

CONSTRUCTING INTEGRABLE
THIRD ORDER SYSTEMS:
THE GAMBIER APPROACH

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Abstract

We present a systematic construction of integrable third order systems based on the coupling of an integrable second order equation and a Riccati equation. This approach is the extension of the Gambier method that led to the equation that bears his name. Our study is carried through for both continuous and discrete systems. In both cases the investigation is based on the study of the singularities of the system (the Painlevé method for ODE's and the singularity confinement method for mappings).

1. INTRODUCTION

The investigation of the integrability of second order differential equations has been one of the most important enterprises in the history of integrable systems. Initiated by Painlevé [1] and completed by Gambier [2], it established the importance of singularity analysis as an integrability criterion. The results of the Painlevé-Gambier investigations are of capital importance since they showed the existence of new transcendents, known since then under the name of Painlevé. Overshadowed by this momentous discovery, the work of Gambier on linearizable systems did not receive the attention it deserved. The recent discovery of integrable *discrete* systems has led naturally to a critical examination of the work of the 19th century masters. In particular, we have shown that it is possible to find discrete forms not only for the Painlevé equations, but, in fact, for every single equation in the Painlevé-Gambier list. The equation #XXVII of the list of 50 canonical equations [3], which we decided to call the Gambier equation, was of course among them. Its discretization necessitated a thorough understanding of the Gambier approach.

The key idea of Gambier (but we are conscious that the historical truth may be different) was to construct an integrable second order equation by suitably coupling two integrable first order ones. The latter were well-known: at first order the only integrable ordinary differential equations are either linear or of Riccati type. The Gambier equation is precisely the coupling of two Riccati in cascade (and it contains as a subcase the coupling involving one or even two linear equations). From the point of view of singularity analysis this coupling of two integrable equations is not harmless. Each of the equations has singlevalued movable singularities. However, the singularities induced on the second equation by the singularities of the solution of the first one (and which would thus look superficially as fixed) may lead to multivaluedness. This feature makes the application of singularity analysis mandatory. Its implementation leads to the (algebraically) integrable forms of the Gambier equation.

In perfect analogy to the continuous case, we have introduced in [4] the Gambier mapping. The latter is a system of two coupled homographic mappings (which play the role of the discrete Riccati) in cascade. The integrable forms were obtained through the application of the discrete integrability criterion that we have proposed under the name of singularity confinement.

In the present work we shall address the question of the construction of integrable third order systems in the spirit of Gambier. Namely we shall start with a second order integrable equation and couple it with a Riccati (or a linear) first order (also integrable) equation. This enterprise may easily assume staggering proportions. While at second order one had only two first order building blocks at one's disposal, at third order there are minimally 24 equations (the Gambier list) to be coupled to the two first order integrable ones. The situation is even more overwhelming in the discrete case since it is well-known that each continuous equation of the Gambier list may possess several discrete avatars. In order to limit the scope of our investigation we shall consider coupled systems where the dependent variable enters only in a polynomial way. This leads naturally to the coupling of a Painlevé (\mathbb{P}) I or II to a Riccati.

Historically the coupling of a \mathbb{P} equation with a Riccati was first considered by Chazy [5]. He examined an additive coupling of P_I with a Riccati. Starting with P_I in the form:

$$w'' = 6w^2 + z,$$

he introduced a Riccati:

$$y' = \alpha y^2 + \beta y + \lambda w + \gamma. \quad (1.1)$$

This coupling is additive as opposed to the one introduced by Gambier which is multiplicative and assumes the form:

$$y' = \alpha y^2 + (\beta + \lambda w)y + \gamma.$$

(In the case of the Gambier coupling w is the solution of a Riccati equation). Since the singularities of P_I are double poles ($6/(z - z_0)^2$), the only coupling that is compatible with integrability is the additive one. Assuming that $\alpha \neq 0$, we can put $\beta = 0$ by a simple translation of y and Chazy found that the only cases where the leading singularity does not induce multivaluedness were when equation (1.1) assumed the form:

$$y' = \frac{1 - k^2}{4} y^2 + w + \gamma, \quad (1.2)$$

where k is an integer not multiple of 6. Thus 5 cases had to be examined as $k = 6m + n$ with $n = 1, \dots, 5$. Chazy

found the following necessary integrability constraints:

$$\begin{aligned}
n = 2 & \quad \gamma = 0 \\
n = 3 & \quad \gamma' = 0 \\
n = 4 & \quad \gamma'' = \mu\gamma^2 + \nu z \\
n = 5 & \quad \gamma''' = \mu\gamma\gamma' + \nu,
\end{aligned}$$

where μ and ν are specific numerical constants. It turns out that for $k = n$ they are also sufficient. For $k = 6m + 1$ the first condition appears at $k = 7$. In this case the constraint reads:

$$\gamma^{(5)} = 48\gamma\gamma''' + 120\gamma'\gamma'' - \frac{2304}{5}\gamma'\gamma^2 - 24z\gamma' - 48\gamma.$$

This equation has the \mathbb{P} property and is thus expected to be integrable. Still it is interesting to point out that this equation is more difficult to solve than the one we started with which is of third order.

Chazy offers only a rapid comment concerning the case $k \geq 8$. In fact, the constraints obtained are necessary, but not sufficient for higher k 's. We have examined the first few cases beyond $k = 7$ using the same method as Chazy, namely singularity analysis (but unlikely Chazy our approach has profited from the existence of computer algebra tools). It turned out that none of the cases we examined satisfied the Painlevé criterion. So, although this is not a proof in a strict sense, we can suppose that no integrable cases exist beyond the 5 identified by Chazy.

2. COUPLING OF INTEGRABLE SECOND ORDER ODE'S WITH A RICCATI

As we have explained in the introduction we shall not attempt an exhaustive treatment of all 24 [2] (or 50 [3], or more [6]) second order equations of the Painlevé/Gambier list with a Riccati. Instead we shall limit ourselves to the simplest case, namely equations where the dependent variable enters in a polynomial way (instead of rational). This limits the research to just three generic equations: P_I (already examined by Chazy), P_{II} and the linearisable, G5 (number 5 of the Gambier list), equation. Both P_I and G5 have dominant singularities that are single poles i.e. $w \sim 1/(z - z_0)$. Thus the adequate coupling is through a multiplicative Riccati. An additive coupling would lead to logarithmic singularities in the Riccati and thus to multivaluedness incompatible with integrability.

2a Coupling P_{II} with a Riccati

We start with the canonical form of P_{II} , namely:

$$w'' = 2w^3 + zw + \mu \tag{2.1}$$

and consider the following multiplicative coupling:

$$y' = \alpha y^2 + (nw + \beta)y + \gamma. \tag{2.2}$$

A gauge transformation on y can be used in order to put β to zero. Next, we proceed to determine α, γ through the application of singularity analysis so as to ensure the Painlevé property for the system. Equation (2.1) has of course the \mathbb{P} property, and the expansion of its solution around a singularity is:

$$w = \frac{\sigma}{z - z_0} + \dots + a_4(z - z_0)^4 + \dots \tag{2.3}$$

where $\sigma^2 = 1$ and a_4 is a free parameter (the second one beside z_0). The coupling of w with y must not lead to multivaluedness. Thus the coefficient n of the coupling must be an integer. This is only the first condition and, by far, not sufficient. In order to proceed further we expand y around the singularity z_0 and assume that y either has a pole at the same location or it is regular. Substituting the expansion form (2.3) we can compute the terms of the series of y and obtain the compatibility conditions for the absence of logarithmic terms in the expansion of y . We find thus the following condition for $n = 1$:

$$\gamma = \alpha = 0. \tag{2.4}$$

Let us point out that $\alpha = \gamma = 0$ works for every value of n : w is simply related to the logarithmic derivative of y . We have furthermore, for $n = 2$:

$$\gamma' = \alpha' = 0. \quad (2.5)$$

For $n = 3$ we obtain

$$\frac{\gamma}{\alpha} = - \left(\frac{2\alpha'' + \alpha z}{\alpha^3} \right) \text{ and } \frac{\alpha}{\gamma} = - \left(\frac{2\gamma'' + \gamma z}{\gamma^3} \right). \quad (2.6)$$

Isolating γ and integrating once, we find:

$$\alpha\alpha''' - 3\alpha'\alpha'' - \alpha'\alpha z - k\alpha^2 = 0 \quad (2.7)$$

and putting $\phi = \frac{\alpha'}{\alpha}$ we find

$$\phi'' = 2\phi^3 + z\phi + k. \quad (2.8)$$

Thus the logarithmic derivative of α satisfies precisely P_{II} (with a free constant k). For $n = 4$ we find a more complicated condition:

$$3\alpha\gamma\gamma' + \alpha'\gamma^2 + 3\alpha(1 + \mu) + 3\gamma'z + 9\gamma''' = 0, \quad (2.9a)$$

$$3\gamma\alpha\alpha' + \gamma'\alpha^2 + 3\gamma(1 - \mu) + 3\alpha'z + 9\alpha''' = 0. \quad (2.9b)$$

Putting $\alpha^2 = \phi'$ we can integrate (2.9b) (multiplied by α) for the quantity $\alpha^3\gamma$ and thus obtain γ . Then (2.9a) gives a 6th order homogeneous equation for ϕ and putting $u = \phi'/\phi$ leads to a 5th order equation. This equation passes the \mathbb{P} test and is thus presumably integrable but its integration is a more complicated task than the equation we started with, which is only of third order.

For $n = 5$ we obtain again as a first condition $\alpha = \gamma = 0$ which as we explained is sufficient. For $n = 6$ a first condition (as in the $n = 2$ case) is $\alpha' = \gamma' = 0$. However a second condition appears. In fact, for $n > 4$ the free parameter of the expression (2.3) a_4 starts appearing in the compatibility condition which must be identically satisfied. Thus for $n = 6$ we find the second the second condition either $\alpha = 0$ and $\mu = -7/6$ or $\gamma = 0$ and $\mu = 7/6$. Thus this coupling works only for some particular case of P_{II} with a specific μ . For $n \geq 7$ we have not been able to find any integrable case, besides the trivial $\alpha = \gamma = 0$ one. In some cases it is even possible to prove the incompatibility of the constraints. We surmise that the multiplicative coupling of P_{II} with a Riccati does not possess any integrable case besides the ones listed above.

2b Coupling the linearisable G5 with a Riccati

The canonical form of the linearisable equation, G5 in the Gambier list is:

$$w'' = -3w'w - w^3 + q(z)(w' + w^2). \quad (2.10)$$

The Cole-Hopf transformation $w = -u'/u$ reduces (2.10) to a linear equation $u''' = q(z)u''$. The function $q(z)$ is completely free. Given this fact one can make two different couplings. The first is the 'standard' one where the solution of (2.10) for given $q(z)$ is injected into a Riccati:

$$y' = \alpha y^2 + nwy + \gamma. \quad (2.11)$$

The condition for Painlevé property for $n < 0$ turns out to be $\alpha = 0$, while for $n > 0$ it is $\gamma = 0$. In both cases (2.10) becomes a linear equation (either for y or for $1/y$) and the remaining free function (γ or α) does not produce multivaluedness.

The second case of coupling is when $q(z)$ is itself proportional to the solution of a Riccati. Thus the coupled system now becomes

$$w' = \alpha w^2 + \beta w + \gamma, \quad (2.12a)$$

$$y'' = -3yy' - y^3 + nw(y' + y^2). \quad (2.12b)$$

Only the case $\alpha \neq 0$ needs to be considered since when $\alpha = 0$ equation (2.12a) is linear and thus does not have any singularities. Since $\alpha \neq 0$ we can take $\alpha = 1$. As previously the coupling enters through nw with integer n since the singularity of (2.12a) is a simple pole. For $n < 0$ the system has always the Painlevé property and thus β and γ are free. On the contrary, for $n > 0$ we have stringent integrability conditions. For $n = 1, 2$ there is no solution

for β, γ leading to the Painlevé property for the system. For $n = 3$ we find as only solution $\beta = \gamma = 0$. For $n = 4$ we obtain the condition:

$$\gamma = -\frac{11}{4}\beta^2 + \beta' \quad (2.13a)$$

and

$$\beta'' = 12\beta\beta' - 16\beta^3 \quad (2.13b).$$

Putting $\beta = -\frac{\phi'}{4\phi}$, (2.13b) reduces to $\phi''' = 0$ and we thus have elementary expressions for β and γ .

For $n \geq 5$ we can obtain the two compatibility conditions in the form of a higher order nonlinear system for β, γ . It turns out that for the first few cases studied this system has the weak \mathbb{P} property [7]. We have not tried to integrate these systems since their integration is more difficult than the problem we started with.

3. COUPLING OF A SECOND ORDER MAPPING WITH A DISCRETE RICCATI EQUATION

Constructing integrable discrete systems in the same spirit as Gambier is quite straightforward once the basic ingredients are available. What is needed is a detailed knowledge of the forms of the equations to be coupled and a reliable integrability detector. The second order mappings which play the role of the \mathbb{P} equations in the discrete domain have been the object of numerous detailed studies and we are now in possession of discrete forms of all the equations of the Painlevé/Gambier classification. The discrete integrability detector is based on the singularity confinement that we discovered in [8] and which has turned out to be of utmost reliability.

The coupling we are going to consider is a homographic mapping (discrete Riccati) for the variable y :

$$\bar{y} = \frac{(\alpha x + \beta)y + (\eta x + \theta)}{(\epsilon x + \zeta)y + (\gamma x + \delta)}, \quad (3.1)$$

(where \bar{y} stands for y_{n+1} , y for y_n and $\alpha, \beta, \dots, \theta$ depend in general on n) the coefficients of which depend linearly on x , the solution of the discrete \mathbb{P} I or II. (We shall not present here the coupling of the discrete analog of the linearisable equation to a Riccati. As a matter of fact this equation is the simplest non-trivial member of the hierarchy of projective Riccati systems, the discretisation of which was presented in full generality in [9]). The mapping (3.1) can be simplified and brought under canonical form through the application of homographic transformations on y . The generic form of the result is:

$$\bar{y} = \frac{(\alpha x + \beta)y + 1}{y + (\gamma x + \delta)}. \quad (3.2)$$

Non-generic cases do exist as well, and foremost among those is the linear relation:

$$\bar{y}(\gamma x + \delta) - y(\alpha x + \beta) - 1 = 0. \quad (3.3)$$

In what follows we shall examine in detail the coupling of (3.2) and (3.3) with either d-P_I or d-P_{II} (under various forms).

How does one apply the singularity confinement criterion to a mapping such as (3.2) when x is given by some discrete equation like d-P_I or d-P_{II}? The first step consists in determining the singularities of (3.2). As we have explained in [10] the singularity manifests itself by the fact that \bar{y} is independent of y . (We say in this case that y “forgets the initial condition” or “loses one degree of freedom”). The condition for \bar{y} to be independent of y is just

$$(\gamma x + \delta)(\alpha x + \beta) = 1. \quad (3.4)$$

This quadratic equation has two roots which we will denote by X_1, X_2 : they can be easily related to $\alpha, \beta, \gamma, \delta$. The confinement condition is for y to recover the lost degree of freedom. This can be done if y assumes an indeterminate form 0/0. This means that x at this stage must again satisfy (3.4) and moreover be such that the denominator (or, equivalently, the numerator) vanishes.

Let us assume now that for some n we have $x_n = X_1$. The confinement requirement is that k steps later $x_{n+k} = X_2$. (We must point out here that we require that x_{n+k} be equal to X_2 and not to X_1 again. If the latter were done this would mean that X_1 is a singularity occurring periodically. Such singularities are not really movable, i.e. their position cannot be freely adjusted by choosing the appropriated initial conditions. Our conjecture is that they do not play any role in integrability, just like the fixed singularities in the continuous case). Starting from $x_n = X_1$ and some initial datum x_{n-1} , we can iterate the mapping for x and obtain x_{n+k} as a complicated function

of x_{n-1} and X_1 . Since x_{n+k} depends on the free parameter x_{n-1} there is no hope for x_{n+k} to be equal to X_2 if X_1 is a generic point for the mapping of x . The only possibility is that both X_1 and X_2 be special values. What are the special values of this equation depends on its details, but clearly in the case of the discrete \mathbb{P} 's we shall examine here, these values can only be the ones related to the singularities. To be more specific, let us examine d-P_{II}:

$$\bar{x} + \underline{x} = \frac{zx + a}{1 - x^2}. \quad (3.5)$$

The only special values of x are the ones related to the singularity $x_n = \pm 1$, $x_{n+1} = \infty$, $x_{n+2} = \mp 1$ while \dots , x_{n-2} , x_{n-1} and x_{n+3} , x_{n+4} , \dots are finite. This means that the two roots of (3.4) must be two of $\{+1, \infty, -1\}$ and moreover that confinement must occur in two steps. The precise implementation of singularity confinement requires that the denominator of (3.2) at $n + 2$ vanishes (and because of (3.4) this ensures that the numerator vanishes as well). Moreover, we must make sure that the lost degree of freedom (i.e. the dependence on y) is indeed recovered through the indeterminate form.

3a Coupling various d-P_I's with a discrete Riccati

In this subsection we are going to analyse the coupling of four different forms of d-P_I to the homographic mapping (3.2) and to a linear equation (3.3). The d-P_I's we are going to consider are the following (presented below together with their singularity pattern):

$$\bar{x} + \underline{x} = \frac{z_n}{x} + \frac{1}{x^2} \quad \{0, \infty, 0\}, \quad (3.6)$$

$$\bar{x} + x + \underline{x} = \frac{z_n}{x} + 1 \quad \{0, \infty, \infty, 0\}, \quad (3.7)$$

$$\bar{x} + \underline{x} = \frac{z_n}{x} + 1 \quad \{0, \infty, 1, \infty, 0\}, \quad (3.8)$$

$$\bar{x}\underline{x} = -\frac{q_n}{x^2} + \frac{1}{x} \quad \{q, 0, \infty, 0, q\}, \quad (3.9)$$

with $z_n = an + b$ and $q_n = q_0\lambda^n$ (a, b, q_0, λ are constants). (More forms of the discrete P_I [11] are known but we shall restrict our analysis of the possible couplings to just these simplest forms).

All the singularity patterns above have as common characteristic that one can enter the singularity through 0 and exit it again through 0. This means that the condition (3.4) can have 0 as a double pole. This results to the following conditions:

$$\begin{aligned} \beta\delta &= 1, \\ \alpha\delta + \beta\gamma &= 0, \end{aligned} \quad (3.10)$$

and since neither α nor γ can vanish (lest the x^2 term disappear) we have $\delta = 1/\beta$ and $\gamma = -\alpha/\beta^2$. One can, of course, consider the case where one (or two) of the roots of (3.4) are equal to ∞ : after all ∞ is part of the special values of the singularity pattern. It has turned out that except for the case (3.8) the consideration of these cases does not lead to any interesting result. (Let us point out that the value 1 appearing in the singularity pattern of (3.8) should not be considered as a special value: it may well occur outside any singularity pattern). Thus the first discrete Riccati we are going to consider is of the form:

$$\bar{y} = \frac{y(\alpha x + \beta) + 1}{y - (\alpha x - \beta)/\beta^2}.$$

In all the cases considered, the first confinement condition, namely that y (at a suitable n) assumes the form 0/0 does not suffice in order to reintroduce the dependence in the initial conditions. It is thus necessary to proceed to the next order and introduce one further constraint (which turns out to be sufficient). Let us work out in detail the case of the d-P_I (3.6). Starting with $\underline{x} = 0$ we obtain $y = \underline{\beta}$ i.e. independent of the value of \underline{y} . For $x = \infty$ we obtain $\bar{y} = -\underline{\beta}\beta^2$ and finally at the next step, $\bar{x} = 0$, we ask that the numerator and denominator of \bar{y} vanish. This leads to the first condition

$$\underline{\beta}\beta^2\bar{\beta} = 1. \quad (3.12)$$

Implementing this constraint leads to a second confinement condition that reads: $\underline{\alpha}/\underline{\beta} = \bar{\alpha}/\bar{\beta}$. This means that $\alpha = c\beta$ where c is a constant with an even-odd dependence. The solution of the constraint (3.12) is straightforward. Taking the logarithm of both members and calling $b = \log \beta$ we find the linear equation

$$\underline{b} + 2b + \bar{b} = 0 \quad (3.13)$$

with solution $b = (p + qn)(-1)^n$. Simple solutions to (3.12) can be obtained from this last solution. On the other hand just by inspection we can obtain solutions to (3.12) where β is constant: $\beta = \pm 1, \pm i$.

The case of the ‘‘standard’’ d-P_I (3.7) can be treated along similar lines. The first confinement condition reads:

$$\beta \beta^2 \bar{\beta}^2 \bar{\beta} = -1 \quad (3.14)$$

while the second becomes too complicated to be exactly solved. We prefer to proceed using one particular solution of (3.14) corresponding to constant β 's, for example $\beta = i$. This leads to a second confinement condition $\alpha/z = \bar{\alpha}/\bar{z}$. Thus $\alpha = cz$ where c is a constant with ternary freedom ($\bar{c} = \underline{c}$). If we implement $\beta = e^{\pm i\pi/6}$ and define $\chi = -\alpha z$, we get, as a second confinement condition, the equation:

$$\bar{\chi} + \chi + \underline{\chi} = 3\beta^2 \frac{z}{\chi} + c, \quad (3.15)$$

where c is a constant of integration. Thus, after considering the coupling with a d-P_I we get a d-P_I of the same type as one of the confinement conditions. This is in perfect parallel to the continuous case of Chazy (coupling (1.2) with $n = 4$) where we get another P_I as the integrability condition for a coupling between a Riccati and a P_I.

The case (3.8) leads to still more complicated equations. One way to simplify them is to choose β satisfying:

$$\beta \beta^2 \bar{\beta} = 1 \quad (3.16)$$

which is sufficient (but not necessary) to satisfy the first confinement condition. We can then implement the solutions $\beta = i$ and $\beta = 1$. If $\beta = i$, the second confinement condition is $\alpha = c/z$ (where c is a constant with quaternary freedom $\underline{c} = \bar{\bar{c}}$). If $\beta = 1$, we define $\chi = -\alpha z$ and we get, as second confinement condition, the following equation:

$$\bar{\chi} + \underline{\chi} = -\frac{4z}{\chi} + c, \quad (3.17)$$

where c is a constant with binary freedom. So, again here, we get a d-P_I of the same type as the one we started with as a confinement condition.

For the case (3.8), it is also possible to consider a coupling where the condition (3.4) has 0 and ∞ as roots. This means that $\alpha = 0$ (we could also choose $\gamma = 0$ but these two cases are equivalent under the homographic transformation $w \rightarrow 1/w$) and $\delta = 1/\beta$. The first confinement condition then is $\gamma = -\delta$. We define $\chi = \bar{\beta}\beta$ and the second integrability condition reads:

$$\bar{\chi} + \underline{\chi} = -\frac{z + c}{\chi} + 1, \quad (3.18)$$

where c is a constant of integration. Thus again we get a d-P_I of the same type as the one we started with. Finally we can also consider the case where the condition (3.4) has ∞ as a double root. We then must have $\alpha = \beta = 0$. The first confinement condition is $\delta = -\gamma$ and we get the following relation for γ :

$$\bar{\gamma}\gamma = \frac{1}{-z + k}, \quad (3.19)$$

(where k is a constant of integration) which can be solved in an elementary way for γ .

In the case of q -P_I (3.9) the full singularity pattern is one where we enter the singularity at q and exit it at q after four steps. However the complete study of this singularity pattern turns out to be untractable. Thus we shall limit ourselves here to the case where we enter the singularity through 0 and exit it through 0 after two steps. In this case the first confinement condition is just (3.12). Once this is implemented the second condition reads $\bar{\alpha}\beta = \lambda\underline{\alpha}\bar{\beta}$ which means $\alpha = c\beta\mu^n$ where c is a constant with binary freedom and $\mu^2 = \lambda$.

Let us now turn to the case of the coupling of d-P_I with a linear equation (3.3). For the special values of d-P_I 0 and ∞ , only three couplings have to be considered:

$$\bar{y} = \alpha y + \frac{1}{\gamma x}, \quad (3.20a)$$

$$\bar{y} = \alpha xy + \frac{1}{\delta}, \quad (3.20b)$$

$$\bar{y} = \frac{\beta y + 1}{\gamma x}. \quad (3.20c)$$

It turns out that in every case examined the second (3.20b) and third (3.20c) are always incompatible with confinement. The only remaining candidate is thus the coupling of the form (3.20a). By the appropriate gauge of y we can bring it to the form:

$$\bar{y} - y = \frac{1}{\gamma x}. \quad (3.21)$$

Let us work out in detail the case of (3.6). A detailed analysis of the singularity pattern shows that if \underline{x} vanishes like ϵ then x diverges like $1/\epsilon^2$ and \bar{x} vanishes like $-\epsilon$. We compute the corresponding y 's and find, at leading order, $y \sim 1/\underline{\gamma}\epsilon$, $\bar{y} \sim 1/\bar{\gamma}\epsilon$ and the condition for \bar{y} to be finite is $1/\underline{\gamma} - 1/\bar{\gamma} = 0$ i.e. γ must be a constant with binary freedom (i.e. even-odd dependence). The analysis of the remaining cases proceeds along similar lines. For (3.7) we have the pattern $\{\epsilon, \underline{z}/\epsilon, -\bar{z}/\epsilon, -\epsilon\bar{z}/\underline{z}\}$ and the condition for \bar{y} to be finite is $\bar{\gamma}\bar{z} = \underline{\gamma}\underline{z}$ i.e. $\gamma = k/z$ where k is a constant with ternary freedom. The case (3.8) is related to the pattern $\{\epsilon, \underline{z}/\epsilon, 1, -\bar{z}/\epsilon, \bar{z}\epsilon/\underline{z}\}$ leading to the confinement condition $\underline{\gamma}\underline{z} = \bar{\gamma}\bar{z}$ i.e. $\gamma = k/z$ where k is a constant with quaternary freedom. Finally the case (3.9) is related to the pattern $\{q + \epsilon, a\epsilon, -\lambda/(a^2\epsilon^2), -\epsilon a/\lambda, \bar{q}\}$ (where a is a free constant). Again we concentrate on the singularity induced by $\underline{x} = 0$ and which confines when $x = 0$ again. This results to the condition $\bar{\gamma} = \lambda\underline{\gamma}$ which means $\gamma = k\mu^n$ where k is a constant with binary freedom and $\mu^2 = \lambda$.

3b Coupling discrete P_{II} 's with a discrete Riccati

In this section we shall examine the coupling of two different discrete forms of P_{II} with a Riccati: a difference one (which is the "standard" d- P_{II})

$$\bar{x} + \underline{x} = \frac{zx + \mu}{1 - x^2}, \quad (3.22)$$

where z is linear in the discrete variable n and μ is a constant, and one of q -type:

$$\bar{x}\underline{x} = \frac{p(x - q)}{x(x - 1)}, \quad (3.23)$$

where $q = q_0\lambda^n$ and $p = p_0\lambda^n$.

Let us start with d- P_{II} (3.22). The singularity pattern of this equation is $\{\pm 1, \infty, \mp 1\}$. This means that the singularity condition (3.4) must have ± 1 as roots (the case when one root is ∞ does not lead to anything interesting). As a result we have that δ and γ are given by $\delta = -\beta/(\alpha^2 - \beta^2)$, $\gamma = \alpha/(\alpha^2 - \beta^2)$. The pattern $\{1, \infty, -1\}$ leads to a confinement condition: $\gamma = (\underline{\alpha} + \underline{\beta})\alpha(\bar{\alpha} - \bar{\beta})$ while the second pattern $\{-1, \infty, 1\}$ leads to $\gamma = (\underline{\alpha} - \underline{\beta})\alpha(\bar{\alpha} + \bar{\beta})$. Equating the two expressions for γ we find $\underline{\alpha}\bar{\beta} = \bar{\alpha}\underline{\beta}$ i.e. $\beta = k\alpha$ where k is a constant with binary freedom which we will ignore from now on. Expressing γ in two possible ways we get finally for α the equation:

$$\underline{\alpha}\alpha^2\bar{\alpha} = \frac{1}{(1 - k^2)^2}. \quad (3.24)$$

This equation can be solved by linearisation just by taking the logarithm of both sides.

The q - P_{II} has also two singularity patterns $\{q, 0, \infty, 1\}$ and $\{1, \infty, 0, q\}$. Requiring 0 and 1 to be roots of (3.4) gives the following expressions for γ , δ : $\delta = 1/\beta$ and $\gamma = -\frac{\alpha}{\beta(\alpha + \beta)}$. Next we obtain the confinement conditions for the two patterns of singularities:

$$\underline{\alpha}\alpha\bar{\beta}\bar{\beta} + \alpha\underline{\beta}\bar{\beta}\bar{\beta} + \underline{\alpha}\beta^2\bar{\beta} + \beta\beta^2\bar{\beta} - 1 = 0, \quad (3.25a)$$

$$\alpha\bar{\alpha}\underline{\beta}\bar{\beta} + \bar{\alpha}\underline{\beta}\beta^2 + \alpha\underline{\beta}\bar{\beta}\bar{\beta} + \beta\beta^2\bar{\beta} - 1 = 0. \quad (3.25b)$$

Subtracting these two equations we obtain $\bar{\alpha}\underline{\beta} = \underline{\alpha}\bar{\beta}$ i.e. $\beta = k\alpha$ where k is a constant with binary freedom which we again ignore. Substituting back to (3.25) we obtain the final condition:

$$\underline{\alpha}\alpha^2\bar{\alpha} = \frac{1}{k^2(k + 1)^2} \quad (3.26)$$

which can be integrated through linearisation as explained above. The case where (3.3) has ∞ and q as roots is equivalent to the one treated above by a homographic transformation.

We now consider the case where (3.4) has 0 and q as roots which impose the relations $\delta = 1/\beta$ and $\gamma = \frac{-\alpha}{\beta(q\alpha+\beta)}$. As first condition we then get that β is a constant with binary freedom. We ignore this freedom and consider β as constant and we get the following relation for α : $\alpha = -\frac{(\beta^2+1)}{\beta q}$. Finally, the case where (3.4) has q and 1 as roots has been studied but the resulting equations are far too complicated to be of any use. There is no other possible coupling of the form (3.2) with the q -P_{II} (3.23).

Let us now turn to the case of a linear coupling given by equation (3.3). In the case of d-P_{II} (3.22) we require that the only singularities of the coupling terms $(\alpha x + \beta)/(\gamma x + \delta)$ be the two singularities ± 1 . This leads to a coupling of the form:

$$\bar{y} = \frac{\alpha(x \pm 1)y + 1}{x \mp 1}, \quad (3.27)$$

where one of the parameters (e.g. γ) has been put to 1 through the appropriate gauge of y . Computing the successive y 's we find that the condition for having a finite \bar{y} , depending on the initial condition y , is just $\bar{\alpha}\alpha = 1$. This means that all even α 's are constant while all odd ones are equal to the inverse of this constant.

For q -P_{II} (3.23), in the case where (3.4) has 0 and 1 as roots, we have two possible couplings:

$$\bar{y} = \frac{\alpha xy + 1}{x - 1} \quad (3.28a)$$

and

$$\bar{y} = \frac{\alpha(x - 1)y + 1}{x}. \quad (3.28b)$$

It turns out that in both cases the confinement condition is the same as in the case of d-P_{II} namely $\bar{\alpha}\alpha = 1$. When the roots are 0 and q , the possible couplings are:

$$\bar{y} = \frac{\alpha xy + 1}{x - q} \quad (3.29a)$$

and

$$\bar{y} = \frac{\alpha(x - q)y + 1}{x}. \quad (3.29b)$$

The condition for integrability in the two cases is $\alpha = 1/\lambda$. Two other couplings are possible when the roots of (3.4) are q and ∞ :

$$\bar{y} = \frac{\alpha y + 1}{x - q}, \quad (3.30a)$$

and

$$\bar{y} = \alpha(x - q)y + 1. \quad (3.30b)$$

The integrability condition for (3.30a) is $\alpha = \underline{q}$ and for (3.30b), it is $\alpha = 1/\underline{q}$. Finally if (3.4) has 1 and q as roots, the possible couplings are:

$$\bar{y} = \frac{\alpha(x - 1)y + 1}{x - q} \quad (3.31a)$$

and

$$\bar{y} = \frac{\alpha(x - q)y + 1}{x - 1}. \quad (3.31b)$$

In the case of (3.31a), the integrability condition is

$$\bar{\alpha}\alpha\underline{\alpha}(-\bar{q} + 1) + \bar{\alpha}(-\bar{q}q + \bar{q} + \underline{q} - 1) + q^2 - \bar{q} = 0 \quad (3.32)$$

and in the case of (3.31b), the condition reads

$$\bar{\alpha}\alpha\underline{\alpha}(\bar{q}^2 - q) + \bar{\alpha}\alpha(-\bar{q}q + \bar{q} + \underline{q} - 1) - \underline{q} + 1 = 0. \quad (3.33)$$

Equations (3.32) and (3.33) are integrable and they belong to the family of linearisable equations [10].

One last remark is necessary at this point since we have seen that almost all the equations we obtained contain terms with binary, ternary or quaternary freedom. The presence of these terms indicates that our systems must be

augmented by adding more components. This will not alter the order of the resulting equation: it just increases the number of its parameters. The continuous limit is, of course, affected by this choice.

4. CONCLUSION

In this work we have presented a systematic approach for the construction of integrable third order systems through the coupling of a second order equation to a Riccati or a linear first order equation. Thus we have extended the Gambier approach (first used in his derivation of the second order ODE that bears his name) to higher order systems. We have applied this coupling method to both continuous and discrete systems (given that we have already presented [4] the discrete equivalent of the Gambier equation).

One point remains to be discussed. It is often argued that, since the Riccati is a linearisable equation, the coupling of the Riccati to another of the same kind or to an integrable second order is always integrable. The (naïve) argument is the following: first solve the first equation, substitute the solution into the second and solve it by linearising it. The argument about singularities is usually swept aside by the statement that one is interested only in solutions on the real-time axis. However the situation is not that simple. What integrability consists in is a global description of the solutions of the equations. The argument about solutions on the real-time axis is not acceptable since it offers just a local description of the solution of the equation. A global representation of the solution of a linear equation (and, thus, also of a Riccati) involves path integrals winding over the complex-time plane. Thus the study of movable singularities is crucial and the Painlevé property a necessary condition for integrability of the systems.

How do these arguments carry over to the discrete setting? One must go back to the way difference equations are formally solved. Given a linear difference (or q -) equation, we can express the solution as an infinite product of matrices, the elements of which depend on the coefficients of the equation. A singularity appears whenever one of the matrices is singular. In this case the solution of the linear difference equation cannot be defined for every n . However it is in general possible to choose the coefficients of the equation so as to avoid these singularities. In the case of a coupling the coefficients depend on the solutions of some other equation. Thus there is no way to control the singularities (which depend on the initial conditions of the first equation). As a consequence the solution of the second equation cannot be defined everywhere unless the confinement property is satisfied. Thus, again, despite the linearisability of the discrete Riccati, whenever we talk about a global description of the solution of the coupled system, the application of the adequate integrability criterion is mandatory.

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