A simple generation of exactly solvable anharmonic oscillators

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Abstract

An elementary finite difference algorithm shortens the Darboux method, permitting an easy generation of families of anharmonic potentials almost isospectral to the harmonic oscillator. Against the common belief, it is possible to associate a SUSY partner to a given Hamiltonian H using a factorization energy *greater* than the ground state energy of H. The explicit 3-SUSY partners of the oscillator potential are found and discussed.

The study of exactly solvable quantum eigenproblems is the most ancient and most recent subject [1, 2, 3, 4]. A powerful tool in this field is the *intertwining technique* [5, 6, 7]. Its key point is the construction of an operator $A^\dagger$ interrelating two different Hamiltonians H and $\widetilde{H}$:

$$\widetilde{H} A^\dagger = A^\dagger H, \tag{1}$$

$$\widetilde{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \widetilde{V}(x), \quad H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x). \tag{2}$$

The relation (1) typically permits to determine the eigenfunctions $\widetilde{\psi}_n$ of $\widetilde{H}$ by applying $A^\dagger$ to the eigenfunctions ψ_n of H. The specially plausible case when $A^\dagger$ is a first order differential operator

$$A^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \alpha(x)\right), \tag{3}$$

covers the standard supersymmetric quantum mechanics (SUSY QM) [8, 9, 10, 11, 12, 13] with $V(x)$ and $\widetilde{V}(x)$ satisfying:

$$\widetilde{V}(x) = \quad V(x) - \alpha', \tag{4}$$

$$\alpha' + \alpha^2 = \quad 2(V(x) - \mathcal{E}), \tag{5}$$

where $\mathcal{E}$ is a constant called the *factorization energy*. If a solution (family of solutions) to (5) is found, (4) can produce a new potential (or a family of potentials) $\widetilde{V}(x)$ with the same spectrum as $V(x)$ except for an additional level at $\mathcal{E}$ (provided that the wavefunctions $\widetilde{\psi}_n \propto A^\dagger\psi_n \neq 0$, $\widetilde{\psi}_{\mathcal{E}} \propto \exp(-\int_0^x \alpha(y)dy)$ have no singularities and are square-integrable).

In spite of simplicity, the applications of (1-5) are far from exhausted, even for the oldest physical model: the harmonic oscillator. A *general* solution of (1-5), with $V(x) = x^2/2$ and $\mathcal{E} = -1/2$, was studied by one of us [3], recovering a family of potentials previously found by Abraham and Moses [14]. Some other interesting sequencies (though not continuous families) of anharmonic oscillators have been generated for the *particular* solutions to (5) with factorization energies $\mathcal{E} = -5/2, -9/2, -13/2, \cdots$ [6] (see also [15]).

Let us therefore notice that the construction can easily be extended. With this aim it is enough to look for solutions $\alpha_k(x)$ to (5) with $V(x) = x^2/2$ and $\mathcal{E}_k = -k - 1/2$ ($k = 0, 1, 2, \ldots$), namely:

$$\alpha_k'(x) + \alpha_k^2(x) = 2\left(V(x) - \mathcal{E}_k\right) = x^2 + 2k + 1. \tag{6}$$

The Riccati equation (6) can be transformed into a stationary Schrödinger equation for the oscillator potential by means of the substitution $\alpha_k = u_k'/u_k$:

$$-\frac{1}{2}\frac{d^2 u_k}{dx^2} + \frac{x^2}{2}u_k = \mathcal{E}_k u_k. \tag{7}$$

where $\mathcal{E}_k < 0$ do not belong to the spectrum of $x^2/2$ and u_k are non-normalizable. Yet, (7) can be converted into a stationary Schrödinger equation in a new variable y for a normalizable eigenfunction of the oscillator (with eigenvalue $-\mathcal{E}_k$) by means of the substitution $y = ix$. Hence, one gets a solution in the form $u_k(x) = h_k(x)\exp(x^2/2)$, where $h_k(x) = (-i)^k H_k(ix)$, $H_k(z)$ is a Hermite polynomial of order k. This immediately provides a particular solution to the Riccati equation (6):

$$\alpha_k^p(x) = x + \frac{d}{dx}\left\{\ln\left[h_k(x)\right]\right\}. \tag{8}$$

Departing from the particular solution (8), the corresponding general solution to (6) can be found by means of two quadratures [3, 16]. It yields:

$$\alpha_k(x) = x + \frac{d}{dx}\left\{\ln\left[g_k(x)\right]\right\}, \tag{9}$$

where

$$\begin{aligned}
g_k(x) \quad &= \quad h_k(x)\left[1 - (-2)^{k+1}k!\nu_k I_k(x)/\sqrt{\pi}\right], \\
I_k(x) = \int \frac{e^{-x^2}}{h_k^2(x)}dx \quad &= \quad \begin{cases} \frac{\sqrt{\pi}}{2}\operatorname{erf}(x) & \text{if } k = 0, \\ \left(-\frac{1}{2}\right)^k \frac{1}{k!}\left(I_0(x) + \frac{p_k(x)}{h_k(x)}e^{-x^2}\right) & \text{if } k > 0, \end{cases} \\
p_k(x) \quad &= \quad \begin{cases} 0 & \text{if } k = 0, \\ \sum_{m=0}^{\left[\frac{k-1}{2}\right]} \frac{2^m(k-m-1)!}{(k-2m-1)!} h_{k-2m-1}(x) & \text{if } k > 0, \end{cases}
\end{aligned} \tag{10}$$

and the ν_k are constants. To show (10), we first use the following recurrence relations on $h_k(x)$:

$$h_k''(x) + 2xh_k'(x) - 2kh_k(x) = 0, \quad h_k'(x) = 2kh_{k-1}(x),$$

to get

$$I_k(x) = -\frac{1}{2k}\left(I_{k-1}(x) + \frac{e^{-x^2}}{h_k(x)h_{k-1}(x)}\right). \tag{11}$$

Iterating (11), we get (10) where we have still to prove that $p_k(x)$ satisfies

$$p_k(x)h_{k+1}(x) - p_{k+1}(x)h_k(x) = -(-2)^k\, k!. \tag{12}$$

This force us to write

$$h_k(x) = p_k(x)h_1(x) + q_k(x)h_0(x), \quad k \geq 2,$$

where

$$\begin{aligned} p_k(x) &= \sum_{m=0}^{[\frac{k-1}{2}]} \frac{2^m(k-m-1)!}{(k-2m-1)!} h_{k-2m-1}(x), \\ q_k(x) &= \sum_{m=0}^{[\frac{k-2}{2}]} \frac{2^{m+1}(m+1)(k-m-2)!}{(k-2m-2)!} h_{k-2m-2}(x). \end{aligned}$$

Since the recurrence relation on $h_k(x)$:

$$h_{k+1}(x) = 2xh_k(x) + 2kh_{k-1}(x)$$

holds also for both $p_k(x)$ and $q_k(x)$, the relation (12) easily follows by iteration.

If one looks for the regular solutions, the next step consists in avoiding the singularities of $\alpha_k(x)$. This is done by restricting the domain of ν_k in order to guarantee that $g_k(x)$ has no zeros for any $x \in$ R. This happens for

$$\begin{cases} |\nu_k| < 1 & \text{if } k \text{ is even,} \\ |\nu_k| > 1 & \text{if } k \text{ is odd.} \end{cases} \tag{13}$$

Within these restrictions, one obtains a non-singular family of 'exactly solvable potentials'

$$V_k(x) = V(x) - \alpha_k'(x) = \frac{x^2}{2} - \frac{d^2}{dx^2}\{\ln[g_k(x)]\} - 1, \tag{14}$$

their spectra are composed of a part isospectral to the oscillator, with eigenvalues $E_n = n + 1/2,\ n = 0, 1, 2, \ldots$, and an additional level at $\mathcal{E}_k = -k - 1/2$. The normalized eigenfunctions are respectively:

$$\psi_n^k(x) = \frac{A_k^\dagger \psi_n(x)}{\sqrt{E_n - \mathcal{E}_k}} = \frac{A_k^\dagger \psi_n(x)}{\sqrt{n+k+1}}, \tag{15}$$

$$\psi_{\mathcal{E}_k}^k(x) = \frac{\sqrt{(-2)^k k!(1-\nu_k^2)}}{\pi^{1/4}} \frac{e^{-x^2/2}}{g_k(x)}. \tag{16}$$

Some particular cases deserve to be mentioned.
i) If $k = 0$, we get the Abraham-Moses potentials [14] (displaced by $\Delta E = -1$ with respect to the standard energy origin):

$$V_0(x) = \frac{x^2}{2} - \frac{d^2}{dx^2}\{\ln[1 + \nu_0\, \mathrm{erf}(x)]\} - 1. \tag{17}$$

ii) If $k = 1$, the recently described potentials $V_1(x)$ of [7] with an incomplete oscillator spectrum are obtained.
iii) If k is even and $k \neq 0$, we get (for $|\nu_k| < 1$) continuous families of anharmonic potentials with spectra having a gap between the ground state level and the part isospectral to the oscillator. The additional level is placed at varying positions, $\{-5/2, -9/2, -13/2, \ldots\}$. Notice that these families include the particular potentials derived by Bagrov and Samsonov [6], which can be obtained from ours by taking $\nu_{k=2m} = 0$ in (14):

$$V_{k=2m}(x) = \frac{x^2}{2} - \frac{d^2}{dx^2}\{\ln[h_{k=2m}(x)]\} - 1 = \frac{x^2}{2} - \frac{d}{dx}\left(\frac{h_{k=2m}'(x)}{h_{k=2m}(x)}\right) - 1. \tag{18}$$

iv) If k is odd, we get (for $|\nu_k| > 1$) the continuous families of anharmonic almost isospectral oscillator potentials including the particular case with $k = 1$ discussed in *ii)*.

Let us point out that the potentials (14) are indeed particular cases of a general two-parametric family derived recently by Junker and Roy by means of techniques different from ours [17]. However, our treatment is so simple that we were unable to resist its explicit presentation.

The first order equations (1-5) are just the starting point of a more general algorithm. An idea of 'higher order SUSY' (HSUSY), with $A^\dagger$ in form of an n-th order differential operator, was put forward by Andrianov et al. [18], significantly advanced by Bagrov and Samsonov [6]. However, it seems curious that even the case of the second order $A^\dagger$ was unexplored until very recently; it covers the super-supersymmetric quantum mechanics (SUSUSY QM [7]). The relationships (3-5) in this case are replaced by:

$$A^\dagger = \frac{1}{2}\left(\frac{d^2}{dx^2} + \beta(x)\frac{d}{dx} + \gamma(x)\right), \tag{19}$$

$$\begin{aligned} \widetilde{V}(x) &= V(x) + \beta', \quad 2\gamma(x) = \beta^2 - \beta' - 4V - d, && (20)\\ 2V(x) &= \frac{\beta''}{2\beta} - \left(\frac{\beta'}{2\beta}\right)^2 - \beta' + \frac{\beta^2}{4} + \frac{c}{\beta^2} - \frac{d}{2}, && (21) \end{aligned}$$

where c and d are constants.

The equations (19-21) can be generally solved for $V(x) = x^2/2,\ c = 1, d = 4$, yielding a SUSUSY generalization of the Abraham-Moses family [7]. An ample class of solutions has been simultaneously constructed by applying the 'determinant formula' of the multiple Darboux process [6]. We shall now show that the algorithm can be still simplified by using a 'finite difference formula' which requires only the knowledge of two different first order intertwinners for two different factorization energies and yields the second order intertwinner $A^\dagger$ already in its factorized form.

Theorem: Suppose that $\alpha_k(x)$ and $\alpha_l(x)$ are two solutions to Riccati equation (5) with two different values of the factorization energy, $\mathcal{E}_k$ and $\mathcal{E}_l$. Then, a solution to the new Riccati equation

$$\alpha'_{kl}(x) + \alpha^2_{kl}(x) = 2\left(V_k(x) - \mathcal{E}_l\right), \tag{22}$$

with $V_k(x) = V(x) - \alpha'_k$, is obtained from the following finite difference formula:

$$\alpha_{kl}(x) = -\alpha_k(x) - 2\frac{\mathcal{E}_k - \mathcal{E}_l}{\alpha_k(x) - \alpha_l(x)}. \tag{23}$$

The proof can be done straighforwardly, by differentiating (23) and then determining $\alpha'_k(x)$ and $\alpha'_l(x)$ from the Riccati equations (5).

Our theorem abbreviates the Darboux interrelation between the two Hamiltonians (2); it generates either particular or general solutions provided one is able to solve the Riccati equation (5) for at least two different values of $\mathcal{E}$. But for $V(x) = x^2/2$, we have derived the general solutions for an infinity of values of $\mathcal{E}_k$. Hence, an infinite number of two-parametric families of potentials almost isospectral to the oscillator, with two new levels at $\mathcal{E}_k$ and $\mathcal{E}_l$ can be immediately written. The explicit expressions are:

$$\begin{aligned} V_{kl}(x) &= V_k(x) - \alpha'_{kl}(x) = V(x) + 2(\mathcal{E}_k - \mathcal{E}_l)\frac{d}{dx}\left(\alpha_k(x) - \alpha_l(x)\right)^{-1}\\ &= \frac{x^2}{2} + 2(l-k)\frac{d}{dx}\left\{\frac{d}{dx}\left[\ln\left(\frac{g_k(x)}{g_l(x)}\right)\right]\right\}^{-1}. \end{aligned} \tag{24}$$

To select regular solutions, we proceed to analyze the difference $\alpha_k - \alpha_l$ trying to avoid that its zeros produce singularities in $V_{kl}(x)$. This means we ask that the Wronskian $W(g_k, g_l) = g_k g'_l - g_l g'_k \neq 0$ for any $x \in \mathrm{R}$, what in turn depends on the parities of k and l. As it is shown in figure 1, four non-equivalent cases are found, each one corresponding to a different region in the plane $\nu_k - \nu_l$.

i) k is even and l is odd: the free singularity region (FSR), i.e., the (ν_k, ν_l) domain in which $V_{kl}(x)$ has no singularity for any $x \in \mathrm{R}$, turns out to be $|\nu_k| < 1$ and $|\nu_l| < 1$.

ii) k is even and l is even: the FSR is $|\nu_k| < 1$ and $|\nu_l| > 1$.

iii) k is odd and l is even: the FSR is $|\nu_k| > 1$ and $|\nu_l| > 1$.

iv) k is odd and l is odd: the FSR is $|\nu_k| > 1$ and $|\nu_l| < 1$.

Let us remark that the classification (13) for the first order case does not permit, in general, to judge the regularity regions of $V_{kl}(x)$: while condition (13) is maintained for the parameter associated to the higher new level (ν_k), it is reversed for the smaller one (ν_l). The normalized eigenfunctions, associated to the eigenvalues $\{E_n, \mathcal{E}_k, \mathcal{E}_l, n = 0, 1, 2, \ldots; k$ and l are fixed$\}$, are respectively:

$$\begin{aligned}
\psi_n^{kl}(x) &= \frac{A_{kl}^\dagger \psi_n^k(x)}{\sqrt{E_n - \mathcal{E}_l}} = \frac{A_{kl}^\dagger A_k^\dagger \psi_n(x)}{\sqrt{(E_n - \mathcal{E}_k)(E_n - \mathcal{E}_l)}}, \\
\psi_{\mathcal{E}_k}^{kl}(x) &= \frac{A_{kl}^\dagger \psi_{\mathcal{E}_k}^k(x)}{\sqrt{\mathcal{E}_k - \mathcal{E}_l}} = \frac{\sqrt{(-2)^{k+1} k!(1-\nu_k^2)(k-l)}}{\pi^{1/4}} \frac{g_l(x) e^{-x^2/2}}{W(g_k, g_l)}, \\
\psi_{\mathcal{E}_l}^{kl}(x) &= \frac{\sqrt{(-2)^{l+1} l!(1-\nu_l^2)(l-k)}}{\pi^{1/4}} \frac{g_k(x) e^{-x^2/2}}{W(g_k, g_l)}.
\end{aligned} \tag{25}$$

By taking $k = 0,\ l = 1$ in (24), one obtains the recently discovered two parametric family of isospectral oscillator potentials [7]. Any other possible combination of k and l will lead to a family of anharmonic exactly solvable oscillators, almost isospectral to the traditional one but having the structure of multiple wells. To illustrate the procedure, we have found one asymmetric case with $k = 1,\ l = 2,\ \nu_k = 100\ \nu_l = -1.2$. Our figure 2 shows one of new solvable potentials $V_{kl}(x)$ and the probability distributions for the three lowest energy bound states. As can be seen, the probability density for the ground state concentrates mainly around one well while the first excited state is attracted to the other. Curiously, the second excited state sits on the very top of the potential barrier around $x = 0$. For larger values of n the probability densities become similar to those of the orthodox oscillator. Note also that our asymmetric double well tends to a symmetric one in the limits $\nu_l \to -\nu_k,\ \nu_k \to \infty$, the case with the probability densities spread symmetrically over the two wells. Our construction makes also obvious that the new levels $\mathcal{E}_k,\ \mathcal{E}_l$ can be implemented at will, thus breaking the common conviction that the factorization energy used to construct a SUSY partner of a given Hamiltonian should be less than or equal to its ground state energy.

The possibility of iterating further the finite difference formula (23) depends on the set of solutions to the initial Riccati equation (5), with different values of $\mathcal{E}$, we have to our disposal. Thus, for equation (6) any triple $(\mathcal{E}_k, \mathcal{E}_l, \mathcal{E}_m)$ generates potentials $V_{klm}(x)$ whose spectra are equal to the spectrum of $V_{kl}(x)$ plus a new level at $\mathcal{E}_m$. The Riccati equation to be solved becomes:

$$\alpha'_{klm}(x) + \alpha_{klm}^2 = 2\left(V_{kl}(x) - \mathcal{E}_m\right). \tag{26}$$

Our algorithm tells now that $\alpha_{klm}(x)$ can be expressed in terms of the two SUSUSY solutions of (22), $\alpha_{kl}(x),\ \alpha_{km}(x)$:

$$\begin{aligned}
\alpha_{klm}(x) &= -\alpha_{kl}(x) - 2\tfrac{\mathcal{E}_l - \mathcal{E}_m}{\alpha_{kl}(x) - \alpha_{km}(x)} \\
&= \alpha_k(x) + 2\tfrac{\mathcal{E}_k - \mathcal{E}_l}{\alpha_k(x) - \alpha_l(x)} + \tfrac{(\mathcal{E}_l - \mathcal{E}_m)[\alpha_k(x) - \alpha_l(x)][\alpha_k(x) - \alpha_m(x)]}{(\mathcal{E}_k - \mathcal{E}_l)[\alpha_k(x) - \alpha_m(x)] - (\mathcal{E}_k - \mathcal{E}_m)[\alpha_k(x) - \alpha_l(x)]}.
\end{aligned} \tag{27}$$

This leads immediately to a new family of 3-SUSY potentials $V_{klm}(x)$:

$$V_{klm}(x) = V_{kl}(x) - \alpha'_{klm}(x) = V(x) - \frac{d}{dx}\left[\alpha_k(x) + \alpha_{kl}(x) + \alpha_{klm}(x)\right] \tag{28}$$

with three new energy levels at $\mathcal{E}_k,\ \mathcal{E}_l,\ \mathcal{E}_m$. The factorized intertwining operator in this algorithm is given explicitely:

$$A^\dagger = \left(\frac{1}{\sqrt{2}}\right)^3 \left(-\frac{d}{dx} + \alpha_{klm}(x)\right)\left(-\frac{d}{dx} + \alpha_{kl}(x)\right)\left(-\frac{d}{dx} + \alpha_k(x)\right) \tag{29}$$

In particular, for $(k, l, m) = (0, 1, 2)$, a three parametric family of potentials exactly isospectral to $x^2/2$ is derived. Two members of this family, for two different values of (ν_k, ν_l, ν_m) are shown in figure 3. As expected, this family includes the oscillator potential for $(\nu_k, \nu_l, \nu_m) = (0, 0, 0)$ as well as some triple wells. It is also the matter of a simple substitution to construct new exactly solvable cases with potentials given explicitly by simple fractions; an example is the following triple 3-SUSY well, almost isospectral to the orthodox harmonic oscillator, except for 3 new energy levels at $(\mathcal{E}_k, \mathcal{E}_l, \mathcal{E}_m) = (-5/2, -7/2, -9/2)$ (see figure 4):

$$V_{klm}(x) = \frac{x^2}{2} - \frac{d}{dx}\left[\frac{3x(2x^2+3)(4x^4-4x^2+7)}{8x^6 - 12x^4 + 18x^2 + 9}\right]. \tag{30}$$

$$\begin{aligned}
A^\dagger = &\left(\tfrac{1}{\sqrt{2}}\right)^3 \left(-\tfrac{d}{dx} + \tfrac{x(2x^2+3)(16x^8 - 16x^6 + 72x^4 - 108x^2 + 45)}{(4x^4+3)(8x^6 - 12x^4 + 18x^2 + 9)}\right) \times \\
&\times \left(-\tfrac{d}{dx} + \tfrac{x(8x^6 + 20x^4 + 22x^2 - 9)}{(2x^2+1)(4x^4+3)}\right)\left(-\tfrac{d}{dx} + \tfrac{x(2x^2+5)}{2x^2+1}\right)
\end{aligned} \tag{31}$$

The finite difference procedure can be continued at will. It might be pertinent to notice that the multiple wells are now an ‘emerging subject’, of interest for the control operations and tunelling effects [19].

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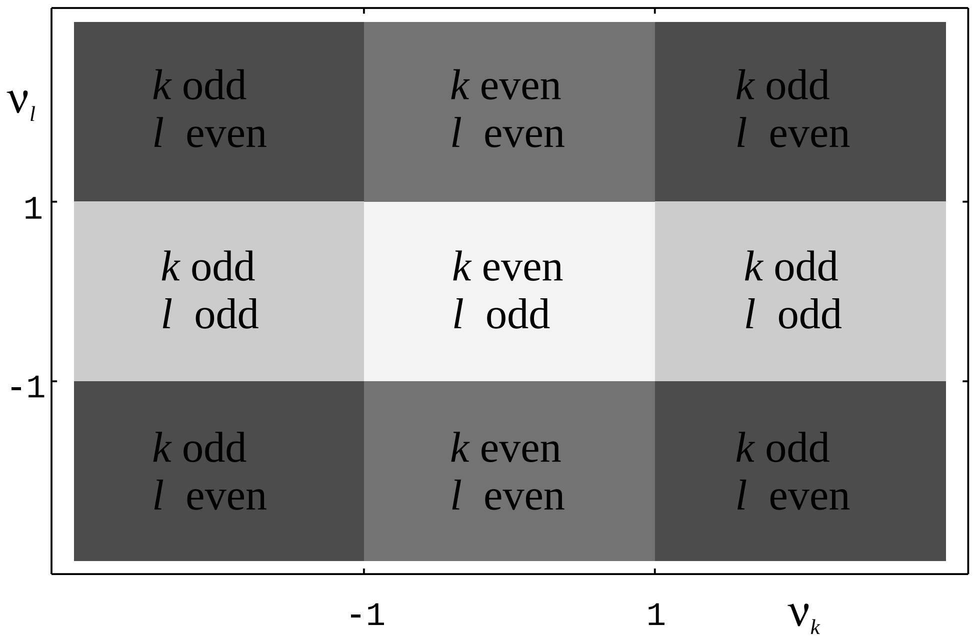

Figure 1: Regions of $\nu_k - \nu_l$ plane where $V_{kl}(x)$ has no singularity, according to the index parities, $l > k$.

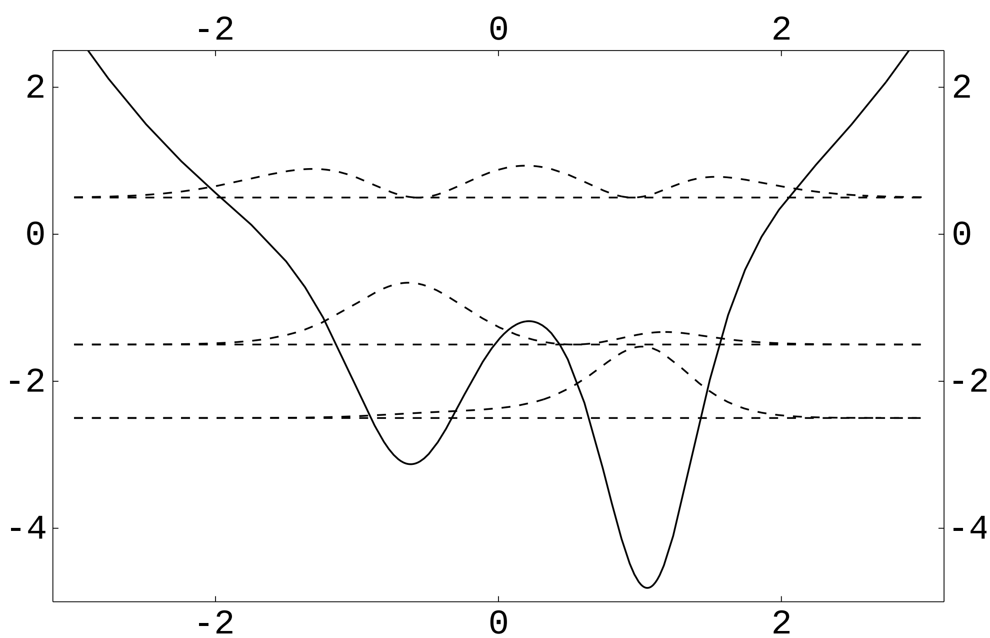

Figure 2: One of the SUSUSY potentials $V_{kl}(x)$ for $k = 1,\ l = 2,\ \nu_k = 100, \nu_l = -1.2$ modifying the $x^2/2 - 3$ potential. One level at $\mathcal{E} = -1/2$ is missing. The corresponding probability densities for the three lowest energy levels $\mathcal{E}_l = -5/2,\ \mathcal{E}_k = -3/2,\ E_0 = 1/2$ are represented by dashed lines.

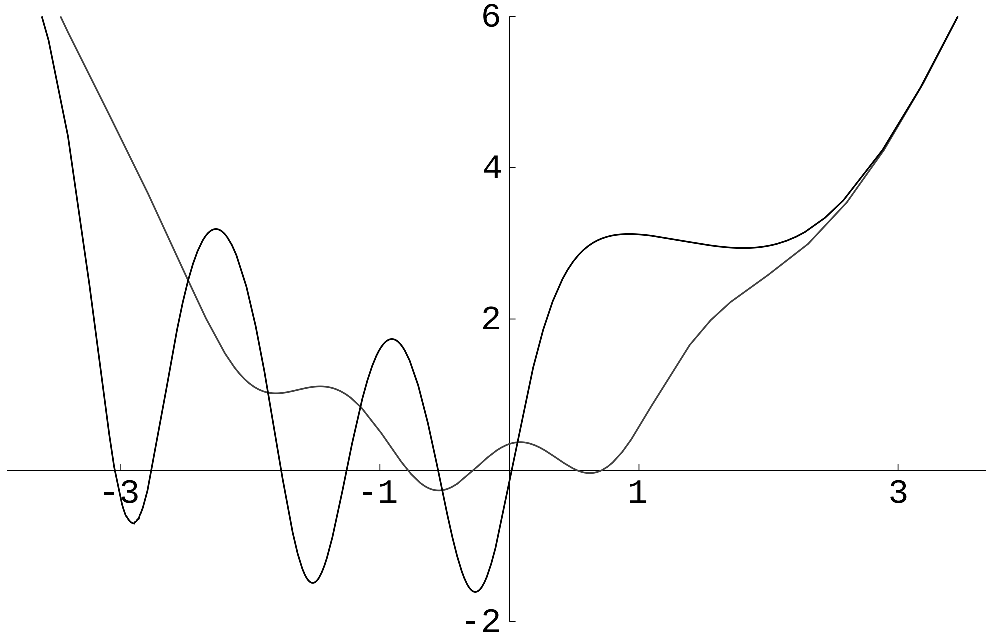

Figure 3: Two new 3-SUSY potentials $V_{klm}(x)+3$ for $k=0,\ l=1,\ m=2$ exactly isospectral to the standard oscillator. The construction was applied for the general solutions of the Riccati equations (6) involving the erf functions. *a)* $(\nu_0,\nu_1,\nu_2)=(0.5,0.5,0.5)$; *b)* $(\nu_0,\nu_1,\nu_2)=(0.99,0.99,0.99)$.

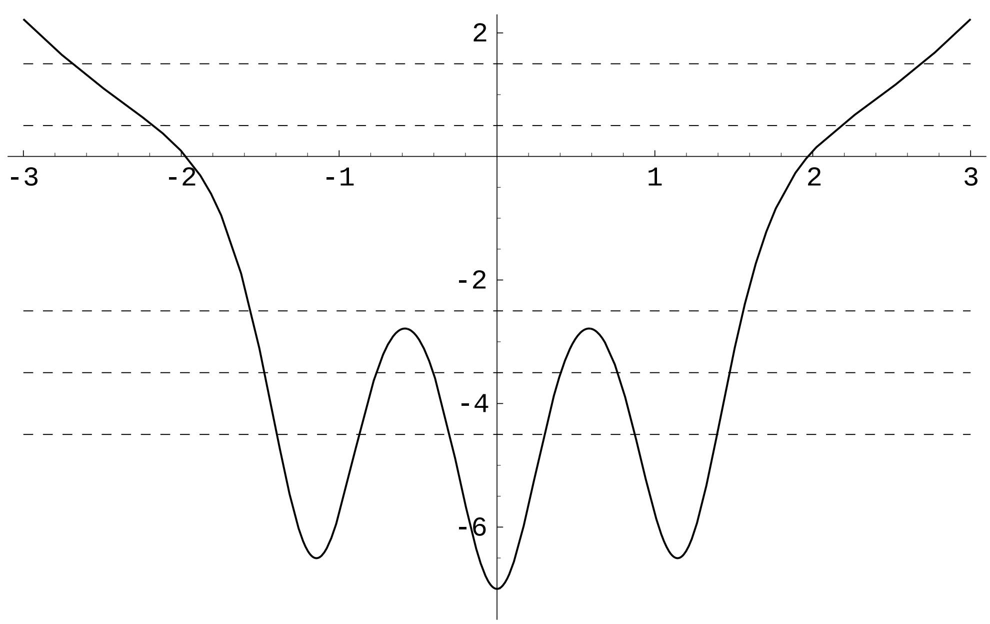

Figure 4: Plot of the triple well 3-SUSY potential given in (30). The five lowest energy levels are shown in dashed lines.