Abstract

In this paper, we introduce a new notion of linking which includes in particular the notions of homotopical linking and local linking. Critical point theorems for continuous functionals on metric spaces are presented. Finally, an application to nonlinear elliptic problems at resonance is given.

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Résumé

Dans ce texte, une nouvelle notion d'enlacement incluant en particulier les notions d'enlacements homotopique et local est introduite. Des théorèmes d'existence de points critiques de fonctionnelles continues définies sur un espace métrique sont obtenus. Finallement, on présente une application à des problèmes elliptiques non linéaires à résonance.

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On a new notion of linking and application to elliptic problems at resonance

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1. Introduction

It is well known that the notion of *linking* is very important in critical point theory. In fact, in the literature, we can find various definitions of linking. Let us simply mention the "homotopically linking", "homologically linking", "linking in the sense of Benci-Rabinowitz", "local linking", ... , see for example [5], [8], [24], [27], [30], [34]. Some attempts were made to unify some of those notions [10], [19].

In this paper, we introduce a new notion of linking on a metric space which precises and includes many notions of linking mentionned above. Moreover, this definition permits us to obtain much more linking sets.

On the other hand, we present as simply as possible, some important notions in critical point theory such as families of sets intersecting a given set, and invariance by deformations. In considering continuous functionals, we state in an abstract setting a deformation property. Then, with those notions and our new notion of linking, we present minimax critical point theorems. We obtain as particular cases, generalizations of many results such as the Mountain Pass Theorem, Saddle Point Theorem, Minimax Theorem [2], [12], [27], [29], [30], and results of Marino, Micheletti and Pistoia [25], [26].

Finally, we present an application to the nonlinear elliptic boundary value problem at resonance

$$\begin{aligned} &\Delta u + \lambda_m u = g(x,u) + h(x), \\ &u|_{\partial\Omega} = 0, \end{aligned}$$

where $\{\lambda_n\}$ is the increasing sequence of eigenvalues of $-\Delta$.

Starting with the celebrated paper of Landesman and Lazer [23], many authors treated this type of problems. Among them, we mention [1], [4], [13], [14], [15], [16], [18], [21], [34] who treated the case where g is bounded.

For the case where g can be unbounded, we mention [3], [15], [20], [22], [31], [32], [33]. To our knowledge, all papers concerning the case where g is unbounded rely on degree theory. Also, many of them considered only the case $\lambda_m = \lambda_1$.

In the literature, there are essentially two types of assumptions: one involving the behavior of g at $+\infty$, and $-\infty$ (Landesman-Lazer type); the other on the sign of $g(x,u)u$ for all $u \in \mathbb{R}$ (for example, see [21]).

Here, we consider the case where $|g(x,u)| \leq a + b|u|^\alpha$ for some $\alpha < 1$, $m \geq 2$, and $h = se_m$ with $s \in \mathbb{R}$ and e_m the eigenvector associated to λ_m. In addition, g satisfies

$$\lim_{u\to\infty} g(x,u) = \infty, \quad \text{and} \quad g(x,u)u \geq 0 \ \text{ for } u \text{ large enough.}$$

However, we have no assumptions on the behavior and on the sign of g as u goes to $-\infty$. Moreover, our existence results rely on critical point theory. We use linking sets according to our definition, which do not link in older senses.

In what follows, for E a Banach space, $A \subset E$, F a subspace of E, and $y \in F$, we denote by $B(y,r)$ the open ball in E centered in y of radius r. We write $B_F(y,r) = B(y,r) \cap F$, and $\partial_F A$ the boundary of A in F with the induced topology. If H is an Hilbert space and H_1 is a subspace of H, the orthogonal complement of H_1 is denoted by $H_1^\perp$.

2. Abstract theory

2.1. Intersecting families.

Let (X,d) be a metric space. For a subset A of X, we denote

$$\Gamma(A) = \{V \subset X : A \subset V \ (V \neq \emptyset \text{ if } A = \emptyset)\}.$$

Definition 2.1. Let Q and A be subsets of X, and Γ_0 a nonempty subset of $\Gamma(A)$. We say that Γ_0 *intersects* Q if $V \cap Q \neq \emptyset$ for every $V \in \Gamma_0$.

Observe that in the previous definition, Q and A can have a nonempty intersection. Also, it is possible to have $A = \emptyset$ or $Q = X$.

Remark 2.2. The previous definition contains as special cases many notions of linking existing in the literature. Let us recall the following notions, and some families of subsets Γ_0 widely used in critical point theory:

(a) Let B be a topological n-ball, A the boundary of B, and $Q \subset X$ such that $A \cap Q = \emptyset$. It is said that A and Q *homotopically link* (see for example [8]) if $\Gamma(B,A)$ intersects Q where

$$\Gamma(B,A) = \{\gamma(B) : \gamma \in C(B,X), \text{ and } \gamma|_A = id\} \subset \Gamma(A).$$

(b) Let B be a topological n-ball, A the boundary of B, and $Q \subset X$ such that $A \cap Q = \emptyset$. It is said that A and Q *homologically link* (see for example [8]) if $\Gamma_c(A)$ intersects Q where

$$\Gamma_c(A) = \{|\tau| : \ \tau \text{ is a singular } n\text{-chain with } \partial\tau = A, \\ \text{where } |\tau| \text{ is the support of } \tau\} \subset \Gamma(A).$$

(c) Let $E = E_1 \oplus E_2$ be a Banach space. Let F be a subspace of E, $B \subset F$, A the boundary of B in F, and $Q \subset E$ such that $A \cap Q = \emptyset$. Denote

$$\begin{aligned}\mathcal{N}_0 = \{\eta \in \ & C(E \times [0,1], E) : \eta = id \text{ on } E \times \{0\} \cup A \times [0,1], \\ & \eta(x,t) = \eta_1(x,t) + \eta_2(x,t) \in E_1 \oplus E_2 \text{ with } x = x_1 + x_2, \\ & \eta_2(x,t) = x_2 - K(x,t), \text{and } K : E \times [0,1] \to E_2 \text{ is compact}\},\end{aligned}$$

and

$$\Gamma_0 = \{\eta(B,1) : \eta \in \mathcal{N}_0\} \subset \Gamma(A).$$

If A and Q *link in the sense of Benci-Rabinowitz* (see [5]), then Γ_0 intersects Q.

(d) Let $A \subset X$ be a compact subset, and Q a closed subset of X such that $A \cap Q = \emptyset$, and let Γ_0 be a subset of $\Gamma(A)$ containing only compact subsets. It is said that A and Q *link via* Γ_0 in the sense of Ghoussoub [19], if Γ_0 intersects Q.

(e) Let $\Gamma_n = \{B \subset X : B \text{ is closed, and } \mathrm{cat}(B;X) \geq n\} \subset \Gamma(\emptyset)$, where $\mathrm{cat}(B;X)$ is the category of B in X.

(f) Let E be a Banach space,

$$\Gamma_n^\pm = \{B \subset E : \ B \text{ is compact symmetric with respect to} \\ \text{the origin, and } \gamma^\pm(B) \geq n\} \subset \Gamma(\emptyset),$$

where $\gamma^+(B)$ and $\gamma^-(B)$ are respectively the genus and the cogenus of B.

2.2. Linking.

It is well known that many critical point theorems rely on the notion of linking sets. For this reason, we are interested to extend this notion. As we can see, in Remark 2.2 (a), (b), (c), there are three sets: Q, B and $A = \partial B$ with A kept fixed; here are three ways of generalizing the notions of linking:

(1) to consider smaller subsets of $\Gamma(A)$;
(2) to consider arbitrary subets (possibly empty) $A \subset B$;
(3) to let the boundary of Q or more generally a subset of Q play a role.

This leads to a new notion of linking which is the main definition of this paper.

For a subset A of X, we denote

$$\mathcal{N}(A) = \{\eta \in C(X \times [0,1], X) : \eta = id \text{ on } X \times \{0\} \cup A \times [0,1]\}.$$

Definition 2.3 (Linking). Let $A \subset B \subset X$, $P \subset Q \subset X$ such that $B \cap Q \neq \emptyset$, $A \cap Q = \emptyset$, and $B \cap P = \emptyset$. Let $\mathcal{N}_0$ be a nonempty subset of $\mathcal{N}(A)$. We say that (B, A) *links* (Q, P) *via* $\mathcal{N}_0$ if for every $\eta \in \mathcal{N}_0$ one of the following statements is satisfied:

(1) $\eta(B, 1) \cap Q \neq \emptyset$;
(2) $\eta(B,]0, 1[) \cap P \neq \emptyset$.

If $\mathcal{N}_0 = \mathcal{N}(A)$, we simply say that (B, A) *links* (Q, P).

Notice that in the previous definition, A and P can be empty. Also, observe that in all definitions of linking given in Remark 2.2, A is nonempty and P is empty. Here, in allowing $A = \emptyset$ and $P \neq \emptyset$, we increase considerably the number of linking sets. It is worthwhile to observe that even when B is a n-topological ball and A is its boundary, it is possible to have (B, A) linking $(Q, \emptyset)$ without having A and Q linking homotopically or homologically.

Example 2.4.

(1) Let $X = \mathbb{R}^2 \backslash B(0,1)$. Take $B = \{(x, y) \in S^1 : y \leq 0\}$, $A = \{(-1, 0), (1, 0)\}$, and $Q = \{0\} \times]-\infty, -1]$. Then (B, A) links $(Q, \emptyset)$ but A and Q do not link homotopically or homologically.
(2) If A and Q link homotopically (see Remark 2.2 (a)) then $(\gamma(B), A)$ links $(Q, \emptyset)$ for every $\gamma \in C(B, X)$ such that $\gamma|_A = id$. Similarly, if A and Q link homologically (see Remark 2.2 (b)) then $(|\tau|, A)$ links $(Q, \emptyset)$ for every singular n-chain τ with $\partial\tau = A$, where $|\tau|$ is the support of τ.
(3) Let $E = E_1 \oplus E_2$ be a Banach space with E_1 finite dimensional. Denote

$$B_i = \overline{B_{E_i}(0, r_i)}, \quad \partial B_i = \partial_{E_i} B(0, r_i), \quad i = 1, 2.$$

Since ∂B_1 and E_2 link homotopically, we have that $(B_1, \partial B_1)$ links $(E_2, \emptyset)$. Also, $(B_1, \partial B_1)$ links $(B_2, \partial B_2)$; in this case, we can not consider the notions of homological or homotopical linkings.
(4) Let $E = E_1 \oplus E_2$ be a Banach space with E_1 finite dimensional, and let $0 \neq e \in E_2$. Denote

$$B_1 = \overline{B(0, r)} \cap (\mathbb{R}e \oplus E_1), \quad \partial B_1 = \partial B(0, r) \cap (\mathbb{R}e \oplus E_1),$$
$$B_2 = \overline{B_{E_2}(e, s)}, \quad \partial B_2 = \partial_{E_2} B(e, s),$$

with $|r - s| < \|e\| < r + s$. We know that ∂B_1 homotopically links ∂B_2; so $(B_1, \partial B_1)$ links $(\partial B_2, \emptyset)$. Moreover, $(\partial B_1, \emptyset)$ links $(B_2, \partial B_2)$.

Example 2.4 (3)(4) are particular cases of the following three situations of linking.

Lemma 2.5. *Let $E = E_1 \oplus E_2$ be a Banach space with E_1 finite dimensional. Let $\mathcal{U}_1$, $\mathcal{U}_2$ be open subsets of E_1 and E_2 respectively, with $\mathcal{U}_1$ bounded and containing 0. Assume that $\phi : \overline{\mathcal{U}_1} \to E$ is a continuous function such that $\phi|_{\partial\mathcal{U}_1} = id$, $\phi^{-1}(E_2 \backslash \overline{\mathcal{U}_2}) = \emptyset$, then $(\phi(\overline{U_1}), \partial U_1)$ links $(\overline{U_2}, \partial U_2)$.*

Proof. Let $\eta \in \mathcal{N}(\partial U_1)$. Define $H : \overline{\mathcal{U}_1} \times [-1, 1] \to E_1$ by

$$H(x, s) = \begin{cases} P_{E_1}(\eta(\phi(x), s)), & \text{if } s \in [0, 1], \\ (1 + s)P_{E_1}(\phi(x)) - sx, & \text{if } s \in [-1, 0[, \end{cases}$$

where P_{E_1} is the projection on E_1. By topological degree theory, there exists a continuum $\mathcal{C} \subset \overline{U_1} \times [-1, 1]$ of zeros of H such that $\mathcal{C} \cap \overline{\mathcal{U}_1} \times \{s\} \neq \emptyset$ for every $s \in [-1, 1]$. Thus, $\{\eta(\phi(x), t) : (x, t) \in \mathcal{C}, t \in [0, 1]\}$ is a connected subset of E_2. Since $\phi^{-1}(E_2 \backslash \overline{U_2}) = \emptyset$, if $\eta(\phi(\mathcal{U}_1), 1) \cap \overline{U_2} = \emptyset$, there exist $t \in]0, 1[$ and $(x, t) \in \mathcal{C}$ such that $\eta(\phi(x), t) \in \partial\mathcal{U}_2$. $\square$

Lemma 2.6. *Let $E = E_1 \oplus E_2 \oplus E_3$ be a Banach space with $E_1 \oplus E_2$ finite dimensional. Let U be an open bounded subset of $E_1 \oplus E_2$, and F a closed subset of $E_2 \oplus E_3$. Assume that $U \cap F \neq \emptyset$, $\partial U \cap F = \emptyset$, and $\partial U \cap E_2$ is a retract of $E_2 \oplus E_3 \backslash F$. Then $(\overline{U}, \partial U)$ links $(F, \emptyset)$.*

Proof. Let $\hat{r} : E_2 \oplus E_3 \backslash F \to \partial U \cap E_2$ be a continuous retraction. Take $p \in U \cap F$, and define $r : E \to E_2$ by

$$r(x) = \begin{cases} \hat{r}(x_2 + x_3), & \text{if } x_2 + x_3 \notin F, \\ p, & \text{otherwise;} \end{cases}$$

where $x = x_1 + x_2 + x_3$, with $x_i \in E_i$, $i = 1, 2, 3$. Let $\alpha : E \to [0, 1]$ be an Uryshon's function such that $\alpha(x) = 0$ if and only if $x \in F$, and $\alpha(x) = 1$ on ∂U.

For $\eta \in \mathcal{N}(\partial U)$, define $H : \overline{U} \times [-1, 1] \to E_1 \oplus E_2$ by

$$H(x,t) = \begin{cases} \eta_1(x,t) + \alpha((\eta_2 + \eta_3)(x,t))(r(\eta(x,t)) - p), & \text{if } t \in [0,1], \\ (1+t)(x_1 + \alpha(x_2)(r(x) - p)) - t(x - p), & \text{if } t \in [-1, 0[; \end{cases}$$

where $\eta(x,t) = (\eta_1 + \eta_2 + \eta_3)(x,t)$. It is easy to check that $0 \notin H(\partial U \times [-1, 1])$. By topological degree theory, there exists $x \in U$ such that $H(x, 1) = 0$, since $H(\cdot, -1) = id - p$, and $p \in U$. Thus, $\eta(U, 1) \cap F \neq \emptyset$. $\square$

Lemma 2.7. *Let $E = E_1 \oplus E_2 \oplus E_3$ be a Banach space with $E_1 \oplus E_2$ finite dimensional. Let U be an open bounded subset of $E_1 \oplus E_2$, and Q a closed subset of $E_2 \oplus E_3$ such that $\partial U \cap Q \neq \emptyset$, $U \cap \partial Q \neq \emptyset$, $\partial U \cap \partial Q = \emptyset$. Assume that $\partial U \cap E_2$ is a retract of $E_2 \oplus E_3 \backslash \partial Q$, and $\partial U \cap E_2 \backslash Q$ is a retract of $E_2 \oplus E_3$. Then $(\partial U, \emptyset)$ links $(Q, \partial Q)$.*

Proof. Let $\hat{r} : E_2 \oplus E_3 \backslash \partial Q \to \partial U \cap E_2$, and $\hat{s} : E_2 \oplus E_3 \to \partial U \cap E_2 \backslash Q$ be continuous retractions. Without lost of generality, we can assume that $\hat{r} = \hat{s}$ on $E_2 \oplus E_3 \backslash Q$. Take $p \in U \cap \partial Q$, and define $r, s : E \to E_2$ by

$$s(x) = \hat{s}(x_2 + x_3), \quad \text{and} \quad r(x) = \begin{cases} \hat{r}(x_2 + x_3), & \text{if } x_2 + x_3 \notin \partial Q, \\ p, & \text{otherwise;} \end{cases}$$

where $x = x_1 + x_2 + x_3$, with $x_i \in E_i$, $i = 1, 2, 3$. Let $\alpha, \beta : E \to [0, 1]$ be continuous functions such that $\alpha(x) = 0$ if and only if $x \in \partial Q$, $\alpha(x) = 1$ on ∂U, and $\beta(x) = 0$ if and only if $x \in Q$.

For $\eta \in \mathcal{N}(\emptyset)$, define $H : \overline{U} \times [-1, 1] \to E_1 \oplus E_2$ by

$$H(x,t) = \begin{cases} \eta_1(x,t) + (1-t)\alpha((\eta_2 + \eta_3)(x,t))(r(\eta(x,t)) - p) \\ \quad + t\Big(1 - \alpha(x)(1 - \beta(\eta(x,t)))\Big)(s(\eta(x,t)) - p), & \text{if } t \in [0,1], \\ (1+t)(x_1 + \alpha(x_2)(r(x) - p)) - t(x - p), & \text{if } t \in [-1, 0[; \end{cases}$$

where $\eta(x,t) = (\eta_1 + \eta_2 + \eta_3)(x,t)$. It is easy to verify that $0 \notin H(\partial U \times [-1, 0])$. Since $H(\cdot, -1) = id - p$, and $p \in U$, by topological degree theory, one of the following statements hold:

(a) there exist $x \in \partial U$, $t \in]0, 1[$ such that $H(x,t) = 0$;
(b) there exists $x \in \overline{U}$ such that $H(x, 1) = 0$.

If (a) holds, $\alpha(x) = 1$, $\eta(x,t) = (\eta_2 + \eta_3)(x,t)$, and

$$0 = \begin{cases} \Big((1-t)\alpha(\eta(x,t)) + t\beta(\eta(x,t))\Big)(r(\eta(x,t)) - p), & \text{if } \eta(x,t) \notin Q, \\ (1-t)\alpha(\eta(x,t))(r(\eta(x,t)) - p), & \text{if } \eta(x,t) \in Q. \end{cases}$$

Thus, $\eta(\partial U \times]0, 1[) \cap \partial Q \neq \emptyset$, so condition (2) of Definition 2.3 is satisfied.

If (b) holds, $\eta(x,t) = (\eta_2(x,t) + \eta_3(x,t))$, and

$$\Big(1 - \alpha(x)(1 - \beta(\eta(x,1)))\Big)(s(\eta(x,1)) - p) = 0.$$

This implies that $\alpha(x) = 1$ and $\beta(\eta(x,1)) = 0$; that is $x \in \partial U$, and $\eta(x,1) \in Q$. $\square$

Remark 2.8. (a) In the three previous lemmas, if we do not assume that E_1 is finite dimensional, we can obtain linking via $\mathcal{N}_0$ with

$$\mathcal{N}_0 = \{\eta \in \mathcal{N}(A) : P_{E_1} \circ \eta(x,t) = x - \zeta(x,t) \text{ with } \zeta \text{ completely continuous}\},$$

where $A = \partial U_1$ and $P_{E_1} \circ \phi = id - \psi$ with ψ compact in Lemma 2.5; and $A = \partial U$ and $A = \emptyset$ in Lemmas 2.6 and 2.7 respectively with E_2 finite dimensional.

(b) To our knowledge, Lemmas 2.6 and 2.7 are the first results of this type allowing dimension of E_2 to be larger than one.

(c) Lemmas 2.6 and 2.7 can be weaken if we introduce the following definition:

> Let $C \subset D$ be two nonempty subsets of a topological space Y. We say that C is a *pseudo-retract of Y relative to D* if there exists a continuous function $r : X \to D$ such that $r(x) = x$ for every $x \in C$.

Lemma 2.6 is true if we assume that $\partial U \cap E_2$ is a pseudo-retract of $E_2 \oplus E_3 \backslash F$ relative to $E_2 \backslash F$. Similarly, Lemma 2.7 is true if $\partial U \cap E_2$ and $\partial U \cap E_2 \backslash Q$ are respectively pseudo-retracts of $E_2 \oplus E_3 \backslash \partial Q$ and $E_2 \oplus E_3$ relative to $E_2 \backslash \partial Q$.

It is clear that from linking sets, other linkings sets can be obtained. In what follows, we use the convention: $d(\emptyset, S) = \infty$.

Lemma 2.9. *Suppose that (B, A) links (Q, P), and assume there exists $\sigma > 0$ such that $\sigma < d(B, P)$ and $\sigma < d(A, Q)$. If $\eta \in \mathcal{N}(\emptyset)$ is such that $d(\eta(x,t), x) \leq \sigma$ for every x and t, then $(\eta(B,1), \eta(A,1))$ links (Q, P).*

Proof. It is obvious that $\eta(B,1) \cap P = \emptyset$, and $\eta(A,1) \cap Q = \emptyset$. Let $\delta > 0$ be such that $\sigma + \delta < d(B, P)$ and $\sigma + \delta < d(A, Q)$, and let $\lambda : X \to [0,1]$ be an Urysohn's function such that $\lambda(x) = 0$ on $\overline{A}$ and $\lambda(x) = 1$ on $X \backslash B(A, \delta/2)$. Define $\widetilde{\eta} : X \times [0,1] \to X$ by $\widetilde{\eta}(x,t) = \eta(x, \lambda(x)t)$. Since $\widetilde{\eta} \in \mathcal{N}(A)$, and (B, A) links (Q, P), we have $\widetilde{\eta}(B,1) \cap Q \neq \emptyset$, and hence $\eta(B,1) \cap Q \neq \emptyset$. Finally, let $\eta_0 \in \mathcal{N}(\eta(A,1))$. Take β an Urysohn's function such that $\beta(x) = 0$ on $\overline{B(A, \delta/2)}$ and $\beta(x) = 1$ on $X \backslash B(A, \delta)$. Define

$$\hat{\eta}(x,t) = \begin{cases} \widetilde{\eta}(x, 2t), & \text{if } t \leq 1/2, \\ \widetilde{\eta}(x, 1), & \text{if } t > 1/2,\ x \in B(A, \delta/2), \\ \eta_0(\eta(x,1), \beta(x)(2t-1)), & \text{otherwise.} \end{cases}$$

Since $\hat{\eta} \in \mathcal{N}(A)$ and (B, A) links (Q, P), we get that $\eta_0(\eta(B,1), 1) \cap Q \neq \emptyset$ or $\eta_0(\eta(B,1),]0,1]) \cap P \neq \emptyset$. $\square$

2.3. Linking in taking into account the functional.

Let $f : X \to \mathbb{R}$ be a continous functional, and let A be a subset of X. Recall that $\mathcal{N}(A)$ is the set of continuous deformations of X keeping A fixed. Taking into account the functional f, we consider the following subset of $\mathcal{N}(A)$:

$$\mathcal{N}_f(A) = \{\eta \in C(X \times [0,1], X) : \ \eta = id \text{ on } X \times \{0\} \cup A \times [0,1] \text{ and } f(\eta(x,t)) \leq f(x) \text{ for all } x \text{ and } t\}.$$

As we will see, using our notion of linking and restricting ourself to deformations η such that $f(\eta(x,t)) \leq f(x)$ will permit us to get more families of sets Γ_0 intersecting some set Q.

Lemma 2.10. *Let A be a subset of X, and $\mathcal{N}_0 \subset \mathcal{N}_f(A)$. Suppose that (B, A) links (Q, P) via $\mathcal{N}_0$. If $f(x) < f(y)$ for every $x \in B$, and every $y \in P$, then condition (2) of Definition 2.3 never holds. In other words, the set*

$$\Gamma_0 = \{\eta(B,1) : \eta \in \mathcal{N}_0\} \subset \Gamma(A)$$

intersects Q.

Observe that the conclusion of this lemma is false if we consider deformations not satisfying $f(\eta(x,t)) \leq f(x)$ when $P \neq \emptyset$.

2.4. Deformation property.

As before, let $f : X \to \mathbb{R}$ be a continuous functional. Let K be a subset of X that we call the set of *critical points of f*. For $c \in \mathbb{R}$, we denote by K_c the set of critical points at level c, that is $K_c = K \cap f^{-1}(c)$.

In this paragraph, we want to define a deformation property for f, (see also [10] for deformation properties in an abstract setting).

Definition 2.11. Let $f : X \to \mathbb{R}$ be a continuous functional, $c \in \mathbb{R}$, K_c the set of critical points of f at level c, $A \subset X$, and let $\mathcal{N}_0$ be a nonempty subset of $\mathcal{N}_f(A)$. We say that f satisfies *property* $\mathcal{D}(c, \mathcal{N}_0)$ if for every $\sigma > 0$, and every open neighborhoods $\mathcal{O}$ of K_c, and $\mathcal{U}$ of A ($\mathcal{O}$ (resp. $\mathcal{U}$) can be empty if K_c (resp. A) is empty), there exist $\eta \in \mathcal{N}_0$, and $\varepsilon > 0$ such that

(1) $f(\eta(x,1)) \leq c - \varepsilon$ for every $x \in \{x \in X \backslash (\mathcal{O} \cup \mathcal{U}) : f(x) \leq c + \varepsilon\}$;
(2) $d(\eta(x,t), x) \leq \sigma$.

Lemma 2.12. *Let X be a metric space, $f : X \to \mathbb{R}$ a continuous functional, $c \in \mathbb{R}$, and let A be a closed subset of X. Assume that f satisfies $\mathcal{D}(c, \mathcal{N}_f(\emptyset))$, then it satisfies the property $\mathcal{D}(c, \mathcal{N}_f(A))$.*

Proof. Let $\sigma > 0$, and $\mathcal{O}$ an open neighborhood of K_c. By assumption there exist $\varepsilon > 0$ and $\eta \in \mathcal{N}_f(\emptyset)$ such that $d(\eta(x,t), x) \leq \sigma$, and $f(\eta(x,1)) \leq c - \varepsilon$ for every $x \in \{x \in X \backslash \mathcal{O} : f(x) \leq c + \varepsilon\}$. Let $\mathcal{U}$ be an open neighborhood of A. Take $\lambda : X \to [0,1]$ an Urysohn's function such that $\lambda(x) = 0$ on A and $\lambda(x) = 1$ on $X \backslash \mathcal{U}$. The function $\widetilde{\eta} \in \mathcal{N}_f(A)$ defined by $\widetilde{\eta}(x,t) = \eta(x, \lambda(x)t)$ is the desired deformation. $\square$

In the literature, we can find many results establishing deformation properties. Many of them are obtained with $A = f^{-1}(]-\infty, c - \varepsilon_0] \cup [c + \varepsilon_0, \infty[)$, see for example [2], [5], [7], [8], [9], [12], [28], [30].

3. Minimax type theorems

3.1. General results.

Using the notions previously introduced, we give the main result of this paper. The proof is similar to the one given in [11], we give it for sake of completness. We use the following conventions: $\sup f(\emptyset) = -\infty$, $\inf f(\emptyset) = \infty$.

Theorem 3.1. *Let X be a metric space, and $f : X \to \mathbb{R}$ a continuous functional. Assume that there exist two pairs (B, A) and (Q, P) such that (B, A) links (Q, P),*

$$f(x) < f(y) \text{ for every } x \in B,\ y \in P, \quad \text{and} \quad \sup f(A) \leq \inf f(Q),$$

with a strict inequality if $d(A, Q) = 0$. Let

$$c = \inf_{\eta \in \mathcal{N}_f(A)} \sup f(\eta(B, 1)).$$

If $c \in \mathbb{R}$, and f satisfies the property $\mathcal{D}(c, \mathcal{N}_f(A))$, then $K_c \neq \emptyset$. Moreover, if $c = \inf f(Q)$, then $d(K_c, Q) = 0$.

Proof. By Lemma 2.10, $c \geq \inf(Q)$. Denote

$$C = \begin{cases} Q, & \text{if } c = \inf f(Q), \\ \displaystyle\bigcup_{\eta \in \mathcal{N}_f(A)} \eta(B, 1), & \text{otherwise.} \end{cases}$$

Assume that $d(K_c, C) \neq 0$. Fix $\sigma > 0$ such that $d(K_c, C) > 2\sigma$, and $d(A, Q) > 2\sigma$ if $c = \sup f(A)$.

Take $\mathcal{O} = B(K_c, \sigma)$, and

$$\mathcal{U} = \begin{cases} B(A, \sigma), & \text{if } c = \sup f(A); \\ f^{-1}(]-\infty, c - \delta[), & \text{if } c > c - \delta > \sup f(A), \text{ for some } \delta > 0. \end{cases}$$

Since f satisfies the property $\mathcal{D}(c, \mathcal{N}_f(A))$, there exist $\varepsilon > 0$ and $\eta \in \mathcal{N}_f(A)$ satisfying the conditions (1) and (2) of Definition 2.11.

Let $\hat{\eta} \in \mathcal{N}_f(A)$ be such that $\sup f(\hat{\eta}(B, 1)) \leq c + \varepsilon$. Since $\eta(\hat{\eta}(x, t), t) \in \mathcal{N}_f(A)$, by Lemma 2.10, we can choose $y \in \hat{\eta}(B, 1)$ such that $\eta(y, 1) \in C$ and $f(\eta(y, 1)) > c - \min\{\varepsilon, \delta\}$. On the other hand, since $d(\eta(y, 1), y) \leq \sigma$ and $f(\eta(y, 1)) \leq f(y)$, we have that $y \notin \mathcal{O} \cup \mathcal{U}$. So, $f(\eta(y, 1)) \leq c - \varepsilon$, which is a contradiction. □

Corollary 3.2. *Let X be a metric space, and $f : X \to \mathbb{R}$ a continuous functional. Assume that there exist two pairs (B, A) and (Q, P) such that (B, A) links (Q, P), $d(A, Q) > 0$, $d(B, P) > 0$,*

$$\sup f(A) \leq a = \inf f(Q) \leq \sup f(B) = b \leq \inf f(P).$$

If a, $b \in \mathbb{R}$ and f satisfies the property $\mathcal{D}(c, \mathcal{N}_f(\emptyset))$ for every $c \in [a, b]$, then $K_c \neq \emptyset$ for some $c \in [a, b]$. Moreover, $c < \inf f(P)$, or $d(K_c, B) = 0$.

Proof. First case: suppose that $b < \inf f(P)$. Without lost of generality, we can assume that A is closed. By Lemma 2.12, we have that f satisfies $\mathcal{D}(c, \mathcal{N}_f(A))$. The previous theorem implies that $K_c \neq \emptyset$ for some $c \in [a, b]$.

Second case: suppose that $b = \inf f(P)$. If $d(K_b, B) \neq 0$, fix $\sigma > 0$ such that $\sigma < d(K_b, B)$, $\sigma < d(B, P)$ and $\sigma < d(A, Q)$. From the deformation property (with $\mathcal{O} = B(K_b, \sigma)$), we deduce that there exist $\varepsilon > 0$, and $\eta \in \mathcal{N}(\emptyset)$ such that $d(\eta(x, t), x) \leq \sigma$, and $f(\eta(x, 1)) \leq b - \varepsilon$ for every $x \in B$. By Lemma 2.9, $(\eta(B, 1), \eta(A, 1))$ links (Q, P). The conclusion follows from the first case. □

Theorem 3.1 can be generalized. For that, we need to introduce the following notion:

Definition 3.3. Let A be a subset of X, $\mathcal{N}_0$ a nonempty subset of $\mathcal{N}(A)$, and Γ_0 a nonempty subset of $\Gamma(A)$. We say that Γ_0 is *invariant with respect to* $\mathcal{N}_0$ if the set $\eta(V, 1) \in \Gamma_0$ for every $V \in \Gamma_0$, and every $\eta \in \mathcal{N}_0$.

Remark 3.4. Similar (but slightly less general) notions already exist in the literature. For example, the notion of "isotopy ambient invariant family" $\Gamma_0 \subset \Gamma(\emptyset)$, introduced by Palais [29]; or the notion of "homotopy stable family with boundary A" $\Gamma_0 \subset \Gamma(A)$, introduced by Ghoussoub [19] in the equivariant context.

Remark 3.5. Therem 3.1 can be stated more generally if we replace $\mathcal{N}_f(A)$ by $\mathcal{N}_0 \subset \mathcal{N}_f(A)$, and if we assume that (B, A) links (Q, P) via $\mathcal{N}_0$, and

$$\Gamma_0 = \{\eta(B, 1) : \eta \in \mathcal{N}_0\}$$

is invariant with respect to $\mathcal{N}_0$.

The two previous results are particular cases of the following theorem. The proof is analogous to the one of Theorem 3.1.

Theorem 3.6. *Let X be a metric space, $f : X \to \mathbb{R}$ a continuous functional, and K its set of critical points. Let $A \subset X$, $\mathcal{N}_0 \subset \mathcal{N}_f(A)$, and $\Gamma_0 \subset \Gamma(A)$ nonempty and invariant with respect to $\mathcal{N}_0$. Assume that there exists $Q \subset X$ such that Γ_0 intersects Q, and*

$$\inf_{V \in \Gamma_0} \sup f(V \cap Q) \geq \sup f(A)$$

with a strict inequality if $d(A, Q) = 0$. Let

$$c = \inf_{V \in \Gamma_0} \sup f(V).$$

If $c \in \mathbb{R}$, and f satisfies the property $\mathcal{D}(c, \mathcal{N}_0)$, then $K_c \neq \emptyset$. Moreover, if

$$c = \inf_{V \in \Gamma_0} \sup f(V \cap Q),$$

then $d(K_c, Q) = 0$.

As corollary, we get the following Minimax Principle in taking $Q = X$.

Theorem 3.7. *Let X be a metric space, and $f : X \to \mathbb{R}$ a continuous functional. Let $\mathcal{N}_0 \subset \mathcal{N}_f(\emptyset)$, $\Gamma_0 \subset \Gamma(\emptyset)$ nonempty and invariant with respect to $\mathcal{N}_0$, and let*

$$c = \inf_{V \in \Gamma_0} \sup f(V).$$

If $c \in \mathbb{R}$, and f satisfies $\mathcal{D}(c, \mathcal{N}_0)$ then $K_c \neq \emptyset$.

3.2. Some particular cases.

In this paragraph, we give some particular cases of the previous theorems with

$$K = \{x \in X : |df|(x) = 0\},$$

where $|df|$ denote the weak slope of f (the reader is referred to [17] or [12] for the definition). Recall that $|df|(x) = f'(x)$ if f is C^1. We give the definition of the Palais-Smale condition which is used to obtain a deformation theorem.

Definition 3.8. Let $c \in \mathbb{R}$. We say that f satisfies the *Palais-Smale condition at level c* ($(PS)_c$) if every sequence $\{x_n\}$ in X such that $f(x_n) \to c$ and $|df|(x_n) \to 0$, has a converging subsequence.

Combining Lemma 2.12, and Theorem 2.14 of [12] gives the following result.

Theorem 3.9. *Let X be a complete metric space, $f : X \to \mathbb{R}$ a continuous functional, $c \in \mathbb{R}$, and let A be a closed subset of X. Assume that f satisfies $(PS)_c$, then it satisfies the property $\mathcal{D}(c, \mathcal{N}_f(A))$.*

The next result is a direct consequence of Lemma 2.5, Corollary 3.2 and Theorem 3.9.

Theorem 3.10. *Let $E = E_1 \oplus E_2$ be a Banach space with E_1 finite dimensional, and let $f : E \to \mathbb{R}$ be a continuous functional. Assume that there exist $\mathcal{U}_1$, $\mathcal{U}_2$ open neighborhoods of 0 in E_1 and E_2 respectively, with $\mathcal{U}_1$ bounded, and*

$$\sup f(\partial \mathcal{U}_1) \leq m = \inf f(\overline{\mathcal{U}_2}) \leq \sup f(\overline{\mathcal{U}_1}) = M \leq \inf f(\partial \mathcal{U}_2).$$

If f satisfies $(PS)_c$ for every $c \in [m, M]$, then f has a critical point.

Remark 3.11. (i) Observe that if $U_2 = E_2$, the previous theorem is the Saddle Point Theorem.
(ii) In the case where $U_i = B_{E_i}(0, r_i)$, the previous theorem generalizes Theorem 8.1 of [26]. Moreover, if $m = M$, this corresponds to a local linking, see [24], and also [6].
(iii) We can replace $\overline{U_1}$ by $\phi(\overline{U_1})$, where ϕ is as in Lemma 2.5.

Theorem 3.12. *Let $E = E_1 \oplus E_2 \oplus E_3$ be a Banach space with $E_1 \oplus E_2$ finite dimensional, and $f : E \to \mathbb{R}$ be a continuous functional. Let U be an open bounded subset of $E_1 \oplus E_2$, and Q a closed subset of $E_2 \oplus E_3$ such that $\partial U \cap Q \neq \emptyset$, $U \cap \partial Q \neq \emptyset$, $\partial U \cap \partial Q = \emptyset$, $\partial U \cap E_2$ is a retract of $E_2 \oplus E_3 \backslash \partial Q$, and $\partial U \cap E_2 \backslash Q$ is a retract of $E_2 \oplus E_3$. Assume that*

$$-\infty < m = \inf f(Q) \leq \sup f(\partial U) \leq \inf f(\partial Q) \leq \sup f(\overline{U}) = M.$$

If f satisfies $(PS)_c$ for every $c \in [m, M]$, then f has at least two critical points.

Proof. By Lemma 2.6, $(\overline{U}, \partial U)$ links $(\partial Q, \emptyset)$. Theorems 3.1 and 3.9 implies that $K_{c_0} \neq \emptyset$ for some $c_0 \geq \inf f(\partial Q)$. Moreover, if $c_0 = \inf f(\partial Q)$ then $K_{c_0} \cap \partial Q \neq \emptyset$.

On the other hand, $(\partial U, \emptyset)$ links $(Q, \partial Q)$ by Lemma 2.7. Again, Corollary 3.2, and Theorem 3.9 implies that $K_{c_1} \neq \emptyset$ for some $c_1 \leq \inf f(\partial Q)$. Moreover, if $c_1 = \inf f(\partial Q)$, $K_{c_1} \cap \partial U \neq \emptyset$. Therefore, f has at least two critical points, since $c_1 < c_0$, or $c = c_1 = c_0$, $K_c \cap \partial U \neq \emptyset$, and $K_c \cap \partial Q \neq \emptyset$. $\square$

Remark 3.13. (i) The Mountain Pass Theorem is a particular case of this theorem with $E_1 = \{0\}$, $E_2 = \mathbb{R}e$ for some $e \neq 0$, $y = e/2$, $r = \|e\|/2$, and $Q = \overline{B(0, s)}$ for some $s < \|e\|$.
(ii) The previous theorem is a generalization of Theorem 8.2 in [26].

4. Application

Let Ω be a bounded smooth domain of $\mathbb{R}^n$, and let $\{\lambda_n\}$ be the nondecreasing sequence of eigenvalues of $-\Delta$, and $\{e_n\}$ the corresponding sequence of eigenvectors such that

$$\int_\Omega |\nabla e_n|^2\,dx = 1 = \lambda_n \int_\Omega e_n^2\,dx.$$

Denote $E_n = span\{e_1, \dots, e_n\}$.

In this section, we want to present an application of Theorem 3.1 to the following problem

$$\begin{aligned} &\Delta u + \lambda_m u = g(x,u) + s e_m, \\ &u|_{\partial\Omega} = 0, \end{aligned} \tag{P}$$

where $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous. Define

$$G(x,u) = \int_0^u g(x,y)\,dy.$$

We make the following assumptions:

(H1) $m \geq 2$, and $\lambda_{m-1} < \lambda_m < \lambda_{m+1}$;

(H2) $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous;

(H3) there exist $1 < \gamma \leq \beta \leq \alpha < 2$, $A_1 \in \mathbb{R}$, and $A_2^\pm$, B_1, $B_2 > 0$ such that

$$A_1 + A_2^+(u^+)^\beta - A_2^-(u^-)^\gamma \leq G(x,u) \leq B_1 + B_2|u|^\alpha;$$

and if $\gamma = \beta$,

$$A_2^+ \int_{\{e_m>0\}} e_m^\beta\,dx > A_2^- \int_{\{e_m<0\}} |e_m|^\beta\,dx;$$

(H4) there exist $1 < \eta < \zeta < 2$, $q \in \mathbb{R}$, and $b_i > 0$, $i = 1, \dots, 4$, such that

$$G(x,u) + g(x,u)(qu + y) \geq b_1(u^+)^\zeta - b_2(u^-)^\eta - b_3|y|^\zeta - b_4;$$

(H4*) there exist $1 < \zeta < 2$, q_j, $b_i^j \in \mathbb{R}$, $j = 1, 2$, $i = 1, \dots, 4$, such that for $j = 1, 2$,

$$G(x,u) + g(x,u)(q_j u + y) \geq b_1^j(u^+)^\zeta - b_2^j(u^-)^\zeta - b_3^j|y|^\zeta - b_4^j;$$

and

$$b_1^j \int_{\{(-1)^j e_m<0\}} |e_m|^\zeta\,dx > b_2^j \int_{\{(-1)^j e_m>0\}} |e_m|^\zeta\,dx.$$

where $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$.

We state the main theorem of this section.

Theorem 4.1. *Assume that (H1)–(H3), and (H4) or (H4*) are satisfied. Then, there exists $s_0 < 0$ such that for every $s < s_0$, the problem (P) has a solution.*

Consider the functional $I : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \int_\Omega \frac{|\nabla u|^2}{2} - \frac{\lambda_m}{2}u^2 + G(x,u) + su\,e_m\,dx,$$

It is easy to ckeck that

$$I'(u)(v) = \int_\Omega \nabla u \cdot \nabla v - \lambda_m uv + g(x,u)v + sv\,e_m\,dx \quad \text{for every } v \in H_0^1(\Omega).$$

Hence, critical points of I are solutions of (P).

To prove Theorem 4.1, we need to establish the Palais-Smale condition. We will use the following lemma.

Lemma 4.2. *Let $\theta > 1$, for every $\varepsilon > 0$, there exists k such that*

$$|u|^\theta + \varepsilon|u|^\theta + k|z|^\theta \geq |u+z|^\theta \geq |u|^\theta - \varepsilon|u|^\theta - k|z|^\theta,$$

for every u, $z \in \mathbb{R}$.

Proposition 4.3. *Under the assumptions $(H1)$–$(H3)$,$(H4)$, or $(H4^*)$ the functional I satisfies $(PS)_c$ for every $c \in \mathbb{R}$.*

Proof. By Proposition B.35 in [30], condition $(PS)_c$ is satisfied if we show that a sequence $\{u_n\}$ such that $I(u_n) \to c$, and $I'(u_n) \to 0$ as $n \to \infty$, is bounded in $H_0^1(\Omega)$. Write $u_n = v_n + t_n e_m + w_n$ with $v_n \in E_{m-1}$, $w_n \in E_m^\perp$, $t_n \in \mathbb{R}$. Let $p_n \in \mathbb{R}$ which will be determined later. We have for n sufficiently large

$$\begin{aligned}
&K + \|v_n\| + |p_n t_n|\,\|e_m\| + \|w_n\| \\
&\quad \geq I(u_n) + I'(u_n)(-v_n + p_n t_n e_m + w_n) \\
&\quad = \int_\Omega \frac{|\nabla u_n|^2}{2} - \frac{\lambda_m}{2} u_n^2 + G(x,u_n) + s t_n e_m^2 \, dx \\
&\qquad - \int_\Omega \nabla u_n \cdot \nabla(v_n - p_n t_n e_m - w_n) - \lambda_m u_n (v_n - p_n t_n e_m - w_n)\, dx \\
&\qquad + \int_\Omega g(x,u_n)(-v_n + p_n t_n e_m + w_n) + s p_n t_n e_m^2 \, dx \\
&\quad = \int_\Omega -\frac{1}{2}(|\nabla v_n|^2 - \lambda_m v_n^2) + \frac{3}{2}(|\nabla w_n|^2 - \lambda_m w_n^2) + s t_n (1+p_n) e_m^2 \, dx \\
&\qquad + \int_\Omega G(x,u_n) + g(x,u_n)(-v_n + p_n t_n e_m + w_n)\, dx \\
&\quad \geq -\frac{1}{2}(1 - \frac{\lambda_m}{\lambda_{m-1}})\|v_n\|^2 + \frac{3}{2}(1 - \frac{\lambda_m}{\lambda_{m+1}})\|w_n\|^2 + s t_n (1+p_n)\|e_m\|_{L^2}^2 \\
&\qquad + \int_\Omega G(x,u_n) + g(x,u_n)(-v_n + p_n t_n e_m + w_n)\, dx.
\end{aligned} \tag{4.1}$$

If $(H4)$ is satisfied, take $0 < \varepsilon < b_1$, and write $\hat{b}_1 = b_1 - \varepsilon$, $\hat{b}_2 = b_2 + \varepsilon$. By Lemma 4.2, there exists k such that

$$\begin{aligned}
&G(x,u_n) + g(x,u_n)(-v_n + q t_n e_m + w_n) \\
&\quad \geq b_1 (u_n^+)^\zeta - b_2 (u_n^-)^\eta - b_3 |(-1-q)v_n + (1-q)w_n|^\zeta - b_4 \\
&\quad \geq (b_1 - \varepsilon)((t_n e_m)^+)^\zeta - (b_2 + \varepsilon)((t_n e_m)^-)^\eta - k(1 + |v_n|^\zeta + |w_n|^\zeta) \\
&\quad = \hat{b}_1((t_n e_m)^+)^\zeta - \hat{b}_2((t_n e_m)^-)^\eta - k(1 + |v_n|^\zeta + |w_n|^\zeta).
\end{aligned} \tag{4.2}$$

In fixing $p_n = q$, inequalities (4.1) and (4.2) imply that for n sufficiently large

$$\begin{aligned}
&K + \|v_n\| + |q t_n|\,\|e_m\| + \|w_n\| \\
&\quad \geq -\frac{1}{2}(1 - \frac{\lambda_m}{\lambda_{m-1}})\|v_n\|^2 + \frac{3}{2}(1 - \frac{\lambda_m}{\lambda_{m+1}})\|w_n\|^2 + s t_n (1+q)\|e_m\|_{L^2}^2 \\
&\qquad + \int_\Omega \hat{b}_1((t_n e_m)^+)^\zeta - \hat{b}_2((t_n e_m)^-)^\eta - k(1 + |v_n|^\zeta + |w_n|^\zeta)\, dx.
\end{aligned}$$

So, $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

On the other hand, if $(H4^*)$ is satisfied, choose $\varepsilon > 0$ small enough such that for $j = 1, 2$,

$$(b_1^j - \varepsilon) \int_{\{(-1)^j e_m < 0\}} |e_m|^\zeta \, dx > (b_2^j + \varepsilon) \int_{\{(-1)^j e_m > 0\}} |e_m|^\zeta \, dx.$$

Write $\hat{b}_i^j = b_i^j + (-1)^i \varepsilon$, $i, j = 1, 2$. By Lemma 4.2, there exists k such that

$$\begin{aligned}
&G(x,u_n) + g(x,u_n)(-v_n + q_j t_n e_m + w_n) \\
&\quad \geq b_1^j (u_n^+)^\zeta - b_2^j (u_n^-)^\zeta - b_3^j |(-1-q_j)v_n + (1-q_j)w_n|^\zeta - b_4^j \\
&\quad \geq (b_1^j - \varepsilon)((t_n e_m)^+)^\zeta - (b_2^j + \varepsilon)((t_n e_m)^-)^\zeta - k(1 + |v_n|^\zeta + |w_n|^\zeta) \\
&\quad = \hat{b}_1^j((t_n e_m)^+)^\zeta - \hat{b}_2^j((t_n e_m)^-)^\zeta - k(1 + |v_n|^\zeta + |w_n|^\zeta) \\
&\quad = |t_n|^\zeta \left(\hat{b}_1^j((sgn(t_n) e_m)^+)^\zeta - \hat{b}_2^j((sgn(t_n) e_m)^-)^\zeta \right) \\
&\qquad - k(1 + |v_n|^\zeta + |w_n|^\zeta).
\end{aligned} \tag{4.3}$$

Now, define

$$p_n = \begin{cases} q_1, & \text{if } t_n \geq 0, \\ q_2, & \text{if } t_n < 0. \end{cases}$$

In combining inequalities (4.1) and (4.3), we get for n sufficiently large

$$\begin{aligned} & K + \|v_n\| + |p_n t_n| \, \|e_m\| + \|w_n\| \\ & \geq -\frac{1}{2}(1 - \frac{\lambda_m}{\lambda_{m-1}})\|v_n\|^2 + \frac{3}{2}(1 - \frac{\lambda_m}{\lambda_{m+1}})\|w_n\|^2 + st_n(1+p_n)\|e_m\|_{L^2}^2 \\ & \quad + |t_n|^\zeta \begin{cases} \hat{b}_1^1 \int_{\{e_m>0\}} e_m^\zeta \, dx - \hat{b}_2^1 \int_{\{e_m<0\}} |e_m|^\zeta \, dx, & \text{if } t_n \geq 0, \\ \hat{b}_1^2 \int_{\{e_m<0\}} |e_m|^\zeta \, dx - \hat{b}_2^2 \int_{\{e_m>0\}} |e_m|^\zeta \, dx, & \text{if } t_n < 0, \end{cases} \\ & \quad - k \int_\Omega (1 + |v_n|^\zeta + |w_n|^\zeta) \, dx. \end{aligned}$$

So, $\{u_n\}$ is bounded in $H_0^1(\Omega)$, and the proof is complete. $\square$

Now, we can prove the main theorem of this section.

Proof of Theorem 4.1. From Proposition 4.3, we know that I satisfies $(PS)_c$ for every $c \in \mathbb{R}$. We will show that there exists $s_0 < 0$ such that for every $s < s_0$, there exist $R > 0$ and $\hat{t} > 0$ such that

$$\begin{aligned} \sup I(\hat{t}e_m + \partial_{E_{m-1}} B(0,R)) < \inf I([0,\infty[\, e_m \oplus E_m^\perp) \\ \leq \sup I(\hat{t}e_m + \overline{B_{E_{m-1}}(0,R)}\,) < \inf I(E_m^\perp). \end{aligned} \tag{4.4}$$

The conclusion will follow from Theorems 3.1 and 3.9, since

$$\big(\hat{t}e_m + \overline{B_{E_{m-1}}(0,R)}\,, \hat{t}e_m + \partial_{E_{m-1}} B(0,R)\big) \quad \text{links} \quad \big([0,\infty[\, e_m \oplus E_m^\perp, E_m^\perp\big).$$

More precisely, we will get u a critical point of I such that $I(u) < \inf I(E_m^\perp)$.

For $w \in E_m^\perp$, we have

$$\begin{aligned} I(w) &= \int_\Omega \frac{|\nabla w|^2}{2} - \lambda_m \frac{w^2}{2} + G(x,w) \, dx \\ &\geq \frac{(\lambda_{m+1} - \lambda_m)}{2} \|w\|_{L^2}^2 + \int_\Omega A_2^+ (w^+)^\beta - A_2^- (w^-)^\gamma \, dx - \bar{k}_0 \\ &\geq \frac{(\lambda_{m+1} - \lambda_m)}{2} \|w\|_{L^2}^2 - \bar{k}_1 \|w\|_{L^2}^\gamma - \bar{k}_2 \\ &\geq K_0. \end{aligned} \tag{4.5}$$

Fix $\varepsilon > 0$. Using Lemma 4.2, there exists k such that for every $v \in E_{m-1}$, and every $t \geq 0$, we have

$$\begin{aligned} I(v + te_m) &= \int_\Omega \frac{|\nabla(v + te_m)|^2}{2} - \lambda_m \frac{(v + te_m)^2}{2} + G(x, v + te_m) + ste_m^2 \, dx \\ &\leq \frac{1}{2}\left(1 - \frac{\lambda_m}{\lambda_{m-1}}\right) \|v\|^2 + st\|e_m\|_{L^2}^2 + \int_\Omega B_1 + B_2 |v + te_m|^\alpha \, dx \\ &\leq \frac{1}{2}\left(1 - \frac{\lambda_m}{\lambda_{m-1}}\right) \|v\|^2 + st\|e_m\|_{L^2}^2 + k_1 + \int_\Omega (B_2 + \varepsilon)|te_m|^\alpha + k_2 |v|^\alpha \, dx \\ &\leq \frac{1}{2}\left(1 - \frac{\lambda_m}{\lambda_{m-1}}\right) \|v\|^2 + k(1 + \|v\|^\alpha) + st\|e_m\|_{L^2}^2 + t^\alpha (B_2 + \varepsilon) \int_\Omega |e_m|^\alpha \, dx \\ &= K_1(\|v\|) + h_s(t) \\ &\leq K_2 + h_s(t), \end{aligned}$$

where

$$K_1(r) = \frac{1}{2}\left(1 - \frac{\lambda_m}{\lambda_{m-1}}\right) r^2 + k(1 + |r|^\alpha) \leq K_2 \quad \text{for all } r \in \mathbb{R},$$

and

$$h_s(t) = st\|e_m\|_{L^2}^2 + t^\alpha (B_2 + \varepsilon) \int_\Omega |e_m|^\alpha \, dx.$$

For every $s < 0$, the function $t \mapsto h_s(t)$ defined on $[0, \infty[$ achieves its minimum at some $t_s > 0$ such that $h_s(t_s) \to -\infty$ as $s \to -\infty$. Fix $s_0 < 0$ such that $h_s(t_s) < K_0 - K_2$ for every $s < s_0$. Now, fix $s < s_0$, and set $\hat{t} = t_s$. Therefore, we obtain

$$I(v + \hat{t}e_m) < K_1(\|v\|) + K_0 - K_2 \leq K_0 \quad \text{for every } v \in E_{m-1}. \tag{4.6}$$

Using $(H3)$, we can choose ε small enough such that the function

$$t \mapsto st\|e_m\|_{L^2}^2 + t^\beta \int_{\{e_m>0\}} (A_2^+ - \varepsilon) e_m^\beta \, dx - t^\gamma \int_{\{e_m<0\}} (A_2^- + \varepsilon)|e_m|^\gamma \, dx$$

is bounded from below on $[0, \infty[$. By Lemma 4.2, there exists $\tilde{k}$ such that for every $w \in E_m^\perp$, and $t \geq 0$, we have

$$\begin{aligned}
I(te_m + w) &= \int_\Omega \frac{|\nabla(te_m + w)|^2}{2} - \lambda_m \frac{(te_m + w)^2}{2} + G(x, te_m + w) + ste_m^2 \, dx \\
&\geq \frac{(\lambda_{m+1} - \lambda_m)}{2} \|w\|_{L^2}^2 + st\|e_m\|_{L^2}^2 - \tilde{k}_0 \\
&\quad + \int_\Omega A_2^+((te_m + w)^+)^\beta - A_2^-((te_m + w)^-)^\gamma \, dx \\
&\geq \frac{(\lambda_{m+1} - \lambda_m)}{2} \|w\|_{L^2}^2 + st\|e_m\|_{L^2}^2 - \tilde{k}_1 \\
&\quad + \int_\Omega (A_2^+ - \varepsilon)((te_m)^+)^\beta - (A_2^- + \varepsilon)((te_m)^-)^\gamma - \tilde{k}_2|w|^\beta \, dx \\
&\geq \frac{(\lambda_{m+1} - \lambda_m)}{2} \|w\|_{L^2}^2 - \tilde{k}(1 + \|w\|_{L^2}^\beta) \\
&\quad + st\|e_m\|_{L^2}^2 + t^\beta \int_{\{e_m>0\}} (A_2^+ - \varepsilon) e_m^\beta \, dx - t^\gamma \int_{\{e_m<0\}} (A_2^- + \varepsilon)|e_m|^\gamma \, dx \\
&\geq K_3.
\end{aligned}$$

So,

$$\inf I([0, \infty[\, e_m \oplus E_m^\perp) \geq K_3. \tag{4.7}$$

To conclude, we fix $R > 0$ such that $K_1(R) + K_0 - K_2 < K_3$. Thus, by (4.6),

$$I(v + \hat{t}e_m) < K_3 \quad \text{for every } v \in \partial_{E_{m-1}} B(0, R). \tag{4.8}$$

Combining (4.5)–(4.8) gives (4.4). □

Some corollaries.

We present some corollaries of Theorem 4.1 in the particular case where the function g satisfies the following growth condition.

(H5) there exist $1 < \gamma \leq \beta \leq \alpha < 2$, a_1, b_1, $b_2^- \in \mathbb{R}$, and $a_2^\pm$, $b_2^+ > 0$, such that

$$a_1 + a_2^+(u^+)^{\beta-1} + b_2^-(u^-)^{\alpha-1} \leq g(x, u) \leq b_1 + b_2^+(u^+)^{\alpha-1} + a_2^-(u^-)^{\gamma-1}.$$

Corollary 4.4. *Assume $(H1)$, $(H2)$, and $(H5)$ with one of the following statements satisfied:*

(i) $\gamma < \beta$;

(ii) $\gamma = \beta$, *and*

$$\frac{a_2^-}{a_2^+} \int_{\{e_m<0\}} |e_m|^\beta \, dx < \int_{\{e_m>0\}} e_m^\beta \, dx < \frac{a_2^+}{a_2^-} \int_{\{e_m<0\}} |e_m|^\beta \, dx.$$

Then there exists $s_0 < 0$ such that for every $s < s_0$, the problem (P) has a solution.

In what follows, we will use the following lemma.

Lemma 4.5. *Let $\theta > 1$, for every $\varepsilon > 0$, there exists k such that*

$$|u|^{\theta-1}(qu+z) \geq qu|u|^{\theta-1} - \varepsilon|u|^{\theta} - k|z|^{\theta}$$

for every u, z, $q \in \mathbb{R}$.

Proof of Corollary 4.4. It is easy to deduce that there exist $A_1 \in \mathbb{R}$, and $A_2^{\pm}$, $B_i > 0$, $i = 1, 2$ such that

$$A_1 + A_2^+(u^+)^{\beta} - A_2^-(u^-)^{\gamma} \leq G(x,u) \leq B_1 + B_2|u|^{\alpha};$$

and if $\gamma = \beta$,

$$\frac{A_2^-}{A_2^+}\int_{\{e_m<0\}} |e_m|^{\beta}\,dx < \int_{\{e_m>0\}} e_m^{\beta}\,dx < \frac{A_2^+}{A_2^-}\int_{\{e_m<0\}} |e_m|^{\beta}\,dx.$$

Thus, $(H3)$ is satisfied.

On the other hand, by Lemma 4.5, we get that for every $\varepsilon > 0$ there exists k such that for every u, $y \in \mathbb{R}$, and every $x \in \Omega$,

$$\begin{aligned}
&G(x,u) + g(x,u)(u+y)\\
&\quad\geq A_2^+(u^+)^{\beta} - A_2^-(u^-)^{\gamma} + a_2^+(u^+)^{\beta-1}(u+y)^+ - b_2^+(u^+)^{\alpha-1}(u+y)^-\\
&\qquad + b_2^-(u^-)^{\alpha-1}(u+y)^+ - a_2^-(u^-)^{\gamma-1}(u+y)^- - k_0 - k_1|u| - k_2|y|\\
&\quad\geq A_2^+(u^+)^{\beta} - A_2^-(u^-)^{\gamma} + a_2^+(u^+)^{\beta-1}(u+y)^+ - a_2^-(u^-)^{\gamma-1}(u+y)^-\\
&\qquad - k_0 - k_1|u| - k_3|y|^{\alpha}\\
&\quad\geq A_2^+(u^+)^{\beta} - A_2^-(u^-)^{\gamma} + (a_2^+ - \varepsilon)(u^+)^{\beta} - (a_2^- + \varepsilon)(u^-)^{\gamma} - k_4 - k_5|y|^{\alpha}\\
&\quad= (A_2^+ + a_2^+ - \varepsilon)(u^+)^{\beta} - (A_2^- + a_2^- + \varepsilon)(u^-)^{\gamma} - k(1+|y|^{\alpha}).
\end{aligned}$$

Therefore, if (i) holds, we deduce $(H4)$, while we deduce $(H4^*)$ with $q_1 = q_2 = 1$ if (ii) holds. The conclusion follows from Theorem 4.1. □

Corollary 4.6. *Assume $(H1)$, $(H2)$, and $(H5)$ with $\gamma = \beta = \alpha$, $b_2^- > 0$, and*

$$\int_{\{e_m>0\}} e_m^{\beta}\,dx > \max\left\{\frac{a_2^-}{a_2^+}, \frac{b_2^+}{b_2^-}\right\}\int_{\{e_m<0\}} |e_m|^{\beta}\,dx.$$

Then there exists $s_0 < 0$ such that for every $s < s_0$, the problem (P) has at least two solutions.

Proof. We deduce that there exist $A_1, B_1 \in \mathbb{R}$, and $A_2^{\pm}$, $B_2^{\pm} > 0$ such that

$$A_1 + A_2^+(u^+)^{\alpha} - A_2^-(u^-)^{\alpha} \leq G(x,u) \leq B_1 + B_2^+(u^+)^{\alpha} - B_2^-(u^-)^{\alpha};$$

and

$$\int_{\{e_m>0\}} e_m^{\beta}\,dx > \max\left\{\frac{A_2^-}{A_2^+}, \frac{B_2^+}{B_2^-}\right\}\int_{\{e_m<0\}} |e_m|^{\beta}\,dx.$$

Therefore, for every $\varepsilon > 0$, and every $q < 0$, there exist $k > 0$ such that for every u, $y \in \mathbb{R}$, and every $x \in \Omega$,

$$\begin{aligned}
&G(x,u) + g(x,u)(qu+y)\\
&\quad\geq A_2^+(u^+)^{\alpha} - A_2^-(u^-)^{\alpha} + a_2^+(u^+)^{\alpha-1}(qu+y)^+ - b_2^+(u^+)^{\alpha-1}(qu+y)^-\\
&\qquad + b_2^-(u^-)^{\alpha-1}(qu+y)^+ - a_2^-(u^-)^{\alpha-1}(qu+y)^- - k_0 - k_1|u| - k_2|y|\\
&\quad\geq A_2^+(u^+)^{\alpha} - A_2^-(u^-)^{\alpha} - b_2^+(u^+)^{\alpha-1}(qu+y)^- + b_2^-(u^-)^{\alpha-1}(qu+y)^+\\
&\qquad - k_0 - k_1|u| - k_3|y|^{\alpha}\\
&\quad\geq A_2^+(u^+)^{\alpha} - A_2^-(u^-)^{\alpha} + (b_2^+q - \varepsilon)(u^+)^{\alpha} - (b_2^-q + \varepsilon)(u^-)^{\alpha} - k(1+|y|^{\alpha})\\
&\quad= \big(A_2^+ + b_2^+q - \varepsilon\big)(u^+)^{\alpha} - \big(A_2^- + b_2^-q + \varepsilon\big)(u^-)^{\alpha} - k(1+|y|^{\alpha}).
\end{aligned}$$

So, we can choose q_1, $q_2 < 0$, and $\varepsilon > 0$ small enough such that for $j = 1, 2$,

$$(A_2^+ + b_2^+q_j - \varepsilon)\int_{\{(-1)^j e_m<0\}} |e_m|^{\alpha}\,dx > (A_2^- + b_2^-q_j + \varepsilon)\int_{\{(-1)^j e_m>0\}} |e_m|^{\alpha}\,dx.$$

Thus $(H4^*)$ is satisfied, and I satisfies $(PS)_c$ for every $c \in \mathbb{R}$ by Proposition 4.3. Theorem 4.1 gives the existence of a solution u of (P) which is such that

$$I(u) < \inf I(E_m^\perp).$$

On the other hand, fix $\varepsilon > 0$ such that

$$\int_{\{e_m<0\}} (B_2^+ + \varepsilon)|e_m|^\alpha \, dx - (B_2^- - \varepsilon) \int_{\{e_m>0\}} |e_m|^\alpha \, dx < 0.$$

Using Lemma 4.2, and arguing as in the proof of Theorem 4.1 give the existence of $\hat{k}$ such that for every $v \in E_{m-1}$, and $t \in \mathbb{R}$, we have

$$\begin{aligned}
I(v + te_m) &= \int_\Omega \frac{|\nabla(v + te_m)|^2}{2} - \lambda_m \frac{(v + te_m)^2}{2} + G(x, v + te_m) + ste_m^2 \, dx \\
&\le \left(1 - \frac{\lambda_m}{\lambda_{m-1}}\right) \frac{\|v\|^2}{2} + st\|e_m\|_{L^2}^2 + \hat{k}_0 \\
&\quad + \int_\Omega B_2^+((v + te_m)^+)^\alpha - B_2^-((v + te_m)^-)^\alpha \, dx \\
&\le \left(1 - \frac{\lambda_m}{\lambda_{m-1}}\right) \frac{\|v\|^2}{2} + \hat{k}(1 + \|v\|^\alpha) + st\|e_m\|_{L^2}^2 \\
&\quad + \int_\Omega (B_2^+ + \varepsilon)((te_m)^+)^\alpha - (B_2^- - \varepsilon)((te_m)^-)^\alpha \, dx \\
&= \hat{K}_1(\|v\|) + \hat{h}_s(t) \\
&\le \hat{K}_2 + \hat{h}_s(t),
\end{aligned}$$

where

$$\hat{K}_1(r) = \left(1 - \frac{\lambda_m}{\lambda_{m-1}}\right) \frac{r^2}{2} + \hat{k}(1 + |r|^\alpha) \le \hat{K}_2 \quad \text{for every } r \in \mathbb{R},$$

and

$$\hat{h}_s(t) = st\|e_m\|_{L^2}^2 + \int_\Omega (B_2^+ + \varepsilon)((te_m)^+)^\alpha - (B_2^- - \varepsilon)((te_m)^-)^\alpha \, dx.$$

We can find $s_0 < 0$ such that for every $s < s_0$, there exist $t_s^1 < 0 < t_s^2$ such that $\hat{K}_2 + \hat{h}_s(t_s^i) < K_0$, $i = 1, 2$, where K_0 is given in (4.5). Fix $s < s_0$. We can choose $R > 0$ such that

$$\hat{K}_1(R) + \hat{h}_s(t) < K_0 \le \inf I(E_m^\perp) \quad \text{for every } t \in [t_s^1, t_s^2].$$

Therefore,

$$\sup I(\partial_{E_m}([t_s^1, t_s^2]e_m + \overline{B_{E_{m-1}}(0, R)}\,) \le \inf I(E_m^\perp) \le \sup I([t_s^1, t_s^2]e_m + \overline{B_{E_{m-1}}(0, R)}\,).$$

Theorem 3.1 implies the existence of a solution $\tilde{u}$ of (P) such that

$$I(\tilde{u}) \ge \inf I(E_m^\perp),$$

since $\big([t_s^1, t_s^2]e_m + \overline{B_{E_{m-1}}(0, R)}, \partial_{E_m}([t_s^1, t_s^2]e_m + \overline{B_{E_{m-1}}(0, R)}\,)\big)$ links $(E_m^\perp, \emptyset)$. $\square$

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