Lie Groups and Difference Equations

P. Winternitz[a]

Abstract

We review some results of a research program, the aim of which is to extend the applicability of Lie group methods from differential equations to difference ones. Emphasis is on two topics. The first is how to find the symmetry group of a system of difference equations. The second is the discretization of nonlinear ordinary differential equations with superposition formulas in such a manner that the obtained difference equations are linearizable and also allow a superposition formula.

1 Introduction

Lie group theory started out as a theory of continuous transformations of differential equations and their solutions. The purpose of this presentation is to review some results of a research program, the aim of which is to extend the applicability of Lie's methods and results to "discrete" equations. These are difference equations, q-difference equations, differential-difference equations and various other equations that involve some variables that vary discretely.

Let us first of all try to sum up what does Lie group theory do for differential equations.

1. Lie groups as *symmetry groups* of differential equations. There are groups of (local) transformations taking solutions into solutions. These can be point transformations

$$\tilde{x} = \Lambda_g(x, u), \quad \tilde{u} = \Omega_g(x, u), \tag{1}$$

contact transformations.

$$\tilde{x} = \Lambda_g(x, u, u_x,), \tilde{u} = \Omega_g(x, u, u_x),$$

or generalized (Lie-Bäcklund) transformations

$$\tilde{x} = \Lambda_g(x, u, u_x, u_{xx}, \ldots), \tilde{u} = \Omega(x, u, u_x, u_{xx}, \ldots).$$

In all cases, if $u = u(x)$ is a solution, then so is $\tilde{u} = \tilde{u}(\tilde{x}, g)$, where g represents group parameters.

Algorithms exist for determining the symmetry group G of a system of differential equations [1,2] and they have been computerized [3,4], at least for point transformations. Once the group G is known, it can be put to good use. For instance:

   (a) To transform known solutions into classes of new ones.

   (b) To perform symmetry reduction. For partial differential equations this amounts to constructing solutions invariant under some subgroup of the symmetry group. This reduces the number of independent variables in the equation to solve. For ordinary differential equations the symmetry group is used to reduce the order of the equation.

   (c) To identify equations that are equivalent, i.e. can be transformed into each other by point (or other) transformations.

2. Lie transformation groups that are *not* symmetry groups can be used to classify equations into conjugacy classes and to choose a representative of a class that is particularly easy to solve. For instance, nonlinear equations can sometimes be transformed into linear ones.

3. Lie theory has been used to generate classes of nonlinear ordinary differential equations with superposition formulas. Or, in other words, equations for which the general solution can be written as a "nonlinear superposition formula" in terms of a finite number of particular solutions [5,6].

4. Lie theory provides criteria of "integrabilitiy" of nonlinear partial differential equations. By this we mean the possibility of either transforming them into linear ones, or constructing Lax pairs [7] for them.

5. Virtually all of special function theory and the theory of the separation of variables in linear partial differential equations, can be best formulated in terms of symmetry groups of differential equations.

The question now is: how does one extend the essentially continuous techniques of Lie theory to discrete equations? This is a topic of considerable concern at this moment. So much so that a series of workshops on it has already emerged, called SIDE (Symmetries and Integrability of Difference Equations), taking place every two years since 1994 (so far in Montreal (Canada), Canterbury (UK) and Sabaudia (Italy).

2 Symmetries of Difference Equations

For differential equations it is usually not fruitful to look directly for symmetry transformations in the form of eq. (1). One of S. Lie's fundamental contributions was that he developed an infinitesimal approach, in which global, or local, point transformations are replaced by infinitesimal ones

$$\tilde{x}_i = x_i + \varepsilon\xi_i(x, u), \quad \tilde{u}_\alpha = u_\alpha + \varepsilon\phi_\alpha(x, u) \tag{2}$$

with $\varepsilon \ll 1$ (so that ε^k, $k \geq 2$ can be ignored). The problems of finding a symmetry group is replaced by that of finding its Lie algebra, the symmetry algebra, realized by vector fields of the form

$$\widehat{X} = \sum_{i=1}^{p} \xi_i(x,u)\partial_{x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\partial_{u_\alpha}. \tag{3}$$

It is this infinitesimal approach that we wish to adapt to the case of difference equations.

Several different methods are being currently investigated.

2.1 The "intrinsic method"

In this method the dependent variables (u) are considered to be continuous, among the independent ones, some are continuous (x), some are discrete (n). The studied equations have the form[8–17]

$$E_n\big(x, n, u(n+k)\,|_{k=a}^{b}, u_{x_i}(n+k)\,|_{k=a_i}^{b_i}, u_{x_i x_k}(n+k)\,|_{k=a_{ik}}^{b_{ik}}, \ldots\big). \tag{4}$$

Thus the equations involve the functions u and their derivatives, with respect to the continuous variables, evaluated at different points $n+k$ of a lattice. An example of such an equation is the two-dimensional Toda lattice [18,19], or Toda field theory [20–22]

$$u(n)_{xt} - e^{u(n-1)-u(n)} + e^{u(n)-u(n+1)} = 0. \tag{5}$$

The Lie algebra of the symmetry group is realized by vector fields of the form

$$\widehat{X} = \sum_{i=1}^{p} \xi_i\big(x, u(n)\big)\partial_{x_i} + \phi\big(n, x, u(n)\big)\partial_{u(n)}. \tag{6}$$

Thus, n is a variable, just like x, and the coefficients ξ_i and ϕ depend on $u(n)$ but not on the function $u(n)$ at other points of the lattice (e.g. $n \pm 1$).

The algorithm for finding the functions ξ_i and ϕ in eq. (6) is similar to that for differential equations. Namely, the vector field $\widehat{X}$ is prolonged so as to act on derivatives of u and, in the case of difference equations, on the function u and its derivatives at all points of the lattice. Thus, we have for instance

$$\begin{aligned} pr^{(2)}\widehat{X} &= \sum_{i=1}^{p} \xi_i\big(x, u(n)\big)\partial_{x_i} + \sum_{k=n+a}^{n+b} \phi\big(k, x, u(k)\big)\partial_{u(k)} \\ &+ \sum_{i=1}^{p} \sum_{k=n+a_j}^{n+b_i} \phi^{x_i}(k)\partial_{u_{x_i}(k)} + \sum_{i,j=1}^{p} \sum_{k=n+a_{ij}}^{n+b_{ij}} \phi^{x_i x_j}(k)\partial_{u_{x_i x_j}(k)} \end{aligned} \tag{7}$$

$$\begin{aligned} \phi^{x_i}(k) &= D_{x_i}\phi(k) - \sum_{j=1}^{p} (D_{x_i}\xi_j)u_{x_j}(k) \\ \phi^{x_i x_j}(k) &= D_{x_j}\phi^{x_i}(k) - \sum_{\ell=1}^{p} (D_{x_j}\xi_\ell)u_{x_i x_\ell} \end{aligned} \tag{8}$$

where e.g. D_{x_i} is a total derivative. The difference between the case of differential equations and that of differential-difference ones are the summations over k in eq. (7), i.e. the summations over different points on the lattice. The invariance condition for the equation is

$$pr\widehat{X} \cdot E|_{E=0} = 0, \tag{9}$$

that is, the prolongation of the vector field $\widehat{X}$ should annihilate the equation on its solution set. The determining equations, i.e. an overdetermined set of linear differential-difference equations for the functions ξ and ϕ can be read off from eq. (9) and solved

For the Toda field equations (5) the result of this procedure is an infinite dimensional Lie algebra, represented by the vector fields [9]

$$\begin{aligned} T(f) &= f(t)\partial_t + f'(t)n\partial_{u_n}, \quad U(k) = k(t)\partial_{u_n} \\ X_g &= g(x)\partial_x + g'(x)n\partial_{u_n}, \quad W(h) = h(x)\partial_{u_n}, \end{aligned} \tag{10}$$

where $f(t)$, $k(t)$, $g(x)$ and $h(x)$ are arbitrary (smooth) functions. The fields $T(f)$ and $X(g)$ correspond to the conformal invariance of the theory, $U(k)$ and $W(h)$ cporrespond to gauge transformations. The structure of the Lie algebra (10) is that of a Kac-Moody-Virasoro algebra. Thus, the Toda system (5) in two continuous and one discrete variables shares an important property of integrable system in three continuous variables, like the Kadomtsev-Petviashvili equation and infinitely many others [23,24]. Namely, their Lie point symmetry algebras are Kac-Moody-Virasoro algebras.

The "intrinsic method" produces purely point symmetries. The vector fields (6) do not depend on derivatives of u, nor on finite differences, i.e. values of u at other points of the lattice than the point n.

More general symmetries could be obtained using the "differential equations method" [9,11,17], in which eq. (4) is viewed as an infinite set of equations for infinitely many dependent functions u_n. Then n is not viewed as a variable, just as a label, enumerating equations and functions. The vector fields realizing the symmetry algebra have the form (3) (or equivalently (6)), but $u = \{u_k\}$ has infinitely many components, so we have $\xi = \xi(x, \{u_k\})$, $\phi_\alpha = \phi_\alpha(x, \{u_k\})$, with $k \in \mathbb{Z}$. The summation in eq. (3) is over $\alpha \in \mathbb{Z}$. The symmetries thus obtained would correspond to point symmetries in the continuous variables x and higher symmetries in the discrete variables n. This method is more difficult to implement than the intrinsic method. Moreover, quite often the two methods give the same result [9,17].

2.2 The "continuous variable" aproach to difference equations

Let us now consider all independent and dependent variables to be continuous, but to sample the independent variables at discrete points of a regular lattice. We introduce shift operators T_i and "discrete derivatives" Δ_{x_i} on an n-dimensional lattice with spacings $\sigma_i \in \mathbb{R}^{\geq 0}$ in the direction x_i:

$$T_i f(x) = f(x_1, \ldots, x_{i-1}, x_i + \sigma_i, x_{i+1}, \ldots, x_n), \tag{11}$$

$$\Delta_{x_i} f(x) = \frac{1}{\sigma_i}(T_i - 1) f(x). \tag{12}$$

In the continuous limit we have $\sigma_i \to 0$, $T_i \to 1$, $\Delta_{x_i} \to \partial_{x_i}$. The equations to be studied have the form

$$E(x, T^\alpha u, T^{\beta_i}\Delta_{x_i} u, T^{\beta_{ij}}\Delta_{x_i}\Delta_{x_j} u, \ldots) = 0, \tag{13}$$

where e.g. the symbol $T^\alpha u$ means that the function $u(x)$ can figure at some finite set of points, e.g. $u(x_1, x_2)$, $u(x_1 + \sigma_1, x_2)$, $u(x_1, x_2 + 2\sigma_2)$, etc.

If eq. (13) is linear, it can be rewritten as

$$Lu = 0, \tag{14}$$

where L is a linear operator of the form

$$L = A_{ik}\Delta_{x_i}\Delta_{x_j} + B_i \Delta_{x_i} + C, \tag{15}$$

where the coefficients A_{ik}, B_i and C depend on x_i and on the shift operators T_i (L can of course be of higher order).

For linear difference equations symmetry algebras can be studied in terms of difference operators X, commuting with the operator L on the solution set of the equation:

$$X = \sum_{i=1}^{q} \xi_i(x, T_j)\Delta_{x_i} + f(x, T_j) \tag{16}$$

$$[L, X] = \lambda L. \tag{17}$$

Floreanini and Vinet [25] have obtained a remarkable result for linear difference equations. Namely, their Lie symmetry algebras realized by difference operators, are isomorphic to the symmetry algebras of their continuous limits, realized by differential operators.

This approach to symmetries of linear difference equations can be reformulated in terms of evolutionary vector fields and then generalized to nonlinear difference equations [26]. Evolutionary vector fields [1] act only on the dependent variables in an equation. Thus, for a system of differential equations, the vector field (3), generating point symmetries, is equivalent to the evolutionary vector field

$$\widehat{X}_e = Q^\alpha \partial_{u_\alpha}, \quad Q^\alpha = \sum_{i=1}^{p} \xi_i(x, u) u_{\alpha, x_i} - \phi_\alpha(x, u). \tag{18}$$

More generally, for generalized symmetries we have

$$Q^\alpha = Q^\alpha(x, u, u_x, u_{xx}, \ldots). \tag{19}$$

For difference equations in general the evolutionary vector field will have the form

$$\widehat{X}_e = \sum_\alpha Q^\alpha(x, T^a u, T^b \Delta_{x_i} u, T^c \Delta_{x_i} \Delta_{x_j} u, \ldots)\partial_{u_\alpha} \tag{20}$$

and the functions Q^α are solutions of determining equations, obtained by requiring that the vector field should annihilate the equation $prX_e E|_{E=0} = 0$.

For linear difference equations a simplified Ansatz for the evolutionary vector fields is sufficient for obtaining all symmetries that reduce to point symmetries in the continuous limit.

This "linear" Ansatz is

$$Q = \Sigma \xi_i(x, T^a u) T^b \Delta_{x_i} u - \phi(x, T^c u). \tag{21}$$

In particular, if ξ_i is independent of u and ϕ is linear in u, this Ansatz provides the commuting operator X of eq. (16), with $f = -\phi/u$.

As an example, consider the discrete heat equation [26],

$$(\Delta_t - \Delta_{xx})u = 0 \tag{22}$$

The symmetry algebra is isomorphic to that of the continuous heat equation [1]. A basis for the algebra is given by the following evolutionary fields, corresponding to time and space translations, multiplication of the solution by a constant, Galilei transformations, dilations and projective transformations, respectively:

$$\begin{aligned}
P_0 &= (\Delta_t u)\partial_u, P_1 = (\Delta_x u)\partial_u, W = u\partial_u, \\
B &= \Big(2tT_t^{-1}\Delta_x u + xT_x^{-1} u + \frac{1}{2}\sigma_x T_x^{-1} u\Big)\partial_u, \\
D &= \Big[2tT_t^{-1}\Delta_t u + xT_x^{-1}\Delta_x u + \Big(1 - \frac{1}{2}T_x^{-1}\Big)u\Big]\partial_u, \\
K &= \Big[t^2 T_t^{-2}\Delta_t u + txT_x^{-1}T_t^{-1}\Delta_x u + \frac{1}{4}x^2 T_x^{-2} u \\
&\quad + t\Big(T_t^{-2} - \frac{1}{2}T_t^{-1}T_x^{-1}\Big)u - \frac{1}{16}\sigma_x^2 T_x^{-2} u\Big]\partial_u.
\end{aligned} \tag{23}$$

For nonlinear difference equations the general form of Q^α in eq. (20) is needed. As an example, consider a discrete Burgers equation [27]. We derive it from the discrete heat equation by introducing a discrete Cole-Hopf transformation. Thus we need

$$\Delta_t \phi = \Delta_{xx}\phi, \tag{24}$$

$$\Delta_x \phi = u\phi. \tag{25}$$

Compatibility of these two equations implies that $u(x,t)$ must satisfy

$$\Delta_t u = \frac{1 + \sigma_x u}{1 + \sigma_t[\Delta_x u + uT_x u]}\Delta_x(\Delta_x u + uT_x u). \tag{26}$$

The continuous limit is the famous Burgers equation

$$u_t = u_{xx} + 2uu_x. \tag{27}$$

Eq. (26) will be called the discrete Burgers equation. Like the usual Burgers equation, it is linearizable, via a (discrete) Cole-Hopf transformation, to the (discrete) heat equation. The symmetries of eq. (26) could be calculated directly. It is however simpler to obtain them from those of the heat equation, making use of the Cole-Hopf transformation[27]. The algebra is 5 dimensional; here we present the 4 simpler evolutionary vector fields, representing time and space translations, Galilei transformations and dilations (for the generator of projective transformations, see the original article[27]):

$$\begin{aligned}
P_0 &= [(1 + \sigma_t v)\Delta_t u)]\partial_u, P_1 = [(1 + \sigma_x u)\Delta_x u]\partial_u, \\
B &= \Big\{(1 + \sigma_x u)\Delta_x\Big[2tT_t^{-1}\frac{u}{1 + \sigma_t v} + \Big(x + \frac{1}{2}\sigma_x\Big)T_x^{-1}\frac{1}{1 + \sigma_x u}\Big]\Big\}\partial_u, \\
D &= \Big\{(1 + \sigma_x u)\Delta_x\Big[2tT_t^{-1}\frac{v}{1 + \sigma_t v} + xT_x^{-1}\frac{u}{1 + \sigma_x u} - \frac{1}{2}T_x^{-1}\frac{1}{1 + \sigma_x u}\Big]\Big\}\partial_u,
\end{aligned} \tag{28}$$

with

$$v = \Delta_x u + uT_x u \tag{29}$$

3 Nonlinear Ordinary Difference Equations with Superposition Formulas

S. Lie proved the following theorem about "nonlinear superposition" for systems of nonlinear ordinary differential equations.

Theorem 1 *The general solution of the system of equations*

$$\dot{y}^\mu = \eta^\mu(y_1, \dots, y_n, t), \quad \mu = 1, \dots, n \tag{30}$$

can be expressed as a function of m particular solutions and n arbitrary constants

$$\vec{y} = \vec{F}(\vec{y}_1, \dots, \vec{y}_m, C_1, \dots, C_n) \tag{31}$$

if and only if

1. *Eq. (30) has the form*

$$\dot{y}^\mu = \sum_{k=1}^{r} Z_k(t)\xi_k^\mu(\vec{y}). \tag{32}$$

2. *The vector fields*

$$X_k = \sum_{\mu=1}^{r} \xi_k^\mu(\vec{y}) \frac{\partial}{\partial y^\mu} \tag{33}$$

generate a finite dimensional Lie algebra L

$$[X_k, X_\ell] = \sum_{m=1}^{r} C_{k\ell}^m X_m. \tag{34}$$

The number m of particular solutions needed satisfies

$$mn \geq r$$

where $r = \dim L$.

A recent series of papers [5,6] was devoted to classifying, constructing and solving indecomposable systems of ordinary differential equations with superposition formulas[5,6,28]. Such equations can be associated with homogeneous spaces $M \sim G/G_0$ where the local action of G on M is transitive, effective and primitive.

The question arises: given a system of ordinary differential equations with a superposition formula, can one discretize in such a way as to obtain difference equations with superposition formulas? The answer is positive [29]. Here we shall just illustrate it with one important example, namely the matrix Riccati equation.

The equation is

$$\begin{aligned} &\dot{W} = A + WB + CW + WDW, \quad TrC = TrB \\ &W, A \in \mathbb{R}^{p\times q}, B \in \mathbb{R}^{q\times q}, C \in \mathbb{R}^{p\times p}, D \in \mathbb{R}^{q\times p} \end{aligned} \tag{35}$$

where the dependent variables are the matrix elements $w_{\alpha a}(t)$ of W and A, B, C, D are given matrix functions of time t.

The vector fields (33) in this case are

$$\begin{aligned} &\widehat{A}_{\alpha a} = \frac{\partial}{\partial w_{\alpha a}}, \quad \widehat{B}_{ba} = \sum_\alpha w_{\alpha b} \frac{\partial}{\partial w_{\alpha a}} \\ &\widehat{C}_{\alpha\beta} = \sum_a w_{\beta a} \frac{\partial}{\partial w_{\alpha a}}, \quad \widehat{D}_{b\beta} = \sum_{\alpha,a} w_{\alpha b} w_{\beta a} \frac{\partial}{\partial w_{\alpha a}} \end{aligned} \tag{36}$$

These vector fields generate the Lie algebra $sl(N, \mathbb{R})$, $N = p + q$, so the conditions of Lie's theorem are satisfied. The number of equations is $n = p \cdot q$.

The superposition formula has the form

$$W(t) = [G_{11}U + G_{12}][G_{21}U + G_{22}]^{-1}, \tag{37}$$

where $U \in \mathbb{R}^{p\times q}$ is a constant matrix, determining the initial conditions. The matrices G_{11}, G_{12}, G_{21} and G_{22} can be reconstructed from a set of m known solutions [6]. For $p = q \geq 2$ we have $m = 5$.

The matrix Riccati equation can be linearized by embedding it into a higher dimensional space, where the group G acts linearly:

$$W = XY^{-1}, \quad X \in \mathbb{R}^{p\times q}, \quad Y \subset \mathbb{R}^{q\times q}, \quad \det Y \neq 0. \tag{38}$$

We put

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} C & A \\ -D & -B \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, TrC = TrB, \tag{39}$$

then W satisfies eq. (35).

This linearization is the key to the appropriate discretization of the matrix Riccati equation. Indeed, let us replace eq. (39) by a system of linear difference equations.

We first define a matrix $G(t) \in SL(n, \mathbb{R})$ satisfying

$$\frac{dG}{dt} = \xi, \quad \xi = \begin{pmatrix} C & A \\ -D & -B \end{pmatrix}, \quad G(0) = I. \tag{40}$$

Next we discretize time, putting

$$\begin{aligned} &X(t) \equiv X, \quad Y(t) \equiv Y, \quad X(t+\delta) = \overline{X}, \\ &Y(t+\delta) = \overline{Y} \quad G(t) \equiv G(t+\delta) = \overline{G} \end{aligned} \tag{41}$$

and replace eq. (39) by

$$\begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{42}$$

The discrete matrix Riccati equation is obtained using the projection (38). We obtain the matrix homographic mapping

$$\overline{W} = (G_{11}W + G_{12})(G_{21}W + G_{22})^{-1}. \tag{43}$$

By construction eq. (43) is a difference equation that is linearizable and has a superposition formula. Its continuous limit is obtained by passing from the Lie group G to its Lie algebra $sl(n, \mathbb{R})$, i.e. putting

$$G_{11} = I + \delta C, \quad G_{12} = \delta A, \quad G_{21} = -\delta B, \quad G_{22} = I - \delta D. \tag{44}$$

For $\delta \to 0$ we have $(\overline{W} - W)/\sigma \to \dot{W}$ and eq. (43) goes into eq. (35).

4 Conclusions

We have shown how two aspects of Lie group analysis for differential equations can be adapted to difference equations: the calculation of symmetry groups and the construction of linearizable equations. Other aspects, like conservation laws, reduction formulas, integrability criteria, exact solutions, etc. also have their discrete analogs.

Acknowledgements

The research reported here was partially supported by research grants from NSERC of Canada, FCAR du Québec and NATO.

References

1. P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1993.
2. P. Winternitz, *Lie Groups and Solutions of Nonlinear Partial Differential Equations*, in Integrable Systems, Quantum Groups and Quantum Field Theory, Kluwer, Dordrecht, 429–495, 1993.
3. B. Champagne, W. Hereman and P. Winternitz, Comp. Phys. Commun. **66**, 319 (1991).
4. W. Hereman, *Symbolic software for Lie symmetry analysis*, in CRC Handbook of Lie Group Anaysis for Differential Equations, Vol. 3, CRC Press, Boca Raton, 367–522, 1996.
5. S. Shnider and P. Winternitz, J. Math. Phys., **25**, 3155 (1984).
6. J. Harnad, P. Winternitz and R. L. Anderson, J. Math. Phys. **24**, 1062 (1983).
7. M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991.
8. D. Levi and P. Winternitz, Phys. Lett. A, **152**, 335 (1991).
9. D. Levi and P. Winternitz, J. Math. Phys. **34**, 3713 (1993).
10. D. Levi and P. Winternitz, **37**, 5551 (1996).

11. S. Lafortune, L. Martina and P. Winternitz, *Symmetries of Toda field theories and their reductions*, Preprint CRM-2566, 1998.
12. D. Gomez Ullate, S. Lafortune and P. Winternitz, *Symmetries of discrete dynamical systems involving two species*, Preprint CRM-2567, 1998.
13. D. Levi and M. A. Rodriguez, J. Phys. A. Math. Gen. **25**, L875 (1992).
14. G. R. W. Quispel, H. W. Capel and R. Sahadevan, Phys. Lett. **A170**, 379–383 (1992).
15. N. Joshi and P. J. Vassiliou, J. Math. Anal. Appl. **195**, 872 (1995).
16. S. Maeda, IMA J. Appl. Math. **38**, 129 (1987).
17. Zhuhan Jiang, Phys. Lett. **A240**, 137 (1993).
18. A. Mikhailov, Pisma Zh. Eksp. Teor. Fiz., 77, 24 (1978).
19. A. P. Fordy and J. Gibbons, Commun. Math. Phys. **77**, 2 (1980).
20. A. N. Leznov and M. V. Saveliev, Lett. Math. Phys. **3**, 489 (1979).
21. A. Bilal and J. L. Gervais, Phys. Lett., **B206**, 412 (1988).
22. D. Olive and N. Turok, Nucl. Phys. **B215**, 470 (1983).
23. D. David, N. Kamran , D. Levi and P. Winternitz, Phys. Rev. Lett. **55**, 2111 (1985).
24. A. Yu. Orlov and P. Winternitz, J. Math. Phys. **38**, 4644 (1997).
25. R. Floreanini and L. Vinet, J. Math. Phys. **36**, 7024 (1995).
26. D. Levi, L. Vinet and P. Winternitz, J. Phys. A. Math. Gen. **30**, 633 (1997).
27. R. Hernandez Heredero, D. Levi, P. Winternitz, *Symmetries of the discrete Burgers equation*, Preprint CRM-2568, 1998.
28. L. Michel and P. Winternitz, *Families of Transitive Primitive Maximal Simple Lie Subalgebras of diff*(n), in Advances in Mathematical Sciences, CRM Proceedings and Lecture Notes, 451–479, AMS, Providence, R.I 1997.
29. B. Grammaticos, A. Ramani and P. Winternitz, *Discretizing families of linearizable equations*, CRM-2456, 1997.