Symmetry Classification of Systems of Differential-Difference Equations

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Abstract

The Lie point symmetries of a coupled system of two nonlinear differential-difference equations are investigated. The equations arise from a model describing the interaction of two long molecular chains. We have classified all symmetry algebras containing an $sl(2,\mathbb{R})$ subalgebra. Examples are given for which the symmetry algebra is 10-dimensional and infinite-dimensional.

Résumé

Nous étudions les symétries d'un système de deux équations différentielles aux différences. Ces equations proviennent d'un modèle pouvant décrire l'intéraction de deux longues chaînes moléculaires. Nous avons classifié toutes les algèbres de symétrie contenant une certaine représentation de $sl(2,\mathbb{R})$ comme sous-algèbre. Nous donnons des exemples pour lesquels l'algèbre de symétrie est de dimension 10 ou de dimension infinie.

1 Introduction

We have performed a symmetry analysis of a system of two coupled differential-difference equations of the form

$$\begin{aligned} E_1 &= \ddot{u}_n - F_n(t, u_{n-1}, u_n, u_{n+1}, v_{n-1}, v_n, v_{n+1}) = 0, \\ E_2 &= \ddot{v}_n - G_n(t, u_{n-1}, u_n, u_{n+1}, v_{n-1}, v_n, v_{n+1}) = 0. \end{aligned} \tag{1.1}$$

where the dots denote time derivatives. The discrete variable n plays the role of a space variable which labels positions along a one-dimensional lattice. The variables u_n and v_n represent deviations from equilibrium positions of two different types of atoms U and V. Each atom at site n interacts with its nearest neighbours on the two molecular chains, at sites $n-1$, n and $n+1$. The functions F_n and G_n represent the interactions between different atoms. Due to the lack of a realistic model, the interactions are a priori unspecified. Our aim is to classify equations of the type (1.1) according to the Lie point symmetries that they allow.

We do not restrict to two-body forces, nor do we impose translational invariance for the chain. We do however assume there is no dissipation, i.e. system (1.1) does not involve first derivatives with respect to time.

Such differential-difference equations typically arise when modeling phenomena in molecular physics, biophysics, or simply coupled oscillations in classical mechanics [1, 2, 3].

A recent article [4] was devoted to a similar problem, but was concerned with a single species, i.e. one dependent variable $u_n(t)$. Several different treatments of Lie symmetries of difference and differential-difference equations exist in the literature [4-9]. The approach adopted here is the "intrinsic method" [4, 5, 6]: we shall consider only symmetries acting on the continuous variables t, u_n and v_n. Transformations of the discrete variable n must then be studied separately.

The Lie algebra of the symmetry group is realized by vector fields of the form

$$\hat{X} = \tau(t, u_n, v_n)\partial_t + \phi_n(t, u_n, v_n)\partial_{u_n} + \psi_n(t, u_n, v_n)\partial_{v_n}. \tag{1.2}$$

The algorithm for finding the functions τ, ϕ_n and ψ_n in (1.2) is to construct the appropriate prolongation $\mathrm{pr}\hat{X}$ of $\hat{X}$ and to impose that it should annihilate the system of equations (1.1) on their solution set

$$\mathrm{pr}\hat{X}E_1|_{E_1=E_2=0} = 0, \quad \mathrm{pr}\hat{X}E_2|_{E_1=E_2=0} = 0. \tag{1.3}$$

Our classification will be up to conjugacy under a group of "allowed transformations". These are fiber preserving locally invertible point transformations

$$u_n = \Omega_n(\tilde{u}_n, \tilde{v}_n, \tilde{t}), \quad v_n = \Gamma_n(\tilde{u}_n, \tilde{v}_n, \tilde{t}), \quad t = t(\tilde{t}) \tag{1.4}$$

that preserve the form of equations (1.1), but not necessarily the functions F_n and G_n.

2 Determining Equations and the Classification Group

To find the Lie point symmetries of the system (1.1) we write the second prolongation of the vector field (1.2) in the form [4, 5, 6]:

$$\begin{aligned} \mathrm{pr}^{(2)}\hat{X} = \tau(t, u_n, v_n)\partial_t &+ \sum_{k=n-1}^{n+1} \phi_k(t, u_n, v_n)\partial_{u_k} \\ &+ \sum_{k=n-1}^{n+1} \psi_k(t, u_n, v_n)\partial_{v_k} + \phi_n^{tt}\partial_{\ddot{u}_n} + \psi_n^{tt}\partial_{\ddot{u}_n}, \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} \phi_n^{tt} &= D_t^2\phi_n - (D_t^2\tau)\dot{u}_n - 2(D_t\tau)\ddot{u}_n, \\ \psi_n^{tt} &= D_t^2\psi_n - (D_t^2\tau)\dot{v}_n - 2(D_t\tau)\ddot{v}_n, \end{aligned} \tag{2.2}$$

where D_t is the total time derivative. The result is that an element $\hat{X}$ of the symmetry algebra must have the form

$$\begin{aligned} \hat{X}_n = \tau(t)\partial_t &+ \left[\left(\frac{\dot{\tau}}{2} + a_n\right)u_n + b_n v_n + \lambda_n(t)\right]\partial_{u_n} \\ &+ \left[c_n u_n + \left(\frac{\dot{\tau}}{2} + d_n\right)v_n + \mu_n(t)\right]\partial_{v_n}, \end{aligned} \tag{2.3}$$

where the dots denote time derivatives. The functions $\tau(t)$, $\lambda_n(t)$, $\mu_n(t)$, a_n, b_n, c_n and d_n satisfy the two remaining determining equations, namely

$$\begin{aligned}
&\frac{\dddot{\tau}}{2}\,u_n + \ddot{\lambda}_n + \left(a_n - \frac{3}{2}\dot{\tau}\right) F_n + b_n G_n - \tau F_{n,t} \\
&\quad - \sum_{k=n-1}^{n+1} F_{n,u_k}\left[\left(\frac{\dot{\tau}}{2} + a_k\right) u_k + b_k v_k + \lambda_k(t)\right] \\
&\quad - \sum_{k=n-1}^{n+1} F_{n,v_k}\left[\left(\frac{\dot{\tau}}{2} + d_k\right) v_k + c_k u_k + \mu_k(t)\right] = 0,
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&\frac{\dddot{\tau}}{2}\,v_n + \ddot{\mu}_n + \left(d_n - \frac{3}{2}\dot{\tau}\right) G_n + c_n F_n - \tau G_{n,t} \\
&\quad - \sum_{k=n-1}^{n+1} G_{n,u_k}\left[\left(\frac{\dot{\tau}}{2} + a_k\right) u_k + b_k v_k + \lambda_k(t)\right] \\
&\quad - \sum_{k=n-1}^{n+1} G_{n,v_k}\left[\left(\frac{\dot{\tau}}{2} + d_k\right) v_k + c_k u_k + \mu_k(t)\right] = 0.
\end{aligned} \tag{2.5}$$

Substituting (1.4) into eq.(1.1) and requiring that the form of these two equations be preserved, we find that the allowed transformations are

$$\begin{gathered}
\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix} = \begin{pmatrix} Q_n & R_n \\ S_n & T_n \end{pmatrix} \dot{\tilde{t}}^{-1/2} \begin{pmatrix} \tilde{u}_n(\tilde{t}) \\ \tilde{v}_n(\tilde{t}) \end{pmatrix} + \begin{pmatrix} \alpha_n(t) \\ \beta_n(t) \end{pmatrix}, \\
\tilde{t} = \tilde{t}(t) \quad , \quad \frac{d\tilde{t}}{dt} \neq 0.
\end{gathered} \tag{2.6}$$

A short-hand notation for the symmetry generator X_n of eq.(2.3) is

$$\left\{\tau(t), A_n, \begin{pmatrix} \lambda_n(t) \\ \mu_n(t) \end{pmatrix}\right\} \quad , \quad A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}. \tag{2.7}$$

Under an allowed transformation (2.6) it transforms into one of the same form with

$$\begin{aligned}
&\tilde{\tau}(\tilde{t}) = \tau(t(\tilde{t}))\dot{\tilde{t}}, \\
&\tilde{A}_n = {M_n}^{-1} A_n M_n, \\
&\begin{pmatrix} \tilde{\lambda}_n(\tilde{t}) \\ \tilde{\mu}_n(\tilde{t}) \end{pmatrix} = {M_n}^{-1} \dot{\tilde{t}}^{\,1/2} \left[(A_n + \tfrac{\dot{\tau}}{2}) \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} - \tau \begin{pmatrix} \dot{\alpha}_n \\ \dot{\beta}_n \end{pmatrix} + \begin{pmatrix} \lambda_n \\ \mu_n \end{pmatrix}\right].
\end{aligned}$$

This transformation will be used to simplify the symmetry generator (2.3). Then we shall find interactions F_n and G_n compatible with such a symmetry, by solving the remaining determining equations (2.4) and (2.5).

3 Classification of Symmetry Algebras

There are 10 equivalence classes of one-dimensional symmetry algebras under allowed transformations [7]. The interactions that allow higher dimensional algebras become more and more specified, and the arbitrary functions involved depend on the joint invariants of the group action. For a certain dimension of the symmetry algebra (in some cases infinity), F_n and G_n are completely specified functions. These are the most interesting cases of the classification. The interested reader is referred to Ref.7 for a more thorough discussion.
We shall concentrate on $sl(2,\mathbb{R})$ symmetry algebras and show that up to equivalence there are 4 different classes. The statement is:
Precisely 4 *classes of* $sl(2,\mathbb{R})$ *algebras can be realized by vector fields of the form (2.3). Any such* $sl(2,\mathbb{R})$ *algebra can*

be taken by an allowed transformation (2.6) into one of the following algebras:

$$\begin{aligned} \mathrm{sl}(2,\mathbb{R})_1 \ : \quad & X_1 = v_n \partial_{u_n} \\ & X_2 = \frac{1}{2}(u_n \partial_{u_n} - v_n \partial_{v_n}) \\ & X_3 = u_n \partial_{v_n} \end{aligned} \tag{3.1}$$

$$\begin{aligned} \mathrm{sl}(2,\mathbb{R})_2 \ : \quad & X_1 = \partial_t \\ & X_2 = t\partial_t + \frac{1}{2}(u_n \partial_{u_n} + v_n \partial_{v_n}) \\ & X_3 = t^2 \partial_t + t(u_n \partial_{u_n} + v_n \partial_{v_n}) \end{aligned} \tag{3.2}$$

$$\begin{aligned} \mathrm{sl}(2,\mathbb{R})_3 \ : \quad & X_1 = \partial_t + v_n \partial_{u_n} \\ & X_2 = t\partial_t + u_n \partial_{u_n} \\ & X_3 = t^2 \partial_t + t u_n \partial_{u_n} + (t v_n - u_n)\partial_{v_n} \end{aligned} \tag{3.3}$$

$$\begin{aligned} \mathrm{sl}(2,\mathbb{R})_4 \ : \quad & X_1 = \partial_t + v_n \partial_{u_n} \\ & X_2 = t\partial_t + (u_n + t)\partial_{u_n} + \partial_{v_n} \\ & X_3 = t^2 \partial_t + (t u_n + 2t^2)\partial_{u_n} + (t v_n - u_n + 2t)\partial_{v_n}. \end{aligned} \tag{3.4}$$

We shall focus on those symmetry algebras which contain $sl(2,\mathbb{R})_1$ as a subalgebra. The classification can be summed up in the following way: in addition to $\mathrm{sl}(2,\mathbb{R})_1$ of eq.(3.1) we have a further algebra L_C (the "complementary" algebra). The structure of each symmetry algebra is

$$L = \mathrm{sl}(2,\mathbb{R})_1 \dot{+} L_C, \quad [\mathrm{sl}(2,\mathbb{R})_1, L_C] \subseteq L_C, \quad [L_C, L_C] \subseteq L_C. \tag{3.5}$$

The symbol $\dot{+}$ denotes a direct sum of vector spaces. Moreover, eq.(3.5) shows that L is either a direct sum or a semidirect one. The algebra L_C is also a representation space for $\mathrm{sl}(2,\mathbb{R})_1$. Irreducible representations in this case can be of dimension 1 or 2. All higher dimensional representations are completely reducible into sums of 1 and 2 dimensional representations. The symmetry algebras can be classified into 4 series, according to the structure of the Lie algebra L_C.

Series A L_C is solvable and each element is an $\mathrm{sl}(2,\mathbb{R})_1$ singlet. There exist 3 Lie algebras in this family for which the interactions are completely specified. Two of them are infinite dimensional, and the other one has dimension 7.

Series B L_C is solvable and contains precisely one $\mathrm{sl}(2,\mathbb{R})_1$ doublet. The algebras are either 5 or 6 dimensional.

Series C L_C contains two $\mathrm{sl}(2,\mathbb{R})_1$ doublets. There are two completely specified interactions in this family, both having symmetry algebras of dimensions 9 and 10.

Series D L_C contains $\mathrm{sl}(2,\mathbb{R})_2$ and (possibly) further elements. There are 4 such algebras of dimensions 6, 7, 8 and 10, the last two having completely specified interactions.

Due to limited space, all the symmetry algebras with their corresponding invariant interactions cannot be listed in this work. We will show two examples of the most interesting cases, and refer the reader to Ref.7 for the complete classification.

Example 1. The following dynamical equations:

$$\begin{aligned} \ddot{u}_n &= u_{n+1} \frac{u_{n-1}v_n - u_n v_{n-1}}{u_{n+1}v_{n-1} - u_{n-1}v_{n+1}} h_n + u_n k_n \\ \ddot{v}_n &= v_{n+1} \frac{u_{n-1}v_n - u_n v_{n-1}}{u_{n+1}v_{n-1} - u_{n-1}v_{n+1}} h_n + v_n k_n \end{aligned} \tag{3.6}$$

with h_n and k_n constant (depending on n) allow an infinite dimensional symmetry algebra

$$A_2(\infty) \sim sl(2,\mathbb{R})_1 \dot{+} \left\{ T(b_n), V(a_{k,n}), \ \ k \in \mathbb{Z}^{>} \right\} \tag{3.7}$$

with

$$\begin{aligned} V(a_n) &= a_n(u_n \partial_{u_n} + v_n \partial_{v_n}), \\ T(b_n) &= \partial_t + b_n(u_n \partial_{u_n} + v_n \partial_{v_n}). \end{aligned}$$

The algebra belongs to the Series A discussed above, the subindex refers to the classification in [7] and the number in brackets is the dimension of the algebra.

Example 2. The following system:

$$\begin{aligned} \ddot{u}_n &= (\xi_{n+1} - \xi_n - \xi_{n-1})^{-2} \left[(u_{n+1} - u_{n-1}) p_n + (u_n - u_{n+1}) q_n \right], \\ \ddot{v}_n &= (\xi_{n+1} - \xi_n - \xi_{n-1})^{-2} \left[(v_{n+1} - v_{n-1}) p_n + (v_n - v_{n+1}) q_n \right], \end{aligned} \tag{3.8}$$

with p_n and q_n constants (depending on n) and

$$\begin{aligned} \xi_n &= u_{n+1} v_{n-1} - u_{n-1} v_{n+1}, \\ \xi_{n+1} &= u_{n+1} v_n - u_n v_{n+1}, \\ \xi_{n-1} &= u_{n-1} v_n - u_n v_{n-1}, \end{aligned}$$

has a 10-dimensional symmetry algebra given by

$$D_4(10) \sim sl(2,\mathbb{R})_1 \oplus sl(2,\mathbb{R})_2 \dot{+} \{Y_u(1), Y_v(1), Y_u(t), Y_v(t)\} \tag{3.9}$$

where $Y_u(t) = t\partial_{u_n}$, $Y_v(t) = t\partial_{v_n}$, $Y_u(1) = \partial_{u_n}$ and $Y_v(1) = \partial_{v_n}$.

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