which is essentially the Schlesinger $n = 2$. On the other hand the Schlesinger $n = 1$ is quite new and we have first discovered it in the discrete case whereupon we looked for (and found) its continuous analogue.

4. THE GAMBIEB MAPPING

The Gambier equation has been examined already in [7,8] and its discrete equivalent has been proposed there. These constructions of the Gambier mapping were *ad hoc* ones, in the sense that we assumed a form and implemented the singularity confinement discrete integrability criterion in order to obtain the integrability conditions. In what follows, we shall use a slightly different approach based on the singularity structure.

Our starting point is the discrete equivalent of the system (2.3). We have thus one equation which is the discrete analogue of the Riccati, i.e. a homographic mapping for y and another homographic mapping for x , the coefficients of which depend linearly on y . Our derivation will be based on the study of singularities of the system. The general homographic equation for y involves three free parameters, but since we have the freedom of choice of a homographic transformation on y , we can always reduce it to $y = \text{constant}$ (i.e. $\bar{y} = y$). However, in the system under study, our aim is to study the singularities of x induced by *special* values of y . One could choose the singularity to enter at point n_0 if the value of y has some special value depending on n_0 , say, $f(n_0)$. This would introduce one function in the homographic mapping. However what is even more convenient is to decide what the special value of y is, say $y = 0$ for all n , at the price of the loss of part of the homographic freedom. Then the special value 0 will occur for some n depending on the particular solution. Of course, if we allow the full homographic freedom we are back to the starting point i.e. with three free functions. However we decide to have only one free function and thus we simplify the mapping by choosing its form so that it presents the pattern $\{-1, 0, 1\}$. This fixes two of the functions and the result is:

$$\bar{y} = \frac{y + c}{y + 1} \quad (4.1)$$

where c is a function of n and we use the notations $y = y(n)$, $\bar{y} = y(n + 1)$.

Next, we turn to the equation for x . This equation is homographic in x . However we require that when y takes the value 0, the resulting value of x be ∞ . Thus the denominator must be proportional to y , and since we can freely translate x we can reduce its form to just xy . The remaining overall gauge factor is chosen so as to put the coefficient of xy of the numerator to unity resulting to the following mapping:

$$\bar{x} = \frac{x(y - r) + q(y - s)}{xy}. \quad (4.2)$$

The system (4.1-2) is a discrete form of the Gambier system. In order to study the confinement of the singularity induced by $y = 0$ we introduce the auxiliary quantity ψ_N which is the N 'th iterate of $y = 0$ in equation (4.1), N times downshifted. Thus $\psi_0 = 0$, $\psi_1 = c$, $\psi_2 = \frac{c+c}{c+1}$, etc... The confinement requirement is that after N steps x becomes 0 in such a way as to lead to $0/0$ at the next step. Thus the mapping (4.2) has in fact the form:

$$\bar{x} = \frac{x(y - r) + q(y - \psi_N)}{xy}. \quad (4.3)$$

Thus when at some step N we have $y = \psi_N$ and $x = 0$. On the view of (4.3) \bar{x} will then be indeterminate of the form $0/0$. However it turns out that, in fact, this value is well-determined and finite. Let us take a closer look at the conditions for confinement. The generic patterns for x and y are:

$$\begin{array}{l} y : \{ \quad \quad 0 \quad \quad \bar{\psi}_1 \quad \quad \bar{\bar{\psi}}_2 \quad \quad \dots \quad \quad \bar{\bar{\bar{\psi}}}_N \quad \quad \} \\ x : \{ \text{free} \quad \quad \infty \quad \quad \frac{\bar{\psi}_1 - \bar{r}}{\bar{\psi}_1} \quad \quad \dots \quad \quad 0 \quad \quad \text{free} \quad \} \end{array}$$

At $N = 1$ it is clearly impossible to confine with a form (4.3) since we do not have enough steps. In this case the only integrable form of the x -equation is a linear one. The first genuinely confining case of the form (4.3) is $N = 2$. From the requirement $\bar{\bar{x}} = 0$ we have $r = \psi_1$ and q free: this is indeed the only integrability condition. For higher N 's we can similarly obtain the confinement condition which takes the form of an equation for r in terms of q .

At this point it is natural to ask whether the mapping (4.1)-(4.3) does indeed correspond to the Gambier equation (2.3). In order to do this we construct its continuous limit. We first introduce:

$$\begin{aligned} c &= \epsilon^2 D \\ y &= \frac{\epsilon D}{Y + H} \end{aligned} \quad (4.4)$$

with $H \approx D'/(2D)$ and obtain the continuous limit of (4.1) for $\epsilon \rightarrow 0$. We find as expected

$$Y' = -Y^2 + C \quad (4.5)$$

i.e. eq. (2.3a), where $C = D - \frac{D''}{2D} + \frac{3}{4} \frac{D'^2}{D^2}$. Using (4.4) and (4.1) we can also compute ψ_N and we find at lowest order:

$$\psi_N = \epsilon^2 N \quad \text{with} \quad N \approx N(D - \epsilon \frac{N+1}{2} D') + \epsilon^2 N. \quad (4.6)$$

where N is an explicit function of D depending on N .

Next we turn to the equation for x and introduce:

$$\begin{aligned} r &= \epsilon^2 R \\ x &\approx \frac{1}{2} + \frac{\epsilon}{2X} - \epsilon \frac{RD'}{4D^2} \\ q &\approx -\frac{1}{4} + \epsilon^2 Q \end{aligned} \quad (4.7)$$

and for the continuous limit of the form (2.3b) to exist in canonical form (i.e. $b=0, \sigma=1$) we find that we must have

$$R \approx \frac{ND}{2} - \epsilon(N+2) \frac{ND'}{8}. \quad (4.8)$$

This leads to the equation for x :

$$X' = AX^2 + NXY + 1 \quad (4.9)$$

with $A = \frac{N}{4}(N/4 + 1) \frac{D'^2}{D^2} - \frac{ND''}{4D} - 4Q$. Moreover the confinement constraint implies a differential relation between D and Q which depends on N . We can verify explicitly in the first few cases that this is indeed the integrability constraint obtained in the continuous case. For instance, for $N = 2$, just imposing (4.8) in order to have the canonical form $b=0, \sigma=1$, is sufficient for integrability.

Once the singularity pattern of the Gambier mapping is established we can use it in order to construct the Schlesinger transformation. Let us first look for a transformation that corresponds to $N = 2$. The idea is that given the N -steps singularity pattern of the equation for x we introduce a variable w with $N + 2$ singularity steps where we enter the singularity one step before x and exit it one step later. The general form of the Schlesinger transformation, which defines w , is:

$$w = X \frac{y - \psi_{N+1}}{y}, \quad (4.10)$$

where X is homographic in x . The presence of the y and $y - \psi_{N+1}$ terms is clear: they ensure that w becomes infinite one step before x , and vanishes one step after x . Next we turn to the determination of X . Since X is homographic in x we can rewrite (4.10) as:

$$w = \frac{\alpha x + \beta}{y} \frac{y - \psi_{N+1}}{\gamma x + \delta}. \quad (4.11)$$

Our requirement is that w becomes infinite when $y = 0$ for every value of x . This statement must be qualified. The numerator $\alpha x + \beta$ will vanish for some x (namely $x = -\beta/\alpha$) so this value of x must be the only one which should *not* occur in the confined singularity. Indeed there is a unique value of x where instead of being confined, the singularity extends to infinity in *both* directions of the independent variable n , while the only nonsingular values of the dependent variable occur in a finite range. The value of x such that \bar{x} is finite and free even though y is zero is such that the numerator $-xr - q\psi_N$ of \bar{x} vanishes. For this value of x , the values of the dependent variable are fixed for $n \leq 0$ and $n \geq N + 1$ and the value can be considered as 'forbidden'. Thus $\alpha x + \beta = xr + q\psi_N$ up to a multiplicative constant. Similarly when $y = \psi_{N+1}$, w must vanish. Thus $\gamma x + \delta$ must not be zero except for the unique value of x that does not occur in the confined singularity. Note that $y = \psi_{N+1}$ means $\underline{y} = \underline{\psi}_N$ and the only value of x that comes from a nonzero \underline{x} in that case is $x = (\underline{\psi}_N - \underline{r})/\underline{\psi}_N$. In that case the values of the dependent variable are fixed for $n \geq 0$ and $n \leq -N - 1$. This value of x being 'forbidden', $\gamma x + \delta$ must be proportional to $\underline{\psi}_N x - (\underline{\psi}_N - \underline{r})$. We now have the first form of the Schlesinger:

$$w = \frac{xr + q\psi_N}{y} \frac{y - \psi_{N+1}}{\underline{\psi}_N x + \underline{r} - \underline{\psi}_N} \quad (4.12)$$

where the proportionality constant has been taken equal to 1 (but any other value would have been equally acceptable). Here w effectively depends on x unless $r(\underline{r} - \underline{\psi}_N) = q\underline{\psi}_N \psi_N$. But in this case the mapping (4.3) is in fact linear in the variable $\xi = (x - 1 + \bar{r}/\bar{\psi}_N)^{-1}$. This case is the analog of the case $a = 0$ in the continuous case where the Schlesinger does not exist.

Let us give an application of the Schlesinger transformation by obtaining the $N = 2$ equation starting from $N = 0$. We have always the equation for y which reads:

$$\bar{y} = \frac{y + c}{y + 1} \quad (4.13)$$

and $\psi_0 = 0$, $\psi_1 = \underline{c}$. For $N = 0$ the equation for x reads:

$$\bar{x} = \frac{x(y - r) + qy}{xy} = \frac{x + q}{x} \quad (4.14)$$

since for integrability $r = 0$ and indeed $N = 0$ means that the x equation does *not* depend on y . We introduce the Schlesinger:

$$w = x \frac{y - \underline{c}}{y} \quad (4.15)$$

(where we first write (4.12) for arbitrary r and since r factors we take the limit $r \rightarrow 0$ afterwards). Using (4.14) and (4.15) to eliminate x we obtain the equation for w :

$$\bar{w} = (1 - c) \frac{yw + q(y - \underline{c})}{(y + c)w}. \quad (4.16)$$

This equation is of the form (4.3) but not quite canonical. We can transform it to canonical form simply by introducing \bar{y} instead of y because indeed w is infinite one step before x , so $w = \infty$ means $\bar{x} = \infty$ i.e. $\bar{y} = 0$. We obtain thus:

$$\bar{w} = \frac{w(\bar{y} - c) + q(1 + \underline{c})(\bar{y} - \bar{\psi}_2)}{\bar{y}w} \quad (4.17)$$

with $\bar{\psi}_2 = (c + \underline{c})/(1 + \underline{c})$ which coupled to (4.13) is indeed a $N = 2$ Gambier mapping.

As we pointed out above the $N = 1$ case is not included in the parametrisation (4.1-4.3): the x -mapping must be linear in order to ensure integrability. Thus we are led to study the linear case separately. For an arbitrary N , the general form of the linear x -mapping can be obtained using confinement arguments in a way similar to what we did for the generic, nonlinear, case. We obtain

$$\bar{x} = \frac{x(y - \psi_N) + g}{y} \quad (4.18)$$

where g is free. The Schlesinger transformation is again given by:

$$w = X \frac{y - \psi_{N+1}}{y} \quad (4.19)$$

and arguments similar to those of the nonlinear case allow us to determine the form of the homographic object X leading to:

$$w = \frac{x\psi_N - g}{y} \frac{y - \psi_{N+1}}{x\underline{\psi}_N - \underline{g}}. \quad (4.20)$$

Thus one can perform a Schlesinger in the linear case. This is not in disagreement with the continuous case. It is, in fact, the analog of the case where $\sigma = 0$ but $a \neq 0$ (which is linear in $1/x$) for which the Schlesinger *can* be performed. The analog of the case $\sigma = 0$ and $a = 0$ is the situation when $g = k\psi_N$ with constant k in which case the mapping rewrites $\bar{\xi} = \xi(y - \psi_N)/y$ with $\xi = x - k$. Then w does not depend on ξ (or x) and (4.20) does not define a Schlesinger in analogy to the case $r(x - \underline{\psi}_N) = q\underline{\psi}_N \psi_N$ in the nonlinear case.

Using this form of the Schlesinger transformation we can, for example, construct the $N = 3$ case starting from the $N = 1$ case. In the case $N = 1$, the mapping for x is given by:

$$\bar{x} = \frac{(y - \underline{c})x + g}{y}. \quad (4.21)$$

Using equation (4.20), we introduce the Schlesinger:

$$w = \frac{x\underline{c} - g}{y} \frac{(\underline{c} + 1)y - \underline{c} - \underline{c}}{x\underline{c} - \underline{g}}. \quad (4.22)$$

In order to simplify the final expression, we define p with $g = \underline{c}p$. We then have the following equation for w :

$$\bar{w} = \frac{c(c-1) \left(\underline{c}w((\bar{p}-\underline{p})y + \underline{c}(\underline{p}-\underline{p})) + \underline{c}(\underline{p}-\bar{p})(y(\underline{c}+1) - (\underline{c}+\underline{c})) \right)}{(p-\underline{p})\underline{c}\underline{c}w(y+c)}. \quad (4.23)$$

To give this equation in the same parametrisation as (4.3), we first write it in terms of \bar{y} and then use a multiplicative gauge $\omega = \phi w$ to put to unity the coefficient of ωy in the numerator of \bar{w} . We then have:

$$\bar{w} = \frac{\omega(\bar{y}-r) + q(\bar{y}-\bar{\psi}_3)}{\omega\bar{y}}. \quad (4.24)$$

where

$$\begin{aligned} q &= \frac{(p-\bar{p})(\underline{p}-\underline{p})(2\underline{c}+\underline{c}+1)}{(\underline{c}(\underline{p}-\underline{p}) + \bar{p}-\underline{p})(\underline{c}(\underline{p}-\underline{p}) + p-\underline{p})} \\ r &= \frac{\underline{c}(\underline{p}-\underline{p}) + c(\bar{p}-\underline{p})}{\underline{c}(\underline{p}-\underline{p}) + \bar{p}-\underline{p}} \\ \bar{\psi}_3 &= \frac{\underline{c}c + c + \underline{c} + \underline{c}}{2\underline{c} + \underline{c} + 1}. \end{aligned} \quad (4.25)$$

Finally, we examine the possibility of the existence of a $N = 1$ Schlesinger. In this case, the structure of the transformation will be obtained by asking that the $N + 1$ case enter the singularity one step before the N case but exit at the same point. The general structure is thus:

$$w = \frac{rx + q\psi_N}{y} \frac{y - \eta}{x - \xi} \quad (4.26)$$

where η and ξ must be determined. We do this by requiring that the equation for w contain no coefficients nonlinear in y . As a result we find that η must satisfy the equation (4.1) for y :

$$\bar{\eta} = \frac{\eta + c}{\eta + 1} \quad (4.27)$$

and ξ the equation (4.3) for x with η instead of y :

$$\bar{\xi} = \frac{\xi(\eta - r) + q(\eta - \psi_N)}{\xi\eta}. \quad (4.28)$$

We remark here the perfect parallel to the continuous case (and as we pointed out the discrete case led the investigation back to the continuous one). Let us point out here that the w obtained through (4.26) does not lead to $w = 0$ at the exit of the singularity (i.e. when $x = 0$, $y = \psi_N$) and a translation is needed. One has in principle to define a new variable

$$\omega = w - w(x=0, y=\psi_N) = w + \frac{q}{\xi}(\psi_N - \eta).$$

We are now going to study the particular case where we construct $N = 1$ starting from $N = 0$. As in the continuous case our starting point is the $N = 0$ nonlinear equation. Thus we are starting with the decoupled equation (4.14) for x . We write the Schlesinger as:

$$w = \frac{y - \eta}{y} \frac{x}{x - \xi}. \quad (4.29)$$

Once more we find that η must satisfy (4.27) while ξ satisfies $\bar{\xi} = 1 + q/\xi$ i.e. the same equation as x . Using (4.27) we can easily obtain the equation for w corresponding to $N = 1$. We write this equation in terms of \bar{y} in order to enter the singularity with $\bar{y} = 0$. The expression of \bar{w} is indeed linear in w and reads

$$\bar{w} = \frac{(q + \xi)w(c - \bar{y}) + q(\bar{y}(\eta + 1) - c - \eta)}{\bar{y}q(\eta + 1)}. \quad (4.30)$$

In order to cast it in the parametrisation of (4.21), we introduce a shift in the w :

$$\omega = w + \omega_0. \quad (4.31)$$

We require that in the numerator of the equation for ω , \bar{y} appears only as a product with ω . We find that ω_0 must satisfy

$$\bar{\omega}_0 = -\frac{\omega_0(q + \xi) + q(\eta + 1)}{q(\eta + 1)} \quad (4.32)$$

and we obtain the following equation for ω

$$\bar{\omega} = \frac{(q + \xi)\omega(c - \bar{y}) - c\omega_0(q + \xi) - q(c + \eta)}{\bar{y}q(\eta + 1)} \quad (4.33)$$

which is in the form of (4.21) up to a multiplicative factor in ω . We note that ω_0 plays the role of \tilde{x}_0 in equation (3.18) in the continuous case. Indeed, ω_0 does satisfy (4.33) for $y = \eta$.

Finally we derive the $N = 1$ Schlesinger for the case of a linear mapping (4.18). We start from:

$$w = \frac{x\psi_N - g}{y} \frac{y - \eta}{x - \xi} \quad (4.34)$$

and again require for w an equation with coefficients linear in y . We find that η must again be a solution of the equation for y i.e. it must satisfy (4.27) and moreover ξ is a solution of (4.18) with $y = \eta$:

$$\bar{\xi} = \frac{\xi(\eta - \psi_N) + g}{\eta}. \quad (4.35)$$

Thus the list of the Schlesinger transformations of the Gambier mapping is complete.

5. CONCLUSION

In this paper we have shown that the Gambier system possesses Schlesinger transformations just like the Painlevé equations. This is a most interesting result given that the Gambier system is C-integrable (in the terminology of Calogero [10]) i.e. integrable through linearisation, and not S-integrable.

We discovered that both the continuous and the discrete systems possess two kinds of Schlesinger transformations: one that allows changes of N by two units and one where the changes of N are by one unit. In the discrete case our approach was based entirely on the singularity confinement approach. We have shown that a study of the singularities allows one to determine the form of the Gambier mapping and at the same time its Schlesinger transformations. This is one more argument in favour of the singularity analysis approach to the study of discrete systems.

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