On the geometric structure of the class of planar quadratic differential systems

Robert Roussarie* Dana Schlomiuk†‡

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*Lab. de Topologie, Université de Bourgogne, 9 avenue Alain-Savary, 21078 Dijon Cedex, France

†Dép. de mathématiques, et statistique & CRM, Université de Montréal, C.P. 6128, succ. Centre-ville, Montréal, QC H3C 3J7, Canada

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1 Introduction

We consider real planar polynomial differential systems, i.e., systems of the form

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{1.1}$$

where P and Q are polynomial with real coefficients. In this work we are interested in the global theory of systems (1.1) with $\max(\deg P, \deg Q) \leq 2$. Such systems will be called here quadratic systems. There are a number of long-standing open problems about polynomial differential systems, the most famous one being Hilbert's 16th problem, formulated by Hilbert in his address to the International Congress of Mathematicians in Paris in 1900. The second part of this problem asks to determine the maximum $\mathrm{H}(n)$ of the numbers of limit cycles which appear in systems of the form (1.1) with $n = \max(\deg P, \deg Q)$, and also their possible relative positions in $\mathbb{R}^2$. As we approach the year two thousand, this problem is still unsolved even for quadratic systems. The difficulty of this problem is due to its twice global nature: analysis of the systems in their whole domain of existence, including the points at infinity and in the whole parameter space. We know (cf. [Ec, I]) that for a given system (1.1) the number of limit cycles is finite. For fixed n, if the coefficients of P and Q vary while $\max(\deg P, \deg Q) \leq n$, we therefore have $\mathrm{H}(n) \leq \aleph_0$. Is $\mathrm{H}(n)$ finite? The answer is not known, even for the quadratic case. In almost one hundred years since the statement of the problem, no example of a quadratic systems was found for which we can prove that we have more than four limit cycles. Due to this, not only is it conjectured that $\mathrm{H}(2)$ is finite but also that $\mathrm{H}(2) = 4$. At the moment work on quadratic systems proceeds in two different directions: first, a program is under way to prove that $\mathrm{H}(2)$ is finite (cf. [DRR1, DRR2]) and secondly, attempts are made to gain insight into the class of quadratic systems by studying specific subclasses or by attempting to understand more of the geometry of this whole class. Our work goes in this last direction. Work on specific classes of quadratic systems usually imply tedious calculations. All too often one becomes aware that there is just not enough mathematical structure around these calculations to make them more transparent. Classification works on special classes of quadratic systems are done in terms of the coefficients appearing in the specific normal forms chosen for the equations (1.1) which are studied. Since coefficients change with coordinate changes and since in general the coefficients have no geometric meaning, the problem of classifying systems in more intrinsic ways and attaching geometric meaning to the parameters arises. The first goal of this work is to introduce some geometrical structure which could help to make calculations more transparent and clarify a number of results and methods such as the "isocline method" or the role of the "rotation parameters." We take this opportunity to open a discussion by asking some relevant questions on the subject.

The article is written so as to be as self contained as possible. In Section 2 we define the space of quadratic equations and consider the action on the affine group on this space. In Section 3 we discuss the notion of isocline and associate a geometric object: a pencil of conics, to a quadratic system. In Section 4 we construct a principal fiber bundle associated to a quadratic equation. In the last section we state a number of questions and make some concluding comments.

2 The space of quadratic differential equations and the action of the affine group on this space

We are interested in the class of real systems (1.1) with $\max(\deg P, \deg Q) \leq 2$. The linear differential systems with constant coefficients are thus included since understanding how systems (1.1) with $\max(\deg P, \deg Q) = 2$ change with respect to parameters also imply the limiting case when the systems become linear. Along with systems (1.1) we may consider its associated vector field, i.e.:

$$D = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}. \tag{2.1}$$

When discussing issues not involving the time variable, we associate to the equation (2.1) the Pfaff form $\omega = P(x, y)\,dx + Q(x, y)\,dy$ and its associated differential equation:

$$P'(x, y)\,dx + Q'(x, y)\,dy = 0, \tag{2.2}$$

where $P'(x, y) = -Q(x, y)$ and $Q'(x, y) = P(x, y)$. Two equations (2.2) differing from one another by multiplication with a non-zero constant, have identical solution curves. Leaving aside the trivial case when both P and Q are identically zero, we consider the space:

$$E = \{P(x, y)\,dx + Q(x, y)\,dy = 0 \mid (P, Q) \in (\mathbb{R}[x, y])^2 \setminus \{0\}, \quad \max(\deg P, \deg Q) \leq 2\} \tag{2.3}$$

E may be thought alternatively as a subspace of $\mathbb{R}^{12}$ or the 11-dimensional real projective space $\mathrm{P}^{11}(\mathbb{R})$, which is compact. Considering this space is also valuable due to the fact that a Pfaff form defines a foliation with singularities on $\mathbb{R}^2$ and this foliation can be extended to its compactification: the foliation with singularities on the projective plane $\mathrm{P}^2(\mathbb{R})$ (cf. [C]).

We consider affine transformations, i.e., maps $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x,y) = (a_{11}x + a_{12}y + b_1, a_{21}x + a_{22}y + b_2)$, for a_{ij} and b_j in $\mathbb{R}$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Let A be the affine group of transformations of $\mathbb{R}^2$:

$$A = \{f\colon \mathbb{R}^2 \to \mathbb{R}^2 \mid f \text{ is an affine transformation}\} \tag{2.4}$$

We consider the action of the affine group A on E induced by substitution on Pfaff forms: $f \cdot \omega = f^* \circ \omega \circ f$ where we have $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\omega\colon \mathbb{R}^2 \to T^*\mathbb{R}^2$, $f^*\colon T^*\mathbb{R}^2 \to T^*\mathbb{R}^2$. We clearly have:

$$(f \circ f').\omega = (f \circ f') * \circ \omega \circ (f \circ f') = f' * \circ (f * \omega f) \circ f'$$

and hence:

$$(f \circ f') \cdot \omega = f' \cdot (f \cdot \omega). \tag{2.5}$$

So A is a right action on 1-forms which induces an right action on E:

$$A \times E \to E(f, \omega = 0) \quad \mapsto (f \cdot \omega) = 0. \tag{2.6}$$

Calculations yield

$$(f \cdot \omega)(x,y) = P_f(x,y)\,dx + Q_f(x,y)\,dy \tag{2.7}$$

where

$$\big(P_f(x,y), Q_f(x,y)\big) = \Big(P\big(f(x,y)\big), Q\big(f(x,y)\big)\Big) M_f \tag{2.8}$$

$M_f = ((a_{ij}))$ being the matrix associated to the transformation f.

We consider the orbit space E/A of E under this group action. Since $\dim(A) = 6$ and $\dim(E) = 11$, it follows that $\dim(E/A) = 5$. We are interested in the classification of the phase portraits of (1.1) under orbital topological equivalence, i.e., equivalence under homeomorphisms which preserve orientation of orbits. Disregarding the orientation of phase curves, we need to look at the adjacent stratification of the space E/A.

3 The method of isoclines. Linear systems of algebraic curves. Pencils of conics associated to quadratic systems

In this paragraph we associate to a quadratic system a geometric object: the set of all its isoclines. First, let us recall the concept of isocline. Consider a real polynomial vector field $D = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y$. Let (a,b) be a nonsingular point of D. The vector $\big(P(a,b), Q(a,b)\big)$ determines the line $l_{(a,b)} = \{\lambda\big(P(a,b), Q(a,b)\big) \mid \lambda \in \mathbb{R}\}$ yielding a point p in the projective space $\mathrm{P}^1(\mathbb{R})$.

Notation 3.1. We shall denote by $[x,y]$ the point in $\mathrm{P}^1(\mathbb{R})$ associated by the canonical projection $\mathbb{R}^2 \setminus \{0\} \to \mathrm{P}^1(\mathbb{R})$ to the point (x,y) of $\mathbb{R}^2 \setminus \{0\}$. More generally a point in $\mathrm{P}^n(\mathbb{R})$ of homogeneous coordinates $(x_0, \ldots, x_n)$ will be denoted by $[x_0, \ldots, x_n]$.

An algebraic curve $C\colon F(x,y) = 0$ such that for all nonsingular points (x,y) of D situated on the curve C, the vector field D has the same direction, i.e., $l_{(x,y)}$ is constant may be called an isocline of D. For all nonsingular points (x,y) and (a,b) of D which are on C we have $[P(x,y), Q(x,y)] = [P(a,b), Q(a,b)]$ and hence $uP(x,y) + vQ(x,y) = 0$ for $u = Q(a,b)$ and $v = -P(a,b)$. This justifies the following formal definition:

Definition 3.1. Let P, Q be polynomials with real coefficients such that at least one of them is not constant. An isocline of the $D = P\partial/\partial x + Q\partial/\partial y$ (or of (1.1)) is a curve C of the form

$$C\colon uP(x,y) + vQ(x,y) = 0 \tag{3.1}$$

where u, v are real numbers, not both zero.

The isoclines are introduced in elementary courses as a convenient method for tracing direction fields. The school of Erugin ([Er1, Er2, G]) used isocline portraits for gaining insight into quadratic systems, as V. Gaiko indicates in [G] where he refers to [Er1, Er2]. This is the so-called "isocline method", a term which according to V. Gaiko was introduced by Nemytsky and Stepanov. In this work we also intend to give a more precise meaning to this term by introducing more mathematical structure in the context of quadratic systems.

We note that if both P and Q are constant polynomials, not both identically zero, the equation (3.1) does not yield a curve and on any algebraic curve in the plane, the field has a constant direction. Leaving aside such cases, the isoclines of a quadratic system are either straight lines, when the equations are linear, or they could be conics. This distinction of cases and also the possibility of having zero or two real singularities in the finite plane and respectively four or two distinct singularities in the complex plane, make us consider complex projective conics associated to a quadratic system. An equation

$$P(x, y) = a_0 + a_1x + a_2y + a_{20}x^2 + a_{21}xy + a_{22}y^2 = 0 \tag{3.2}$$

where a_0, a_1, a_2, a_{20}, a_{21}, a_{22} are not all zero, determines an algebraic curve of order at most two. To such an affine curve we associate its projective completion which is the conic:

$$P^*(x, y, z) = a_0z^2 + a_1xz + a_2yz + a_{20}x^2 + a_{21}xy + a_{22}y^2 = 0. \tag{3.3}$$

Thus the correspondence $P(x, y) \to P^*(x, y, z)$ associates to any P in $\mathbb{R}[x, y]\setminus\{0\}$ of degree at most two a homogeneous polynomial $P^*[x, y, z]$ of degree two in $\mathbb{R}[x, y, z] \setminus \{0\}$.

The curves (3.3) correspond bijectively to elements of the five dimensional real projective space $\mathrm{P}^5(\mathbb{R})$ via the correspondence

$$i\colon \{C \mid C\colon P^*(x, y, z) = 0\} \to \mathrm{P}^5(\mathbb{R}) \tag{3.4}$$

where $i(C) = [a_0, a_1, a_2, a_{20}, a_{21}, a_{22}]$.

Remark 3.1. We observe that even in the case when both P and Q are constant, not both identically zero, the equation (3.3) corresponding to an isocline (3.1) which becomes

$$uP^*(x, y, z) + vQ^*(x, y, z) = 0, \tag{3.5}$$

still represents a conic: the double line at infinity. If we consider curves in the projective space $\mathrm{P}^2(\mathbb{R})$, then any quadratic system (1.1) with P, Q not both identically zero, yields a system of conics (3.1) which when considered as points in $\mathrm{P}^5(\mathbb{R})$ form either a zero or a one-dimensional linear variety, regardless of the degrees of P and Q. Thus viewing the curves (3.1) as curves (3.3) in the projective space unifies the discussion.

To any quadratic system (1.1) such that P and Q are not both identically zero, we can associate the set $\mathrm{p}(P, Q)$ of conics in $\mathrm{P}^2(\mathbb{R})$, i.e.:

$$\mathrm{p}(P, Q) = \{uP^*(x, y, z) + vQ^*(x, y, z) = 0 \mid (u, v) \neq 0, u, v \in \mathbb{R}\}. \tag{3.6}$$

The coefficients of the conics in $\mathrm{p}(P, Q)$ yield (via the correspondence (3.4)) a subset of $\mathrm{P}^5(\mathbb{R})$ which forms a linear projective variety, or subspace of $\mathrm{P}^5(\mathbb{R})$. In general, a linear projective variety or linear subspace of the n-dimensional projective space $\mathrm{P}^n(k)$ over a field k, is the image under canonical projection $k^{n+1} \setminus \{0\} \to \mathrm{P}^n(k)$ of a non-zero subspace of k^{n+1}. A set of points in $P^n(k)$ is projectively independent if they are generated by points in k^{n+1} which are linearly independent. A linear subspace V of $\mathrm{P}^n(k)$ has dimension r if there are $r + 1$ projectively independent points which generate V. One-dimensional subspaces of $\mathrm{P}^n(k)$ are called a pencils, two-dimensional subspaces are called nets and three-dimensional subspaces are called webs.

More generally any projective curve of order n in $\mathrm{P}^2(k)$ with equation $P^*(x, y, z) = 0$ with P^* a homogeneous polynomial over k of degree n, $P^*(x, y, z) = \sum a_{ijm}x^iy^jz^m$, $i+j+m = n$ determines uniquely up to a constant factor its sequence of coefficients, hence the coefficients a_{ijm} can be considered as homogeneous coordinates in a projective space $\mathrm{P}^N(k)$ with $N = n(n+3)/2$. Below we sometimes identify a curve in $\mathrm{P}^2(k)$ with its associated point in $\mathrm{P}^N(k)$ given by its sequence of coefficients.

Definition 3.2. Let us suppose that we have curves $F_i = 0$, $i = 1, 2,\ldots, R + 1$, of degrees at most n which yield $R+1$ independent points in $\mathrm{P}^N(k)$. Then the system of curves $\left\{\sum u_iF_i = 0 \mid (u_1, \ldots, u_{R+1}) \in k^{R+1} \setminus \{0\}\right\}$ is called an R-dimensional linear system of curves in $\mathrm{P}^2(k)$ (cf. [W]).

Examples of linear systems of curves are: the hyperplane in $\mathrm{P}^5(k)$ formed by the set of all conics passing through a given point in $\mathrm{P}^2(k)$; the set of conics passing through two distinct points in $\mathrm{P}^2(k)$ which is a web in $\mathrm{P}^5(k)$; the set of all conics passing trough 3 distinct points which is a net. The set of all conics passing through 4 distinct points not all on a line in the plane forms a pencil of conics. The following proposition is easily obtained:

Proposition 3.1. *Through four distinct collinear points in* $\mathrm{P}^2(k)$ *there pass a net of conics, each one of them reducible with the line passing through the four points as one of its components. Through four distinct points in* $\mathrm{P}^2(k)$ *only three of them collinear, there pass a pencil of reducible conics, each formed by the line through the three points and a line passing through the fourth point. Through four distinct points in the plane, no three of which are collinear there pass a pencil of conics.*

Definition 3.3. A nondegenerate pencil of conics is a pencil of conics which contains at least one irreducible conic.

We aim at stratifying the orbit space E/A by using the zero and one-dimensional subspaces in $\mathrm{P}^5(\mathbb{R})$ associated to the set of conics (3.6).

A pencil of conics has the property that all of its conics pass through the points of intersection of two distinct conics of the pencil.

Definition 3.4. The common points of all the conics in a pencil of projective conics are called the base points of the pencil.

Theorem 3.1 (Bezout's theorem). *Let* $P(x,y,z)=0$ *and* $Q(x,y,z)=0$ *be two distinct conics in the projective plane* $\mathrm{P}^2(k)$ *where* k *is an algebraically closed field. Let us suppose that these curves do not have a common component. If* $\deg P = n$, $\deg Q = m$, *then the curves have at most* nm *points in common in* $\mathrm{P}^2(k)$ *and exactly* nm *if we take into account their multiplicities.*

As a consequence of Bezout's theorem two complex projective conics without a common component, have at most four distinct points in common and exactly four if we take into account their multiplicities of intersection.

Roughly speaking the intersection multiplicity of two algebraic curves at a point indicates how many points the curves have in common at that point. For example the intersection multiplicity of $y=0$ with $y-x^2=0$ at $(0,0)$ is two, since the line is tangent to the parabola at $(0,0)$. At the same time the intersection multiplicity of $y^2=0$ with $y-x^2=0$ at $(0,0)$ is four since $y=0$ is a double component of $y^2=0$. The intersection multiplicity of $xy=0$ with $y-x^2=0$ at $(0,0)$ is three since $x=0$ and $y=0$ are components of $xy=0$ and $x=0$ intersects the parabola $y-x^2=0$ transversely while $y=0$ is tangent at zero.

In the case of conics, the discussion of multiplicity of intersection is simple. Placing the intersection point at the origin for both conics and taking $x=0$ as one of the two possible tangent lines at 0 for one of the conics we can write the two equations in the affine plane as being:

$$\begin{aligned} C\colon P(x,y) &= a_0x^2+a_1xy+a_2y^2+a_3x=0 \\ C'\colon Q(x,y) &= b_0x^2+b_1xy+b_2y^2+b_3x+b_4y=0. \end{aligned} \tag{3.7}$$

which yield:

$$\begin{aligned} P^*(x,y,z) &= a_0x^2+a_1xy+a_2y^2+a_3xz=0 \\ Q^*(x,y,z) &= b_0x^2+b_1xy+b_2y^2+b_3xz+b_4yz=0. \end{aligned} \tag{3.8}$$

If $a_3=0$, the first conic is reducible, formed by two lines $l_i\colon c_ix-d_iy=0$ over $\mathbb{C}$, $i=1$, 2. These lines could be tangent to the second conic or cut the second conic transversely. If a line l_i is tangent to the conic C' then the intersection multiplicity $\mathrm{I}_p(l_i,C)$ at the common point p of l_i and C is 2 and if the line l_i is not tangent to the conic (we say that it intersects the conic transversely), we write $\mathrm{I}_p(l,C)=1$. The intersection multiplicity of the two conics at p will in this case be the sum of the intersection multiplicities of the lines with the conic. An analogous situation occurs when $a_2=0$.

In case $a_2a_3\neq 0$ in (3.7) (or (3.8)), $C=0$ is irreducible, otherwise x would be a component. $(0,0)$ is the only point of this curve situated on $x=0$. So C, C' will not have an intersection point on $x=0$ other than zero. Looking for the other intersection points we may thus assume that $x\neq 0$. Then from the first equation (3.8) we have $z=-(a_0x^2+a_1xy+a_2y^2)/(a_3x)$ for $x\neq 0$ and replacing this into the second equation we obtain:

$$a_3x(b_0x^2+b_1xy+b_2y^2)-(b_3x+b_4y)(a_0x^2+a_1xy+a_2y^2)=0. \tag{3.9}$$

This is a homogeneous polynomial of degree three in x, y and hence it can be factored over $\mathbb{C}$ yielding three lines $c_ix-d_iy=0$ over $\mathbb{C}$, $i=1$, 2, 3. The solutions (d_i,c_i) of the equation (3.8) will yield three intersection points of the conics which may or not have multiplicities according to whether or not the above cubic equation factors in three factors with or without multiplicities. If $b_4\neq 0$ then $x=0$ is not tangent to $C'=0$ and C and C' intersect transversely at $(0,0)$.

For a quick understanding of the concept of intersection multiplicity in the general case see for example [K].

Notation 3.2. We shall denote by $\mathrm{I}_p(P,Q)$ (or by $\mathrm{I}_p(C,C')$) the intersection multiplicity of the algebraic (respectively projective) curves $C\colon P=0$ and $C'\colon Q=0$ at a point p in the affine (respectively projective) plane over a field k.

We give below some basic facts about pencils of conics:

Theorem 3.2. *Through five distinct points of the projective plane over an algebraically closed field k such that no four of them lie on a line, there pass exactly one conic.*

Proof. Suppose that C, C' are two projective conics which pass through five distinct points such that no four of them lie on a line. In view of Bezout's theorem C and C' must have a common component. If three of the five points are on a line l, then l must be a common component. Hence $C = l \cup l'$ and $C' = l \cup l''$ for some lines l' and l''. The remaining two common points must lie on both l' and l'' and hence $l' = l''$. If no three of the five points are on a line, then neither C nor C' could be reducible and so the common component is $C = C'$. □

Notation 3.3. Let us denote by $\sigma(\alpha)$ the set of base points of a pencil of conics α. Let us denote by F_4 the set of pencils of projective conics a such that $\sigma(\alpha)$ has four points and no three points of $\sigma(\alpha)$ are collinear. Let us denote by SP_4 the set of all sets of four distinct points in the complex projective plane such that no three of them are collinear.

Proposition 3.2. *Assume that α is a pencil of conics such that $\sigma(\alpha)$ is made of four distinct points a, b, c, d. Then no three of the four base points are collinear and α contains only three reducible conics, i.e., the conics formed by the lines ab and cd, ac and bd, ad and bc. All other conics in the pencil are irreducible and the pencil is nondegenerate.*

Proof. If three of the points are collinear, the line passing through them is a component of all conics in the pencil, so we cannot have just four base points. □

Corollary 3.1. *The map $\sigma\colon F_4 \to SP_4$ is a bijection.*

Proof. This map is a surjection. Indeed let $S \in SP_4$. Then S determines three distinct reducible projective conics. Taking two of these conics, say $P^* = 0$ and $Q^* = 0$, they yield two independent points in the projective space and hence we have the pencil of conics $\mathrm{p}(P^*, Q^*)$ and $\sigma\big(\mathrm{p}(P^*, Q^*)\big) = S$. σ is also injective since through any point R not in S there passes a single conic which contains S. Assuming $\big(P(R), Q(R)\big) \neq 0$, the conic $C\colon -Q(R)P(x, y) + P(R)Q(x, y) = 0$ passes through all points of S and through R. Hence if $S = \sigma(\alpha)$, any conic in α is of the form C and hence belongs to $\mathrm{p}(P^*, Q^*)$. □

Definition 3.5. Let $C\colon P^* = 0$ and $C'\colon Q^* = 0$ be two distinct projective conics and let p be a point of intersection of C with C'. We say that the conics have a contact of order one (resp. two, three) at p if $\mathrm{I}_p(P', Q') = 2$ (resp. 3, 4).

Proposition 3.3. *Let P and Q be irreducible polynomials over $\mathbb{C}$, $\deg P = \deg Q = 2$. Let us suppose that the conics $C\colon P = 0$ and $C'\colon Q = 0$ have a point of contact which without loss of generality may be assumed to be the origin, so $\mathrm{I}_0(P, Q) > 1$. The equations of the two conics may be assumed to be (3.8) with $b_4 = 0 \neq a_2a_3$ and $b_2 \neq 0$. Then C has a contact of order one (resp. two, three) at zero with C' if and only if the equation of C' is the first (resp. second, third) equation below:*

$$C'\colon Q*(x, y, z) = b_0x^2 + b_1xy + b_2y^2 + ua_3xz = 0 \tag{3.10}$$

$$C'\colon Q*(x, y, z) = b_0x^2 + b_1xy + ua_2y^2 + ua_3xz = 0 \tag{3.11}$$

$$C'\colon Q*(x, y, z) = b_0x^2 + ua_1xy + ua_2y^2 + ua_3xz = 0. \tag{3.12}$$

Proof. All points of intersection of C with C' must satisfy the equation:

$$(a_0b_3 - b_0a_3)x^2 + (a_1b_3 - a_3b_0)xy + (a_2b_3 - a_3b_2)y^2 = 0$$

This is the equation of two lines passing through the origin. One (resp. two) of these lines coincides with $x = 0$ if and only if $a_2b_3 - a_3b_2 = 0$ (resp. $a_2b_3 - a_3b_2 = 0 = a_1b_3 - a_3b_0$). In the first case $\mathrm{I}_0(P, Q) = 3$ and C' has the equation (3.11) and in the second $\mathrm{I}_0(P, Q) = 4$ and C' has the equation (3.11) as canonical form. □

This above proposition immediately yields

Corollary 3.2. *If two distinct conics, irreducible over $\mathbb{C}$ have a contact of order one, two or three at a point p then any two distinct conics in the pencil have a contact of order one, two or three at p.*

Proof. We consider only one of the three cases, namely the case when C and C' that a contact of order two at the origin, the other cases being analogous.

$$\begin{aligned} C\colon P^*(x, y, z) &= a_0x^2 + a_1xy + a_2y^2 + a_3xz = 0 \\ C'\colon Q^*(x, y, z) &= b_0x^2 + b_1xy + ua_2y^2 + ua_3xz = 0. \end{aligned} \tag{3.13}$$

Then a conic of $\mathrm{p}(P^*, Q^*)$ will have the equation $vP^* + wQ^* = 0$ which is

$$(va_0 + wb_0)x^2 + (va_1 + wb_1)xy + a_2(v + uw)y^2 + a_3(v + uw)xz = 0.$$

Clearly any two such conics have the coefficients of y^2 and xz proportional. □

Proposition 3.4. *If $P' = 0$ and $Q' = 0$ are two distinct projective conics over $\mathbb{C}$ conics, then through any point p in $\mathrm{P}^2(\mathbb{C})$ which is not a base point, there passes exactly one conic of the pencil $\mathrm{p}(P', Q')$.*

This yields the classification of all pencils of conics with respect to base points.

The classification of nondegenerate pencils of conics can be done according to the number and multiplicities of the base points. We have:

Theorem 3.3. *There are five types of nondegenerate pencils of conics given by the types of base points and multiplicities:*

- *four distinct base points each being simple, i.e., with multiplicity one*
- *three base points, one double (with multiplicity 2) and another two points each with multiplicities one,*
- *two base points, each a double one (with multiplicity two),*
- *two base points, one triple (with multiplicity 3), one with multiplicity one,*
- *one base point which is quadruple, i.e., with multiplicity four.*

The generic one among these cases is the case with four distinct points over the complex numbers.

For the stratification of E/A we shall use F/A and in particular for E_4 we shall use E_4/A.

4 Quadratic equations structured as a principal fiber bundle over pencils of conics

In this paragraph we consider a subset of the set E of real quadratic equations $P\,dx + Q\,dy = 0$, namely the subset $E_{4\mathrm{f}}$ of E formed by equations for which the curves $P = 0$ and $Q = 0$ have four distinct common points in the real finite plane, no three of which are collinear. It is clear that the group A acts on $E_{4\mathrm{f}}$ yielding the orbit space $E_{4\mathrm{f}}/A$. We shall organize the space $E_{4\mathrm{f}}/A$ as a principal fiber bundle over a base space to be defined below and with structure group $\mathrm{Aut}(\mathrm{P}^1(\mathbb{R})) = \mathrm{PGL}(2, \mathbb{R})$. Considering the embedding $i\colon \mathbb{R}^2 \to \mathrm{P}^2(\mathbb{R})$ where $i(x, y) = [x, y, 1]$ and identifying $\mathbb{R}^2$ with $i(\mathbb{R}^2)$, $z = 0$ may be called the line at infinity of $\mathbb{R}^2$ (in $\mathrm{P}^2(\mathbb{R})$). Let us denote by F the set of pencils of conics of $\mathrm{P}^2(\mathbb{R})$ and by $F_{4\mathrm{f}}$ the subset of F of pencils with four distinct base points in the real finite plane, no three of them collinear. We have the map:

$$\Phi_{4\mathrm{f}}\colon E_{4\mathrm{f}} \to F_{4\mathrm{f}} \tag{4.1}$$

which assigns to any equation $P\,dx + Q\,dy = 0$ in $E_{4\mathrm{f}}$, the set of conics:

$$\mathrm{p}(P, Q) = \{uP^*(x, y, z) + vQ^*(x, y, z) = 0 \mid u, v \in R, (u, v) \neq (0, 0)\}. \tag{4.2}$$

The affine group A of transformations of $\mathbb{R}^2$ acts also on F by substitution (pull-back): $A \times F \to F$. More precisely if $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transformation $f(x, y) = (a_{11}x + a_{12}y + b_1, a_{21}x + a_{22}y + b_2)$, we consider the induced projective transformation on $\mathrm{P}^2(\mathbb{R})$: $f^*(x, y, z) = (a_{11}x + a_{12}y + b_1z, a_{21}x + a_{22}y + b_2z, z)$. If $C\colon R(x, y, z) = 0$ is a homogeneous equation of a conic in $\mathrm{P}^2(\mathbb{R})$ which belongs to a pencil p then we put $f \cdot C\colon R \circ f^* = 0$. This induces a right action on $F_{4\mathrm{f}}$, i.e., $A \times F_{4\mathrm{f}} \to F_{4\mathrm{f}}$, $(f, p) \to f \cdot p$ where $f \cdot p = \{f \cdot C | C \in p\}$. Indeed:

$$\begin{aligned}(f \circ f') \cdot p &= \{(f \circ f') \cdot C \mid C \in p\} = \{R \circ (f^* \circ f'^*) = 0 \mid C \in p\}\\ &= \{(R \circ f^*) \circ f'^* = 0 \mid C \in \mathrm{p}\} = f' \cdot \{R \circ f^* = 0 \mid C \in p\}\\ &= f' \cdot (f \cdot p).\end{aligned}$$

The above action on $F_{4\mathrm{f}}$ yields the orbit space $F_{4\mathrm{f}}/A$.

Definition 4.1. Let G be a group. A right (resp. left) G-set is a set X together with a right (resp. left) group action $G \times X \to X$, $(g, x) \mapsto g \cdot x$ with $(gh) \cdot x = h \cdot (g \cdot x)$ (resp. $(gh) \cdot x = g \cdot (h \cdot x)$) and $e \cdot x = x$ for all x in X and e the neutral element of G. A map of G-sets is a map $u \colon X \to Y$ such that the diagram below commutes:

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ {\scriptstyle 1\times u}\downarrow & & \downarrow{\scriptstyle u} \\ G \times Y & \longrightarrow & Y, \end{array}$$

Proposition 4.1. *$\Phi_{4\mathrm{f}} \colon E_{4\mathrm{f}} \to F_{4\mathrm{f}}$ is a map of A-sets, i.e., the following diagram is commutative:*

$$\begin{array}{ccc} A \times E_{4\mathrm{f}} & \longrightarrow & E_{4\mathrm{f}} \\ {\scriptstyle 1\times F_{4\mathrm{f}}}\downarrow & & \downarrow{\scriptstyle F_{4\mathrm{f}}} \\ A \times F_{4\mathrm{f}} & \longrightarrow & F_{4\mathrm{f}}, \end{array} \tag{4.3}$$

Proof. The action on E induced by: $(f, \omega) \to f \cdot \omega$ where ω is a 1-form and $(f \cdot \omega)(x, y) = P_f(x, y)\, dx + Q_f(x, y)\, dy$ with

$$\big(P_f(x, y), Q_f(x, y)\big) = \big((P \circ f)(x, y), (Q \circ f)(x, y)\big) M_f,$$

$M_f = ((a_{ij}))$ being the 2×2 matrix of f. We have

$$\begin{aligned} \Phi\big(f \cdot (\omega = 0)\big) &= \mathrm{p}\Big(\big((P \circ f)(x, y), (Q \circ f)(x, y)\big) M_f\Big) \\ \Phi\big(f \cdot (\omega = 0)\big) &= \big\{u\big(a_{11}(P \circ f)(x, y) + a_{21}(Q \circ f)(x, y)\big)^* \\ &\qquad + v\big(a_{12}(P \circ f)(x, y) + a_{22}(Q \circ f)(x, y)\big)* = 0 \\ &\qquad\qquad \big| \; u, v \in \mathbb{R}, (u, v) \neq (0, 0)\big\} \\ \Phi\big(f \cdot (\omega = 0)\big) &= \{(ua_{11} + va_{12})(P \circ f)^*(x, y, z) \\ &\qquad + (ua_{21} + va_{22})(Q \circ f)^*(x, y, z) = 0 \\ &\qquad\qquad \mid u, v \in \mathbb{R}, (u, v) \neq (0, 0)\}. \end{aligned} \tag{4.4}$$

We consider now $f \cdot \big(\mathrm{p}(P, Q)\big)$. We have

$$\mathrm{p}(P, Q) = \{wP^*(x, y, z) + sQ^*(x, y, z) = 0 \mid w, s \in \mathbb{R}, (w, s) \neq (0, 0)\}.$$

Now

$$f \cdot \big(\mathrm{p}(P, Q)\big) = \{w(P^* \circ f^*)(x, y, z) + s(Q^* \circ f^*)(x, y, z) = 0 \qquad \mid w, s \in \mathbb{R}, (w, s) \neq (0, 0)\}.$$

But $P^* \circ f^* = (P \circ f)^*$ hence

$$f \cdot \big(\mathrm{p}(P, Q)\big) = \{w(P \circ f)^*(x, y, z) + s(Q \circ f)^*(x, y, z) = 0 \qquad \mid w, s \text{ real}, (w, s) \neq (0, 0)\}.$$

Since the matrix of f is invertible, we have:

$$\Phi\big(f \cdot (\omega = 0)\big) = f \cdot \big(\mathrm{p}(P, Q)\big). \tag{4.5}$$

□

The map of A-sets $\Phi_{4\mathrm{f}} \colon E_{4\mathrm{f}} \to F_{4\mathrm{f}}$ induces a map on the A-orbit spaces:

$$\Phi_{4\mathrm{f}}/A \colon E_{4\mathrm{f}}/A \to F_{4\mathrm{f}}/A. \tag{4.6}$$

We intend to use this map to give meaning to the parameters of the family of differential equations E_4.

Notation 4.1. Let us denote by $S_{4\mathrm{f}}$ the set of all sets of four distinct points in the real affine plane such that no three of them are collinear. Let $\sigma_{4\mathrm{f}} \colon F_{4\mathrm{f}} \to S_{4\mathrm{f}}$ where $\sigma_{4\mathrm{f}}(p) = \sigma(p)$ for all p in $F_{4\mathrm{f}}$. We shall denote by S_{4fo} the set of all ordered sets of four distinct points $(B, <)$ where B is in $S_{4\mathrm{f}}$.

Proposition 4.2. *A acts on $S_{4\mathrm{f}}$ by substitution. $\sigma_{4\mathrm{f}}\colon F_{4\mathrm{f}} \to S_{4\mathrm{f}}$ is a map of A-sets, i.e., the following diagram is commutative:*

$$\begin{array}{ccc} A \times F_{4\mathrm{f}} & \longrightarrow & F_{4\mathrm{f}} \\ {\scriptstyle 1_A \times \sigma_{4\mathrm{f}}} \downarrow & & \downarrow {\scriptstyle \sigma_{4\mathrm{f}}} \\ A \times S_{4\mathrm{f}} & \longrightarrow & S_{4\mathrm{f}} \end{array} \tag{4.7}$$

$S_{4\mathrm{fo}}$ is an open subset of $\mathbb{R}^8$. $S_{4\mathrm{fo}}$ has also a structure of an A-set and the forgetful map $\phi\colon S_{4\mathrm{fo}} \to S_{4\mathrm{f}}$ is a map of A-sets. The dimension of $S_{4\mathrm{fo}}/A$ is two.

Proof. The first part of the proposition is clear. $\dim(S_{4\mathrm{fo}}/A) = 2$ since for $q \in S_{4\mathrm{fo}}$, $q = (q_1, q_2, q_3, q_4)$ in $S_{4\mathrm{fo}}$, (q_1, q_2, q_3) forms an affine frame in $\mathbb{R}^2$ and hence there is a unique affine transformation ${}_fq\colon \mathbb{R}^2 \to \mathbb{R}^2$ which sends q_1 to $(0,0)$, q_2 to $(1,0)$ and q_3 to $(1,0)$. Hence the point q in $S_{4\mathrm{fo}}$ determines a unique point $f_q(q_4)$ in $\mathbb{R}^2$. This gives rise to a homeomorphism $f'\colon S_{4\mathrm{fo}}/A \to \mathbb{R}^2 \setminus \{xy = 0\}$ where $f'([q]) = f_q(q_4)$. □

We recall that by the Corollary 3.1 the map $\sigma_4\colon F_4 \to SP_4$ is a bijection which yields:

Corollary 4.1. *The map $\sigma_{4\mathrm{f}}\colon F_{4\mathrm{f}} \to S_{4\mathrm{f}}$ is an isomorphism of A-sets yielding a bijection $\sigma_{4\mathrm{f}}/A\colon F_{4\mathrm{f}}/A \to S_{4\mathrm{f}}/A$.*

In view of the above, fibering $E_{4\mathrm{f}}/A$ over $F_{4\mathrm{f}}/A$ is equivalent to fibering $E_{4\mathrm{f}}/A$ over $S_{4\mathrm{f}}/A$.

For each equation $P\,dx + Q\,dy = 0$ in $E_{4\mathrm{f}}$ we consider $\sigma_{4\mathrm{f}}\big(\mathrm{p}(P,Q)\big)$ and the map

$$\pi = \sigma_{4\mathrm{f}} \circ \Phi_{4\mathrm{f}}\colon E_{4\mathrm{f}} \to S_{4\mathrm{f}} \tag{4.8}$$

of A-sets yielding a map

$$\pi/A\colon E_{4\mathrm{f}}/A \to S_{4\mathrm{f}}/A. \tag{4.9}$$

$E_{4\mathrm{f}}$ is a an open subspace of $\mathrm{P}^{11}(\mathbb{R})$. Hence $E_{4\mathrm{f}}/A$ is a topological space with the quotient topology. The topology of $\mathbb{R}^2$ induces a topology on $S_{4\mathrm{f}}$. Indeed, let $\{p_i \mid i \le 4\} \in S_{4\mathrm{f}}$ and let U_{p_i} be open sets in $\mathbb{R}^2$ such that for every set $\{p'_i \mid i \le 4\}$ such that $p'_i \in U_{p_i}$ we have that $\{p'_i \mid i \le 4\} \in S_{4\mathrm{f}}$. This is possible since $\{p_i \mid i \le 4\} \in S_{4\mathrm{f}}$. Then a basis of open sets in $S_{4\mathrm{f}}$ is obtained by taking the following as open sets: $U_{\{p_i\}_{i\le 4}} = \big\{\{p'_i \mid i \le 4\} \mid p'_i \in U_{p_i}\big\}$. The map $\pi\colon E_{4\mathrm{f}} \to S_{4\mathrm{f}}$ is clearly continuous and it induces a continuous map $\pi/A\colon E_{4\mathrm{f}}/A \to S_{4\mathrm{f}}/A$. To $(E_{4\mathrm{f}}/A, \pi/A, S_{4\mathrm{f}}/A)$ we can put a structure of a principal fiber bundle with fiber and structure group $\mathrm{Aut}\big(\mathrm{P}^1(\mathbb{R})\big) = \mathrm{PGL}(2,\mathbb{R})$. To show this we shall use the following:

Lemma 4.1. *Let $\alpha \in F_{4\mathrm{f}}$. If $a \in \sigma(\alpha)$, then a is not a singular point of any conic in α. We can therefore consider the map*

$$t_a\colon \alpha \to P(T_a\mathbb{R}^2) \tag{4.10}$$

where $t_a(C)$ is the one-dimensional subspace of $T_a\mathbb{R}^2$ determined by the unique tangent line to C at the point a. Then t_a is an isomorphism.

Proof. Let $\sigma(\alpha) = \{a, b, c, d\}$. In what follows we identify the one-dimensional subspace of $T_a\mathbb{R}^2$ detremined by the tangent line to C at the point a with this tangent line. We note that we have:

$$t_a(C) = \begin{cases} ac, & \text{if } C = \{ac, bd\}, \\ ad, & \text{if } C = \{ad, bc\}, \\ ab, & \text{if } C = \{ab, cd\}. \end{cases} \tag{4.11}$$

We show that t_a is a surjective map. Let $l \in \mathrm{P}(T_a\mathbb{R}^2)$. If l is one of the lines ac, ad, ab then there is a unique reducible conic in a, having l as a component. Assume now that l is not one of these lines. Let $p \in l$, $p \ne a$. Then in view of Theorem 3.1 there is a unique conic passing through a, b, c, d, and p. Let $C(p)$ be this conic. This conic corresponds to a point in the one-dimensional subvariety V of $\mathrm{P}^5(\mathbb{R})$ associated to the pencil of conics α. V is a closed subset of $\mathrm{P}^5(\mathbb{R})$ and hence if p_n tends to a as n tends to infinity, there is a subsequence $\{p_{n_i}\}_{i\ge 0}$ of $\{p_n\}_{n\ge 0}$ such that $C(p_{n_i})$ tends to a conic C as i tends to infinity. Clearly C belongs to α and we have $t_a(C) = l$. Hence t_a is a surjective map. t_a is also injective since if $t_a(C') = t_a(C'')$, C' and C'' have two coincident points at a and hence C' and C'' have fewer than four distinct common points, contrary to our hypothesis. □

From this lemma it follows that given a set $\sigma_{4\mathrm{f}}(\alpha)$ with $\alpha = \mathrm{p}(P,Q)$, specifying a conic in this pencil is equivalent to specifying a line $l \in P(T_a\mathbb{R}^2)$.

We shall now show that $\big(E_{4f}/A, \pi/A, S_{4f}/A, \mathrm{Aut}(\mathrm{P}^1(\mathbb{R}))\big)$ has the structure of a principal fiber bundle. We consider an open covering of S_{4f}/A formed by sets $[U]$ defined by representatives of the form $U = U_{\{p_i\}_{i\leq 4}}$ considered before, where $\{p_i \mid i \leq 4\} \in S_{4f}$. For each such U we need a homeomorphism

$$\Psi_U \colon (U/A) \times \mathrm{Aut}\big(\mathrm{P}^1(\mathbb{R})\big) \to (\pi/A)^{-1}(U/A) \tag{4.12}$$

such that $(\pi/A)\big(\Psi_U([\xi], s)\big) = [\xi]$ where $[\xi] \in U/A$, and $\xi \in U$. It will suffice to define a homeomorphism

$$\psi_U \colon U \times \mathrm{Aut}\big(\mathrm{P}^1(\mathbb{R})\big) \to (\pi)^{-1}(U) \tag{4.13}$$

such that $(\pi \circ \psi_U)(\xi, s) = \xi$. Let $\xi \in U$ and let $a \in \xi$. We need to define a bijection

$$\psi_{U,a}(\xi, -)\colon\ \mathrm{Aut}\big(\mathrm{P}^1(\mathbb{R})\big) \to (\pi)^{-1}(\xi). \tag{4.14}$$

or equivalently we need to define a bijection

$$\psi'_{U,a}(\xi, -)\colon (\pi)^{-1}(\xi) \to \mathrm{Aut}\big(\mathrm{P}^1(\mathbb{R})\big), \tag{4.15}$$

which is the inverse of $\psi_{U,a}(\xi, -)$. To construct $\psi'_{U,a}(\xi, -)$ we associate to a quadratic equation $e\colon Q\,dx + P\,dy = 0$ belonging to E_{4f} with $\xi = \sigma\big(\mathrm{p}(P, Q)\big) \in S_{4f}$ and $a \in \xi$, a projectivity $s_{\xi,e,a}\colon\ \mathrm{P}^1(\mathbb{R}) \to \mathrm{P}^1(\mathbb{R})$. We do this by associating to a point $[u, v]$ in $\mathrm{P}^1(\mathbb{R})$ the direction of the vector field $X = P\partial/\partial x + Q\partial/\partial y$ on the isocline $uP + vQ = 0$.

First we note that if $C\colon k(x, y) = 0$ belongs to $\mathrm{p}(P, Q)$, for $e\colon Q\,dx + P\,dy = 0$, then a is a nonsingular point of the conic C and hence $[(\partial k/\partial x)(a), (\partial k/\partial y)(a)]$ belongs to $\mathrm{P}^1(\mathbb{R})$ and the map

$$t'_a\colon\ \mathrm{p}(P, Q) \to \mathrm{P}^1(\mathbb{R}) \tag{4.16}$$

with $t'_a(C) = [(\partial k/\partial x)(a), (\partial k/\partial y)(a)]$ is a bijection (cf. Lemma 4.1). Using this map we define the projectivity $s_{\xi,e,a}$ by using the invertible linear transformation $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ which sends the ordered basis $\{g, h\}$, where

$$g = \left(\frac{\partial P}{\partial x}(a), \frac{\partial P}{\partial y}(a)\right), \quad h = \left(\frac{\partial Q}{\partial x}(a), \frac{\partial Q}{\partial y}(a)\right),$$

to the canonical basis. The projectivity $s_{\xi,e,a}\colon\ \mathrm{P}^1(\mathbb{R}) \to \mathrm{P}^1(\mathbb{R})\colon [w', w''] \mapsto [f(w', w'')]$. For $[w', w''] \in \mathrm{P}^1(\mathbb{R})$ there exists a unique conic $C\colon k(x, y) = 0$ in $\mathrm{p}(P, Q)$ such that $[w', w''] = t'_a(C) = [(\partial k/\partial x)(a), (\partial k/\partial y)(a)]$. In particular

$$s_{\xi,e,a}\left(\left[\frac{\partial P}{\partial x}(a), \frac{\partial P}{\partial y}(a)\right]\right) = \left[f\left(\frac{\partial P}{\partial x}(a), \frac{\partial P}{\partial y}(a)\right)\right] = [f(g)] = [1, 0].$$

The map $s_{\xi,e,a}$ thus associates to the line $(\partial P/\partial x)(a)(x - a_1) + (\partial P/\partial y)(a)(y - a_2) = 0$, the line $x = 0$ which is the direction of the vector field $X = P\partial/\partial x + Q\partial/\partial y$ on $P = 0$. Analogously this map associates to the line $(\partial Q/\partial x)(a)x + (\partial Q/\partial y)(a)y = 0$, the line $y = 0$ which is the direction of the field X on $Q = 0$. Now consider a conic in the pencil $\mathrm{p}(P, Q)$. This conic has an equation of the form $uP + vQ = 0$ with u, v real not both zero. This conic is uniquely determined by its tangent line at a whose equation is:

$$\left(u\frac{\partial P}{\partial x}(a) + v\frac{\partial Q}{\partial x}(a)\right)(x - a_1) + \left(u\frac{\partial P}{\partial y}(a) + v\frac{\partial Q}{\partial y}(a)\right)(y - a_2) = 0 \tag{4.17}$$

This line determines a point

$$\left[\left(u\frac{\partial P}{\partial x}(a) + v\frac{\partial Q}{\partial x}(a)\right), \left(u\frac{\partial P}{\partial y}(a) + v\frac{\partial Q}{\partial y}(a)\right)\right] \tag{4.18}$$

in $\mathrm{P}^1(\mathbb{R})$. We have

$$\left(u\frac{\partial P}{\partial x}(a) + v\frac{\partial Q}{\partial x}(a), u\frac{\partial P}{\partial y}(a) + v\frac{\partial Q}{\partial y}(a)\right) = ug + vh.$$

$f(ug+vh) = (u, v)$. Hence $s_{\xi,e,a}([ug+vh]) = [f(ug+vh)] = [uf(g)+vf(h)] = [u, v]$. The point (u, v) determines a line in $\mathbb{R}^2$ of equation is $ux+vy = 0$. Let us consider now a point $(x_0, y_0) \notin \xi$ situated on the conic $uP(x, y)+vQ(x, y) = 0$. We have $uP(x_0, y_0) + vQ(x_0, y_0) = 0$. (u, v) are the coordinates of the line having the direction of the vector $\big(P(x_0, y_0), Q(x_0, y_0)\big)$ and the projective map $s_{\xi,e,a}\colon\ \mathrm{P}^1(\mathbb{R}) \to \mathrm{P}^1(\mathbb{R})$ associates to the direction of the tangent to the conic $uP + vQ = 0$ at a the direction of the vector field on this isocline of the system. We have thus defined the map (4.12) by putting

$$\psi'_{U,a}(\xi, e) = s_{\xi,e,a}. \tag{4.19}$$

This map is a bijection since we can define its inverse map (4.11) as follows: Let $\xi \in U$, $a \in \xi$ and let s be a projective transformation $s\colon \mathrm{P}^1(\mathbb{R}) \to \mathrm{P}^1(\mathbb{R})$ defined by a linear map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. Let $g = f^{-1}(1,0)$ and $h = f^{-1}(0,1)$. Then $\{g, h\}$ forms an ordered basis of $\mathbb{R}^2$ and $\{(a,g),(a,h)\}$ is an ordered basis of $T_a\mathbb{R}^2$. We consider the two distinct lines l_a, l'_a in $\mathrm{P}(T_a\mathbb{R}^2)$ passing through a whose equations are respectively

$$l_a\colon \langle g, (x-a_1, y-a_2)\rangle = 0 \tag{4.20}$$

$$l'_a\colon \langle h, (x-a_1, y-a_2)\rangle = 0. \tag{4.21}$$

In view of the lemma, there exist unique distinct conics $C\colon P(x,y) = 0$ and $C'\colon Q(x,y) = 0$ passing through all points of ξ such that l_a, l'_a are respectively the tangent lines at a to the conics C, C'. Since the the tangent line to a conic $C^*\colon k(x,y) = 0$ passing through a has the equation:

$$\frac{\partial k}{\partial x}(a)(x-a_1) + \frac{\partial k}{\partial y}(a)(y-a_2) = 0, \tag{4.22}$$

in view of (4.20) and (4.21), we may assume

$$\left(\frac{\partial P}{\partial x}(a), \frac{\partial P}{\partial y}(a)\right) = g \quad \text{and} \quad \left(\frac{\partial Q}{\partial x}(a), \frac{\partial Q}{\partial y}(a)\right) = h.$$

If C passes through all points of ξ, a is a nonsingular point of ξ, so

$$\left(\frac{\partial P}{\partial x}(a), \frac{\partial P}{\partial y}(a)\right) \quad \text{and} \quad \left(\frac{\partial Q}{\partial x}(a), \frac{\partial Q}{\partial y}(a)\right)$$

are both nonzero and each defines an element of $\mathrm{P}^1(\mathbb{R})$. Since the tangent at a to C uniquely determines the curve C and vice-versa, the conic C is entirely determined by the point $[(\partial k/\partial x)(a), (\partial k/\partial y)(a)]$ in $\mathrm{P}^1(\mathbb{R})$. Let

$$\psi_{U,a}(\xi, s) = (Q\,dx + P\,dy = 0). \tag{4.23}$$

Then $\psi_{U,a}(\xi, -)$ is invertible and it is the inverse of $\psi'_{U,a}(\xi, -)$.

If instead of a we take $b \in \sigma\big(\mathrm{p}(P,Q)\big)$, then for the same equation $e\colon Q\,dx + P\,dy = 0$, we get a map $s_{\xi,e,b}\colon \mathrm{P}^1(\mathbb{R}) \to \mathrm{P}^1(\mathbb{R})$. The two maps $s_{\xi,e,a}$ and $s_{\xi,e,b}$, are isomorphic since there is a projectivity $s_{a,b}\colon \mathrm{P}^1(\mathbb{R}) \to \mathrm{P}^1(\mathbb{R})$ such that $s_{\xi,e,a} = s_{\xi,e,b}s_{a,b}$. The map $s_{a,b}$ may be viewed just as a coordinate change in $\mathrm{P}^1(\mathbb{R})$which does not affect the value of the isomorphisms $s_{\xi,e,a}$, $s_{\xi,e,b}$ at corresponding points via $s_{a,b}$.

Since P, Q are polynomials and the maps f used in the above construction are linear, the construction yields a continuous map

$$\psi_U\colon U \times \mathrm{Aut}(\mathrm{P}^1(\mathbb{R})) \to (\pi)^{-1}(U) \tag{4.24}$$

for which $\psi_{U,a}(\xi, -)$ is a homeomorphism.

To have a fiber bundle we also need that the group, in this case $\mathrm{Aut}\big(\mathrm{P}^1(\mathbb{R})\big)$ be a transformation group of the fiber which is again $\mathrm{Aut}\big(\mathrm{P}^1(\mathbb{R})\big)$. This is clear if we identify an element of $\mathrm{Aut}\big(\mathrm{P}^1(\mathbb{R})\big)$ with the left translation it engenders. Passing to the quotient spaces $E_{4\mathrm{f}}/A$, $S_{4\mathrm{f}}/A$, we thus obtain:

Theorem 4.1. *The ordered tuple* $\big(E_{4\mathrm{f}}/A, \pi/A, S_{4\mathrm{f}}/A, \mathrm{Aut}\big(\mathrm{P}^1(\mathbb{R})\big)\big)$ *forms a principal fiber bundle.*

In Section 3 we discussed pencils of conics classifying them according to the number of base points and the respective multiplicities of these base points. We have considered a generic part $F_{4\mathrm{f}}$ of the class of pencils of conics and showed how $F_{4\mathrm{f}}/A$ fibers $E_{4\mathrm{f}}/A$. In subsequent work we shall show how we can extend this to all the other types of pencils of conics.

5 Open questions and concluding comments

The above fibering separates the parameters of the space $E_{4\mathrm{f}}/A$ in those of the base space $S_{4\mathrm{f}}/A$ and those of the fiber. Hence the five dimensional space $E_{4\mathrm{f}}/A$ splits into a two-dimensional space (the base) and the three-dimensional space of the fiber which is homeomorphic to $\mathrm{Aut}\big(\mathrm{P}^1(\mathbb{R})\big)$. As seen in Proposition 4.2 the two parameters of the base space can be viewed as the coordinates of the fourth singular point of an equation in the frame given by the first three singular points.

Question 5.1. What is the significance of the remaining three other parameters of the space $E_{4\mathrm{f}}/A$?

Work of the school of Erugin separates the five parameters into the dynamic parameters and static parameters, the dynamic parameters being interpreted as rotation parameters. These last parameters correspond in the above principal bundle structure to the three parameters of the fiber, in other words the three parameters of the group $\mathrm{Aut}\big(\mathrm{P}^1(\mathbb{R})\big)$.

Question 5.2. In what sense can we interpret the parameters of the projective linear group $\mathrm{Aut}\big(\mathrm{P}^1(\mathbb{R})\big)$? Could we interpret them as rotation parameters?

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