Free Boundary Value Problems for Analytic Functions in the Closed Unit Disk

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Abstract

We shall prove (a slightly more general version of) the following theorem: *Let Φ be analytic in the closed unit disk $\overline{\mathbb{D}}$ with $\Phi\colon [0,1] \to (0,1]$, and $B(z)$ a finite Blaschke product. Then there exists a function h satisfying: i) h analytic in the closed unit disk $\overline{\mathbb{D}}$, ii) $h(0) > 0$, iii) $h(z) \neq 0$ in $\overline{\mathbb{D}}$, such that*

$$F(z) := \int_0^z h(t)B(t)\,dt$$

satisfies

$$|F'(z)| = \Phi(|F(z)|^2), \quad z \in \partial\mathbb{D}.$$

This completes a recent result of Kühnau [7] (compare also Fournier & Ruscheweyh [5]) for $\Phi(x) = 1 + \alpha x$, $-1 < \alpha < 0$, where this boundary value problem has a geometrical interpretation, namely that $\beta(\alpha)F(r(\alpha)z)$ preserves hyperbolic arc length on $\partial\mathbb{D}$ for suitable $\beta(\alpha)$, $r(\alpha)$. For these important choices of Φ we also prove that the corresponding functions h are uniquely determined by B, and that $zh(z)$ is univalent in $\mathbb{D}$. Our work is related to Beurling's [3] and Avhadiev's [2] on conformal mappings solving free boundary value conditions in the unit disk.

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1 Introduction and statement of the results

In his work on Riemann's mapping theorem A. Beurling [3] studied existence and uniqueness of conformal mappings f of the unit disk $\mathbb{D} := \{z : |z| < 1\}$, normalized by $f(0) = 0$, $f'(0) = 1$, and satisfying

$$|f'(z)| = \Phi(f(z)), \quad |z| = 1, \tag{1.1}$$

or rather

$$\liminf_{|z|\uparrow 1} (|f'(z)| - \Phi(f(z))) = 0, \tag{1.2}$$

where Φ is a bounded continuous positive real function in the complex plane $\mathbb{C}$. This concept was extended by F.G. Avhadiev [2], who replaced the domain $\mathbb{D}$ of the functions under consideration by other domains, and was consequently lead to modified conditions replacing (1.1), (1.2).

These considerations were made for univalent 'solutions' defined in $\mathbb{D}$ (or the other domain), with no reference made to the possible analyticity of these functions on or across the boundaries. In the present note we look for analytic solutions in $\overline{\mathbb{D}}$, and with the zeros of f' inside $\mathbb{D}$ prescribed. It is clear that the desired analyticity requires corresponding restrictions on Φ, which we shall impose. The prescribed zeros will be looked at in terms of corresponding finite Blaschke products

$$B(z) := \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}, \quad z_k \in \mathbb{D},$$

and we shall make use of the quantity

$$\mu(B) := \max_{z\in\overline{\mathbb{D}}} \int_0^1 |B(tz)|\, dt.$$

We write $\mathcal{H}(\Omega)$ for the set of analytic functions in a set Ω and $\mathbb{D}_r := \{z : |z| < r\}$. Our main result is then

Theorem 1.1. *Let $\Phi \in \mathcal{H}(\overline{\mathbb{D}})$, $\Phi : [0, 1] \to (0, \frac{1}{\mu(B)})$, for some finite Blaschke product B. Then there exists $F \in \mathcal{H}(\overline{\mathbb{D}})$ with $F(0) = 0$ such that $F' = Bh$ with $h \in \mathcal{H}(\overline{\mathbb{D}})$, $h(z) \neq 0$ in $\overline{\mathbb{D}}$, $h(0) > 0$, and*

$$|F'(z)| = \Phi(|F(z)|^2), \quad |z| = 1. \tag{1.3}$$

It is clear that the existence of such solutions is guaranteed for *arbitrary* B if $\Phi(x) < 1$ for $0 \le x \le 1$. This can be slightly refined.

Theorem 1.2. *Let $\Phi \in \mathcal{H}(\overline{\mathbb{D}})$, with $\Phi(x) > 0$ for $0 \le x \le 1$. If*

$$\min_{0<x\le 1} \frac{\max_{0\le t\le x} \Phi(t^2)}{x} \le 1, \tag{1.4}$$

then the conclusion of Theorem 1.1 holds for every Blaschke product B. If, on the other hand,

$$\Phi(x^2) > x, \quad 0 \le x \le 1, \tag{1.5}$$

then there is no locally univalent solution for (1.3) (i.e., for $B \equiv 1$ there is no solution).

It should be observed that the latter fact does not contradict Beurling's result, since we are dealing with functions Φ on $[0, 1]$ only.

The next question related to Theorem 1.1 concerns uniqueness of the solutions F. In general this will not be the case. It is easy to construct admissible functions Φ for which (1.3) has, for instance, various monomial solutions of the form cz, $c > 0$.

The case $n = 0$ of the following theorem is due to R. Kühnau [7].

Theorem 1.3. *Let the function Φ of Theorem 1.1 be such that $\Phi(x^2)/x$ is strictly decreasing. Let the Blaschke product be of the form $B(z) = z^n$, $n \in \mathbb{N}_{\nvdash}$. Then there exists at most one solution F in the sense of Theorem 1.1, and it is of the form $F(z) = cz^{n+1}$, $c > 0$.*

Note that this covers the cases

$$\Phi_\alpha(x) := \begin{cases} 1 + \alpha x, & -1 < \alpha < 0, \\ \sqrt{\alpha}(1 + x), & 0 < \alpha \le \frac{1}{4}. \end{cases} \tag{1.6}$$

It has been observed by R. Kühnau [7] that the solution of equation (1.3) for Φ_α corresponds to the identification of analytic functions $f \in \mathcal{H}(\overline{\mathbb{D}_r})$ with $f(\overline{\mathbb{D}_r}) \subset \mathbb{D}$ which preserve the hyperbolic ($\alpha < 0$) respectively spherical ($\alpha > 0$) length of the arcs on $|z| = r$ when mapped by f. Here α and r are related by

$$\alpha = \frac{\pm r^2}{(1 \pm r^2)^2}.$$

Note that (1.5) holds for $\alpha > \frac{1}{4}$.

The 'hyperbolic' case is the only one where we can prove general unicity for the solutions F discussed in Theorem 1.1. This result was conjectured by Kühnau [7].

Theorem 1.4. *Let $\Phi(x) = 1 + \alpha x,\ -1 < \alpha < 0$. Then for every Blaschke product B the solution F in Theorem* 1.1 *is uniquely determined.*

In his paper, Kühnau proved the existence of solutions as in Theorem 1.1 for Φ_α with $-1 < \alpha \le \frac{1}{4}$ and Blaschke products of the form $B(z) = (z - z_0)/(1 - \overline{z_0}z)$ and small $|z_0|$. His proof is constructive (although numerically prohibitively complex), and was the first one to establish the existence of such functions. This also gave a negative answer to the following question (related to Φ_α with $\alpha = -\frac{1}{4}$, compare R. Fournier & St. Ruscheweyh [5]):

Let F be analytic in the closed unit disk $\overline{\mathbb{D}}$ with $F(0) = 0$ and

$$2|F'(z)| = 1 - |F(z)|^2, \quad z \in \partial\mathbb{D}. \tag{1.7}$$

Does this imply $F(z) = cz^n$ for some $c \in \mathbb{C},\ n \in \mathbb{N}$?

The functions satisfying (1.7) play an important rôle in a general multiplier conjecture for univalent functions, see [5].

In spite of Kühnau's examples and the results in Theorems 1.1 and 1.4 the discussion of the functions solving (1.7) is by far not complete. We observe that the 'question' has an affirmative answer for the Blaschke products $B(z) = z^n$, $n = 0, 1, \ldots$, a result also obtained by M. Agranovski & T. Bandman [1] (in a more general version). They also proved that no normalized entire functions, except for monomials, can satisfy (1.7). On the other hand it is a consequence of general results on Ricatti equations in the complex plane (cf. Bieberbach [4]) that any solution of (1.7), related to a Blaschke product B and properly normalized, extends meromorphically into $\overline{\mathbb{C}}$, except for possible branch points in the poles of B. This, however, requires a more thorough investigation.

In this context we confine ourselves to pointing out a mapping property of the functions zh corresponding to the solutions of F of (1.3) in the case of (1.6).

Theorem 1.5. *Let $-1 < \alpha < 0$ and $\Phi = \Phi_\alpha$. Let $F' = Bh$ for a solution F of* (1.3) *in the sense of Theorem* 1.1. *Then*

$$\operatorname{Re} \frac{(zh(z))'}{h(z)} > 1 + \frac{\alpha}{2}, \quad z \in \mathbb{D}. \tag{1.8}$$

In particular: zh is starlike univalent (of order $1 + \alpha/2$) in $\mathbb{D}$.

2 The general case.

The proof for Theorem 1.1 uses ideas from Kühnau's work [7]. However, we have to replace the constructive portions by a non-constructive fixed point argument. We shall make use of the notations

$$\|f\|_\Omega := \sup_{z \in \Omega} |f(z)|,$$

and

$$\mathcal{W}_\Omega := \{f \in \mathcal{H}(\Omega) : \|f\|_\Omega \le 1\}, \quad \mathcal{W}_\delta := \mathcal{W}_{\mathbb{D}_{1+\delta}} \quad (\delta > 0).$$

Lemma 2.1. *Let $\Phi \in \mathcal{H}(\overline{\mathbb{D}})$, with $\Phi(x) > 0$ for $x \ge 0$, and let $g \in \mathcal{W}_\Omega$ for some domain $\Omega \supset \overline{\mathbb{D}}$. Then there exists a uniquely determined function $h \in \mathcal{H}(\Omega)$ with $h(z) \ne 0$ for $z \in \overline{\mathbb{D}}$, $h(0) > 0$, such that*

$$|h(z)| = \Phi(|g(z)|^2), \quad |z| = 1.$$

Proof. Define

$$h(z) := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} (\log \Phi(|g(e^{i\tau})|^2)) \frac{1 + e^{-i\tau}z}{1 - e^{-i\tau}z} d\tau\right\}, \quad z \in \mathbb{D},$$

so that $h \in \mathcal{H}(\mathbb{D})$ and $|h|$ extends continuously (with $|h| \neq 0$) into $\overline{\mathbb{D}}$. By assumption we have $\Phi(g(z)\overline{g(1/\bar{z})}) \in \mathcal{H}(A_\rho)$, where $A_\rho := \{z : \rho < |z| < \frac{1}{\rho}\}$ for some $\rho < 1$, but close to 1. Hence

$$Q(z) := \frac{\Phi(g(z)\overline{g(1/\bar{z})})}{h(z)} \in \mathcal{H}(\mathbb{D} \cap A_\rho),$$

and $|Q(z)|$ extends continuously to $\overline{\mathbb{D}} \cap A_\rho$, with $|Q(z)| = 1$ on $\partial\mathbb{D}$. By the (extended) Schwarz' reflection principle we deduce that Q has an analytic extension across $\partial\mathbb{D}$, so that also $h \in \mathcal{H}(\mathbb{D}_r)$ for some $r > 1$. We can now use the relation

$$h(z) = \frac{\Phi^2(g(z)\overline{g(1/\bar{z})})}{\overline{h(1/\bar{z})}}$$

to analytically extend h into the whole of $\Omega \setminus \overline{\mathbb{D}}$. The assertion concerning unicity of h is obvious. □

Proof (Theorem 1.1). We define $\delta > 0$ to be the solution of

$$\mu(B)M(\Phi) + \delta\frac{\|\Phi\|_{\mathbb{D}}^2}{m(\Phi)}\|B\|_{\mathbb{D}_{1+\delta}} = 1,$$

where

$$0 < m(\Phi) := \min_{0\le x\le 1} \Phi(x) \le \max_{0\le x\le 1} \Phi(x) =: M(\Phi) < \frac{1}{\mu(B)}.$$

We now define $T\colon \mathcal{W}_\delta \to \mathcal{H}(\mathbb{D}_{1+\delta})$ by

$$T\colon g \mapsto F, \quad F(z) := \int_0^z B(t)h(t)\,dt,$$

where h is the function corresponding to g in the sense of Lemma 2.1. It is immediately clear that T is a continuous (in the topology of locally uniform convergence in $\mathcal{H}(\mathbb{D}_{1+\delta})$) operator, acting on the convex and compact subspace $\mathcal{W}_\delta$. Furthermore,

$$T(\mathcal{W}_\delta) \subset \mathcal{W}_\delta. \tag{2.1}$$

In fact, for $F = T(g)$ we have $|F'(z)| = |B(z)||h(z)| \le |B(z)|M(\Phi)$ in $\mathbb{D}$ by the maximum principle, and therefore

$$|F(z)| \le \max_{z\in\overline{\mathbb{D}}} \int_0^1 |F'(tz)|dt \le \mu(B)M(\Phi), \quad z \in \overline{\mathbb{D}}.$$

Using the minimum principle we obtain

$$|h(z)| \ge m(\Phi), \quad z \in \overline{\mathbb{D}}. \tag{2.2}$$

Furthermore, for $1 < |z| < 1 + \delta$,

$$h(z) = \frac{\Phi^2(g(z)\overline{g(1/\bar{z})})}{\overline{h(1/\bar{z})}},$$

which together with (2.2) gives

$$|h(z)| \le \frac{\|\Phi\|_{\mathbb{D}}^2}{m(\Phi)}, \quad z \in \mathbb{D}_{1+\delta}.$$

Hence,

$$\begin{aligned} |F(z)| &\le |F(\tfrac{z}{|z|})| + |F(z) - F(\tfrac{z}{|z|})| \\ &\le \mu(B)M(\Phi) + \textstyle\int_1^{1+\delta} \max_{\phi\in\mathbb{R}} |h(te^{i\phi})B(te^{i\phi})|\,dt \\ &\le \mu(B)M(\Phi) + \delta\tfrac{\|\Phi\|_{\mathbb{D}}^2}{m(\Phi)}\|B\|_{\mathbb{D}_{1+\delta}} = 1. \end{aligned}$$

This implies $F \in \mathcal{W}_\delta$ as required.

Schauder's principle provides us now with a fixed point $F = T(F) \in \mathcal{W}_\delta$. Clearly F satisfies (1.3). □

Proof (Theorem 1.2). Our assumption yields some $x_0 \in (0,1]$ with $\Phi(x_0^2)/x_0 \le 1$, which in turn shows the existence of some $c \in (0, x_0]$ with $\Phi(c^2) = c$. Thus $F(z) = cBz$ solves (1.3) for $|B| \equiv 1$.

For non-constant B we have $\mu(B) < 1$. Then

$$\Psi(x) := \frac{\Phi(x_0^2 x)}{x_0}, \quad x \in [0,1],$$

satisfies $0 < \Psi(x) \le 1 < 1/\mu(B)$, and therefore we can apply Theorem 1.1 to find a solution F^* corresponding to Ψ and B. Clearly $F := x_0 F^*$ solves the same problem for Φ and B.

Next assume (1.5) and let $B \equiv 1$, so that the corresponding $F(z)$ should be locally univalent in $\mathbb{D}$, with $F(0) = 0$. This implies

$$\left.\frac{zF'(z)}{F(z)}\right|_{z=0} = 1,$$

and therefore

$$\sup_{|z|\uparrow 1} \left|\frac{F(z)}{zF'(z)}\right| \ge 1, \text{ or } \inf_{|z|\uparrow 1} \left|\frac{zF'(z)}{F(z)}\right| \le 1.$$

This contradicts

$$\left|\frac{zF'(z)}{F(z)}\right| = \frac{\Phi(|F(z)|^2)}{|F(z)|} > 1, \quad |z| = 1.$$

□

In the proof of Theorem 1.3 we shall use the following well-known result.

Lemma 2.2 (Jack [6]). *Let $w \in \mathcal{H}(\overline{\mathbb{D}})$ have a n-fold zero in the origin and assume that $|w(z_0)| \ge |w(z)|$ holds for all z with $|z| \le |z_0| \le 1$. Then $z_0 w'(z_0)/w(z_0) \ge n$.*

Proof (Theorem 1.3). Choose $z_0 \in \partial\mathbb{D}$ with $|F(z_0)| \ge |F(z)|$, $z \in \overline{\mathbb{D}}$. From our assumptions and Lemma 2.2 we get

$$\left.\frac{zF'(z)}{F(z)}\right|_{z=0} = n+1, \quad \frac{z_0 F'(z_0)}{F(z_0)} \ge n+1.$$

We distinguish two cases, related to the quantity

$$\tau := \min_{|z|=1} \left|\frac{zF'(z)}{F(z)}\right| = \left|\frac{z_1 F'(z_1)}{F(z_1)}\right|,$$

where $z_1 \in \partial\mathbb{D}$. If $\tau \ge n+1$ we find, by the maximum principle,

$$\left\|\frac{F(z)}{zF'(z)}\right\| \le \frac{1}{n+1} = \left.\frac{F(z)}{zF'(z)}\right|_{z=0},$$

which implies $zF'(z)/F(z) \equiv$ const., the assertion. On the other hand, if $\tau < n+1$, then

$$\frac{\Phi(|F(z_1)|^2)}{|F(z_1)|} = \left|\frac{z_1 F'(z_1)}{F(z_1)}\right| < \left|\frac{z_0 F'(z_0)}{F(z_0)}\right| = \frac{\Phi(|F(z_0)|^2)}{|F(z_0)|},$$

which contradicts our assumption, since $|F(z_1)| < |F(z_0)|$. □

3 The hyperbolic case

Our proof of Theorem 1.4 is based on the following uniqueness statement for Dirichlet's problem for (3.1) (compare Sato [8, 9] for somewhat stronger versions).

Lemma 3.1. *Let $J\colon \mathbb{D} \times \mathbb{R} \to \mathbb{R}$ be such that for arbitrary $z \in \mathbb{D}$ the function $J(z,t)$ is non-decreasing for $t \in \mathbb{R}$. Then there exists at most one solution $u = u(z, \bar{z})$ of*

$$u_{z\bar{z}} = J(z,u), \quad z \in \mathbb{D}, \tag{3.1}$$

which extends continuously into $\overline{\mathbb{D}}$, and assumes prescribed continuous boundary values on $\partial\mathbb{D}$.

Proof (Theorem 1.4). Let B be an arbitrary finite Blaschke product, and assume that F_j, $j = 1, 2$, are solutions such that $F_j' = Bh_j$ with non-vanishing functions h_j in $\mathbb{D}$, $h_j(0) > 0$. Define

$$u_j(z, \bar{z}) := \log\left(\frac{|h_j(z)|}{1 + \alpha|F_j(z)|^2}\right).$$

Then each u_j is continuous in $\overline{\mathbb{D}}$ and a simple calculation shows that

$$(u_j)_{z\bar{z}} = -\alpha|B(z)|^2 e^{2u_j}.$$

Furthermore, $u_j = 0$ on $\partial\mathbb{D}$. Lemma 3.1 gives now

$$\frac{|F_1'(z)|}{1 + \alpha|F_1(z)|^2} = \frac{|F_2'(z)|}{1 + \alpha|F_2(z)|^2}, \quad z \in \mathbb{D}.$$

This latter relation is easily seen to imply $F_1 = F_2$, taking into account the normalizations used. □

Proof (Theorem 1.5). We need to prove (1.8) for $z \in \partial\mathbb{D}$ only. Let $u \in \partial\mathbb{D}$ be fixed. If $F(u) = 0$ we see that the function $|zh(z)|$ assumes a maximum w.r.t. $\overline{\mathbb{D}}$ in u, and an application of Lemma 2.2 yields (1.8). Hence we may assume $F(u) \neq 0$. We define

$$w(z) := zh(z) + \mu G(z), \qquad \text{where } G(z) := -\alpha\frac{F^2(z)}{z},\ \mu := \frac{uh(u)}{|h(u)|}\frac{|G(u)|}{G(u)},$$

so that, by assumption,

$$w \in \mathcal{H}(\overline{\mathbb{D}}), \quad \|w\|_{\mathbb{D}} = 1, \quad w(u) = \frac{uh(u)}{|h(u)|}.$$

Note that Lemma (2.2) applies to w at $z_0 = u$. We now get

$$\begin{aligned}
\operatorname{Re}\frac{(uh(u))'}{h(u)} &= \frac{w'(u)}{h(u)} - \mu\frac{G'(u)}{h(u)} \\
&= \frac{1}{|h(u)|}\frac{uw'(u)}{w(u)} + 2\alpha\operatorname{Re}\left(\mu\frac{F'(u)}{h(u)}\frac{F(u)}{u}\right) + \frac{1}{|h(u)|} - 1 \\
&\geq \frac{2}{|h(u)|} + 2\alpha|F(u)| - 1 \\
&= \frac{2\alpha^2|F(u)|^4}{1 + \alpha|F(u)|^2} - \frac{\alpha}{2}(1 - 2|F(u)|)^2 + 1 + \frac{\alpha}{2} \\
&\geq 1 + \frac{\alpha}{2},
\end{aligned}$$

which is our assertion. □

4 Open problems

There is a number of intriguing open problems in the context of this note. We explain some of them.

1 Unicity

There is a large gap between Theorems 1.3 and 1.4. What are the right conditions on Φ to ensure the uniqueness of the solutions in Theorem 1.1? The unicity statement in Beurling's paper may hint in the right direction.

2 The hyperbolic case

1. Let $f \in \mathcal{H}(\overline{\mathbb{D}_r})$ for some $r < 1$, and $f(\overline{\mathbb{D}_r}) \subset \mathbb{D}$, be such that f maps arcs $\Gamma \in \partial\overline{\mathbb{D}_r}$ onto arcs in $\mathbb{D}$ with the same hyperbolic length. As indicated above this leads to the condition

$$|F'(z)| = 1 - \beta^2|F(z)|^2, \quad |z| = 1, \tag{4.1}$$

for $F(z) := \frac{1}{\beta}f(\frac{z}{r})$, $\beta = \frac{r}{1-r^2}$. Obviously, the methods developed in this paper work only for $r < (\sqrt{5} - 1)/2$, while for larger values of r additional difficulties arise.

2. Even for the cases where our methods work (and B is 'non-trivial') there is not a single 'example' for such a solution. No numerical method seems available to calculate even a single value of any such function in any given point $z \neq 0$.

3. The case $\beta = 0$ in (4.1) can be looked at as the Euclidean limiting case of this hyperbolic situation, where all solutions are given by $F' = B$, i.e. F' rational. For $\beta \neq 0$ the existence of solutions F with F' rational is not known.

4. Is it true that every solution F in the hyperbolic case (or beyond) can have singularities (namely branch points) only in the poles of the corresponding B?

5. Assume that $F' \in \mathcal{H}(\mathbb{D})$, continuous in $\overline{\mathbb{D}}$, and that F satisfies (4.1) on $\partial\mathbb{D}$. Then the unicity statement of Theorem 1.4 implies that F must be identical (up to a constant factor) with one of the functions of Theorem 1.1 with suitable B, i.e. F has an analytic extension onto $\overline{\mathbb{D}}$. This is, for $\alpha = 0$, some form of the extended Schwarz reflection principle, but now for our 'free' boundary conditions. It is an interesting open question whether this extends to proper subarcs of $\partial\mathbb{D}$, namely: *assume $F' \in \mathcal{H}(\mathbb{D})$ is continuous in $\mathbb{D} \cup \Gamma$, where Γ is a proper subarc of $\partial\mathbb{D}$, and that F satisfies (4.1) on Γ. Does F extend analytically across Γ?*

3 The spherical case

Here we deal with meromorphic functions f in $\overline{\mathbb{D}_r}$ for some $r > 0$, which preserve the spherical length of arcs on $\partial\mathbb{D}_r$. It is clear that a number of new problems arise in comparison with the hyperbolic case. The methods described in this note handle (partially) only the cases $r \leq 1$ and $f \in \mathcal{H}(\overline{\mathbb{D}_r})$, and show that for $r > 1$ we cannot expect solutions for arbitrary choices of B. The questions described above for the hyperbolic case are open for these cases as well.

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