Quadratic irrationalities and geometric properties of one-dimensional quasicrystals

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CRM-2565

September 1998

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Abstract

Scaling properties and tile lengths are described for 1-dimensional ('cut and project type') quasicrystals with quadratic irrationalities. Two infinite sequences of quadratic unitary Pisot numbers considered in this article correspond to algebraic equations $x^2 = mx \pm 1$, where m is a positive integer. An uncountable set of quasicrystals is related to any such Pisot number. A generic quasicrystal for any of these irrationalities has 3 types of tiles. Length of tiles and their ordering in a quasicrystal depend on the type of irrationality. We define binary operations analogous to quasiaddition of Berman and Moody [**?**] and study the possibility of generating all quasicrystal points by such operations. Only quasicrystals with the three lowest irrationalities can be generated using just one of the available quasiaddition operations from a few seed points. Curiously, these are the quasicrystals with Pisot numbers found in physical materials, namely $\frac{1}{2}(1+\sqrt{5})$, $1+\sqrt{2}$, and $2+\sqrt{3}$.

Résumé

Les quasi-cristaux unidimensionnels (de type ■cut and project■) avec les irrationnels quadratiques sont définis. On considère deux familles des nombres de Pisot quadratiques unitaires donnés par les équations $x^2 = mx+1$, où m est un entier positif. Un ensemble non dénombrable de quasi-cristaux correspond à chacun de ces nombres de Pisot quadratiques. Généralement un quasi-cristal a deux ou trois types de pavés. On décrit la longueur des pavés et leur ordre dans le pavage quasi-cristalique en fonction du nombre de Pisot considéré.

Sur les quasi-cristaux on définit des opérations binaires analogues à la quasi-addition de Berman et Moody. On étudie la possibilité de générer tous les points d'un quasi-cristal par de telles opérations. Les nombres de Pisot les plus bas des deux familles, notamment $\frac{1}{2}(1+\sqrt{5})$, $1+\sqrt{2}$, et $2+\sqrt{3}$, ont la propriété exceptionnelle suivante : Seulement des quasi-cristaux basés sur de tels nombres irrationnels peuvent être générés par juste une des opérations admissibles.

1 Introduction

In all cases what is called a quasicrystal in mathematics is either an infinite point set, or tiling of a corresponding space. For a recent overview of the field we refer to [**?**, **?**].

Quasicrystals take their origin in physics. The logic of the name is often expressed as 'aperiodic crystal'. They were discovered in 1984 [**?**] and eversince they have been a subject of intensive experimental and theoretical studies, (see for instance the earlier literature in [**?**] and proceedings from the most recent world quasicrystal conferences [**?**, **?**]). In spite of that there is no generally accepted definition of a quasicrystal which would satisfy a mathematician. In physics, with specific properties of real quasicrystals in mind, the absence of mathematical precision usually represents no difficulty.

In mathematics, the quasicrystals appeared about a decade before the discovery of quasicrystalline materials in physics under the name of Penrose tilings of the plane [**?**]. Their algebraic theory was then developed by de Bruijn [**?**]. Much later [**?**, **?**, **?**] it was recognized that quasicrystals can be understood as a special case in the general theory of Y. Meyer [**?**, **?**]. The earliest reference we know which links Meyer's theory to quasicrystals is [**?**].

The origins of quasicrystals in physics and mathematics are closely related to $\sqrt{5}$. Indeed, that is the irrationality in the Penrose tilings, and that is the irrationality which appeared in the diffraction spectra from which the existence of quasicrystals was first recognized. Other irrationalities were introduced into the theory of quasicrystals rather naturally later on as well.

It is convenient for our discussion to single out several ways of defining quasicrystals. First is the family of geometric definitions [**?**, **?**], where a slab is cut in a higher dimensional crystallographic lattice and its points are projected onto a suitable lower dimensional subspace. In this case the irrationality involved comes from the orientation of the subspace with respect to the lattice directions. The second family of definitions calls for quasicrystals to be point sets with Fourier transform producing Bragg peaks [**?**, **?**] in a suitable subspace. The irrationality here is established either indirectly or from the symmetry of the diffraction pattern. The third definition is algebraic. It is the one used in this paper. Typically the stage for a quasicrystal is an algebraic module over a ring of integers and it exploits the Galois automorphism of the corresponding algebraic field. A specific quasicrystal is determined by a position, shape and size of a bounded region, called an acceptance window. The earliest definition of this kind was presented in [**?**].

Quasicrystals carry a number of interesting lattice-like properties with the obvious exclusion of periodicity, namely self-similarities or relative discreteness and uniform density (Delone property), and possibly others. For years these attributes have been used in modelling quasicrystals. In this article we extend the detailed study of some geometric properties of quasicrystals with the golden mean $\tau = \frac{1}{2}(1+\sqrt{5})$ published in [**?**, **?**, **?**] to other quadratic irrationalities. In [**?**] a complete description of inflation centers inside and outside of a given quasicrystal in any dimensions was provided. The minimal distances between points of quasicrystals as functions of linear dimensions of the acceptance window were described in [**?**]. In the third paper of the series [**?**], sets invariant under a certain binary operation (quasiaddition) are studied and identified with quasicrystals.

The aim of this article is to generalize the results of [**?**, **?**, **?**], obtained for $\sqrt{5}$, to quasicrystals with irrationalities determined by two families of quadratic equations, namely

$$\begin{array}{lll} x^2 = mx + 1\,, & m = 1, 2, 3, \ldots & {}^{+}\text{family}\,, \\ x^2 = mx - 1\,, & m = 3, 4, 5, \ldots & {}^{-}\text{family}\,. \end{array} \tag{1}$$

It is convenient to distinguish the $^{+}$family and $^{-}$family of quasicrystal irrationalities.

The greater of the solutions of each of the equations for a fixed m is a Pisot number β. For any such β there are uncountably many non equivalent quasicrystals in the algebraic ring $\mathbb{Z} + \mathbb{Z}\beta$. Note that different Pisot numbers with different coefficients in the quadratic equation (1) may involve the same irrationality, (cf. (4)). The corresponding algebraic rings $\mathbb{Z} + \mathbb{Z}\beta$ are however different. Generally they do not coincide with the ring of integers of the quadratic field.

It is curious to notice that the lowest value of m in both families, leads to the same algebraic rings $\mathbb{Z} + \mathbb{Z}\tau$, and therefore to identical quasicrystals based on the golden mean irrationality. In physics, the

global reflection symmetry of diffraction pattern of observed quasicrystalline materials is related to an algebraic number. The only exparimentally observed ones, connected to quadratic Pisot numbers, are 5, 8, 10 and 12-fold symmetry. Quasicrystals with 5-fold (10-fold) symmetry are related to the golden ratio, the others correspond to the second lowest members of $^{\pm}$families (Pisot numbers $1+\sqrt{2}$ and $2+\sqrt{3}$ solutions of $x^2 = 2x+1$ and $x^2 = 4x-1$ respectively). One of the results of this article is the conclusion (Proposition **??**) that these irrationalities are singled out among other quadratic Pisot numbers also by their unique property with respect to quasiaddition operations.

The subject of this paper is the study of inflation properties and the determination of minimal distances in 1-dimensional quasicrystals with irrationalities from the $^{\pm}$family. Similar properties of higher dimensional quasicrystals are very interesting, too. For that the results in one dimension are an indispensable tool. Some of the higher dimensional questions are solved in [**?**, **?**] for the golden ratio, using the fact that any straight line through a quasicrystal in $\mathbb{R}^n$ is a rescaled 1-dimensional subquasicrystal. The problems for other irrationalities become considerably more complicated when the corresponding algebraic ring is not a unique factorization domain.

In Section 2 some number theoretical facts are recalled, and 1-dimensional quasicrystals are defined. The structure of 1-dimensional tilings corresponding to quasicrystals based on different irrationalities of the $^{\pm}$families are studied in Section 3. We determine the distances between adjacent points in a 1-dimensional quasicrystal (length of possible tiles) as functions of the length of acceptance window, and find some of the rules how the tiles can be ordered in the tiling sequence. The properties of quasicrystals corresponding to $^{+}$family and $^{-}$family are described in Propositions 3.4 and 3.5. The two families are compared and some of the properties are illustrated on examples. In Section **??** we define certain binary operations, under which quasicrystals are invariant. In this paper we determine when a quasicrystal can be generated by these inflation operations. Conversely, we decide whether a set invariant under such an operation can be identified with a quasicrystal. Main results of Section **??** are formulated as Propositions **??** and **??**. Proofs for some of the necessary statements concerning divisors of unity in algebraic rings are found in the Appendix.

2 Preliminaries

A quasicrystal Λ, considered as a generalization of classical periodic crystals, should have several attributes. First of all, $\Lambda \subset \mathbb{R}^n$ is a Delone set. It means that (i) distances between quasicrystal points are greater than a uniform positive constant, and that (ii) the entire space can be covered by balls of fixed radius, centered in quasicrystal points. Another property usually required of quasicrystalline point sets is to be an 'almost lattice', i.e. $\Lambda - \Lambda \subset \Lambda + F$, where F is a finite subset of $\mathbb{R}^n$. If, moreover, one requires Λ to have a self-similarity $\theta\Lambda \subset \Lambda$ for $\theta > 1$, then according to [**?**], θ is a Pisot or Salem number.

A Pisot number β is a solution of an algebraic equation

$$x^n = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0\,, \qquad a_i \in \mathbb{Z}\,,$$

such that $\beta > 1$ and all other solutions of the equation above are in modulus smaller than 1. Definition of a Salem number admits that some of the conjugated roots have modulus equal to 1.

In this article we restrict our attention to quadratic irrationalities. For a quadratic irrationality $\beta > 1$, the modulus of the conjugated root never equals to 1. Thus it suffices to consider quadratic equations $x^2 = mx \pm n$, with Pisot numbers as solutions.

Let us now recall some general number theoretical facts related to quadratic irrationalities. For further details we refer to [**?**, **?**]. Consider the extension $\mathbb{Q}[\sqrt{D}]$ of the rational numbers by $\sqrt{D}$, where D is a square free positive integer. In $\mathbb{Q}[\sqrt{D}]$ one may define the Galois automorphism $'$: $\mathbb{Q}[\sqrt{D}] \to \mathbb{Q}[\sqrt{D}]$, given by $a + b\sqrt{D} \mapsto a - b\sqrt{D}$. Using the automorphism $'$, one introduces two mappings $\mathbb{Q}[\sqrt{D}] \to \mathbb{Q}$,

$$\begin{aligned} \mathrm{Tr}(x) &:= x + x'\,, \\ N(x) &:= xx'\,, \end{aligned}$$

called trace and norm, respectively. Then the ring of integers of the algebraic field $\mathbb{Q}[\sqrt{D}]$ is the set

$$I[\sqrt{D}] := \{x \in \mathbb{Q}[\sqrt{D}] \mid \mathrm{Tr}(x) \in \mathbb{Z}\,,\ N(x) \in \mathbb{Z}\}\,.$$

In $I[\sqrt{D}]$, two elements play a special role: the fundamental unit η and the basis element ω.

The group of units of the ring $I[\sqrt{D}]$ is formed by all elements $x \in I[\sqrt{D}]$, such that $x^{-1} \in I[\sqrt{D}]$. In the ring of integers there exists the so called fundamental unit η, such that the group of units of $I[\sqrt{D}]$ coincides with the set $\{\pm\eta^k \mid k \in \mathbb{Z}\}$.

The basis element ω of the ring of integers has the property that $I[\sqrt{D}] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$. In general, a fundamental unit is not a basis element of the ring of integers. For any square free integer D, a basis element can be found according to the following prescription,

$$\omega_0 := \begin{cases} \dfrac{1+\sqrt{D}}{2}\,, & \text{for } D \equiv 1 \pmod 4\,, \\ \sqrt{D}\,, & \text{for } D \equiv 2, 3 \pmod 4\,. \end{cases}$$

Using ω_0, one may always construct a basis element ω of the ring of integers $I[\sqrt{D}]$, which is a Pisot number.

Since for any $r \in \mathbb{Z}$, the number $\omega_0 + r$ is also a basis element, we can put $\omega = \omega_0 + [\omega_0]$. One may easily verify that ω is a Pisot number, satisfying $x^2 = mx + n$, with $m = \omega + \omega' \in \mathbb{Z}$ and $n = -\omega\omega' \in \mathbb{Z}$. Note that the coefficients satisfy the conditions

$$m, n \in \mathbb{Z}\,, \quad m > 0\,, \quad 1 - m < n < 1 + m\,, \tag{2}$$

but they are not independent. Allowing the coefficients m, n of $x^2 = mx + n$ to be any pair of integers satisfying (2), one obtains all quadratic Pisot numbers. Some of them are basis elements of the corresponding ring of integers as above; some of them (for $n = \pm 1$) are divisors of unity in the ring of integers. Such Pisot numbers one usually calls 'unitary'.

Let β be a Pisot number, i.e. the greater of the roots of the equation $x^2 = mx + n$, with m, n satisfying (2). Denote by $\mathbb{Z}[\beta]$ the set

$$\mathbb{Z}[\beta] := \{a + b\beta \mid a, b \in \mathbb{Z}\}\,. \tag{3}$$

Obviously, $\mathbb{Z}[\beta]$ is a ring. Recall the well known Weyl theorem, which tells that for any irrational number γ, the values of the fractional part $\{b\gamma\} := b\gamma - [b\gamma]$ cover densely the interval $(0, 1)$ as b runs over all integers. Using the theorem it is possible to show that the ring $\mathbb{Z}[\beta]$ is dense in the set $\mathbb{R}$ of all real numbers.

If β is a basis element of the ring of integers of $\mathbb{Q}[\sqrt{D}]$, then $\mathbb{Z}[\beta]$ coincides with $I[\sqrt{D}]$. However, considering all m, n from (2), the resulting equations provide also infinitely many subrings of $I[\sqrt{D}]$. Let us now describe a transparent example of what happens.

The best known and most frequently studied quadratic Pisot number is the golden ratio τ, solution of the equation $x^2 = x + 1$. It is the basis element of the ring of integers $I[\sqrt{5}]$ of $\mathbb{Q}[\sqrt{5}]$,

$$\tau = \frac{1+\sqrt{5}}{2}\,.$$

Since the coefficient n in the quadratic equation is equal to 1, τ is simultaneously the fundamental unit of $I[\sqrt{5}]$. Consider the sequence of quadratic equations

$$x^2 = (2F_k + F_{k+1})x + (-1)^k\,, \qquad k \in \mathbb{Z}^{\geq}\,. \tag{4}$$

The coefficients F_k denote the Fibonacci sequence, given by the recursion

$$F_{k+2} = F_{k+1} + F_k\,, \qquad F_0 = 0, F_1 = 1\,.$$

The solutions of the quadratic equations (4) are Pisot numbers τ^{k+1}, and their conjugates $(\tau')^{k+1}$. The discriminant of the equation (4) is equal to

$$d = (2F_k + F_{k+1})^2 + 4(-1)^k = 5F_{k+1}^2\,,$$

i.e. $\sqrt{d} = F_{k+1}\sqrt{5}$. Therefore the corresponding algebraic field is always $\mathbb{Q}[\sqrt{5}]$. The ring $\mathbb{Z}[\tau^{k+1}] = \mathbb{Z} + \mathbb{Z}\tau^{k+1}$, coincides with the ring of integers of $\mathbb{Q}[\sqrt{5}]$ only for $k = 0, 1$. The later cases provide equations

$$\begin{array}{rcll} x^2 & = & x+1\,, & \text{solutions} \quad \tau\,,\tau'\,, \\ x^2 & = & 3x-1\,, & \text{solutions} \quad \tau^2\,,(\tau')^2\,. \end{array}$$

In all other cases ($k > 1$) we have

$$\mathbb{Z}[\tau^{k+1}] \subsetneqq I[\sqrt{5}]\,.$$

In this article we study all unitary Pisot numbers. They arise from equations $x^2 = mx + n$, with m, n satisfying conditions (2), where $n = \pm 1$. These conditions all together provide two families of equations (1). According to the sign in the equation (1), we speak about the $^+$family or $^-$family of irrationalities respectively. Note that the conditions on m, n in (2) imply for $n = -1$ that only $m \geq 3$ is admissible. Therefore the equations $x^2 = x + 1$ and $x^2 = 3x - 1$ are the lowest members of the $^+$family and $^-$family, respectively. Rings $\mathbb{Z}[\beta]$ related to these two equations coincide with the ring of integers in $\mathbb{Q}[\sqrt{5}]$.

Let us consider the two roots of the equations (1),

$$\beta = \frac{m + \sqrt{m^2 \pm 4}}{2}\,, \qquad \beta' = \frac{m - \sqrt{m^2 \pm 4}}{2}\,. \tag{5}$$

Following properties of β and β' for the two families are obvious:

$$[\beta] = \begin{cases} m\,, & \text{for } x^2 = mx + 1 \\ m-1\,, & \text{for } x^2 = mx - 1 \end{cases}\,, \qquad \beta' \begin{cases} < 0\,, & \text{for } x^2 = mx + 1 \\ > 0\,, & \text{for } x^2 = mx - 1 \end{cases}\,.$$

The ring $\mathbb{Z}[\beta]$, defined by (3), can be characterized using β-expansions [**?**]. A β-expansion of a real number $x \geq 0$ is defined for any real $\beta > 1$ as an infinite sequence $(x_i)_{k \geq i > -\infty}$ given by the 'greedy' algorithm in the following way:

$$x_k := \left[\frac{x}{\beta^k}\right]\,,$$

where k satisfies $\beta^k \leq x < \beta^{k+1}$. Denote $r_k = x/\beta^k - x_k$. Numbers x_{i-1} and r_{i-1} are computed from x_i and r_i by prescription:

$$x_{i-1} := [\beta r_i]\,, \qquad r_{i-1} := \beta r_i - x_{i-1}\,.$$

Clearly $x_i \in \{0, 1, \ldots, [\beta]\}$ for each i. In [**?**] it is proven that

$$x = \sum_{i=-\infty}^{k} x_i \beta^i\,,$$

i.e. the sum converges for each positive real x and for each $\beta > 1$.

Parry in [**?**] answered the question for which sequences $(x_i)_{k \geq i > -\infty}$ there exists a positive real x, such that (x_i) is its β-expansion. Let

$$1 = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \frac{a_3}{\beta^3} + \cdots\,, \qquad a_i \in \{0, 1, \ldots, [\beta]\}\,,$$

and let

$$x = \sum_{i=-\infty}^{k} x_i \beta^i\,, \qquad x_i \in \{0, 1, \ldots, [\beta]\}\,.$$

Then the sequence $(x_i)_{k\geq i>-\infty}$ is a β expansion of x if and only if for any integer $j \leq k$, the sequence $x_j x_{j-1} x_{j-2} \dots$ is lexicographically strictly smaller than sequence $a_1 a_2 a_3 \dots$. Let us apply the rule on β-expansions for β belonging to the two families of irrationalities, (see (5)).

For $\beta^2 = m\beta + 1$, we have $1 = m/\beta + 1/\beta^2$, so that $a_1 = m$, $a_2 = 1$, $a_3 = 0$, $\dots$. It means that $x = \sum x_i \beta^i$ is a β-expansion iff any $x_i = m$ occuring in the sequence $(x_i)_{k\geq i>-\infty}$ is followed by $x_{i-1} = 0$.

For $\beta^2 = m\beta - 1$, we have

$$1 = (m-1)/\beta + (m-2)\sum_{k=2}^{\infty} \frac{1}{\beta^k}\,.$$

In this case $(x_i)_{k\geq i>-\infty}$ is a β-expansion of some $x > 0$ iff $x_i x_{i-1} x_{i-2} \dots$ is strictly lexicographically smaller than $(m-1)(m-2)(m-2)\dots$ for any $i \leq k$.

Let $x \in \mathbb{R}$. If x is negative we put $x_i = -|x|_i$, negatives of β-expansion coefficients for $|x|$. If the β-expansion of an $x \in \mathbb{R}$ ends in infinitely many zeros, it is said to be finite, and the zeros at the end are omitted. Denote the set

$$\mathrm{Fin}(\beta) := \{\varepsilon x \mid x \in \mathbb{R}_0^+\,,\ \varepsilon = \pm 1\,,\ x \text{ has a finite } \beta\text{-expansion}\}\,.$$

In the following, in particular in Section **??** of the article, it will be useful to use the relation between the set $\mathrm{Fin}(\beta)$ of all numbers with finite β-expansion and the ring $\mathbb{Z}[\beta]$. In [**?**] it is proved that for β root of the equation $x^2 = mx + 1$, the set $\mathrm{Fin}(\beta)$ is a ring. It follows immediately that

$$\mathbb{Z}[\beta] = \mathrm{Fin}(\beta)\,.$$

If β belongs to the $^-$family of irrationalities, i.e. is a root of $x^2 = mx - 1$, the situation is different. In this case $\mathrm{Fin}(\beta)$ is not closed under addition, (for example $\beta - 1 \notin \mathrm{Fin}(\beta)$), while $\mathbb{Z}[\beta]$ is, therefore

$$\mathbb{Z}[\beta] \supsetneqq \mathrm{Fin}(\beta)\,.$$

Burdík et al. in [**?**] prove that for β belonging to the $^-$family of irrationalities, the set $\mathrm{Fin}(\beta)$ can be characterized as the set of those elements x of $\mathbb{Z}[\beta]$, for which $N(x) = xx'$ is non negative.

After all preparations, we are at the position to introduce the notion of cut and project quasicrystals.

Definition 2.1. A cut and project quasicrystal is the set

$$\Sigma(\Omega) := \{x \in \mathbb{Z}[\beta] \mid x' \in \Omega\}\,,$$

where Ω, a bounded interval in $\mathbb{R}$ with non empty interior, is called the acceptance window of the quasicrystal $\Sigma(\Omega)$.

One can study properties of quasicrystals, dependingly on their acceptance windows. Often, while studying quasicrystals in general dimension, one restricts the consideration to convex acceptance windows. In dimension 1 we take the acceptance interval always connected, hence convex. If Ω is bounded and has non empty interior, the quasicrystal $\Sigma(\Omega)$ is a Delone set.

One can find several different definitions of quasicrystals in the literature. As an example of this, consider the set $\mathbb{Z}_\beta$ of all β-integers, defined in [**?**] by

$$\mathbb{Z}_\beta := \left\{ x \in \mathrm{Fin}(\beta) \;\middle|\; |x| = \sum_{i=0}^{k} x_i \beta^i \text{ is the } \beta\text{-expansion} \right\}\,. \tag{6}$$

Note that $\mathbb{Z}_\beta = -\mathbb{Z}_\beta$. Obviously, the set of β-integers is a Delone set. In fact, it can be recast in the frame of the cut and project definition. The following relations between β-integers and cut and project quasicrystals has been shown in [**?**]. Let β be the root of $x^2 = mx + 1$. Then

$$\mathbb{Z}_\beta \cap [0, +\infty) = \Sigma\big((-1, \beta]\big) \cap [0, +\infty)\,.$$

For β root of $x^2 = mx - 1$, one has

$$\mathbb{Z}_\beta \cap [0, +\infty) = \Sigma\big([0, \beta)\big) \cap [0, +\infty)\,.$$

Practically, the set $\mathbb{Z}_\beta$ of β-integers is just a particular example of a 1-dimensional cut and project quasicrystal. It will be used to illustrate the results of the Section 3.

It can be shown that the algebraic definition of quasicrystals corresponds to the cut and project scheme. However, there is a difference for the two cases of $^\pm$families of irrationalities. The ring $\mathbb{Z}[\beta]$ can be interpreted as projection of the lattice $\mathbb{Z}^2$ using two straight lines with slopes β and β' respectively. For β root of $x^2 = mx + 1$, the projection is orthogonal, in the case of $x^2 = mx - 1$, the cosin of the angle of straight lines on which we project is equal to $\frac{2}{m}$. The question, whether the ring $\mathbb{Z}[\beta]$ coincides with the ring of integers of the corresponding quadratic field is irrelevant for the cut and project scheme.

In this article, we study properties of 1-dimensional quasicrystals based on quadratic unitary Pisot numbers belonging to the $^\pm$families. In the first part (Section 3) we provide the description of their structure. In particular, we determine the values of distances between adjacent points of the quasicrystal, and find some rules for ordering and repetition of tiles. In the second part of the paper (Section **??**), we are interested mainly in scaling (inflation) symmetries of quasicrystals and in the possibility of generating the quasicrystal points by a binary 'quasiaddition' operations.

3 Tiling in 1-dimensional quasicrystals

In this Section let us focus on structural properties of 1-dimensional quasicrystals. The quasicrystal points are considered to divide the real axis to tiles. We are interested in the 1-dimensional tilings determined by quasicrystals from the two families of irrationalities. It turns out that properties of a tiling, such as length of the tiles, ratio of adjacent tiles and ordering in the tiling sequences, depend on the length of the acceptance window, and not on its position on the real axis.

Such properties were described in [**?**] for the quasicrystal based on the golden mean τ. In here, we describe the tilings corresponding to Pisot numbers of (1) in general. The special role of τ as the lowest member of both $^\pm$families of irrationalities follows from our results.

Let us now recall the structural properties of 1-dimensional quasicrystals based on the irrationality τ. The information, namely possible distances between adjacent points, their respective ratio, ordering of tiles of different length in the quasicrystal, etc. is brought together in the following proposition. For its proof and more detailed formulations of all phenomena which may occur, see [**?**].

Proposition 3.1. *Let $\Sigma(\Omega)$ be a quasicrystal with open acceptance window Ω of the length d, where $d \in (\tau^{k-1}, \tau^k]$ for some integer k. Then:*

(a) The minimal distance in $\Sigma(\Omega)$ is equal to τ^{1-k}.

(b) If $d \neq \tau^k$, then $\Sigma(\Omega)$ has three types of tiles of length τ^{1-k}, τ^{2-k}, and τ^{3-k}. All of the types occur infinitely many times in $\Sigma(\Omega)$.

(c) If $d = \tau^k$, then the quasicrystal $\Sigma(\Omega)$ has only two types of tiles with lengths τ^{1-k} and τ^{2-k}. An exception occurs if Ω has boundary points in $\mathbb{Z}[\tau]$. Then there is one exceptional tile of the length τ^{3-k}.

(d) The ratio of adjacent tiles takes values τ, 1, and τ^{-1}.

(e) In $\Sigma(\Omega)$, two tiles with the length equal to the minimal distance are never adjacent.

(f) In $\Sigma(\Omega)$, a non-trivial string formed only by tiles of the type τ^{2-k} has length at most two and occurs if $d \in (2\tau^{k-2}, \tau^k]$.

(g) *In $\Sigma(\Omega)$, a non-trivial string formed only by tiles of the type τ^{3-k} has length at most two and occurs if $d \in (\tau^{k-1}, 2\tau^{k-2})$.*

In the rest of the Section we study analogues of properties in the general case of irrationalities of the $^{\pm}$families. We compare our results with corresponding properties of τ-quasicrystals. It is the lowest case in both families of Pisot numbers.

The properties of a quasicrystal $\Sigma(\Omega)$ of interest to us, depend on the length d of the acceptance interval Ω rather than on its position on the real axis. A very important fact, which is often used, is that we need not study all posible lengths $d > 0$. It suffices to restrict our consideration to the values of d within the range $(\beta^{-1}, 1]$. Note that the positive real half-axis can be divided into disjoint union

$$\mathbb{R}+ = \bigcup_{k\in\mathbb{Z}} (\beta^k, \beta^{k+1}] \,.$$

In analogy with $\sqrt{5}$, the quasicrystals based on β solution of $x^2 = mx \pm 1$ satisfy the 'rescaling condition' (7) in the following lemma.

Lemma 3.2. *Let β be the solution of the quadratic equation $x^2 = mx \pm 1$. Let Ω be a bounded interval in $\mathbb{R}$. Then*

$$\Sigma\left(\beta\Omega\right) = \beta'\Sigma(\Omega) = \mp\frac{1}{\beta}\Sigma(\Omega)\,. \tag{7}$$

Proof. One has the following equivalences,

$$x \in \Sigma\left(\beta\Omega\right) \quad\Leftrightarrow\quad x' \in (\beta\Omega)\cap\mathbb{Z}[\beta] \quad\Leftrightarrow\quad \frac{x'}{\beta} \in \Omega \cap \left(\frac{1}{\beta}\mathbb{Z}[\beta]\right) \quad\Leftrightarrow$$
$$\Leftrightarrow\quad \left(\frac{x}{\beta'}\right)' \in \Omega\cap\mathbb{Z}[\beta] \quad\Leftrightarrow\quad \frac{x}{\beta'} \in \Sigma(\Omega) \quad\Leftrightarrow\quad x \in \beta'\Sigma(\Omega)\,.$$

Note that $1/\beta\mathbb{Z}[\beta] = \mathbb{Z}[\beta]$, since β is a divisor of unity. □

For τ, solution of $x^2 = x + 1$, Berman and Moody in [**?**] introduced a remarkable binary operation on $\mathbb{Z}[\tau]$, called quasiaddition, under which quasicrystals are invariant. This operation can be defined for any $\mathbb{Z}[\beta]$ in the following way,

$$x \vdash y := \begin{cases} -\beta x + (1+\beta)y\,, & \text{for } \beta^2 = m\beta + 1\,, \\ \beta x + (1-\beta)y\,, & \text{for } \beta^2 = m\beta - 1\,. \end{cases} \tag{8}$$

Since the expression $(x \vdash y)' = x'/|\beta| + y'(1 - 1/|\beta|)$ is a convex combination of x', y', (for both families of irrationalities), a quasicrystal with convex acceptance window is closed under quasiaddition $\vdash$. The binary operation (8) is closely related to scaling properties of quasicrystals. Indeed, if $0 \in \Omega$, i.e. $0 \in \Sigma(\Omega)$, the quasicrystal has a self-similarity property

$$(-\beta)\Sigma(\Omega) = \{x \vdash 0 \mid x \in \Sigma(\Omega)\} \subset \Sigma(\Omega)\,,$$

if β belongs to the $^{+}$family of irrationalities; or similarly $\beta\Sigma(\Omega) \subset \Sigma(\Omega)$ in the other case. In Section **??** we show how other scaling properties of quasicrystals are connected to other binary operations.

As a consequence of Lemma 3.2, we may limit our attention to quasicrystals whose acceptance windows have length $d \in (\beta^{-1}, 1]$, and put aside all other values of d. For them, the statements can be easily derived using Lemma 3.2. From now on, consider quasicrystals $\Sigma(c, c+d]$, where $d \in (\beta^{-1}, 1]$. Note that adding/removing the extreme point to/from the acceptance window may cause only adding/removing of one quasicrystal point. It is obvious from the algebraic definition of a cut and project quasicrystal that

$$\Sigma[c, c+d] = \begin{cases} \Sigma(c, c+d]\,, & \text{for } c \notin \mathbb{Z}[\beta]\,, \\ \Sigma(c, c+d] \cup \{c'\} & \text{for } c \in \mathbb{Z}[\beta]\,, \end{cases}$$

and similarly for the other extreme point of the acceptance interval Ω.

As an example let us now study the quasicrystal whose acceptance window has length equal to 1. Such quasicrystals can be written as an increasing sequence of points. This statement is formulated in Lemma 3.3. It will be used also further to determine the distances between adjacent points in quasicrystals with general acceptance window.

Lemma 3.3. *Let β be the root of the quadratic equation $x^2 = mx \pm 1$. The quasicrystal $\Sigma(c, c+1]$ can be written as a sequence*

$$\Sigma(c, c+1] = \{[c+1-b\beta'] + b\beta \mid b \in \mathbb{Z}\}\,.$$

Proof. Let us recall the definition of the quasicrystal $\Sigma(c, c+1]$,

$$\Sigma(c, c+1] := \{x \in \mathbb{Z}[\beta] \mid c < x' \leq c+1\}\,.$$

Consider $x \in \mathbb{Z}[\beta]$, i.e. $x = a + b\beta$, $a, b \in \mathbb{Z}$. Then if x belongs to the quasicrystal, we have

$$\begin{array}{rcl} c < & a + b\beta' & \leq c+1 \\ c - b\beta' < & a & \leq c+1-b\beta'\,. \end{array}$$

Note that the later inequalities have precisely one integer solution a for any fixed $b \in \mathbb{Z}$. This solution can be written as integer part $[c+1-b\beta']$. Therefore the point x is of the form

$$x = a + b\beta = [c+1-b\beta'] + b\beta\,,$$

what was to show. □

At this point it is useful to divide the study of the two families of irrationalities into separate subsections. The similarities and differences of the two families will be pointed out.

3.1 The $^+$family

As a simple consequence of the Lemma 3.3, let us determine the lengths of tiles in the quasicrystal $\Sigma(c, c+1] \subset \mathbb{Z}[\beta]$, where β is the solution of $x^2 = mx + 1$. Clearly, in the increasing sequence from the lemma, two consecutive points determine a tile. Therefore the possible lengths of tiles in the quasicrystal $\Sigma(c, c+1]$ are equal to

$$\left[c+1+\frac{b+1}{\beta}\right] + (b+1)\beta - \left[c+1+\frac{b}{\beta}\right] - b\beta = \begin{cases} \beta\,, \\ \beta+1\,. \end{cases}$$

Note that for the $^+$family of irrationalities one has $\beta' = -1/\beta$, which has been used here. The fact that both of the possibilities β and $\beta+1$ may occur can be justified using the Weyl theorem, or by the statement that the set $\mathbb{Z}[\beta]$ is dense on the real axis.

Let us now determine all the possible distances between adjacent points of a generic quasicrystals with acceptance window $(c, c+d]$, with $d \in (1/\beta, 1]$.

Proposition 3.4. *Let β be the larger of the solutions of $x^2 = mx + 1$. The distances in the quasicrystal $\Sigma(c, c+d]$ depending on the length $d \in (1/\beta, 1]$ of the acceptance window, are the following:*

(a) If $d \in (1/\beta, 1-(m-1)/\beta)$ then the quasicrystal has tiles of length β, $m\beta+1$ and $(m+1)\beta+1$.

(b) If $d \in (1-j/\beta, 1-(j-1)/\beta)$, for $j = 1, \ldots, m-1$, then the quasicrystal has tiles of length β, $j\beta+1$ and $(j+1)\beta+1$.

(c) If $d = 1-(j-1)/\beta$, for $j = 1, \ldots, m$, there are only two tiles β, $j\beta+1$.

Proof. According to the definition, quasicrystals are ordered by inclusion, if their acceptance windows are ordered in the same way. Therefore we have

$$\Sigma\left(c, c+\frac{1}{\beta}\right] \subset \Sigma(c, c+d] \subset \Sigma(c, c+1]\,.$$

Recall that the distances between adjacent points in the quasicrystal $\Sigma(c, c+1]$ are β and $\beta+1$. Similarly, according to the Lemma 3.2, the distances in the quasicrystal $\Sigma(c, c+1/\beta]$ are $\beta^2 = m\beta+1$ and $\beta^2+\beta = (m+1)\beta+1$. Since the tiles of the quasicrystal $\Sigma(c, c+d]$ arise as a union of tiles of the length β or $\beta+1$, and in the same time by division of tiles of the length $m\beta+1$ of $(m+1)\beta+1$, their only possible lengths are $k\beta$ and $p\beta+1$, for some integers k, p. Let us examine, which k and p are admissible.

Suppose that $x \in \Sigma(c, c+d]$ and $x+k\beta \in \Sigma(c, c+d]$, i.e. $x', x'-k/\beta \in (c, c+d]$. This however implies that $x'-1/\beta \in (c, c+d]$, and therefore $x < x+\beta \leq x+k\beta$ are quasicrystal points. In order that $x+k\beta$ is a neighbour of x, one has to have $k=1$.

It is possible to describe the set of those $x \in \Sigma(c, c+d]$, which are followed by the tile of length β,

$$\left\{x \in \Sigma(c, c+d] \;\middle|\; x' > c+\frac{1}{\beta}\right\}\,. \tag{9}$$

Note that the maximal length of a string formed only by tiles of length β is equal to $r \in \mathbb{N}$, where

$$\frac{r}{\beta} < d \leq \frac{r+1}{\beta} \tag{10}$$

Suppose that a quasicrystal point x is followed by the tile of length $p\beta+1$, i.e. the point x is not followed neither by β, nor by $(p-1)\beta+1$. According to (9), this means that

$$c < x' \leq c+\frac{1}{\beta}\,,$$
$$c+d < x'-\frac{p-1}{\beta}+1 \leq c+d+\frac{1}{\beta}\,.$$

This implies

$$c+d+\frac{p-1}{\beta}-1 < x' \leq c+\frac{1}{\beta}\,,$$
$$c < x' \leq c+d+\frac{p}{\beta}-1\,,$$

which gives

$$\beta(1-d) < p < 2+\beta(1-d)\,.$$

Thus for fixed d, the tiles of the quasicrystal can be of length β, and of one or two other values of the type $p\beta+1$.

If $\beta(1-d)$ is an integer, precisely one value of p is allowed. Since $1/\beta < d \leq 1$, we have $0 \leq \beta(1-d) < \beta-1$. Therefore $j := \beta(1-d)$ may take integer values only among $\{0, 1, 2, \ldots, m-1\}$. For the length $d = 1-j/\beta$ the quasicrystal has tiles of two different lengths: β and $(j+1)\beta+1$.

The statement of the proposition for $d \neq 1-j/\beta$ follows easily. □

Let us now make several comments to results of Proposition 3.4.

(i.) In quasicrystals based on golden ratio τ, adjacent tiles are always in ratio 1 or τ (see Proposition 3.1). For $\beta \neq \tau$ one may find in quasicrystals adjacent tiles whose lengths have ratio

$$1\,,\ \frac{\beta+1}{\beta}\,,\ \frac{2\beta+1}{\beta}\,,\ \ldots\,,\ \frac{m\beta+1}{\beta} = \beta\,.$$

However, the following property remains valid : For fixed d a quasicrystal has only 2 or 3 tiles.

(ii.) For $\beta = \tau$ the 2-tile quasicrystals occured only for $d = \tau^k$, which is a unit in the ring $\mathbb{Z}[\tau]$. For other irrationalities, there are also other 2-tile quasicrystals. One of those is just the set $\mathbb{Z}_\beta$ of β-integers, defined by (6) in Section 2. The lengths of tiles in $\mathbb{Z}_\beta$ can be calculated independently of Proposition 3.4. It can be seen directly from (6) that two adjacent β-integers can have the difference only 1, or $1/\beta$. Let us see that this agrees with Proposition 3.4. According to Lemma 3.2, one has

$$\mathbb{Z}_\beta \cap (0, +\infty) = \Sigma(-1, \beta) \cap (0, \infty) = (\beta')^2 \Sigma\left(-\frac{1}{\beta^2}, \frac{1}{\beta}\right) \cap (0, \infty)\,.$$

The later quasicrystal has the acceptance window of length

$$\frac{1}{\beta} < d = \frac{1}{\beta} + \frac{1}{\beta^2} = 1 - \frac{m-1}{\beta} \leq 1\,,$$

where the last equality occurs for $\beta = \tau$ only. For such d, we can apply Proposition 3.4 to find that there are tiles of only two lengths β and $m\beta + 1 = \beta^2$. Therefore, considering the rescaling by $(\beta')^2$, one gets the desired $1/\beta$ and 1.

(iii.) The fact that the 1-dimensional tiling determined by a quasicrystal has only three different types of tiles justifies the statement that quasicrystals are Delone sets. The Delone property of a set $\Lambda \subset \mathbb{R}$ implies existence of the minimal distance defined as $\inf\{|x - y| \mid x, y \in \Lambda\}$; and of the covering radius, given by $\sup\{r > 0 \mid \exists\, x \in \mathbb{R},\, (x - r, x + r) \cap \Lambda = \emptyset\}$. For a quasicrystal whose acceptance window is of length $d \in (1/\beta, 1]$, the minimal distance is equal to β, and the value of the covering radius is one half of the length of largest tile in the quasicrystal.

(iv.) From the proof of the Proposition 3.4 it can be seen that in a fixed quasicrystal, the string formed by tiles of only one type β, is either of the length $[d\beta]$, or $[d\beta] - 1$, (cf. eq. (10)). Recall the expression (9) for the set of all points of $\Sigma(c, c + d]$, which are followed by a tile of the length β. In fact, this set is again a quasicrystal, in this case the acceptance interval is $(c + 1/\beta, c + d]$, i.e. has the length $d - 1/\beta$. Using results of [**?**], concerning densities of quasicrystal point sets, one may deduce that the density of the shortest tiles in the tiling sequence is proportional to the ratio of lengths of the two acceptance windows,

$$\rho = \frac{d - \frac{1}{\beta}}{d} = 1 - \frac{1}{d\beta}\,.$$

The quasicrystals with maximal density of the shortest tiles are the 2-tile ones, where the acceptance window is of the length 1. Here one has

$$\rho = 1 - \frac{1}{\beta}\,.$$

It is obvious that with growing m, the density of the shortest tile increases. Let us denote the tile of length β by the letter S (=short) and $\beta + 1$ by L (=long). The quasicrystal corresponds to an infinite aperiodic word from the alphabet $\{S, L\}$. Since for different irrationalities β the density of occurence of the letter S in the infinite word is different, with a higher member of the $^+$family we obtain essentially new combinatorial structure.

(v.) Let us study other non-trivial strings of tiles (of length ≥ 2), namely those, which do not contain the shortest tile β.

We shall discuss the problem for β root of $x^2 = mx + 1$, $m \geq 2$. The case $m = 1$ is described in Proposition 3.1. Since $2/\beta < 1$, for the sum of two consecutive tiles in $\Sigma(c, c + 1]$ one has

$$\left[c + 1 + \frac{b+2}{\beta}\right] + (b+2)\beta - \left[c + 1 + \frac{b}{\beta}\right] - b\beta \leq 2\beta + 1\,.$$

Thus tiles of length $\beta + 1$ are never adjacent in the quasicrystal $\Sigma(c, c+1]$. (Note that for golden ratio τ, the string of two tiles of length $\tau + 1$ in $\Sigma(c, c+1]$ is not forbidden, since $2/\tau > 1$.) Similarly, one may show for

quasicrystals having the acceptance window of length $1/\beta$ that a 2-tile sequence $\big((m+1)\beta+1,(m+1)\beta+1\big)$ never appears.

The generic quasicrystal $\Sigma(c,c+d]$ with $1/\beta < d < 1$ is obtained by chopping tiles $m\beta+1$ and $(m+1)\beta+1$ of $\Sigma(c,c+1/\beta]$ into smaller ones. Therefore we have immediately, that *2-tile sequence* $\big((m+1)\beta+1,(m+1)\beta+1\big)$ *never appears in* $\Sigma(c,c+d]$.

Suppose that a tile T of length $k\beta+1$ with end points x and $x+k\beta+1$ is followed by a tile R of length $k\beta+1$, or $(k+1)\beta+1$, with $1\le k\le m-1$. Then at least $m-k$ predecessors of the sequence TR are tiles of length β. (Indeed, at least $m-k$ tiles of length β should be added to $k\beta+1$ to obtain $m\beta+1$ in $\Sigma(c,c+1/\beta]$.) Therefore $x-(m-k)\beta$ and $x-(m-k-1)\beta$ are elements of the quasicrystal $\Sigma(c,c+d]$. Recall that a quasicrystal is closed under quasiaddition $\vdash$. This implies that

$$(x-(m-k)\beta)\vdash(x-(m-k-1)\beta)=x+(k+1)\beta+1\in\Sigma(c,c+d]\,.$$

But this is a contradiction, because no quasicrystal point should lie inside of the tile R. Therefore *2-tile fragments of a quasicrystal not containing* β *are only of the form* $\big(m\beta+1,m\beta+1\big)$ *and* $\big(m\beta+1,(m+1)\beta+1\big)$.

3.2 The $^-$family

Let us now study the quasicrystals related to quadratic equations $x^2=mx-1$, $m\ge 3$. Similarly as for the $^+$family of irrationalities, we start with determining the lengths of tiles in a quasicrystal with acceptance window of length 1. In Lemma 3.3, a quasicrystal $\Sigma(c,c+1]$ is written as a sequence of points of the form $[c+1-b\beta']+b\beta$. Unlike the $^+$family, we have here β' positive, since $\beta\beta'=1$. However, the sequence remains increasing, since $\beta'<1<\beta$. The distances between adjacent points of $\Sigma(c,c+1]$ are found as differences of two consecutive points of the sequence,

$$\left[c+1-\frac{b+1}{\beta}\right]+(b+1)\beta-\left[c+1-\frac{b}{\beta}\right]-b\beta=\begin{cases}\beta\,,\\ \beta-1\,.\end{cases}$$

The fact that both of the possibilities β and $\beta-1$ may occur can be justified using the statement that the set $\mathbb{Z}[\beta]$ is dense on the real axis.

The following proposition provides description of distances between adjacent points in a quasicrystal with a generic acceptance window.

Proposition 3.5. *Let* β *be the larger of the solutions of* $x^2=mx-1$. *The distances in the quasicrystal* $\Sigma(c,c+d]$ *depending on the length* $d\in(1/\beta,1]$ *of the acceptance window, are the following.*

(a) If $d\in(1/\beta,1-(m-2)/\beta)$ *then the quasicrystal has tiles of length* β, $m\beta-1$ *and* $(m-1)\beta-1$.

(b) If $d\in(1-j/\beta,1-(j-1)/\beta)$, *for* $j=1,\dots,m-2$, *then the quasicrystal has tiles of length* β, $j\beta-1$ *and* $(j+1)\beta-1$.

(c) If $d=1-(j-1)/\beta$, *for* $j=1,\dots,m-1$, *there are only two tiles* β, $j\beta-1$.

Note the following interesting fact: for $d\in(1-1/\beta,1]$ the minimal distance is equal to $\beta-1$ and not to β as for the $^+$family.

Let us now list some of the properties of tiling sequences for the $^-$family of quasicrystals. They can be easily justified using the definition of quasicrystals and their inflation property. We have chosen only the most remarkable rules for tiling sequences.

- A string formed only by tiles of length β has $[d\beta]$ or $[d\beta]-1$ members, where d is the length of the acceptance interval.
- 2-tile strings not containing any tile of length β have either of the forms $\big(m\beta-1,m\beta-1\big)$, $\big(m\beta-1,(m-1)\beta-1\big)$ or $\big((m-1)\beta-1,(m-1)\beta-1\big)$.

Figure 1: Vertically aligned samples of nine quasicrystals based on irrationality related to the equation (**??**). The length of the acceptance window belongs to the fundamental range $(1/\beta, 1]$, except for the last one, whose acceptance window has length equal to its lower bound $1/\beta$.

- The maximal length of a string built only from tiles of type $(m-1)\beta - 1$ is two.
- The maximal length of a string built only from tiles of type $m\beta - 1$ is m.

It is possible to find other rules for ordering of tiles in quasicrystals.

3.3 Example

Let us consider an example from the $^{-}$family of quasicrystals. This family starts with quasicrystals based on golden ratio τ corresponding to the equation $x^2 = 3x - 1$. As we have shown in Section 2, it is common for both families. Further, the equation $x^2 = 4x - 1$ corresponds to $\sqrt{3}$, which is one of the three cases related to experimental observations. For our example we choose the first next case, involving the irrationality $\sqrt{21}$, which arises from the quadratic equation

$$x^2 = 5x - 1\,. \tag{11}$$

Consider a set of quasicrystals $\Sigma[0, d)$, where the lenght d of the acceptance interval belongs to the fundamental range $[\beta^{-1}, 1]$. Our set of nine examples involves quasicrystals, for which d takes the following values,

$$1\,,\, 1 - \frac{1}{2\beta}\,,\, 1 - \frac{1}{\beta}\,,\, 1 - \frac{3}{2\beta}\,,\, 1 - \frac{2}{\beta}\,,\, 1 - \frac{5}{2\beta}\,,\, 1 - \frac{3}{\beta}\,,\, \frac{1}{2} - \frac{1}{\beta}\,,\, \frac{1}{\beta}\,. \tag{12}$$

The values of d are chosen in such a way, that the five quasicrystals with odd numbers on Figure **??** are 2-tile, while the others are 3-tile quasicrystals, (see Proposition 3.5).

Let us make several comments to Figure **??**. First of all note that the quasicrystals are ordered by inclusions, according to inclusions of their acceptance windows. The next more dense quasicrystal arises from the previous one only by spliting of the longest tile into two in a suitable ratio.

Note that the tile of length β is present in all of the samples, but the last one, whose acceptance window is of lenght $1/\beta = d \notin (1/\beta, 1]$. The density of occurence of this tile increases with d. For first two samples the tile of length β is *not* the minimal distance. According to Proposition 3.5 there is a shorter tile of length $\beta - 1$.

The first and last quasicrystals $\Sigma[0, 1)$ and $\Sigma[0, 1/\beta)$ respectively, are identic up to a scale, since

$$\Sigma\left[0, \frac{1}{\beta}\right) = \beta\Sigma[0, 1)\,.$$

In the same sense they are equivalent to the quasicrystal of β-integers $\mathbb{Z}_\beta$, (for definition see (6)). Other examples of 2-tile quasicrystals are found in numbers 3, 5, and 7.

Note that only tiles of certain lengths may built non trivial monotype strings. Maximal number of tiles in such a string is $m - 1 = 4$.

4 Inflation properties

Quasicrystals based on the golden mean τ reveal rich structure of scaling symmetries. There are infinitely many centers of such symmetries (inflation centers), and to each of the inflation centers, there correspond infinitely many different scaling factors. A complete description of all inflation centers of cut and project quasicrystals in any dimensions has been given in [**?**], assuming only the convexity of the acceptance window.

Quasicrystals as aperiodic point sets cannot be closed under ordinary addition. Nevertheless, one is interested to have a binary operation under which the quasicrystal set would be invariant, and which, therefore, could be in some sense used for its growth. An example of such a binary operation is the quasi-addition of Berman and Moody established for the case of Fibonacci quasicrystals, (for its generalization to other irrationalities see (8) in Section 3). With regard to the scaling invariances of quasicrystals, one may define infinitely many of such operations and study their relations.

For $s \in \mathbb{Z}[\beta]$, we define the so called s-inflation by

$$x \vdash_s y := sx + (1-s)y\,. \tag{13}$$

Note that the Berman and Moody's quasiaddition in $\mathbb{Z}[\tau]$ correspond to our $\vdash_s$ for $s = -\tau$. It is easy to see that any quasicrystal $\Sigma(\Omega)$ with convex acceptance window is invariant under $\vdash_s$ for any $s \in \Sigma[0,1]$. Indeed, take any $x, y \in \Sigma(\Omega)$, i.e. $x', y' \in \Omega \cap \mathbb{Z}[\beta]$. Then $(x \vdash_s y)' = s'x' + (1-s')y' \in \mathbb{Z}[\beta]$. The later is a convex combination of points in a convex set Ω, therefore it belongs to it, thus $(x \vdash_s y) \in \Sigma(\Omega)$. Note that the factors s and $1-s$ in the definition (**??**) are interchangeble. Indeed, $s \in \Sigma[0,1]$ if and only if $1 - s \in \Sigma[0,1]$.

In the study of inflation properties of quasicrystals, it turns sometimes useful to consider the quasiaddition on the side of the acceptance window. The $'$ image of the result of quasiaddition of two points $x, y \in \Sigma(\Omega)$ is a convex combination of x', and y'. For convenience, we introduce the following notation. For $s \in \mathbb{Z}[\beta]$, such that $0 \le s \le 1$ we put,

$$x \Vdash_s y := sx + (1-s)y\,.$$

Formally, the prescription coincides with that for $\vdash_s$, however, we use another symbol to point out that we work with a convex combination of points. In fact, one has

$$(x \vdash_s y)' = x' \Vdash_{s'} y'\,.$$

Any quasicrystal with convex acceptance window is invariant under $\vdash_s$, $s \in \Sigma[0,1]$. One may ask about all sets invariant under this operation. Can they be identified with cut and project quasicrystals? For the case of $s = -\tau$ an affirmative answer is given in [**?**]. More precisely, we have proven that any uniformly discrete set $\Lambda \subset \mathbb{R}^n$ invariant with respect to $\vdash_{-\tau}$ is an affine image of a cut and project quasicrystal based on $\mathbb{Z}[\tau]$. This already implies the other half of the Delone property of Λ: being relatively dense. A question arises immediately, whether the scaling factor $-\tau$ can be replaced by another $s \in \Sigma[0,1] \subset \mathbb{Z}[\tau]$. The negative answer to this question is a consequence of a more general statement which will be proven in this Section.

Let us now study the problem simultaneously for all irrationalities belonging to the $^{\pm}$families. One of the questions which will be answered here is the following:

Question 1. *Given β, solution of the equation $x^2 = mx \pm 1$ and $s \in \Sigma[0,1] \subset \mathbb{Z}[\beta]$. Can any uniformly discrete set $\Lambda \subset \mathbb{R}^n$ invariant under $\vdash_s$ be identified with a cut and project quasicrystal?*

For equations $x^2 = mx+1$, $m > 2$ and $x^2 = mx-1$, $m > 4$, there does not exist any $s \in \mathbb{Z}[\beta]$, such that the invariance of Λ under $\vdash_s$ implies being a quasicrystal. More precisely, for each $s \in \mathbb{Z}[\beta]$, $0 < s' < 1$, we find as a counter example a uniformly discrete Λ invariant under $\vdash_s$ which cannot be mapped onto a quasicrystal.

Our counter example does not apply for the irrationalities $\beta = \tau$, $\beta = 1 + \sqrt{2}$, and $\beta = 2 + \sqrt{3}$. Surprisingly, those are precisely the cases which correspond to experimentally observed reflection symmetries in quasicrystallography, connected with quadratic irrationalities. (Recall that $x^2 = x + 1$ and $x^2 = 3x - 1$ lead to identic quasicrystals.)

In [**?**], we have shown that invariance of a set $\Lambda \subset \mathbb{R}^n$ under the quasiaddition ($\vdash_s$ for $s = -\tau$) implies that Λ is a quasicrystal in the ring $\mathbb{Z}[\tau]$. In this Section we shall prove that no other $s \in \mathbb{Z}[\tau]$ has this property. For the irrationality $\beta = 1 + \sqrt{2}$ an afirmative answer to Question **??** will be given for sets in dimension 1. The corresponding scaling factor is $s = -\beta$ and no other admissible s exists. Since the demonstration of this fact is not completely an analogue of that for τ, we include it in this article. It is likely that similar proof could be given also for the third case (irrationality $\sqrt{3}$ and $s = \beta = 2 + \sqrt{3}$). The demonstration stems from an embedding of an arbitrary uniformly discrete $\Lambda \subset \mathbb{R}$ into the ring $\mathbb{Z}[\beta]$, and then using the Galois automorphism $'$, finding an acceptance interval for the embedded set.

Note that for $\beta = \tau$, $\beta = 1 + \sqrt{2}$, and $\beta = 2 + \sqrt{3}$, the suitable inflation is precisely the quasiaddition introduced in (8), denoted by $\vdash$. Using this operation, 1-dimensional quasicrystals can be generated, starting from only a finite set of seed points.

In order to be able to speak about the growth of quasicrystal in a more precise way, let us introduce the notion of the 'inflation closure'. Let A be an arbitrary set in $\mathbb{R}^n$. The smallest set invariant under $\vdash_s$ containing A is called the s-inflation closure of A. It is denoted by $A^{\vdash_s}$. For a set Λ closed under s-inflation, one has $\Lambda^{\vdash_s} = \Lambda$. Similarly we define $\Lambda^{\Vdash_s}$.

The nice property of the three lowest irrationalities can be now formulated as follows. Given a quasicrystal $\Sigma(\Omega) \subset \mathbb{Z}[\beta]$, there exist a set of at most three seed points a, b, c, such that $\Sigma(\Omega)$ is the inflation closure of the set $\{a, b, c\}$ under the operation $\vdash_s$. For that one needs Ω to be a closed interval in $\mathbb{R}$ with extreme points in $\mathbb{Z}[\beta]$.

In analogy with this statement, one may ask about the possibility to grow quasicrystals based on the other irrationalities belonging to the $^{\pm}$families using some inflation operations. Due to results answering Question **??**, which are to be shown in this Section, one cannot expect the existence of one s-inflation operation allowing the growth of the quasicrystal. It turns out that one needs to consider inflation closure under several s-inflations simultaneously.

Question 2. *Given a 1-dimensional quasicrystal $\Sigma(\Omega) \subset \mathbb{Z}[\beta]$. Does there exist a finite set of seed points $a_1, a_2, \ldots, a_k$ and a finite set of scaling factors $s_1, s_2, \ldots, s_j$, such that the quasicrystal $\Sigma(\Omega)$ can be written as the inflation closure $\{a_1, \ldots, a_k\}^{\vdash_{s_1} \cdots \vdash_{s_j}}$?*

In what follows, we shall proceed step by step formulating all necessary statements into lemmas and propositions.

The answer to Question **??** can be given by a contra example. This is possible in majority of cases: For a given β comming from $x^2 = mx \pm 1$ and for given $s \in \Sigma[0, 1] \subset \mathbb{Z}[\beta]$ we find a uniformly discrete set Λ, closed under $\vdash_s$, which cannot be identified with any quasicrystal.

Consider $\Lambda = \{0, 1\}^{\vdash_s}$. Clearly, since s belongs to the quasicrystal $\Sigma[0, 1] \subset \mathbb{Z}[\beta]$, we have $\Lambda' \subset [0, 1] \cap \mathbb{Z}[\beta]$. With regard to this, if Λ should coincide with a cut and project quasicrystal $\Sigma(\Omega)$, necessarily $\Omega = [0, 1]$. Since Λ is a subset of the quasicrystal $\Sigma[0, 1]$, it is uniformly discrete. The question about being relatively dense remains open. Let us find conditions, under which Λ coincides with $\Sigma[0, 1]$. The fact that $\{0, 1\}^{\vdash_s} = \Sigma[0, 1] \subset \mathbb{Z}[\beta]$ is equivalent to

$$\{0, 1\}^{\Vdash_{s'}} = [0, 1] \cap \mathbb{Z}[\beta] \,.$$

Lemma 4.1. *Let $\{0, 1\}^{\Vdash_s} = \mathbb{Z}[\beta] \cap [0, 1]$. Then for any $x \in \mathbb{Z}[\beta] \cap [0, 1]$, the scaling factor s divides either x or $x - 1$ in the ring $\mathbb{Z}[\beta]$.*

Proof. Let us observe that all points $x \in \{0,1\}^{\Vdash_s}$ can be written in the form

$$x = \sum_{i=0}^{n} b_i s^i \,, \qquad b_i \in \mathbb{Z}\,,\; b_0 = 0\,,\; \text{or } 1\,, \tag{14}$$

for some non negative integer n. Clearly, any z is a polynomial in s. We prove the above relation by induction on the degree of the polynomial. Consider $x, y \in \{0,1\}^{\Vdash_s}$, for which the polynomials have degree p, q respectively,

$$z = sx + (1-s)y = s\sum_{i=0}^{p} c_i s^i + (1-s)\sum_{i=0}^{q} d_i s^i \,.$$

Without loss of generality put $p = q$. Then

$$z = d_0 + s\sum_{i=0}^{p}(c_i - d_i + d_{i+1})s^i \,.$$

Using the induction hypothesis, we get $c_i - d_i + d_{i+1} \in \mathbb{Z}$ and d_0 either 0 or 1. Consequently, the relation (**??**) is valid. If b_0 is equal to 0, then x is divisible by s, otherwise s divides $x - b_0 = x - 1$, what was to show. □

Corollary 4.2. *If s satisfies $\{0,1\}^{\Vdash_s} = \mathbb{Z}[\beta] \cap [0,1]$, then both s and $1-s$ divide 2 in the ring $\mathbb{Z}[\beta]$.*

Proof. Note that the roles of s and $1-s$ are symmetric. Thus it suffices to show that s divides 2. If s is a divisor of unity then the assertion is true. Assume the opposite, i.e. s is not a divisor of unity and s does not divide 2. Choose $x = 1/\beta^2$. Since x is a divisor of unity, the scaling factor s does not divide neither x, nor $2x$. According to Lemma **??** this implies $s|(x-1)$ and $s|(2x-1)$. Therefore s divides $(2x-1)-(x-1) = x$, which is a contradiction. □

If s is a divisor of 2 in the ring $\mathbb{Z}[\beta]$, the same is true for s'. Therefore the norm $N(s) = ss'$ can take only values ± 1, ± 2, or ± 4. It can be shown (see the Appendix) that all numbers in $\mathbb{Z}[\beta]$ with the norm equal to ± 1, i.e. divisors of unity, are of the form $\pm\beta^k$, for some integer k. The only exception is the case when β comes from the equation $x^2 = 3x - 1$, i.e. the lowest member of the $^-$family of irrationalities. In the Section 2 we have shown that $\mathbb{Z}[\beta]$ in this case coincides with $\mathbb{Z}[\tau]$. All divisors of unity in $\mathbb{Z}[\tau]$ are $\pm\tau^k$, $k \in \mathbb{Z}$, from which only $\pm\tau^{2k}$, $k \in \mathbb{Z}$, are powers of $\beta = \tau^2$.

Proposition 4.3. *Let β be the root of $x^2 = mx \pm 1$ and let the relation $\{0,1\}^{\Vdash_s} = \mathbb{Z}[\beta] \cap [0,1]$ be satisfied for some s. Then either of the possibilities below is true:*

- *$\beta = \tau$ (root of $x^2 = x + 1$) and $s = 1/\tau$.*
- *$\beta = \tau^2$ (root of $x^2 = 3x - 1$) and $s = 1/\tau^2$.*
- *$\beta = 1 + \sqrt{2}$ (root of $x^2 = 2x + 1$) and $s = 1/\beta = \sqrt{2} - 1$.*
- *$\beta = 2 + \sqrt{3}$ (root of $x^2 = 4x - 1$) and $s = 1/\beta = 2 - \sqrt{3}$.*

Note that the four possibilities are the two lowest members of each of the $^\pm$families of irrationalities. Moreover, the two lowest cases lead to same quasicrystals with same inflations, since $1/\tau^2 = 1 - 1/\tau$. Unlike the majority of our proofs, here we shall not distinguish between the $^\pm$families. Instead, we consider separately m odd and m even.

Proof. Let us first consider equations $x^2 = mx \pm 1$ with m odd. We show that both $s = 1/\beta^k$ and $1 - s = 1/\beta^l$, for some positive integers k, l. Indeed, let $s \neq 1/\beta^k$. Then s does not divide $1/\beta$, thus s divides $1 - 1/\beta$, and thus also $\beta^2 - \beta = (m-1)\beta \pm 1$. Since $s|2$ (according to Corollary **??**), we have also $s|2\beta$. This all together implies that s divides the difference $(m-1)\beta \pm 1 - 2\beta(m-1)/2 = \pm 1 = \pm\beta^0$. This

is a contradiction with the assumption that $s \neq 1/\beta^k$. The situation for $1-s$ is the same. Since both s and $1-s$ are between 0 and 1, we consider k, l to be positive integers.

We have $s = 1/\beta^k$ and $1-s = 1/\beta^l$ for some $k, l \in \mathbb{Z}^{>0}$. Without loss of generality assume that $0 < k \leq l$. The sum of s and $1-s$ is equal to 1,

$$1 = \frac{1}{\beta^k} + \frac{1}{\beta^l} \leq \frac{2}{\beta^k}\,,$$

which implies $\beta^k \leq 2$. For both of the families one has $m-1 < \beta$, therefore we want to find k and m odd, such that $(m-1)^k < 2$. This condition implies necessarily $m = 1$ (hence $\beta = \tau$). In order to satisfy $\tau^k \leq 2$, we have only $k = 1$. Then $1 - s = 1 - 1/\tau = 1/\tau^2$.

Note that we have used the assumption that a divisor of unity is of the form $\pm\beta^k$, $k \in \mathbb{Z}$. This is not valid for the equation $x^2 = 3x - 1$. However, this case leads to the same results as that of $x^2 = x + 1$.

Let us now study the irrationalities arising from equations $x^2 = mx \pm 1$, where m is even. According to Corollary **??**, s and $1-s$ are divisors of 2 in the ring $\mathbb{Z}[\beta]$. Therefore the norms $N(s) = ss'$ and $N(1-s) = (1-s)(1-s')$ divide 4 in $\mathbb{Z}$.

Let $s = a + b\beta$. Then

$$\begin{aligned} ss' &= a^2 + mab \mp b^2\,, \\ (1-s)(1-s') &= 1 - 2a - mb + a^2 + mab \mp b^2\,. \end{aligned}$$

Therefore if m is even, precisely one of integers ss' and $(1-s)(1-s')$ is odd and one of them even, taking values from the possibilities $\{\pm1, \pm2, \pm4\}$. Without loss of generality assume that $ss' = \pm1$, thus s is a divisor of unity. Thus $s = 1/\beta^k$, for some integer $k > 0$. The integer $(1-s)(1-s')$ takes one of the values $\pm2, \pm4$. Consider $s + s' = 1 + ss' - (1-s)(1-s')$. The maximal possible absolute value of $s + s'$ is 6. One has therefore the following inequality

$$(m-1)^k - 1 < \beta^k - 1 \leq \left|\frac{1}{\beta^k} + (\mp1)^k\beta^k\right| = |s + s'| \leq 6\,.$$

(Recall that $\beta > m-1$ for both $^{\pm}$family of irrationalities.) Let us find integers m even, and $k > 0$ satisfying the inequality $(m-1)^k - 1 < 6$.

Assume that $k \geq 2$, then necessarily $m = 2$. For β root of the equation $x^2 = 2x + 1$ the inequality implies $k = 2$. This gives $s = 1/\beta^2$ and $1 - s = 2/\beta$. According to Lemma **??** if the s-inflation should generate the entire quasicrystal $\Sigma[0,1]$, then $1-s$ divides either x, or $x-1$ for any $x \in [0,1] \cap \mathbb{Z}[\beta]$. Put $x = 1/\beta$. Clearly, $2/\beta$ does not divide $1/\beta$. Thus $2/\beta$ divides $1 - 1/\beta$. Therefore $-4 = N(2/\beta)$ divides $N(1/\beta - 1) = 2$ in integers, which is a contradiction.

Consider $k = 1$. In such a case the given even m satisfies $m - 2 < 6$, therefore m can be equal to 2, 4 or 6. The corresponding scaling factor s has the value $1/\beta$. The number $1 - s = 1 - 1/\beta$ should divide 2, i.e. there exist $c, d \in \mathbb{Z}$, such that

$$2 = (c + d\beta)\left(1 - \frac{1}{\beta}\right) = \begin{cases} c - d + mc + \beta(d - c)\,, & \text{for } x^2 = mx + 1\,, \\ c - d - mc + \beta(d + c)\,, & \text{for } x^2 = mx - 1\,. \end{cases}$$

This implies for $^{+}$family of irrationalities that $c = d$ and $mc = 2$. Since m is even, the only possibility is $m = 2$. For the $^{-}$family of irrationalities, one has $c = -d$ and $(2-m)c = 2$, hence the only possible m even is $m = 4$. The corresponding scaling factors is in both cases $s = 1/\beta$, what was to show. □

The above proposition states that the quasicrystal $\Sigma[0,1] \subset \mathbb{Z}[\beta]$ could be generated using one s-inflation only in three different cases of irrationalities $\beta = \tau$, $\beta = 1 + \sqrt{2}$, and $\beta = 2 + \sqrt{3}$, with $s = -\tau$, $s = -1 - \sqrt{2}$, and $s = 2 + \sqrt{3}$ respectively. (We shall not consider the quasicrystals arising from $x^2 = 3x - 1$ any more.) However, it has not been yet proven, that the relation $\{0,1\}^{\vdash_s} = \Sigma[0,1]$ is valid indeed. This will be justified as a special case of a general answer to Question **??**.

For that we introduce the notion of an inflation closure. Let $S \subset \mathbb{Z}[\beta]$ and $A \subset \mathbb{R}$. The smallest set containing A and closed under $\vdash_s$ for all $s \in S$ is called an S-inflation closure of A.

Proposition 4.4. *Let β be the root of the quadratic equation $x^2 = mx \pm 1$, and let $\Sigma(\Omega) \subset \mathbb{Z}[\beta]$ be a 1-dimensional quasicrystal, $\Omega \subset \mathbb{R}$ bounded. Then there exists a finite set P of seed points and a finite set S of scaling factors, such that $\Sigma(\Omega)$ is equal to the S-inflation closure of P, if and only if Ω is a closed interval with extreme points being elements of the ring $\mathbb{Z}[\beta]$.*

In particular, let $\Omega = [a, b]$, $a, b \in \mathbb{Z}[\beta]$. Then the set of seed points is the set

$$P = \{a', b', c'\},$$

where $c' = a' + \beta^k$ with k sufficiently small to have $a' < c' < b'$; the set S of scaling factors can be chosen in a way that

$$S' = \left\{ \frac{1}{\beta}, \frac{2}{\beta}, \ldots, \frac{\left[\frac{m\pm 1}{2}\right]}{\beta} \right\}.$$

The proof of the above proposition is based on several lemmas which we prove separately.

Lemma 4.5. *Let β be the root of $x^2 = mx \pm 1$. Then the quasicrystal $\Sigma[0, 1]$ is an S-inflation closure of $\{0, 1\}$, where*

$$S' = \left\{ \frac{i}{\beta} \,\middle|\, i = 1, 2, \ldots, \left[\frac{m \pm 1}{2}\right] \right\}.$$

Proof. Consider the operations $\Vdash_s$ with $s = i/\beta$. For simplicity, denote

$$x \Vdash_i y := \frac{i}{\beta} x + \left(1 - \frac{i}{\beta}\right) y, \qquad i = 1, 2, \ldots$$

The statement of the lemma is equivalent to the fact that any point $\mathbb{Z}[\beta]$ in the interval $[0, 1]$ can be generated from 0, and 1 using inflations $x \Vdash_i y$, $i = 1, 2, \ldots, [(m \pm 1)/2]$. We first show that $[0, 1] \cap \mathbb{Z}[\beta]$ can be generated using all inflations $\Vdash_1, \ldots, \Vdash_r$, with $r = m$ for $^+$family and $r = m - 1$ for $^-$family of irrationalities. In the second step we find expression for $\Vdash_i$, $i = [(m \pm 1)/2] + 1, \ldots, r$, as a combination of operations $\vdash_i$, $i = 1, \ldots, [(m \pm 1)/2]$.

There arise an interesting complication for $x^2 = mx - 1$, therefore we shall consider the irrationalities belonging to the $^\pm$families separately.

(a) Consider β to be the solution of the equation $x^2 = mx + 1$. Any $x \in \mathbb{Z}[\beta] \cap [0, 1)$ has a finite β-expansion

$$x = \sum_{i=1}^{k} \frac{x_i}{\beta^i}, \qquad k \in \mathbb{Z}.$$

We proceed by induction on k. Clearly, for the first step of induction we have

$$\frac{j}{\beta} = 1 \Vdash_j 0, \tag{15}$$

for any $j = 1, \ldots, m$. Suppose that the β-expansion of a point $x \in \mathbb{Z}[\beta] \cap [0, 1]$ is of the form

$$x = \frac{j}{\beta} + \sum_{i=2}^{k} \frac{x_i}{\beta^i}, \qquad j = 0, \ldots, m - 1.$$

Then x can be rewritten as the combination

$$x = \frac{j}{\beta} + \frac{z}{\beta}, \qquad z = \sum_{i=1}^{k-1} \frac{x_{i+1}}{\beta^i}.$$

By induction hypothesis, z could be generated from 0 and 1 using the given inflation operations; written schematicaly, one has $z \in \{0,1\}^{\Vdash}$. Since

$$(x \Vdash_r y) + c = (x+c) \Vdash_r (y+c)\,, \quad \text{and} \quad \frac{x}{\beta} \Vdash_r \frac{y}{\beta} = \frac{1}{\beta}(x \Vdash_r y)\,, \qquad \text{for any } r = 1,\dots,m\,, \tag{16}$$

one has the following relation,

$$\left\{\frac{j}{\beta}, \frac{j+1}{\beta}\right\}^{\Vdash} = \frac{j}{\beta} + \frac{1}{\beta}\{0,1\}^{\Vdash}\,.$$

Therefore

$$x = \frac{j}{\beta} + \frac{z}{\beta} \in \frac{j}{\beta} + \frac{1}{\beta}\{0,1\}^{\Vdash} = \left\{\frac{j}{\beta}, \frac{j+1}{\beta}\right\}^{\Vdash} \subset \{0,1\}^{\Vdash}\,.$$

The later inclusion is valid due to the first step of induction (see (**??**)).

Assume that the coefficient x_1 of x in the β-expansion at β^{-1} is equal to m. Necessarily, x_2 is equal to 0, see Section 2. In this case we use

$$x = \frac{m}{\beta} + \frac{z}{\beta^2} \in \frac{m}{\beta} + \frac{1}{\beta^2}\{0,1\}^{\Vdash} = \left\{\frac{m}{\beta}, 1\right\}^{\Vdash} \subset \{0,1\}^{\Vdash}\,.$$

Thus we have shown that all points in $[0,1] \cap \mathbb{Z}[\beta]$ can be generated starting from 0, and 1, using the inflation operations $\Vdash_i$, $i = 1,\dots,m$. The following relation can be used to reduce the number of necessary operations from m to $[(m+1)/2]$. The relation is valid for any k, $k = 1,\dots,m-1$, independently of x, y.

$$(x \Vdash_{k+1} y) \Vdash_1 (x \Vdash_k y) = (1 - \frac{m-k}{\beta})x + \frac{m-k}{\beta}y = y \Vdash_{m-k} x\,.$$

(b) Let now β belong to the $^-$family of irrationalities, i.e. it is the root of $x^2 = mx - 1$. In this case, if we want to use the β-expansions of numbers, we have to encounter a complication: There exist points in the ring $\mathbb{Z}[\beta]$ which do not have a finite β-expansion.

In [**?**], it is shown that an $x \in \mathbb{Z}[\beta]$ has a finite β-expansion if and only if $xx' \geq 0$. For us, only $x \in [0,1]$ are of interest. The following statement is valid. Let $x \in \mathbb{Z}[\beta] \cap (0,1)$. Then either x or $(1-x)$ has a finite β-expansion. Indeed, since $N(x) = xx'$ is an integer and $x < 1$, necessarily $|x'| > 1$. Therefore $x' > 0$ if and only if $1 - x' < 0$.

Now we can proceed, similarly as for the $^+$family of irrationalities, by induction on the length of the β-expansion in order to show that any element of $\mathrm{Fin}(\beta) \cap [0,1]$ can be generated by corresponding inflations $\Vdash_i$, $i = 1,\dots,m-1$. For elements $x \in [0,1] \cap (\mathbb{Z}[\beta] \setminus \mathrm{Fin}(\beta))$, we have $1 - x \in [0,1] \cap \mathrm{Fin}(\beta)$. Therefore $1 - x$ can be generated from 0 and 1 using the inflation operations. In the combination of quasiadditions we replace all 0 by 1 and vice versa, which gives the desired combination for element x.

The coefficients in a β-expansion take one of the values $0,\dots,m-1$. For an $x \in \mathrm{Fin}(\beta) \cap [0,1]$, whose $x_1 = j$, $j = 0,\dots,m-2$, the procedure is the same as in the case of $^+$family of irrationalities. Consider an

$$x = \frac{m-1}{\beta} + \frac{x_2}{\beta^2} + \sum_{i=3}^{k} \frac{x_i}{\beta^i} = 1 - \frac{1}{\beta} + \frac{x_2+1}{\beta^2} + \sum_{i=3}^{k} \frac{x_i}{\beta^i} = \frac{1}{\beta}z + \left(1 - \frac{1}{\beta}\right)1 = z \Vdash_1 1\,.$$

Since any segment $x_1x_2\dots x_k$ of a β-expansion is strictly lexicographically smaller than the sequence $(m-1)(m-2)(m-2)\dots$ (see Section 2), the string $x_2x_3\dots x_k$ is strictly smaller than $(m-2)(m-2)\dots$. Therefore $(x_2+1)x_3\dots x_k$ is the β-expansion of z. By induction hypothesis $z \in \{0,1\}^{\Vdash}$, therefore $x = z \Vdash_1 1 \in \{0,1\}^{\Vdash}$.

By that we have shown that any $x \in [0,1] \cap \mathbb{Z}[\beta]$ can be generated by given inflation operations $\Vdash_i$, $i = 1,\dots,m-1$. The number of operations can be reduced from $m-1$ to $[(m-1)/2]$ by the following relation, which is valid for any k, $k = 1,\dots,m-2$, for all x, y.

$$(x \Vdash_k y) \Vdash_1 (x \Vdash_{k+1} y) = \left(1 - \frac{m-k-1}{\beta}\right)x + \frac{m-k-1}{\beta}y = y \Vdash_{m-k-1} x\,.$$

□

Due to (**??**), the Lemma **??** can be generalized in the following way. If $c \in \mathbb{Z}[\beta]$, we have

$$\{c, c+\beta^k\}^{\Vdash} = [c, c+\beta^k] \cap \mathbb{Z}[\beta]\,. \tag{17}$$

Here we are using the simplified notation of $\Vdash$ for all S'-inflation closures, where s' is the set $S' = \{1/\beta, \ldots, [m \pm 1]/\beta\}$. Indeed,

$$\{c, c+\beta^k\}^{\Vdash} = c + \beta^k\{0,1\}^{\Vdash} = c + \beta^k([0,1] \cap \mathbb{Z}[\beta]) = [c, c+\beta^k] \cap \mathbb{Z}[\beta]\,. \tag{18}$$

In the last equality we have used that β is a divisor of unity, and therefore $\beta^k\mathbb{Z}[\beta] = \mathbb{Z}[\beta]$. Consider an interval of length $d \neq \beta^k$. A relation similar to (**??**) is not valid, however, since the inflation operations have purely geometrical meaning, one may conclude, that the set $\{c, c+d\}^{\Vdash}$ is dense in $[c, c+d]$.

Lemma 4.6. *Let β be the root of $x^2 = mx \pm 1$ and let S be the finite set described in Proposition* **??**. *If $\Lambda \subset \mathbb{Z}[\beta]$ is invariant under $\vdash_s$, for any $s \in S$, and if Λ contains the quasicrystal $\Sigma[0,1]$, then $\Lambda = \Sigma(\Omega)$, where Ω is the convex hull $\langle\Lambda'\rangle$.*

Proof. Since $\Sigma[0,1] \subset \Lambda$, one has $\Lambda' \supset ([0,1] \cap \mathbb{Z}[\beta])$. Since Λ is invariant with respect to S-inflations, we have to show that any element of $\mathbb{Z}[\beta] \cap \langle\Lambda'\rangle$ can be generated using S'-inflations starting from points of Λ'.

First note that since Λ' is invariant under S'-inflations, i.e. convex combinations, its points cover densely the interval $\langle\Lambda'\rangle$. Consider $x \in \mathbb{Z}[\beta] \cap \langle\Lambda'\rangle$. If $x \in [0,1]$, then x belongs to Λ', due to the assumption of the lemma. Without loss of generality let $x > 1$. Since Λ' is dense in the convex hull $\langle\Lambda'\rangle$, one can find a finite sequence of intervals $[y_1 - 1, y_1], \ldots, [y_k - 1, y_k]$, such that $y_i \in \Lambda'$, satisfying $0 < y_1 - 1 < 1$, $y_{i+1} - 1 < y_i$ and $y_k - 1 < x \leq y_k$. According to (**??**), one has

$$\Lambda' \supset \{y_i - 1, y_i\}^{\Vdash} = [y_i - 1, y_i] \cap \mathbb{Z}[\beta]\,.$$

Therefore x can be generated from points of Λ' by $\Vdash$, hence it is contained in Λ'. □

Proof of Proposition **??**. The proposition states that a finite set P of seed points and a finite set S of scaling factors for a quasicrystal $\Sigma(\Omega)$ exists, if and only if Ω is a closed interval with boundary points in $\mathbb{Z}[\beta]$. For Ω satisfying these conditions, the proposition shows possible choice of P and S. The fact that this choice satisfies the required property, i.e. $\Sigma(\Omega)$ is an S-inflation closure of P, has been justified by Lemma **??** and Lemma **??**.

Let us now explain why for other acceptance windows the sets S and P do not exist. Suppose that Ω is an interval either not closed, or with at least one boundary point outside of $\mathbb{Z}[\beta]$. Assume that P and S are such that $\Sigma(\Omega)$ is an S-inflation closure of P. It follows then from Lemma **??** that $\Omega = \langle P'\rangle$. However, since P is a finite subset of $\mathbb{Z}[\beta]$, its convex hull is a closed interval with boundary points in $\mathbb{Z}[\beta]$, which gives a contradiction. □

Let us now make several remarks concerning Proposition **??**. Note that the number of inflation operations, suggested in Proposition **??** to generate the given quasicrystal, depends on the Pisot number. More precisely, for β arising from $x^2 = mx \pm 1$, it is equal to $[(m \pm 1)/2]$. This number leads to 1 precisely in the four cases named in Proposition **??**. In that sense they are exceptional among members of the $^{\pm}$families of irrationalities.

On the other hand, we can choose the set P of seed points for a given quasicrystal in Proposition **??** in many ways. Obviously, P has to contain the $'$ images of boundary points of Ω. However, there remains an ambiguity in the choice of the third seed point. We have not excluded the existence of a 'better' choice for S, namely the possibility of reducing the number of necessary inflation operations, by adding more elements to the set of seed points P. Such a problem is closely related to the question about properties of sets invariant under $\vdash_s$, which are not closed with respect to other inflation operations.

Let us focus on the exceptional lowest cases of $^{\pm}$families of irrationalities. Among the several reflection symmetries which have been observed in the X-ray diffraction on physical quasicrystals, only the 5-fold (or 10), 8-fold, and 12-fold correspond to a quadratic irrationality, namely to $\sqrt{5}$, $\sqrt{2}$, $\sqrt{3}$. The best known quasicrystals based on golden ratio τ have been many times studied in the literature. A description of their scaling symmetries and properties related to quasiaddition are provided in [**?**, **?**].

Proposition **??** states that quasicrystals can be generated using the quasiaddition $\vdash$ starting from a set of seed points. A stronger assertion was given in [**?**] for quasicrystals based on τ, in any dimension: Any set $\Lambda \subset \mathbb{R}^n$, Delone and invariant under $\vdash$, can be identified with a quasicrystal. We shall prove now a similar statement for 1-dimensional set invariant under inflation operation with $\beta = 1 + \sqrt{2}$. The proof is not a direct analogue of the one for τ in [**?**], although it follows a similar path. Therefore we found useful to put it in this article. It is likely that a proof of similar proposition concerning $\beta = 2 + \sqrt{3}$ could be done in the same way. Equally, we expect the generalization to higher dimensions to hold, although it remains unproven.

Proposition 4.7. *Let β be the root of the quadratic equation $x^2 = 2x + 1$, and let $\vdash$ be a binary operation defined for $x, y \in \mathbb{R}$ by $x \vdash y := -\beta x + (1+\beta)y$. Let $\Lambda \subset \mathbb{R}$ be a Delone set invariant under $\vdash$. Then there exists an affine mapping Φ and a bounded interval Ω, such that $\Phi\Lambda = \Sigma(\Omega)$.*

Proof. The proof of the proposition uses facts shown in this article. It is divided into two steps. First we find an affine embedding Φ of a given Λ into $\mathbb{Z}[\beta]$ in such a way that the points 0 and 1 are elements of Λ. Then, according to Lemma **??**, we identify $\Phi\Lambda$ with a quasicrystal.

Let $\Lambda \subset \mathbb{R}$ be Delone, invariant under $\vdash$. We find all possible ratios of adjacent tiles in Λ, which allow us to embed it into the ring $\mathbb{Z}[\beta]$. The Delone property of Λ implies that the neighbours in Λ are well defined. Therefore take three adjacent points in Λ, without loss of generality say -1, 0, and c, with $c \geq 1$. Since Λ is closed under $\vdash$, we have $-1 \vdash 0 = \beta \in \Lambda$, therefore we may consider only $1 \leq c \leq \beta$.

First, let us show that the ratio c of lengths of the two adjacent tiles in Λ between -1, 0, and c can take only one of three values: 1, β, or $(\beta+1)/\beta$. Since Λ is invariant under $\vdash$, all points in $\{-1, 0, c\}^{\vdash} \subset \Lambda$. We shall show that for any c, up to exceptions 1, β, and $(\beta+1)/\beta$, the set $\{-1, 0, c\}^{\vdash}$ has a point, say x inside of the interval $(-1, 0)$, or $(0, c)$, which therefore contradicts the assumption that -1, 0 and c are neighbours. It is easy to verify that:

- If $c \in \big(1, 2\beta/(\beta+2)\big)$, then
$$x = (0 \vdash -1) \vdash (\beta \vdash c) \in (0, c)\,.$$
- If $c = 2\beta/(\beta+2)$, then
$$x = \Big(((\beta \vdash c) \vdash -1) \vdash -1\Big) \vdash c = -\frac{1}{\beta+2} \in (-1, 0)\,.$$
- If $c \in \big(2\beta/(\beta+2), (4\beta+1)/(3\beta+2)\big]$, then
$$x = \big((0 \vdash -1) \vdash (\beta \vdash c)\big) \vdash c \in (-1, 0)\,.$$
- If $c \in \big((4\beta+1)/(3\beta+2), (\beta+1)/\beta\big)$, $c \neq (\beta+2)/(\beta+1)$, then
$$x = (\beta \vdash c) \vdash -1 \in (-1, 0) \cup (0, c)\,.$$
- If $c = (\beta+2)/(\beta+1)$, then
$$x = \big((0 \vdash -1) \vdash (\beta \vdash c)\big) \vdash c = \frac{1}{\beta+1} \in (0, c)\,.$$

- If $c \in \big((\beta+1)/\beta, \beta\big)$, $c \neq (2\beta+1)/(\beta+1)$, then

$$x = \beta \vdash c \in (-1,0) \cup (0,c)\,.$$

- If $c = (2\beta+1)/(\beta+1)$, then

$$x = \big((c \vdash 0) \vdash (0 \vdash -1)\big) \vdash -1 = \frac{2\beta}{\beta+1} \in (0,c)\,.$$

By that we have shown that the ratio of two adjacent tiles in a Delone set Λ invariant under $\vdash$ may take only values 1, β, or $(\beta+1)/\beta$. Note that these are precisely the values allowed in a generic quasicrystal based on the irrationality $\beta = 1+\sqrt{2}$.

The Delone property of Λ implies that $r = \inf\{|y-x| \mid x,y \in \Lambda\}$ is a positive number. Put $\varepsilon = 1/10$. There exists a pair of neighbouring points $x_0, y_0 \in \Lambda$, such that $r \leq y_0 - x_0 < r(1+\varepsilon)$. Take an affine Φ, which maps x_0 to 0 and y_0 to 1. By that we assure that the infimum $\tilde{r} = \inf\{|y-x| \mid x,y \in \Phi\Lambda\}$ satisfies $1-\varepsilon < \tilde{r} \leq 1$. Therefore all tiles in $\Phi\Lambda$ are longer than $1-\varepsilon$.

Since Λ is closed under $\vdash$, the same holds for $\Phi\Lambda$. Due to Lemma **??**, the quasicrystal $\Sigma[0,1]$ is a subset of $\Phi\Lambda$. Recall Proposition 3.4, which states that tiles of such a quasicrystal have length either β or $\beta+1$. (The exception of the tile between 0 and 1 is caused by the fact that we consider closed interval $[0,1]$.) Therefore all tiles in the sets $\Phi\Lambda$ arise by division of tiles β and $\beta+1$ into smaller ones in such a way that the ratio of adjacent tiles would have values 1, β, or $(\beta+1)/\beta$.

First realize that the tile of length β cannot be divided into more than two, since otherwise at least one of the arising tiles would have length smaller than $1-\varepsilon$. Similarly, the tile of length $\beta+1$ cannot be divided into mode than three pieces. After a simple computation we find following possibilities:

- β is divided in the ratio $1:1$ to the pair $\big(\beta/2, \beta/2\big)$.
- β is divided in the ratio $1:(\beta+1)/\beta$ to the pair $\big(1, (\beta+1)/\beta\big)$.
- $\beta+1$ is divided in the ratio $1:1$ to the pair $\big((\beta+1)/2, (\beta+1)/2\big)$.
- $\beta+1$ is divided in the ratio $1:\beta$ to the pair $\big(1, \beta\big)$.
- $\beta+1$ is divided in the ratio $1:(\beta+1)/\beta$ to the pair $\big((\beta+1)/\beta, 2\big)$.
- $\beta+1$ is divided in the ratio $1:1:1$ to the triplet $\big((\beta+1)/3, (\beta+1)/3, (\beta+1)/3\big)$.
- $\beta+1$ is divided in the ratio $1:1:(\beta+1)/\beta$ to the triplet $\big(1, 1, (\beta+1)/\beta\big)$.

All other possibilities are excluded, since there would arise tiles of length smaller than $1-\varepsilon < \tilde{r} = \inf\{|y-x| \mid x,y \in \Phi\Lambda\}$.

Let us assume that there exist adjacent points in $\Phi\Lambda$, with the distance $(\beta+1)/3$. The sequence of tiles with this length is necessarily finite. This means that the ratio of length of the last tile in the sequence and its first neighbour does not take the admissible value 1, β, or $(\beta+1)/\beta$. Therefore tiles of length $(\beta+1)/3$ do not occur in $\Phi\Lambda$. In the same way we can exclude tiles having 2 in the denominator.

After such an analysis, one arrives to conclusion that only tiles of length 1, $(\beta+1)/\beta$, 2 and β may occur in $\Phi\Lambda$. Therefore it is easy to see that $\Phi\Lambda \subset \mathbb{Z}[\beta]$.

As the last step of the proof, use Lemma **??** to conclude that $\Phi\Lambda$ is a quasicrystal, whose acceptance window is the interval $\langle(\Phi\Lambda)'\rangle$. $\square$

5 Concluding remarks and open problems

1. Points of cut and project quasicrystals, according to Definition 2.1, are found in an algebraic ring $\mathbb{Z}[\beta] := \mathbb{Z} + \mathbb{Z}\beta$, where β is a Pisot number, solution of the quadratic equation $x^2 = mx \pm 1$. Generally, the ring $\mathbb{Z}[\beta]$ does not coincide with the ring of integers in the quadratic field $\mathbb{Q}[\beta]$: it is its proper subring. Different quadratic equations $x^2 = mx \pm 1$ may lead to the same algebraic field, but the Pisot numbers involved lead to different subrings and therefore to different quasicrystals. With each quadratic equation we obtain essentially new aperiodic structures. Their distinctness can be justified using densities of tiles in different tilings, (see comment (iv.) in Section 3).

2. It is useful to underline common features as well as differences between quasicrystals related to distinct members of the $^{\pm}$families. In any single quasicrystal (acceptance window always connected) there are at most 3 distinct types of tiles. For each Pisot number there are as well 2-tile quasicrystals. The lengths of acceptance windows of 2-tile quasicrystals form a discrete series of values. Quasicrystals with different Pisot numbers differ by the relative lengths of their tiles and their tiling adjacency rules are distinct. These properties are determined by the value of m within each family.

3. Given a sufficiently long segment of a quasicrystal, we are able to determine the quadratic equation and bounds on the length of acceptance window of the quasicrystal. This is true, of course, only provided we have the precise coordinates of points. As a curious example we may consider a quadratic equation $x^2 = mx + 1$ with a very large m. Let the acceptance window be simply the interval $[0, 1]$. Such a quasicrystal has two tiles: β and $\beta + 1$. Since $[\beta] = m \gg 1$, the ratio of lengths is almost equal to 1. In such a quasicrystal one finds side-by-side up to m or $m - 1$ equal tiles (cf. (10)) of length β. Therefore it may happen that a given fragment of the quasicrystal is indistinguishable from a piece of a periodic sequence.

4. Quasicrystals as physical materials, more precisely their diffraction patterns, reveal non crystallographic reflection symmetries. Among those, which were observed in physics there are 5,8, and 12-fold symmetry. They correspond to quadratic Pisot numbers $\frac{1}{2}(1 + \sqrt{5})$, $1 + \sqrt{2}$ and $2 + \sqrt{3}$, respectively. It is perhaps not an accident that the same quasicrystals (in the algebraic rings $I[\sqrt{5}] = \mathbb{Z}[\tau]$, $I[\sqrt{2}] = \mathbb{Z}[1 + \sqrt{2}]$, and $I[\sqrt{3}] = \mathbb{Z}[2 + \sqrt{3}]$) turn out to be exceptional among others of the $^{\pm}$families: As it was shown In Section **??**, such quasicrystals can be generated from a finite set of seed points using just one, namely the lowest, inflation operation, the quasiaddition. Nevertheless, even for these irrationalities it remains to show, for $\sqrt{2}$ in dimension greater than 1, and for $\sqrt{3}$ in any dimensions, that a Delone set Λ closed under such a quasiaddition is always identificable with a quasicrystal.

5. For quasicrystals with Pisot numbers from higher members of the $^{\pm}$families, we show that a quasicrystal can be generated using several quasiaddition operations. More precisely, one needs $[(m \pm 1)/2]$ operations, (Proposition **??**). There remain some open problems:

- Is it possible to reduce the number of necessary operations to generate a quasicrystal, say $\Sigma[0, 1]$, allowing more elements in the finite set of seed points? In fact, this question is closely connected to the next one:

- We have shown in Proposition **??** that (up to the three exceptions) $\Lambda = \{0, 1\}^{\vdash_s}$ cannot be identified with any quasicrystal. What are the properties of the resulting Λ for general scaling factor s, $0 < s' < 1$, in general $\mathbb{Z}[\beta]$? Since Λ is a subset of the quasicrystal $\Sigma[0, 1]$, which is Delone, it is uniformly discrete. Is Λ also relatively dense? For which factors s?

- Assume that $\Lambda = \{0, 1\}^{\vdash_s}$ is both uniformly discrete and relatively dense. This means that we have a completely new aperiodic structure with Delone property. What are other properties of such sets? For example, is Λ a Meyer set? (Does there exists F finite such that $\Lambda - \Lambda \subset \Lambda + F$?)

6. It this article, we were interested in the study of quadratic irrationalities. Other irrationalities are also interesting. In turned out that quasicrystals in rings corresponding to a quadratic field have many properties similar to those for the golden ratio. Nevertheless, they are aperiodic structures with new

attributes. Naturally, one can generalize the algebraic Definition 2.1 also for irrationalities of higher degree, allowing more than one acceptance window. What are general features of quasicrystals related to irrationalities of a fixed degree? Are there some essentially new properties? Which of the attributes of quasicrystals with quadratic irrationalities are lost? The geometrical cut and project scheme for defining aperiodic structures allow a general irrational number, even transcendental ones. Obviously, the aperiodic structure arising from the cut and project scheme using for example the irrationality π, would miss many important algebraic properties, for example the scaling symmetries.

Acknowledgements

We are grateful to Drs. Frank W. Lemire and Alfred Weiss for useful suggestions and comments. We acknowledge partial support from the National Science and Engineering Research Council of Canada, FCAR of Quebec, and by the Grant Agency of the Czech Republic. Two of the authors (Z.M. and E.P.) are greateful for the hospitality of the Centre de recherches mathématiques, Université de Montréal, Québec, Canada, and of the Joint Institute for Nuclear Research in Dubna, Russia.

A Divisors of unity in the ring $\mathbb{Z}[\beta]$

We find all the divisors of unity in the ring $\mathbb{Z}[\beta] = \mathbb{Z} + \mathbb{Z}\beta$, $\beta^2 = m\beta \pm 1$. In particular, we show that they are all the powers of β taken with both signes.

According to the definition, an element $u \in \mathbb{Z}[\beta]$ is a divisor of unity, if there exists $y \in \mathbb{Z}[\beta]$, such that $uy = 1$. Applying the Galois automorphism to this equality, we obtain $u'y' = 1$ and together $(uu')(yy') = N(u)N(y) = 1$. Since the norms $N(u)$ and $N(y)$ take integer values, we have $N(u) = uu' = \pm 1$. On the other hand, if $N(u) = \pm 1$, then u is a divisor of unity by the definition. (The role of y is played by u', or $-u'$ respectively.) Writing explicitly the norm $N(u)$ for $u = a + b\beta$, $a, b \in \mathbb{Z}$, we have proven

Lemma A.1. *(i) Let β be the solution of $x^2 = mx + 1$. Then $u = a + b\beta$ is adivisor of unity in $\mathbb{Z}[\beta]$ if and only if*

$$N(u) = a^2 + abm - b^2 = \pm 1\,. \tag{19}$$

(ii) Let β be the solution of $x^2 = mx - 1$. Then $u = a + b\beta$ is adivisor of unity in $\mathbb{Z}[\beta]$ if and only if

$$N(u) = a^2 + abm + b^2 = \pm 1\,. \tag{20}$$

We have transformed the task to find all divisors of unity in the ring $\mathbb{Z}[\beta]$ to a task of solving Diophantine equations, which will enable us to prove Propositions **??** and **??**.

Proposition A.2. *In the ring $\mathbb{Z}[\beta] = \mathbb{Z} + \mathbb{Z}\beta$, where β is the solution of $x^2 = mx + 1$, $m > 0$, a number u is a divisor of unity, if and only if $u = \pm\beta^k$, for some integer k.*

Proof. According to Lemma **??** above, we are interested in all solutions of (**??**). It can be easily checked that if a pair of integers (a, b) is a solution of (**??**), corresponding to $u = a + b\beta$, then

$$(-a, -b) \quad \text{is a solution, corresponding to} \quad -a - b\beta = -u\,, \tag{21}$$

$$(-b, a) \quad \text{is a solution, corresponding to} \quad -b + a\beta = \frac{\pm\beta}{u}\,, \tag{22}$$

$$(b - am, a) \quad \text{is a solution, corresponding to} \quad b - am + a\beta = \frac{u}{\beta}\,. \tag{23}$$

The statements (**??**), (**??**) imply, that it suffices to consider only the case when $a, b \geq 0$. Assume that we have a solution (a, b) of (**??**), such that $a, b \geq 0$. We repeat the transformation from the step (**??**), while

both components are non negative. We stop at (a_0, b_0), where both a_0 and b_0 are non negative, but which verify $b_0 - ma_0 < 0$, which is equivalent to

$$1 + b_0 \leq ma_0 \,. \tag{24}$$

Since (a_0, b_0) is a solution (see (**??**)), one has

$$\pm 1 = a_0^2 + ma_0 b_0 - b_0^2 \geq a_0^2 + (b_0 + 1)b_0 - b_0^2 = a_0^2 + b_0 \,,$$

which implies either $a_0 = 0$ and $b_0 = 1$, which contradicts to (**??**), or $a_0 = 1$ and $b_0 = 0$. This corresponds to number $1 + 0\beta = 1$. It means that all solutions (a, b) with $a, b \geq 0$ can be obtained from the solution $(1, 0)$ corresponding to $u = 1$ by inverse transformation to that of (**??**). All (a, b), $a, b \geq 0$ correspond to numbers β^k, $k > 0$. Solutions with negative a, b correspond, according to (**??**), to units $-\beta^k$, $k > 0$. The transformation (**??**) then gives the rest of solutions $\pm\beta^{-k}$, $k > 0$. □

Similar results are for the ring $\mathbb{Z}[\beta]$, where β corresponds to quadratic equation $x^2 = mx - 1$. Such an equation gives a Pisot number for $m > 2$. From the reasons we have explained in Section 2, the case $m = 3$ lead to identic quasicrystals as the golden ratio. We shall see in the proof of the proposition below, that $m = 3$ is an exceptional case.

Proposition A.3. *In the ring $\mathbb{Z}[\beta] = \mathbb{Z} + \mathbb{Z}\beta$, where β is the solution of $x^2 = mx - 1$, $m > 3$, a number u is a divisor of unity, if and only if $u = \pm\beta^k$, for some integer k.*

Proof. According to Lemma **??**, one searches for all numbers $u = a + b\beta \in \mathbb{Z}[\beta]$, satisfying (**??**). It can be easily checked that if a pair of integers (a, b) is a solution of (**??**), corresponding to $u = a + b\beta$, then

$$(-a, -b) \quad \text{is a solution, corresponding to} \quad -a - b\beta = -u \,, \tag{25}$$

$$(b, a) \quad \text{is a solution, corresponding to} \quad b + a\beta = \frac{\pm\beta}{u} \,, \tag{26}$$

$$(am + b, -a) \quad \text{is a solution, corresponding to} \quad am + b - a\beta = \frac{u}{\beta} \,. \tag{27}$$

The statements (**??**), (**??**) imply, that it suffices to consider only the case when $b \geq |a| \geq 0$.

If both a and b are equal to 0, the pair (a, b) does not solve (**??**). If one of them is equal to 0, we get

$$a = 0 \quad \Longrightarrow \quad b = \pm 1 \,,$$
$$b = 0 \quad \Longrightarrow \quad a = \pm 1 \,.$$

The first case gives $a + b\beta = \pm\beta$, the second one $a + b\beta = \pm 1$. Both are of the type $\pm\beta^k$, $k \in \mathbb{Z}$.

Assume that $b = |a| > 0$, hence $a = \pm b$. The equation (**??**) implies that

$$2b^2 \pm mb^2 = b^2(2 \pm m) = \pm 1 \,.$$

This is possible only if $a = 1$ and $m = 3$, (since m is positive greater than 2). However, the case $m = 3$ coincides with the famous τ case, where all the units are known. Moreover, our theorem does not apply to the case $m = 3$.

Assume that we have a solution (a, b) of (**??**), which satisfies $b > |a| > 0$. Clearly, if a and b are both non zero, they cannot have the same sign. Therefore without loss of generality consider $b > -a > 0$. Using the transformation from the step (**??**), we diminish the second component ($b \to -a$), however, it remains positive. Therefore the first component cannot get positive,

$$am + b \leq 0 \,.$$

Let us show that the first conponent comes in absolute value smaller than the second one $-(am + b) \leq -a$. Suppose the opposite $-(am + b) > -a$, hence

$$a^2 + abm + b^2 = a^2 + b(am + b) < a^2 + ba = a(a + b) \,.$$

Since a is negative and $b > -a$, the later $(a(a+b))$ is necessarily negative, therefore $a^2 + abm + b^2$ is less than -1, which is a contradiction. Having this, one finds that the repeated transformation (**??**), applied on a solution (a, b) of (**??**), which satisfies $b > |a| > 0$, gives a solution $(\tilde{a}, \tilde{b})$ with $\tilde{b} \geq -\tilde{a} \geq 0$. Sooner or later, but after a finite number of steps, we find that either both components are equal, or one of the components becomes 0. In the first case we get contradiction as explained above, in the second case we obtain solutions ± 1, or $\pm\beta$. From these solutions we then generate by the inverse transformation to (**??**) and by the transformation (**??**) the new solutions $\pm\beta^k$, $k \in \mathbb{Z}$. □

Note that for $x^2 = mx - 1$ we have $\beta\beta' = 1$. Therefore $N(\beta^k) = +1$, for any integer k. This implies, that in the ring $\mathbb{Z}[\beta]$, where β is the solution of $x^2 = mx - 1$, $m > 3$, there is no number with the norm -1.

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