Group invariant solutions for the $N = 2$ super Korteweg-de Vries equation

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Abstract

The method of symmetry reduction is used to solve Grassmann-valued differential equations. The $(N = 2)$ supersymmetric Korteweg-de Vries equation is considered. It admits a Lie superalgebra of symmetries of dimension 5. A 2-dimensional subsuperalgebra is chosen to reduce the number of independent variables in this equation. We are then able to give different types of exact solutions, in particular soliton solutions.

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1 Introduction

The study of integrability and conservation laws for systems of Grassmann-valued differential equations (SGVDEs) has known a wide expansion in the area of supersymmetry[1−13]. Many of these equations have been constructed[2−6] in order to combine bosonic and fermionic degrees of freedom in a way such that these equations are invariant under supersymmetry transformations. This means that in these cases, there exists a symmetry relating bosonic and fermionic fields.

Many authors have proven integrability by finding Lax pairs and conservation laws. For example, Yung[12] has studied the supersymmetric Boussinesq hierarchies and his results were confirmed later by Bellucci *et al*[13]. Mathieu[3,4] has investigated the supersymmetric (SUSY) Korteweg-de Vries (KdV) equation for $N = 1, 2$ odd independent variables and has found that integrability occurs for special values of a parameter figuring in the superequations.

These SUSY KdV equations are the starting point of our approach and more particularly the case[4] $N = 2$. These equations contain both the KdV and modified KdV equation in the limit where odd Grassmann dependent variables are set equal to zero. Moreover, the supersymmetry group is richer in the $N = 2$ case than for $N = 1$ and it contains significant subgroups[14]. We will use the technique of symmetry reduction adapted to the super case to give some solutions of this ($N = 2$) SUSY KdV equation. This technique does not depend on the integrability of the equation and consists of a systematic application of group theory to reduce the SGVDE to a system of ordinary differential equations (ODEs).

The problem of computing solutions for SGVDEs has recently received a large amount of attention[15−19]. In these approaches, the SGVDEs are decomposed and give rise to systems of classical partial differential equations (PDEs) which can be solved. Our approach is based on symmetries and supersymmetries and does not require that we start with such a decomposition.

The Lie-point symmetries of a SGVDE have been obtained using an extension to Grassmann variables of the procedure described e.g. by Olver[20]. The $N = 2$ SUSY KdV admits a symmetry superalgebra with three even and two odd generators[14]. The equation, its invariance superalgebra and the corresponding group will be given in Section 2. The technique of symmetry reduction allows us to consider solutions which are invariant under subsuperalgebras with at least one supersymmetric (or odd) generator. In Section 3, a subsuperalgebra of dimension 2 will be used to derive the reduced equations and the solutions of the bosonic and fermionic part of the superfield. Some conclusions are drawn in Section 4.

2 The supersymmetry groups of the $N = 2$ SUSY KdV equation

Let us recall that a SGVDE is a system of s partial differential equations of order $k = (k_1; k_2)$ of the form

$$\Delta_\nu(\mathrm{X}, \Theta; \mathrm{A}^{(k_1)}, \Gamma^{(k_2)}) = 0, \qquad \nu = 1, \dots, s \tag{1}$$

with m independent even variables $\mathrm{X} = \{x_1, ..., x_m\}$, n independent odd variables $\Theta = \{\theta_1, ..., \theta_n\}$, q dependent even variables $\mathrm{A} = \{A^1, A^2, ..., A^q\}$ and p dependent odd variables $\Gamma = \{\Gamma^1, \Gamma^2, ..., \Gamma^p\}$. Note that odd variables η_i must satisfy

$$\eta_i\eta_j = -\eta_j\eta_i, \;\; \eta_i^2 = 0, \; 1 \leq i, j \leq r.$$

The equation we are interested in is precisely of this type. It takes the form[4]:

$$A_t = -A_{xxx} + 3(AD_1D_2A)_x + \frac{(a-1)}{2}\left(D_1D_2A^2\right)_x + 3aA^2A_x, \tag{2}$$

where $D_1,\ D_2$ are the covariant superderivatives

$$D_i = \theta_i\partial_x + \partial_{\theta_i},\ i = 1, 2. \tag{3}$$

Eq. (2) thus represents a one-parameter ($a \in I\!R$) family of Grassmann-valued partial differential equations having four (two even and two odd) independent variables $(x, t; \theta_1, \theta_2)$ and one dependent variable A which is supposed to be a bosonic (or even) superfield. It has been constructed by Mathieu *et al*[4] as a nontrivial SUSY equation which contains both the KdV and the modified KdV as nonSUSY limits.

A more suitable form of the eq. (2) is given using partial derivatives, i.e.

$$\begin{aligned} A_t &+ A_{xxx} - 3a\theta_1\theta_2A_xA_{xx} - (a+2)\theta_1AA_{xx\theta_2} - (a+2)\left\{\theta_1\theta_2AA_{xxx} - \theta_2AA_{xx\theta_1}\right\} \\ &+ (2a+1)\theta_2A_xA_{x\theta_1} + (a+2)\left\{A_xA_{\theta_1\theta_2} + AA_{x\theta_1\theta_2}\right\} \\ &- (2a+1)\theta_1A_xA_{x\theta_2} - (a-1)\left\{\theta_1A_{\theta_2}A_{xx}\right. \\ &- \left.\theta_2A_{\theta_1}A_{xx} + A_{\theta_1}A_{x\theta_2} - A_{\theta_2}A_{x\theta_1}\right\} - 3aA^2A_x = 0. \end{aligned} \tag{4}$$

Since the superfield A is bosonic, it can be decomposed into

$$A(x, t; \theta_1, \theta_2) = u(x, t) + \theta_1\rho^1(x, t) + \theta_2\rho^2(x, t) + \theta_1\theta_2v(x, t), \tag{5}$$

where u and v are even functions of (x, t) while ρ^1 and ρ^2 are odd functions of the same independent even variables. It is then easy to see that the superequation (4) reduces to the following system of two bosonic and two fermionic equations:

$$\begin{aligned} &u_t + u_{xxx} - 3au^2u_x + (a+2)(uv)_x - (a-1)(\rho^1\rho^2)_x = 0, \\ &v_t + v_{xxx} + 6vv_x - 3au_xu_{xx} - (a+2)uu_{xxx} - 3\rho^2\rho^2_{xx} - (a+2)\rho^1\rho^1_{xx} \\ - &3a(u^2v_x + 2uu_xv - 2u\rho^1\rho^2_x + 2u\rho^2\rho^1_x - 2u_x\rho^1\rho^2) = 0, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\rho^1_t + \rho^1_{xxx} + (a+2)(\rho^1v)_x - (a-1)(\rho^1v)_x - (a+2)u\rho^2_{xx} \\ - &(2a+1)u_x\rho^2_x - (a-1)\rho^2u_{xx} - 3a(u^2\rho^1_x + 2uu_x\rho^1) = 0, \\ &\rho^2_t + \rho^2_{xxx} + (a+2)(\rho^2v)_x + (a+2)u\rho^1_{xx} + (2a+1)u_x\rho^1_x \\ + &(a-1)\rho^1u_{xx} - (a-1)(\rho^2v)_x - 3a(u^2\rho^2_x + 2uu_x\rho^2) = 0. \end{aligned} \tag{7}$$

We see that the first bosonic equation reduces to the mKdV equation when $v = 0$ and $\rho^1 = \rho^2 = 0$. The second one reduces to the KdV equation when $u = 0$ and $\rho^1 = \rho^2 = 0$. So we expect to find, for example, supersolitonic solutions for the complete supersymmetric system. Concerning the fermionic equations, they form a system of coupled linear equations in ρ^1 and ρ^2 once u and v are known. We will see later that the method of symmetry reduction will decouple these equations and will help in its resolution.

The Lie superalgebra of symmetries for the equation (4) has been computed making use of a MAPLE program **GLie**[21]. It is a $(3|2)-$dimensional superalgebra with basis

$$\begin{aligned} \mathcal{P}_1 &= \partial_x,\ \ \mathcal{P}_0 = \partial_t, \\ \mathcal{D} &= x\partial_x + 3t\partial_t + \frac{1}{2}\theta_1\partial_{\theta_1} + \frac{1}{2}\theta_2\partial_{\theta_2} - A\partial_A, \\ \mathcal{Q}_1 &= \theta_1\partial_x - \partial_{\theta_1},\ \ \mathcal{Q}_2 = \theta_2\partial_x - \partial_{\theta_2}, \end{aligned} \tag{8}$$

where $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{D}$ are the three even generators and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are the two odd ones. This result is true independently of the value of the parameter a entering in the superequation. The supercommutator table of the Lie superalgebra is given in Table 1, where, as usual, commutation relations are satisfied for even-even and even-odd products while anticommutation relations are satisfied for odd-odd products.

As usual in super Lie group theory, starting with a Lie superalgebra, we obtain the corresponding Lie group by exponentiation. The group G of Lie-point symmetries for the equation (4) is generated by the elements $g = (x_0, t_0, d; \eta_1, \eta_2)$ where x_0, t_0, d are even Grassmann numbers and η_1, η_2 odd ones. They satisfy the composition law

$$g \equiv (x_0, t_0, d; \eta_1, \eta_2) = (x_0^2, t_0^2, d_2; \eta_1^2, \eta_2^2)(x_0^1, t_0^1, d_1; \eta_1^1, \eta_2^1) = g_2 g_1 \tag{9}$$

with

$$\begin{aligned} & d = d_2 + d_1,\ t_0 = t_0^2 + e^{3d_2} t_0^1, \\ & x_0 = x_0^2 + e^{d_2} x_0^1 + e^{(d_2 + \frac{d_1}{2})}(\eta_1^2 \eta_1^1 + \eta_2^2 \eta_2^1), \\ & \eta_1 = \eta_1^1 + e^{-\frac{d_1}{2}} \eta_1^2,\ \eta_2 = \eta_2^1 + e^{-\frac{d_1}{2}} \eta_2^2. \end{aligned} \tag{10}$$

The identity element is $e = (0, 0, 0; 0, 0)$ and the inverse g^{-1} of g is

$$g^{-1} = (-e^{-3d} t_0, -e^{-d} x_0, -d; -e^{-d/2} \eta_1, -e^{-d/2} \eta_2). \tag{11}$$

This group acts on the superspace $(x, t; \theta_1, \theta_2)$ and on the superfield A as

$$g(x, t; \theta_1, \theta_2) = (e^d (x + \eta_1 \theta_1 + \eta_2 \theta_2) - x_0, e^{3d} t - t_0; e^{\frac{d}{2}}(\theta_1 - \eta_1), e^{\frac{d}{2}}(\theta_2 - \eta_2)) \tag{12}$$

and

$$gA(x, t; \theta_1, \theta_2) = e^{-d} A\left(g^{-1}(x, t; \theta_1, \theta_2)\right). \tag{13}$$

The even parameters x_0 and t_0 correspond to space and time translations respectively while d corresponds to a dilatation. The two odd parameters η_1 and η_2 correspond to supersymmetric transformations mixing even and odd variables.

3 Invariant solutions

Different subgroups may be chosen to get invariant solutions of the SUSY KdV equation (4). The more interesting ones contain at least one SUSY transformation. In fact, since both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are like square roots of the translation generator $\mathcal{P}_1$, they come together in a subgroup structure and lead to invariant solutions constant in the variable x. We are not interested in such trivial solutions. Nevertheless, a combination like $\mathcal{Q}_+ = \mathcal{Q}_1 + i\mathcal{Q}_2$ is allowed giving $(Q_+)^2 = 0$. The subgroup $G_1 = \{g_0 = (cb, b, 0; \eta, i\eta)\}$ will be considered since it has the superalgebra $\mathcal{G}_1 = \{\mathcal{P}_0 - c\mathcal{P}_1, \mathcal{Q}_+\}$ and will give rise to traveling-wave solutions of eq. (4). In order to perform the symmetry reduction using such a subgroup of the symmetry group G, we have first to find the invariants of the action of this subgroup on the independent and dependent variables and then rewrite the equations in terms of them. This will reduce the number of independent variables and we will get superequations with one even and one odd variables.

3.1 The reduced superequations

>From eqs. (12) and (13), we see that the subgroup G_1 acts on the independent and dependent variables as follows

$$g_0(x,t;\theta_1,\theta_2) = (x+\eta(\theta_1+i\theta_2)-cb, t-b; \theta_1-\eta, \theta_2-i\eta)\,,$$
$$g_0 A(x,t;\theta_1,\theta_2) = A\left(g_0^{-1}(x,t;\theta_1,\theta_2)\right).$$

It is then easy to compute the invariants of this action. They are

$$y = x+ct+i\theta_1\theta_2,\;\; \theta=\theta_1+i\theta_2. \tag{14}$$

Now, if we take $A = A(y,\theta)$, we get, from eq. (4) and the fact that $D_1 = \partial_\theta+\theta\partial_y$, $D_2 = i(\partial_\theta-\theta\partial_y)$, the reduced superequation

$$\begin{aligned} cA_y &= -A_{yyy}+3aA^2A_y - i(a+2)(A_y)^2 - i(a+2)AA_{yy}+2i(2a+1)\theta A_yA_{y\theta} \\ &+ 2i(a+2)\theta AA_{yy\theta}+2i(a-1)\theta A_\theta A_{yy}. \end{aligned} \tag{15}$$

Such an equation may be integrated once with respect to y and gives

$$A_{yy} - aA^3 + i(a+2)AA_y - 2i(a+2)\theta AA_{y\theta} - 2i(a-1)\theta A_\theta A_y + cA + c_1 - \theta k = 0, \tag{16}$$

where c_1 and k are even and odd integration constants respectively. In fact, it is again a PDE but much simpler than the original one. Indeed, expanding now the bosonic superfield A as

$$A(y,\theta) = u(y)+\theta\rho(y),$$

we finally get the system of ODEs:

$$u'' - au^3 + i(a+2)uu' + cu + c_1 = 0, \tag{17}$$
$$\rho'' - i(a+2)u\rho' + (c-3au^2+i(4-a)u')\rho = k. \tag{18}$$

The prime ($'$) means that we differentiate with respect to y. We notice that the first equation is a second order non-linear differential equation in u which does not depend on ρ. For $a=-2$, we recover a reduction of the mKdV equation. Since u is an even function in y, the method of resolution of eq. (17) is a standard one and we will use the classification given in Ince's book[22] to get explicit solutions. We will see that this will give a selection of admissible values for the parameter a. The second equation is linear in ρ once u is known and it can be solved by the usual techniques for linear equations, once we take $\rho(y) = \psi f(y)$, where ψ is an odd constant parameter and $f(y)$ an even function.

3.2 Solution of the equation for u

It has been noted that eq. (17) reduces to the mKdV equation for the special value $a=-2$ and then has well-known solutions such as the soliton solution $u = 1/\cosh y$. Requiring that all solutions of eq. (17) be free of movable critical point, we will see that this value of a appears in a list of only five admissible values, namely $a = -4, -2, -1, 1$ and 4. Here let us mention that the complete integrability of eq. (2), in terms of the existence of Lax pairs, has been proven by Mathieu *et al*[4] only for $a=-2$ and 4. Conservation laws have been found for $a=1$ but not a Lax pair.

To make use of the classification of second order ODEs without movable critical points given in Ince's book[22], we first make the following change of variables

$$u(y) = \alpha(y)w(z(y)) + \beta(y), \quad \alpha(y) \neq 0, \quad z'(y) \neq 0, \tag{19}$$

in order to take eq. (17) to one of the canonical forms of the equation

$$w_{zz} = (A(z)w + B(z))w_z + C(z)w^3 + D(z)w^2 + E(z)w + F(z). \tag{20}$$

This corresponds to the case 14.31 i) listed by Ince. The functions A, B, C, D, E, F are written as follows

$$\begin{aligned}
A(z) &= -i(a+2)\frac{\alpha}{z'}, \quad B(z) = -\frac{1}{\alpha(z')^2}\left(2\alpha' z' + \alpha z'' + i(a+2)\beta\alpha z'\right), \\
C(z) &= \frac{a\alpha^2}{(z')^2}, \quad D(z) = \frac{1}{(z')^2}\left(3a\alpha\beta - i(a+2)\alpha'\right), \\
E(z) &= \frac{1}{\alpha(z')^2}\left(3a\alpha\beta^2 - i(a+2)(\alpha'\beta + \alpha\beta') - \alpha'' - \alpha c\right), \\
F(z) &= \frac{1}{\alpha(z')^2}\left(a\beta^3 - c\beta - c_1 - i(a+2)\beta'\beta - \beta''\right).
\end{aligned}$$

In order to obtain an equation with solutions free from movable critical points, the pair of functions A and C must belong to the following list (after a suitable choice of the functions α, β and z):

$$\begin{array}{llll}
(i.a) & A = 0,\ C = 0; & (i.b) & A = -2,\ C = 0; \\
(i.c) & A = -3,\ C = -1; & (i.d) & A = -1,\ C = 1; \\
(i.e) & A = 0,\ C = 2. & &
\end{array}$$

The first two cases cannot occur in our context. Indeed, no value of a satisfies $(i.a)$. For the case $(i.b)$, a must be zero and $\alpha = -iz'$. Continuing with the classification[23], we have to satisfy at least one of the following conditions:

$$(i) \quad B = D \text{ and } E = 0; \quad (ii) \quad D = F = 0 \text{ and } E = B',$$

to get an integrable equation. This is impossible for $c \neq 0$.

The three other cases effectively occur and give rise to a selection of the admissible values for a. This leads to $a = 1$ or $a = 4$ for the case $(i.c)$, $a = -4$ or $a = -1$ for the case $(i.d)$ and finally $a = -2$ for the case $(i.e)$. Let us now specify the classification process and exhibit the solutions for all these cases.

Case $(i.c)$: $a = 1$, or $a = 4$.

We have to take $\alpha = -3iz'/(a+2)$ and the functions B, D, E and F must verify $B = D$ and $E = F = 0$. The first equality is trivially satisfied when $a = 1$ or $a = 4$ and the other constraints lead to the following equations on β and z':

$$\beta'' - a\beta^3 + i(a+2)\beta\beta' + c\beta + c_1 = 0, \tag{21}$$

$$z''' + i(a+2)(z'\beta)' - 3a\beta^2 z' + cz' = 0. \tag{22}$$

Equation (21) is the same as the original eq. (17) but now we only need one particular solution. It is obtained as the constant solution

$$\beta = \left(\frac{-c_1}{2a}\right)^{1/3} = \left(\frac{c}{3a}\right)^{1/2}, \tag{23}$$

which gives a relation between c and the integration constant c_1. With such a solution β, the function z satisfying eq. (22) can be simply taken as $z(y) = y$.

Thus, equation (20) reduces to the canonical form

$$w_{yy} = -3ww_y - w^3 + q(w_y + w^2), \tag{24}$$

where the constant q is given by

$$q = -i(a+2)\left(\frac{c}{3a}\right)^{1/2} = -i(3c)^{1/2}, \tag{25}$$

which is the same for $a = 1$ and $a = 4$. The solution of eq. (24) is easily computed using the substitution $w = v_y/v$, where v satisfies the linear equation $v''' = qv''$, i.e.

$$v(y) = c_2 e^{qy} + c_3 y + c_4. \tag{26}$$

Finally, since $u(y) = (-3i/(a+2))\, w(y) + \beta$, we get

$$u(y) = \frac{i}{a+2}\left[q - 3\frac{v_y}{v}\right]. \tag{27}$$

We see that v given by eq. (26) depends on three integration constants but u depends only on two independent constants. Indeed for $c_2 \neq 0$, we get

$$u(y) = \frac{i}{a+2}\left[q - 3\frac{qe^{qy} + c_3}{e^{qy} + c_3 y + c_4}\right]. \tag{28}$$

For $c_2 = 0, c_3 \neq 0$, the solution is

$$u(y) = \frac{i}{a+2}\left[q - \frac{3}{y + c_4}\right], \tag{29}$$

and finally for $c_2 = c_3 = 0$, we get the constant solution $u(y) = \beta$. Thus we have obtained the general solution of eq. (17) with $a = 1$, or $a = 4$ subject to the constraint (23) on c and c_1.

Case $(i.d)$: $a = -4$, or $a = -1$.

We choose $\alpha = -iz'/(a+2)$ and all solutions of equation (20) will be free from movable critical points if $B = D = 0$ and $F = -E'$. These conditions are satisfied for $\beta = 0$, $z'' = 0$ and this implies that the integration constant c_1 must be zero. Hence equation (20) reduces to the canonical form

$$w_{zz} = -ww_z + w^3 - pw, \tag{30}$$

where $p = -c/(z')^2$ is a constant. The standard way of solving eq. (30) is to introduce a new function $v(z)$ that satisfies

$$v_z^2 = p_3 v^3 + p_2 v^2 + p_1 v + p_0, \tag{31}$$

for some constants $p_i (i = 0, 1, 2, 3)$ and to write $w = v_z/(v-1)$. From eq. (31), we immediately get

$$v_{zz} = \frac{1}{2}(3p_3 v^2 + 2p_2 v + p_1) \text{ and } v_{zzz} = (3p_3 v + p_2)v_z. \tag{32}$$

Inserting now the new expression of w in eq. (30) and using the expressions (32), we get the admissible values for the constants p_1, p_2 and p_3. A canonical choice gives

$$p_3 = \frac{p}{3},\ p_2 = 0,\ p_1 = -p.$$

The solution of eq.(31) is expressed in terms of the P-Weierstrass elliptic function, $\mathcal{P}(z, g_2, g_3)$, (see e.g. Byrd and Friedman[23]) and is written as follows

$$v(z) = \left(\frac{12}{p}\right)^{1/3} \mathcal{P}(z), \tag{33}$$

where

$$g_2 = (12p^2)^{1/3} \text{ and } g_3 = -p_0.$$

Finally, for the simplest choice $z(y) = y$, we obtain the solution of equation (4) as

$$u(y) = -\frac{i}{(a+2)}\left(\frac{v'(y)}{v(y)-1}\right) = -\frac{i}{(a+2)}\left(\frac{\mathcal{P}'(y)}{\mathcal{P}(y) - \left(\frac{p}{12}\right)^{1/3}}\right) \tag{34}$$

where $\mathcal{P}'(y) = -\sqrt{4\mathcal{P}^3(y) - g_2\mathcal{P}(y) + p_0}$.
Case $(i.e)$: $a = -2$.
This corresponds to the mKdV equation and all solutions are free from movable critical points. We set $\alpha^2 = -(z')^2$ and have $B = D = 0$. We choose the auxiliary quantities to be $\alpha = i, \beta = 0$ and $z = y$ so that from eq. (19), we see that $u(y) = iw(y)$. Hence, we can solve directly the equation (17) that reduces to

$$u'' + 2u^3 + cu + c_1 = 0. \tag{35}$$

Multiplying it by u' and integrating once, we get

$$(u')^2 = -(u^4 + cu^2 + 2c_1u - c_2) = H(u), \tag{36}$$

where c_2 is an integration constant. This is a well-known equation solvable in terms of elliptic functions and their degenerate cases. Let us give some particular solutions.

We first consider the solitonic solution

$$u_S(y) = u_2 + \frac{\omega^2}{2u_2 + (u_1 + u_2)\cosh(\omega y)}, \tag{37}$$

where $\omega = \sqrt{(u_1 - u_2)(u_1 + 3u_2)}$, u_1, u_2 being roots of the polynomial $H(u)$ in eq. (36). Indeed, if we denote by u_1, u_2, u_3 and u_4 the four roots of $H(u)$, the solution u_S corresponds to the case where $u_1 \geq u_S > u_2 = u_3 > u_4$ (note that $u_4 = -u_1 - u_2 - u_3$). The constants c and c_1 (written now as c_S and c_{1S}) are then given by

$$c_S = -(u_1^2 + 2u_1u_2 + 3u_2^2),\ c_{1S} = -2u_2(u_1^2 + u_1u_2 + u_2^2).$$

The second solution we are interested in, is the rational one

$$u_R(y) = u_1 - \frac{4u_1}{4u_1^2y^2 + 1}. \tag{38}$$

It corresponds to the case where three roots of $H(u)$ are equal and such that $u_1 = u_2 = u_3 > u_R > u_4$. Again the constants c and c_1 in eq. (36) may be written as

$$c_R = -6u_1^2,\ c_{1R} = -8u_1^3.$$

3.3 Solutions of the equation for ρ

Now we will use the solutions u of the PDE (17) to solve eq. (18) that is linear in ρ. We can take $\rho(y) = \psi f(y)$ and $k = \psi k_1$, where ψ is an odd constant. We get a linear equation satisfied by the even function f:

$$f''(y) + p_1 f'(y) + p_0 f(y) = k_1, \tag{39}$$

where

$$p_1(y) = -i(a+2)u,\; p_0(y) = c - 3au^2 + i(4-a)u^{'} \tag{40}$$

and u is a solution of eq. (17). For general solutions $u(y)$, eq. (39) cannot be solved in terms of elementary functions, nor the standard special functions. Some particular solutions for u lead to simple forms of eq. (39) that can be solved explicitly.

For the case $a = 1$, or $a = 4$, the general solution $u(y)$ was given by eq. (27) with v satisfying the expression (26). It is easy to see that p_1 and p_0 in eq. (40) are now given by

$$p_1(y) = \left(q - 3\frac{v_y}{v}\right),\; p_0(y) = -2q\frac{v_y}{v} + (a-1)\left(\frac{v_y}{v}\right)^2. \tag{41}$$

We see that $p_1(y)$ does not depend on a and that $p_0(y)$ has been simplified taking into account the admissible values of a. We now distinguish the two values of a. For $a = 1$, we use the particular solution (29) which leads to the linear equation

$$f'' + \left(q - \frac{3}{y}\right) f' - \frac{2q}{y} f = k_1 \tag{42}$$

where without loss of generality we put the constant c_4 equal to zero. The general solution is

$$f(y) = C_1(6 - 4qy + q^2y^2) + C_2(qy+3)e^{-qy} + \frac{k_1}{2q^2}(3 - 2qy), \tag{43}$$

C_1 and C_2 being arbitrary integration constants. For $a = 4$, the same solution $u(y)$ inserted in eq. (39) gives

$$f'' + \left(q - \frac{3}{y}\right) f' + \left(\frac{-2q}{y} + \frac{3}{y^2}\right) f = k_1, \tag{44}$$

which admits the general solution

$$f(y) = qy[C_1(1 - qy) + C_2 e^{-qy}] - \frac{k_1 y}{q}, \tag{45}$$

where again C_1 and C_2 are arbitrary constants.

We now turn to the case $a = -2$ which leads to the solitonic solution $u_S(y)$ of eq. (37). Equation (39) takes the form

$$f''(y) + p_0(y) f(y) = k_1 \tag{46}$$

with

$$\begin{aligned} p_0(y) \;&=\; \frac{1}{[(u_1+u_2)\cosh(\omega y) + 2u_2]^2}\left\{-(u_1^2 + 2u_1u_2 + 3u_2^2)((u_1+u_2)^2\cosh(\omega y)\right. \\ &+\; 2u_2)^2 + 6(u_2(u_1+u_2)\cosh(\omega y) + u_1^2 + 2u_1u_2 - u_2^2)^2 \\ &-\; \left. 6i(u_1-u_2)(u_1+u_2)(u_1+3u_2)\omega\sinh(\omega y)\right\}. \end{aligned} \tag{47}$$

Suitable changes of variables lead to a hypergeometric equation that can be solved exactly. Indeed, we first consider the homogeneous equation and use the change of variable $s = e^{\omega y}$ to get the new equation

$$f_{ss} + P(s) f_s + Q(s) f = 0, \tag{48}$$

where

$$P(s) = \frac{1}{s},\ Q(s) = \frac{-(s^2 + 10\kappa s + (\kappa)^2)}{s^2 (s-\kappa)^2},$$

with $\kappa = -(2u_2 - i\omega)/(u_1 + u_2)$. A second change of variable will bring us to the hypergeometric equation

$$\varphi_{zz} + \frac{3(2z-1)}{z(z-1)}\varphi_z - \frac{6}{z(z-1)}\varphi = 0, \tag{49}$$

where $z = s/(s-\kappa)$ and $\varphi = \frac{1}{z(z-1)} f$. One solution of eq. (49) is easily found as $\varphi_1(z) = 2z - 1$ which leads us to a solution for the original homogeneous equation

$$f_1(y) = \frac{\kappa e^{\omega y}(e^{\omega y} + \kappa)}{(e^{\omega y} - \kappa)^3}. \tag{50}$$

The second linearly independent solution $\varphi_2(z)$ is obtained from $\varphi_1(z)$ using the formula

$$\varphi_2(z) = \varphi_1(z) \int \frac{1}{\varphi_1^2(z)} e^{-\int P(z)dz} dz, \tag{51}$$

or, explicitly,

$$\varphi_2(z) = 32 + \frac{7(2z-1)^2}{z(z-1)} - \frac{(2z-1)^2}{2z^2(z-1)^2} + 30(2z-1)\ln\left(\frac{z-1}{z}\right). \tag{52}$$

We then get the other solution of the original homogeneous equation by replacing z in terms of y, i.e

$$f_2(y) = \frac{32\kappa e^{\omega y} + 7(e^{\omega y} + \kappa)^2}{(e^{\omega y} - \kappa)^2} - \frac{(e^{\omega y} + \kappa)^2}{2\kappa e^{\omega y}} + 30 f_1(y)(-\omega y + \ln(\kappa)). \tag{53}$$

The corresponding solution of the inhomogeneous equation is obtained by the method of variation of constants and it is the linear combination of f_1 and f_2 and a particular solution given by

$$f_p(z(y)) = \frac{k}{\omega^2}\left(-1 - 3z + 33z^2 - 30z^3 - 6z(1 - 3z + 2z^2)\ln(z - \frac{1}{z})\right). \tag{54}$$

Finally let us take the rational solution u_R given by eq. (38) and solve

$$f'' + \left(\frac{48u_1^2}{(1 - 2iu_1 y)^2}\right) f = k_1. \tag{55}$$

Once we put $r = 1 - 2iu_1 y$, we get an Euler type equation of the form

$$r^2 f_{rr} - 12 f = \frac{k_1 r^2}{-4u_1^2}. \tag{56}$$

We easily see that the general solution of eq. (55) is

$$f(y) = C_1(1 - 2iu_1 y)^4 + C_2(1 - 2iu_1 y)^{-3} + \frac{k_1}{40u_1^2}(1 - 2iu_1 y)^2, \tag{57}$$

with C_1 and C_2 arbitrary integration constants.

3.4 Super traveling-wave solution

Returning to the original super KdV equation (4), let us give the expression for the solutions A which are invariant under G_1. Let us recall that we have solved eq. (4) with $A = A(y,\theta) = u(y) + \theta\rho(y)$ where y is an even Grassmann variable given by eq. (14). Developping u and ρ, we get (in view of the nilpotent character of θ_1 and θ_2)

$$A = u(x+ct) + i\theta_1\theta_2 \frac{du}{d\xi}|_{\xi=x+ct} + (\theta_1 + i\theta_2)\rho(x+ct). \tag{58}$$

This means that the components of A in eq. (5) are not independent. Indeed we have

$$v = \frac{du}{d\xi}|_{\xi=x+ct},\ \rho^2 = i\rho^1 = i\rho(\xi). \tag{59}$$

So we can view the solutions u and ρ as functions of $\xi = x+ct$ instead of y and $A = u(\xi) + i\theta_1\theta_2\frac{du}{d\xi} + (\theta_1 + i\theta_2)\rho(\xi)$.
For example, we have for $a = 1$, the solutions

$$u(\xi) = \frac{i}{3}\left[q - \frac{3}{\xi}\right], \tag{60}$$

$$\rho(\xi) = \psi\left[C_1(6 - 4q\xi + q^2\xi^2) + C_2(q\xi + 3)e^{-q\xi} + \frac{k_1}{2q^2}(3 - 2q\xi)\right]. \tag{61}$$

For $a = 4$, we get

$$u(\xi) = \frac{i}{6}\left[q - \frac{3}{\xi}\right], \tag{62}$$

$$\rho(\xi) = \psi\left[q\xi[C_1(1 - q\xi) + C_2 e^{-q\xi}] - \frac{k_1\xi}{q}\right]. \tag{63}$$

Let us now develop the supersolitonic solution occuring for $a = -2$. First, we take some particular values for the constants appearing in the solution (37). Thus, with $u_1 = 1, u_2 = 0$ and $c = -1$, we get

$$u(x-t) = \frac{1}{\cosh(x-t)}, \tag{64}$$

the well-known solitonic solution of the mKdV equation. The first solution for the homogeneous equation in ρ becomes very simple. It takes the form $\rho_1(x-t) = \psi f_1(x-t)$ where

$$f_1(x-t) = \frac{1}{\cosh^3(x-t)}\left[\sinh(x-t) + \frac{i}{2}(1 - \sinh^2(x-t))\right]. \tag{65}$$

In Fig 1, we see the behaviour of u, $Re(f_1)$ and $Im(f_1)$ as functions of $\xi = x-t$. As function of x and t, we have the graphs of $Re(f_1)$ and $Im(f_1)$ in Fig 2 and 3. The solution (65) has a very interesting behaviour, discussed earlier by Ibort *et al*[16] for a different supersymmetric equation. They had a real solution which they called a *solitino* and its graph was similar to that of $Re(f_1)$ on Fig. 2. Here we get a complex solution for which the norm of the corresponding even function $f_1(x-t)$ is $|f_1(x-t)| = \frac{1}{2}u(x-t)$. We also see that the real part of f_1 is the derivative of the usual soliton solution (up to a multiplication by a constant factor) of the KdV equation, while the imaginary part is related to u by $Im f_1(\xi) = \frac{d^2u(\xi)}{d\xi^2}$.

Let us emphasize that our soliton type solution of the SUSY mKdV equation provides a complete solution of eq. (4), since we have obtained all components of A (see eq. (58)). The behaviour of the other independent solutions $\rho_2(\xi)$ and $\rho_p(\xi)$ is more complex, but we can compute it in the particular case where u is given by eq. (64). They take the form

$$f_2(\xi) = Re(f_2(\xi)) + iIm(f_2(\xi)) \tag{66}$$

with

$$Re(f_2(\xi)) = -6 + 15\pi\frac{\sinh(\xi)}{(\cosh(\xi))^3} + 30\frac{1}{(\cosh(\xi))^2} - 30\xi\frac{\sinh(\xi)}{(\cosh(\xi))^3}, \tag{67}$$

$$\begin{aligned} Im(f_2(\xi)) &= -\sinh(\xi) + 15\frac{\xi}{\cosh(\xi)} - 30\frac{\xi}{(\cosh(\xi))^3} - \frac{15}{2}\pi\frac{1}{\cosh(\xi)} \\ &+ 15\pi\frac{1}{(\cosh(\xi))^3} - 30\frac{\sinh(\xi)}{(\cosh(\xi))^2}. \end{aligned} \tag{68}$$

Finally, the particular solution f_p is

$$f_p(\xi) = Re(f_p(\xi)) + iIm(f_p(\xi)) \tag{69}$$

with

$$\begin{aligned} Re(f_p(\xi)) &= k_1\left(-1 - \frac{3}{2}\frac{e^\xi}{\cosh(\xi)} + \frac{33}{4}\frac{(-1+e^{2\xi})}{(\cosh(\xi))^2} - \frac{15}{4}\frac{(-3e^\xi + e^{3\xi})}{(\cosh(\xi))^3}\right. \\ &- \left.6(\frac{\pi}{2} - \xi)\left[\frac{e^\xi}{2\cosh(\xi)} - \frac{3}{4}\frac{(-1+e^{2\xi})}{(\cosh(\xi))^2} + \frac{(-3e^\xi + e^{3\xi})}{4(\cosh(\xi))^3}\right]\right), \end{aligned} \tag{70}$$

$$\begin{aligned} Im(f_p(\xi)) &= k_1\left(-\frac{3}{2\cosh(\xi)} + \frac{33}{2}\frac{e^\xi}{(\cosh(\xi))^2} - 15\frac{(-1+3e^{2\xi})}{4(\cosh(\xi))^3}\right. \\ &- \left.6(\frac{\pi}{2} - \xi)\left[\frac{1}{2\cosh(\xi)} - \frac{3e^\xi}{2(\cosh(\xi))^2} + \frac{(-1+3e^{2\xi})}{4(\cosh(\xi))^3}\right]\right). \end{aligned} \tag{71}$$

All these functions are represented in Fig 4 and 5 as functions of ξ and for $k_1 = 1$.

4 Conclusion

Starting from the $N = 2$ SUSY KdV equation (4), we have used a subgroup of the supersymmetry group to obtain Grassmann valued solutions depending on one even and one odd Grassmann independent variables. Such an equation contains a lot of information since if we decompose the bosonic superfield in terms of its components, as in eq. (5), it produces a set of four partial differential equations. Our way of determining symmetries has the advantage of avoiding that decomposition, of working with the concise equation (4) and of producing a superalgebra (8) of symmetries. In the search for solutions using the method of symmetry reduction, we work with Grassmann variables until we get a PDE with one even and one odd Grassmann variables. At this stage, the nilpotency of the odd variable leads, by expansion of the dependent variable, to a set of two ODEs.

The invariant solutions that we have obtained are based on the choice of G_1 as a subgroup of the SUSY group. Let us recall that it contains a SYSY transformation and a combination of both spatial and temporal translations. It gives rise to interesting solutions such as the supersolitonic ones. Another group G_2 containing the same SUSY transformation and the dilation generated by

$\mathcal{D}$ may also be considered. Here the difficulty is to solve explicitly the linear equation for the odd field ρ. Indeed, if we take the group $G_2 = \{g_0' = (0,0,d;\eta,i\eta)\}$, it acts on the independent and dependent variables as

$$g_0'(x,t;\theta_1,\theta_2) = \left(e^d(x+\eta(\theta_1+i\theta_2)), e^{3d}t; e^{d/2}(\theta_1-\eta), e^{d/2}(\theta_2-i\eta)\right),$$
$$g_0'A(x,t;\theta_1,\theta_2) = e^{-d}A((g_0')^{-1}(x,t;\theta_1,\theta_2))$$

and the invariants of this action are

$$y = t^{-1/3}(x+i\theta_1\theta_2),\ \theta = \theta_1 + i\theta_2,\ W = t^{1/3}A. \tag{72}$$

It is easy to show that the reduced equation in terms of $W = W(y,\theta)$ may be developed as a set similar to eqs. (17) and (18) where the constant c is replaced by $-y/3$. This means that the solution for u essentially follows the lines described in section 3. In particular, this implies the selection of the same values for a, an important information for the integrability of the SUSY KdV equation. We plan to return to the study of self-similar SUSY solutions in the near future.

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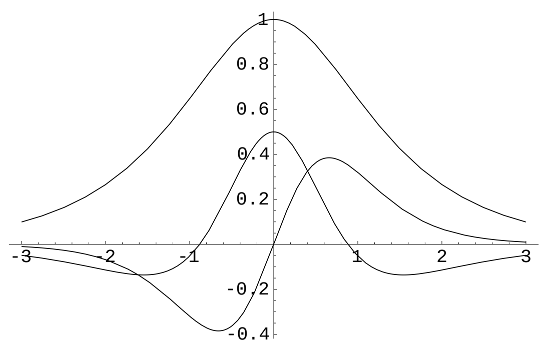

Figure 1: The functions $u(\xi), Re(f_1(\xi))$ and $Im(f_1(\xi))$

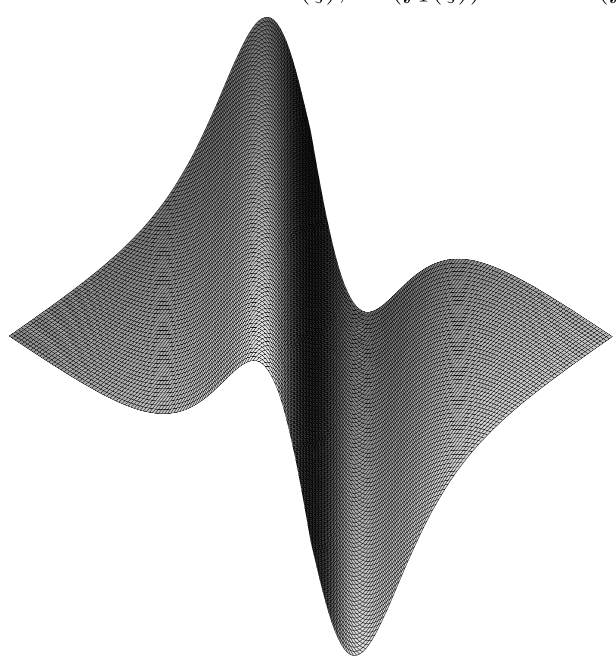

Figure 2: The function $Re(f_1(x-t))$.

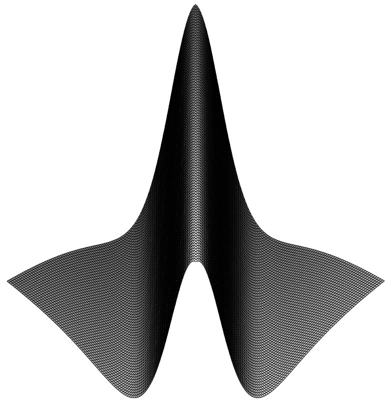

Figure 3: The function $Im(f_1(x-t))$.

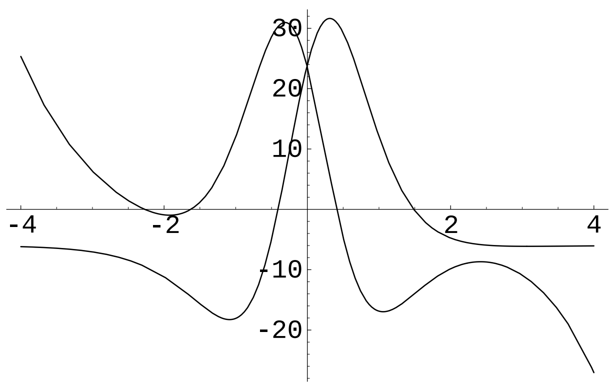

Figure 4: The functions $Re(f_2(\xi))$ and $Im(f_2(\xi))$

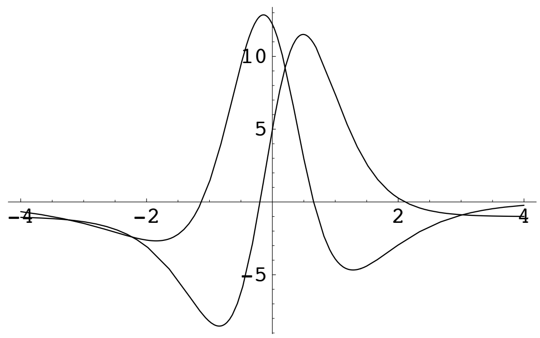

Figure 5: The functions $Re(f_p(\xi))$ and $Im(f_p(\xi))$

Table 1: The supercommutator table of the SUSY KdV equation ($N=2$).

	$\mathcal{P}_1$	$\mathcal{P}_0$	$\mathcal{D}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$
$\mathcal{P}_1$	0	0	$\mathcal{P}_1$	0	0
$\mathcal{P}_0$	0	0	$3\mathcal{P}_0$	0	0
$\mathcal{D}$	$-\mathcal{P}_1$	$-3\mathcal{P}_0$	0	$(-1/2)\mathcal{Q}_1$	$(-1/2)\mathcal{Q}_2$
$\mathcal{Q}_1$	0	0	$(1/2)\mathcal{Q}_1$	$-2\mathcal{P}_1$	0
$\mathcal{Q}_2$	0	0	$(1/2)\mathcal{Q}_2$	0	$-2\mathcal{P}_1$