Multidimensional Dyadic Iterative Interpolation and Fourier Multipliers on Lebesgue Spaces*

Lubomir T. Dechevsky† Serge Dubuc‡

CRM-2549

June 1998

*The two authors were supported in part by a research grant of the Natural Sciences and Engineering Research Council of Canada.

†Département de mathématiques et de statistique, Université de Montréal, C.P. 6128, Succ. Centre-ville, Montréal, QC H3C 3J7, Canada; dechevsk@DMS.UMontreal.CA

‡Département de mathématiques et de statistique, Université de Montréal, C.P. 6128, Succ. Centre-ville, Montréal, QC H3C 3J7, Canada; dubucs@DMS.UMontreal.CA

Abstract

Iterative interpolation processes are studied by applying the theory of Fourier multipliers on Lebesgue spaces; the main results are formulated in terms of Besov spaces. In the case of interpolatory refinement schemes, the Besov regularity of solutions of relevant functional equations is found. The regularity of the fundamental interpolant is being studied under minimal assumptions about the characteristic polynomial of the interpolation process; in particular, the authors study iterative interpolation processes preserving positivity. Error analysis of iterative interpolation and some aspects of factorization are also discussed.

Mathematics Subject Classification: 41A05, 42A45, 42B15, 46E30, 65D05

Keywords and Phrases: Multipliers, interpolation, Besov spaces

Résumé

On fait l'étude des procédés d'interpolation itérative à l'aide de la théorie des multiplicateurs de Fourier pour des espaces de Lebesgue ; les principaux résultats s'expriment dans le cadre des espaces de Besov. Dans le cas des schémas de subdivision interpolatoires, les auteurs déterminent la régularité de Besov des solutions des équations fonctionnelles associés. La régularité de l'interpolante fondamentale est étudiée sous des hypothèses minimales pour le polynôme caractéristique du procédé d'interpolation ; en particulier les auteurs étudient les procédés d'interpolation itérative qui préservent la positivité. On discute aussi de l'analyse de l'erreur de l'interpolation itérative et de quelques aspects de la factorisation.

Introduction

We suggest a new approach to studying iterative interpolation processes "from the Fourier side". The main new tool is the theory of Fourier multipliers on Lebesgue spaces; the main results are formulated in terms of regularity in Besov spaces.

Considerations involving Fourier multipliers on L_p have been essential part of techniques for obtaining highly refined results in the study of stability, apriori estimates and error analysis of numerical methods for solving individual ODE and PDE and systems of ODE and PDE ever since the 1960s. In this paper, our aim is to systematically adapt and modify elements of this theory, so as to make it a powerful tool in the study of iterative interpolation, refinement schemes, and regularity of solutions of relevant functional equations. The present article deals with stationary dyadic refinement schemes for multivariate functions only. However, it should be considered just as a first step into developing a general theory for refinement schemes and iterative interpolation of *multivariate vector-fields*, stationary or non-stationary, the refinement being with respect to general contracting linear transformation(s).

Section 2 contains the main results: in 2.1 the regularity of the fundamental interpolant is being studied under minimal assumptions about the characteristic polynomial of the interpolation process; 2.2 is dedicated to the study of iterative interpolation processes preserving positivity; 2.3 deals with error analysis of iterative interpolation; in 2.4 some aspects of factorization are discussed. In section 3, we consider some topics for further research. All proofs are in section 4. Subsection 1.1 includes basic notations and is expected to be used only as a reference source while reading the rest of the manuscript.

Taking in consideration that, to our best knowledge, Fourier multipliers on L_p and Besov spaces with negative smoothness indices are relatively new tools in the study of refinement schemes, iterative interpolation and related functional equations, we would have liked, for the readers's convenience, to include a sufficiently self-consistent survey of the relevant basic results in subsection 1.2. Unfortunately, such a survey turned out to be too spacious. In its present form, 1.2 is comprised of short comments on the varied types of basic results to be utilized in the sequel, accompanied by detailed references selected from as few sources as was possible. We hope that, so organized, subsection 1.2 is reasonably concise yet sufficiently helpful for the reader's orientation.

1 Preliminaries

1.1 Basic notations

$A \hookrightarrow B$ ($B \hookleftarrow A$): A, B are normed spaces and A is continuously embedded in B, i.e., $A \subset B$ and $\exists c \in (0, \infty) : ||a||_B \leq c||a||_A, \forall a \in A$.

$A \rightleftharpoons B$: A is isomorphic to B, i.e., $A \hookrightarrow B$ and $A \hookleftarrow B$. (If the constants in the two embeddings are both equal to 1, A will be called isometric to B.)

$|| \cdot ||_A \sim || \cdot ||_B$: equivalence between norms (and, more generally, non-negative quantities). In the case $A \rightleftharpoons B$ we may also assume that on A two equivalent norms are given.

Let A be a set in a topological space. $\bar{A}$: completion of A; $\overset{\circ}{A}$: open interior of A; $X \subset\subset A (A \supset\supset X)$: $X \subset \overset{\circ}{A}$ and X is compact.

Throughout the sequel $d \in \mathbb{N}$ will denote dimension, in particular, of $\mathbb{Z}^{\mathrm{d}}$, $\mathbb{R}^{\mathrm{d}}$, $\mathbb{C}^{\mathrm{d}}$ and $[-\pi, \pi]^d$.

For $\lambda \in \mathbb{C}$, $\lambda\mathbb{Z}^{\mathrm{d}} := \{(\lambda\nu_1, \cdots, , \lambda\nu_{\mathrm{d}}) : (\nu_1, \cdots, \nu_{\mathrm{d}}) \in \mathbb{Z}^{\mathrm{d}}\}$.

$\mathbb{Z}^{\mathrm{d}}_+ := \{(\nu_1, \cdots, \nu_{\mathrm{d}}) \in \mathbb{Z}^{\mathrm{d}} : \nu_{\mathrm{k}} \geq 0, \mathrm{k} = 1, \cdots, \mathrm{d}\}$: the positive cone in $\mathbb{Z}^{\mathrm{d}}$.

$\langle z_1, z_2 \rangle, z_1, z_2 \in \mathbb{C}^{\mathrm{d}}$: the usual scalar product in $\mathbb{C}^{\mathrm{d}}$; the same notation is preserved for its restriction on $\mathbb{R}^{\mathrm{d}}$. $|\cdot|$: the respective Hilbert ℓ_2-norm in $\mathbb{C}^{\mathrm{d}}$, $\mathbb{R}^{\mathrm{d}}$ and $\mathbb{Z}^{\mathrm{d}}$. $|\cdot|_1$: the ℓ_1-norm $|\alpha|_1 := \sum_{\nu=1}^{d} |\alpha_\nu|$ on $\mathbb{Z}^{\mathrm{d}}$ (the multiindex norm).

$D^\alpha := \dfrac{\partial^{|\alpha|_1}}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}$, $\alpha \in \mathbb{Z}_+^{\mathrm{d}}$.

Let $\Omega \subset \mathbb{R}^{\mathrm{d}}$; $C(\Omega) = C^0(\Omega) := \{f : \Omega \to \mathbb{C}, f$ - continuous on $\Omega\}$.

Let $\Omega \subset \mathbb{R}^{\mathrm{d}}$ be open or closed with non-void interior;

$$C_0^0(\Omega) := \{f \in C^0(\Omega) : f(x) \to 0, x \to x_1 \in \partial\Omega\},$$

where $\partial\Omega = \bar{\Omega} \setminus \overset{\circ}{\Omega}$ is the boundary of Ω.

Let $\Omega \subset \mathbb{R}^{\mathrm{d}}$ be open; $C^\infty(\Omega) := \{f \in C^0(\Omega) : D^\alpha f \in C^0(\Omega), \forall \alpha \in \mathbb{Z}_+^{\mathrm{d}}\}$.

Let $\Omega \subset \mathbb{R}^{\mathrm{d}}$ be closed; $C^\infty(\Omega) := \{f = f_1|_\Omega : f_1 \in C^\infty(\Omega_1), \Omega_1\text{-open}, \Omega_1 \supset \Omega\}$.

Let $\Omega \subset \mathbb{R}^{\mathrm{d}}$ be open or closed with non-void interior; $C_0^\infty(\Omega) := \{f \in C^\infty(\Omega) : \operatorname{supp} f \subset\subset \Omega\}$, $\operatorname{supp} f$: the support of f.

$\mathcal{D} = \mathcal{D}(\Omega)$: $C_0^\infty(\Omega)$ endowed with topology as in [10], Ch.I, §1.2.

$\mathcal{D}' = \mathcal{D}'(\Omega)$: the dual space of all continuous functionals on $\mathcal{D}(\Omega)$.

$\mathcal{S} = \mathcal{S}(\mathbb{R}^{\mathrm{d}})$: L. Schwartz space ([1], 6.1, p.134; [7], 1.5, p.31; [8], V.3; [10], Ch.I, §5.1).

$\mathcal{S}' = \mathcal{S}'(\mathbb{R}^{\mathrm{d}})$(the space of all moderate distributions): the dual of $\mathcal{S}(\mathbb{R}^{\mathrm{d}})$([1], 6.1, p.134; [7], 1.5, p.34; [8], V.3; [10], Ch.I, §5.2).

Let $\Omega \subset \mathbb{R}^{\mathrm{d}}$ be open or closed with non-void interior;

$$L_{1,\mathrm{loc}}(\Omega) := \{f \in \mathcal{D}'(\Omega) : f|_{\Omega_1} \in L_1(\Omega_1), \forall \Omega_1 \subset\subset \Omega\};$$

$f \in L_{1,\mathrm{loc}}(\Omega)$: a regular distribution in $\mathcal{D}'(\Omega)$; $f \in \mathcal{D}'(\Omega) \setminus L_{1,\mathrm{loc}}(\Omega)$: a singular distribution in $\mathcal{D}'(\Omega)$; in the sequel we shall be concerned with regular and singular distributions which are in $\mathcal{S}'(\mathbb{R}^{\mathrm{d}})$.

δ: the delta-function on $\mathbb{R}^{\mathrm{d}}$, concentrated at the origin.

For $f, g \in \mathcal{S}'(\mathbb{R}^{\mathrm{d}})$, $f * g$ will denote their convolution, if it exists in $\mathcal{D}'(\mathbb{R}^{\mathrm{d}})$ ([10], Ch.I, §4.1.(1.3-6)); in particular, if $f, g \in L_1(\mathbb{R}^{\mathrm{d}})$, then $f * g \in L_1(\mathbb{R}^{\mathrm{d}})$ and

$$f * g(x) = \int_{\mathbb{R}^{\mathrm{d}}} f(x-y)g(y)dy = \int_{\mathbb{R}^{\mathrm{d}}} f(y)g(x-y)dy,$$

(Lebesgue-)a.e. $x \in \mathbb{R}^{\mathrm{d}}$.

Let $f \in \mathcal{S}'(\mathbb{R}^{\mathrm{d}})$. $\mathcal{F}f = \hat{f}$: the Fourier transform of f; $\mathcal{F}^{-1}f = \check{f}$: the inverse Fourier transform of f; in particular, if $f \in L_1(\mathbb{R}^{\mathrm{d}})$:

$$\hat{f}(\xi) = \int_{\mathbb{R}^{\mathrm{d}}} f(x)e^{-i<x,\xi>}dx,$$

a.e. $\xi \in \mathbb{R}^{\mathrm{d}}$;

$$\check{f}(\xi) = (2\pi)^{-d}\int_{\mathbb{R}^{\mathrm{d}}} f(\xi)e^{i<x,\xi>}dx,$$

a.e. $x \in \mathbb{R}^{\mathrm{d}}$.

Let $\sigma(\xi)$, $\xi \in \mathbb{R}^{\mathrm{d}}$, be an algebraic polynomial, i.e., $\sigma(\xi) = \sum_{\alpha\in\mathbb{Z}_+^{\mathrm{d}}} c_\alpha \xi^\alpha$, where $\xi^\alpha := \xi_1^{\alpha_1}\cdots\xi_d^{\alpha_d}$, $\xi \in \mathbb{R}^{\mathrm{d}}$, and $\operatorname{supp} c_\alpha := \{\alpha \in \mathbb{Z}_+^{\mathrm{d}} : \mathrm{c}_\alpha \neq 0\}$ is finite; the (total) degree of σ is

$$\deg\sigma := \min\{m \in \mathbb{N}\bigcup\{0\} : \operatorname{supp} \mathrm{c}_\alpha \subset \{\alpha \in \mathbb{Z}_+^{\mathrm{d}} : |\alpha|_1 \leq \mathrm{m}\}\}.$$

Let $f(\xi) = f(\xi_1, \cdots, \xi_d)$, $\xi \in \mathbb{R}^{\mathrm{d}}$, be periodic in ξ_ν with period $T_\nu \in (0, \infty)$, $\nu = 1, \cdots, d$; f will be called $T_1 \cdots T_d$-periodic for short ($(2\pi)^d$-periodic, if $T_1 = \cdots = T_d = 2\pi$). Let $P(\xi)$, $\xi \in \mathbb{R}^{\mathrm{d}}$, be a trigonometric

polynomial of the form $P(\xi) = \sum_{\alpha\in\mathbb{Z}^{\mathrm{d}}} c_\alpha e^{i<\alpha,\xi>}$, where $\operatorname{supp} c_\alpha := \{\alpha \in \mathbb{Z}^{\mathrm{d}} : \mathrm{c}_\alpha \neq 0\}$ is finite; clearly P is a $(2\pi)^d$-periodic function in $C^\infty(\mathbb{R}^{\mathrm{d}})$; the (total) degree of P is

$$\deg P := \min\{m \in \mathbb{N} : \operatorname{supp} \mathrm{c}_\alpha \subset \{\alpha \in \mathbb{Z}^{\mathrm{d}} : |\alpha|_1 \leq \mathrm{m}\}\}.$$

1.2 Basic theory

1.2.1 Iterative interpolation.

We consider iterative interpolation processes as defined in [6], section 1. In that section's notations, we shall be considering the partial case when the subgroup G of $\mathbb{R}^{\mathrm{d}}$ is $\mathbb{Z}^{\mathrm{d}}$ and the linear bijective transformation $T \in L(\mathbb{R}^{\mathrm{d}})$ is $T = \frac{1}{2}\mathrm{Id}_{\mathbb{R}^{\mathrm{d}}}$, $\Delta = \det(T) = 2^{-d}$. ($L(X)$ is the space of all linear operators on the normed space X, bounded in the uniform operator topology; Id_X is the identity on X.) We note that all statements in section 2 have their generalizations for an arbitrary closed discrete subgroup of $\mathbb{R}^{\mathrm{d}}$, such that G_∞ (defined in [6], section 1) is dense on $\mathbb{R}^{\mathrm{d}}$ and the spectral radius of T is less than 1. However, the general case is more technically involved and we intend to consider it in a separate study later on.

The characteristic polynomial of the interpolation process is (cf.[6], section 3):

$$P(\xi) = \sum_{\alpha\in\mathbb{Z}^{\mathrm{d}}} c_\alpha e^{i<\xi,\alpha>}, \xi \in \mathbb{R}^{\mathrm{d}}, \tag{1}$$

where $c_\alpha := w(\alpha/2)$, w being the weight function defined in [6], section 1.

All the results in this study are valid if w is finite, i.e., if there exists R: $0 < R < \infty$ such that $\{x \in \mathbb{Z}^{\mathrm{d}}/2 : \mathrm{w(x)} \neq 0\} \subset \{\mathrm{x} \in \mathbb{R}^{\mathrm{d}} : |x| \leq R\}$. (In the sequel this case is denoted by $R < \infty$ for short.) However, part of the results are valid when w is not necessarily finite and is such that $P(\xi)$ defined by (1) is a $(2\pi)^d$-periodic function in $C^\infty(\mathbb{R}^{\mathrm{d}})$. (We shall be denoting this more general case by $R \leq \infty$ for short.) It is easy to see that the minimal R for which $\{|x| \leq R\}$ contains $\operatorname{supp} w$ and $\deg P$ are respectively the ℓ_2- and ℓ_1-norm of a certain multiindex α and, therefore, $R \sim \deg P$, $\forall P$ satisfying (1) for some $\{c_\alpha\}$, with constants of equivalence equal to the equivalence constants between the ℓ_2- and ℓ_1-norm in d-dimensional space. We also notice that (cf., e.g., [2], p.19) $P \in C^\infty(\mathbb{R}^{\mathrm{d}})$ and P-$(2\pi)^d$-periodic together imply $\sum_{\alpha\in\mathbb{Z}^{\mathrm{d}}} |c_\alpha| < \infty$.

The characteristic function (of the fundamental interpolant-defined below) of the interpolation process satisfies the functional equation (cf.[6], (3.1)):

$$\Phi(\xi) = 2^{-d}\cdot P(-\xi/2)\cdot \Phi(\xi/2), \xi \in \mathbb{R}^{\mathrm{d}}, \Phi(0) = 1, \tag{2}$$

with unique solution

$$\Phi(\xi) = \prod_{k=1}^{\infty} Q(2^{-k}\xi), \xi \in \mathbb{R}^{\mathrm{d}}, \tag{3}$$

where the infinite product is in the sense of $\lim_{n\to\infty} \Phi_n(\xi)$, with

$$\Phi_n(\xi) = \prod_{k=1}^{n} Q(2^{-k}\xi), n \in \mathbb{N}, \tag{4}$$

$$Q(\xi) = 2^{-d}P(-\xi), \tag{5}$$

and $\lim_{n\to\infty}$ is interpreted in the sense of uniform convergence on compact subsets of $\mathbb{R}^{\mathrm{d}}$. In the sequel we shall be making frequent use of the notations $Q(\xi)$ and $\Phi_n(\xi)$.

The first part of the proof of Theorem 3.1 in [6] for $R < \infty$, and its generalization for $R \leq \infty$ (see Theorem 2.1.2) shows that for $R \leq \infty$ and $Q(0) = 1$, Φ is a continuous function of $\xi \in \mathbb{R}^{\mathrm{d}}$ with polynomial

growth as $|\xi| \to \infty$. In particular, $\Phi \in \mathcal{S}'$. If, moreover, $R < \infty$ and $\Phi = \Phi(\zeta)$, $\zeta \in \mathbb{C}^{\mathrm{d}}$, satisfies (2) for any $\zeta \in \mathbb{C}^{\mathrm{d}}$, where $P(\zeta)$, $\zeta \in \mathbb{C}^{\mathrm{d}}$, is the analytic extension of $P(\xi)$, $\xi \in \mathbb{R}^{\mathrm{d}}$, onto $\mathbb{C}^{\mathrm{d}}$, then (3-5) are true for any $\zeta \in \mathbb{C}^{\mathrm{d}}$and $\Phi(\zeta)$ is an entire function of exponential type with polynomial growth as $|\zeta| \to \infty$.

The (generalized) fundamental interpolant of the interpolation process can be defined as $F = \check{\Phi}$. Since $\Phi \in \mathcal{S}'$, $\mathcal{F}^{-1}(\mathcal{S}') = \mathcal{S}'$ implies $F \in \mathcal{S}'$(cf. also [4], Theorem 6.1,2). In the case $R < \infty$, by the Paley-Wiener-Schwartz' theorem (see [9], Theorem IX.12 and 1.2.2 below) the support of $F \in \mathcal{S}'$ is compact (and contained in $\{x \in \mathbb{R}^{\mathrm{d}} : |x| \leq R\}$). (We shall also be using the notation $F_{(n)}$: $F_{(n)} = \mathcal{F}^{-1}(\Phi_n) \in \mathcal{S}'$, $n \in \mathbb{N}$.) In the applications, however, $F_1(x)$ is being defined (cf. [4-6]) on $G_\infty := \{t = (t_1, \cdots, t_d) \in \mathbb{R}^{\mathrm{d}}$, $t_l = m_l 2^{-n_l}, m_l \in \mathbb{Z}$, $n_l \in \mathbb{N} \bigcup \{0\}, \mathrm{l} = 1, \cdots, \mathrm{d}\}$ as the unique solution of the functional equation

$$F_1(x) = \sum_{\alpha \in \mathbb{Z}^{\mathrm{d}}} c_\alpha F_1(2x - \alpha), x \in G_\infty. \tag{6}$$

Because G_∞ is dense on $\mathbb{R}^{\mathrm{d}}$, one can consider the completion $\bar{F}_1(x)$, $x \in \mathbb{R}^{\mathrm{d}}$, of F_1. In general, $\bar{F}_1(x)$ would be a *multivalued* mapping, not necessarily a function. We notice that any distribution in $\mathcal{S}'$ can also be represented pointwise as a multivalued mapping (in Section 4 we give a precise formulation of this representation). While F_1 is the mapping of interest with respect to applications, the importance of F lies in the fact that it can be studied via Fourier methods and that, under fairly general assumptions about P, one can assume $F = \bar{F}_1$ in a certain meaningful sense, not only weak, but also pointwise. Here we specify the meaning of $F = \bar{F}_1$ in the case when $\bar{F}_1$ is a continuous function.

Proposition 1.2.1 *(i)* $\bar{F}_1 \in C^0(\mathbb{R}^{\mathrm{d}}) \iff F_1 \in C^0(G_\infty)$*;*
(ii) if $R < \infty$ *and* $\bar{F}_1 \in C^0(\mathbb{R}^{\mathrm{d}})$*, then*

$$F \in \mathcal{S}' \textit{ is regular } (\textit{even Riemann} - \textit{integrable}), \tag{7}$$

and

$$F \equiv \bar{F}_1, \tag{8}$$

(Lebesgue-)a.e. on $\mathbb{R}^{\mathrm{d}}$*.*

In other words, F corresponds to the class of equivalence modulo a set of zero d-dimensional Lebesgue measure, in which class there is a unique continuous representative, and this continuous representative is exactly $\bar{F}_1$.

For a detailed study of the case $\bar{F}_1 \in C^0$, we refer to [4-6].

In this paper, we shall be studying regularity properties of F depending on analytic properties of P (not on algebraic ones, i.e., no assumptions about the manifold $\{\xi \in \mathbb{R}^{\mathrm{d}}$: $P(\xi) = 0\}$ will be imposed).

1.2.2 The range of the Fourier transform on $\mathcal{S}'(\mathbb{R}^{\mathrm{d}})$.

Some familiar classical identities and inequalities will be useful in the sequel: Plancherel's theorem (e.g., [9], Theorem IX.6), Riemann-Lebesgue's lemma ([9], Theorem IX.7) and Hausdorff-Young's inequality ([9], Theorem IX.8; [1], 1.2.1). In connection with Riemann-Lebesgue's lemma, we recall that

$$C_0^0(\mathbb{R}^{\mathrm{d}}) \setminus \mathcal{F}(\mathrm{L}_1(\mathbb{R}^{\mathrm{d}})) \neq \emptyset$$

(cf., e.g., [9], Problem 16 after Ch.IX).

Next we consider the cone of positive definite distributions and its Fourier image. Positive definite (bounded continuous) functions on $\mathbb{R}^{\mathrm{d}}$ are defined in [9], IX.2 (see also [10], Ch.II, §8.1). Some of their intrinsic properties ([9], IX. 2.(1-3); [10], Ch.II, §8.1.(1.1)) will be used later in the text. Their relationship with bounded positive measures on $\mathbb{R}^{\mathrm{d}}$ is established by Bochner's theorem ([9], Theorem IX.9; [10]. Ch.II, §8.2, Corollary 2). We need also to consider the more general notion of positive definite distributions in

$\mathcal{D}'(\mathbb{R}^d)$([9], IX.2; [10], Ch.II, §8.1, (1.3)). They form a cone $K_+(\mathcal{S}'(\mathbb{R}^d)) \subset \mathcal{D}'(\mathbb{R}^d)$. In the sequel, we shall be interested in $K_+(\mathcal{S}'(\mathbb{R}^d)) := \mathcal{S}'(\mathbb{R}^d) \bigcap \mathrm{K}_+(\mathcal{D}'(\mathbb{R}^d)$. Bochner's theorem admits a generalization for $K_+(\mathcal{D}'(\mathbb{R}^d))$: the Bochner-Schwartz' theorem ([9], Theorem IX.10; [10], Ch.II, §8.2). This theorem implies, in particular, that $K_+(\mathcal{D}'(\mathbb{R}^d)) \subset \mathcal{S}'(\mathbb{R}^d)$, i.e.,

$$K_+(\mathcal{S}'(\mathbb{R}^d)) = K_+(\mathcal{D}'(\mathbb{R}^d))$$

(see also [10], Ch.I, §5.3 for the definition and properties of non-negative measures with polynomial growth at infinity). Moreover (see the proposition at the end of IX.2 in [9]), $\Phi \in K_+(\mathcal{S}'(\mathbb{R}^d))$ is *regular* and *bounded* if and only if (iff) Φ is equal to a "classical" positive definite function Lebesgue-a.e. on $\mathbb{R}^d$.

Next, periodic distributions in $\mathcal{D}'(\mathbb{R}^d)$ are considered. Their properties are studied in [10], Ch.II, §7. With every such distribution P a unique formal Fourier series can be associated - see [10], Ch.II, §7.2, (2.1). The theorem in the same paragraph of [10] shows that every periodic distribution $P \in \mathcal{D}'(\mathbb{R}^d)$ is in fact in $\mathcal{S}'(\mathbb{R}^d)$ and its formal Fourier series converges to it in the topology of $\mathcal{S}'$([10], Ch.II, §7.2, (2.4)). If, moreover, the distribution is regular, then ([10], Ch.II, §7.2, Example 1) its formal Fourier series (i.e., with Fourier coefficients given by (2.1) in §7.2, Ch.II of [10]) turns into the classical Fourier series (i.e., with Fourier coefficients given by (1.6) in §7.1, Ch.II of [10]). For the purposes of the present paper it will be enough to notice the following three facts only:

(i) the periodic distribution $P \in \mathcal{S}'(\mathbb{R}^d)$ is in $K_+(\mathcal{S}'(\mathbb{R}^d))$ iff $c_\alpha \geq 0$, $\alpha \in \mathbb{Z}^d$, ([10], Ch.II, §8.3 g), where c_α are the Fourier coefficients in its formal Fourier series;

(ii) if the periodic distribution P is a function in $C^\infty(\mathbb{R}^d)$, then its Fourier series is uniformly convergent to $P(\xi)$, $\forall \xi \in \mathbb{R}^d$, and $\sum_{\alpha \in \mathbb{Z}^d} |c_\alpha| < \infty$ ([2], Section 1.4, p.19);

(iii) under the conditions of (ii), P is a bounded function on $\mathbb{R}^d$ and so is any of its derivatives of any order (cf.,e.g., [2], the proof of Theorem 6.2.1, p.136).

The following consideration of entire functions of exponential type is included here only in reference to our Theorem 2.1.4 which we formulate for $R < \infty$. A modification of the same theorem is in fact true also in the case $R \leq \infty$, but we need a special lemma (Lemma 2.1.1) to estimate the growth with n of the derivatives of $\Phi_n(2^n\xi)$, $\xi \in \mathbb{R}^d$, while in the most important case $R < \infty$ this lemma is a straightforward consequence of Bernstein's inequality for entire functions of exponential type. We shall say that $G \in \mathcal{S}'$ is an entire function of exponential type iff

$$\operatorname{supp} \hat{G} \bigcup \operatorname{supp} \check{G} \subset \{x \in \mathbb{R}^d : |x| \leq R\}$$

for some $R < \infty$. The Paley-Wiener-Schwartz' theorem ([9], IX.12; cf. also [10], Ch.II, §12.3, (3.1) and [7], 3.1.5) establishes the equivalence of the above definition with the usual definition of G as the restriction on $\mathbb{R}^d$ of an entire function of $\mathbb{C}^d$ satisfying the bound in [9], (IX.9) in the formulation of Theorem IX.12. The version of Bernstein's inequality which we shall use in the proof of Theorem 2.1.4 can be obtained, e.g., from [7], 3.2.2.(8) (for $p = \infty$ in the notations there), Paley-Wiener-Schwartz' theorem and the two-sided inclusions between the sets of entire functions of exponential "rectangular" and "spherical" type which are bounded on $\mathbb{R}^d$ - see [7], 3.2.6, p.120.

1.2.3 Multiplier theory.

$\psi \in C^\infty(\mathbb{R}^d)$ is called a *pointwise* multiplier on $\mathcal{S}(\mathbb{R}^d)$, if $\forall \varphi \in \mathcal{S}(\mathbb{R}^d) \Rightarrow \psi\varphi \in \mathcal{S}$. The linear space of all pointwise multipliers on $\mathcal{S}$ is denoted by $\theta_M = \theta_M(\mathbb{R}^d) = \theta_M^d$. In other words, θ_M is the largest subspace $\theta \subset C^\infty$ such that $\theta.\mathcal{S} = \mathcal{S}$. It is well known that θ_M consists exactly of those $\psi \in C^\infty(\mathbb{R}^d)$ which have polynomial growth at infinity ([7], 1.5; [8], V.3, Example 7; [10], Ch.I, §5.1, (1.7)). For our purposes we need to recall that $\mathcal{S}' * \mathcal{S} \subset \theta_M$([9], Theorem IX.4 a); [10], Ch.I, §5.6 c), (6.3, 3', 4)).

Definition ([1], 6.1.1; [7], 1.5.1.1; cf. [2], section 1.1, p.7). Let $1 \leq p \leq \infty$. $g \in \mathcal{S}'(\mathbb{R}^d)$ is called a *Fourier*

multiplier on $L_p(\mathbb{R}^{\mathrm{d}})$ if $\forall f \in \mathcal{S}(\mathbb{R}^{\mathrm{d}}) \Rightarrow \check{g} * f \in L_p(\mathbb{R}^{\mathrm{d}})$ and

$$||g||_{M_p(\mathbb{R}^{\mathrm{d}})} = \sup\{||\check{g} * f||_{L_p(\mathbb{R}^{\mathrm{d}})} : ||f||_{L_p(\mathbb{R}^{\mathrm{d}})} = 1, f \in \mathcal{S}(\mathbb{R}^{\mathrm{d}})\} < \infty.$$

The linear space $M_p = M_p(\mathbb{R}^{\mathrm{d}})$ of all such $g \in \mathcal{S}'(\mathbb{R}^{\mathrm{d}})$ is normed space with respect to $||.||_{M_p}$.

Since $\mathcal{S}$ is dense on L_p, $1 \le p < \infty$, the continuous linear mapping

$$\check{g}_* : f \in \mathcal{S} \longmapsto \check{g} * f \in L_p$$

can be extended to an element of $L(L_p)$ with the same norm by standard continuity/density argument. We preserve the notation also for the extended mapping. Notice that $||g||_{M_p} = ||\check{g}_*||_{L(L_p)}$. The comment after the proof of Theorem 1.2.3 in [2], p.9, explains how $\check{g}_*$ can be extended on L_p with preservation of the norm $||g||_{M_p}$ also for $p = \infty$, but in this case another argument is being used instead of density.

There follows a list of properties of M_p to be utilized later on.

(A) Properties of $||.||_{M_p}$ as a function of $p : 1 \le p \le \infty$:
(i)symmetry of $||.||_{M_p}$ with respect to dual indices([1], 6.1.2, (1); [2], Theorem 1.2.1);
(ii)logarithmic convexity of $||.||_{M_p}$ ([1], 6.1.2, (4); cf. [2], Theorem 1.2.5);
(iii) monotonicity of $||.||_{M_p}$ on $[1, 2]$ and $[2, \infty]$ ([1], 6.1.2, (5); [2], Theorem 1.2.4).

(B) $\forall p : 1 \le p \le \infty$, M_p is a Banach algebra under pointwise multiplication ([1], the comment after the proof of 6.1.2; [2], Theorem 1.2.4; [7], 1.5.1.1.(9)).

(C) For "extreme" p, $||.||_{M_p}$ can be characterized explicitly:
(i) for the maximal space in the space scale ($p = 2$) - see [1], 6.1.2, (3); [2], Theorem 1.2.2);
(ii) for the minimal space in the space scale ($p = 1, \infty$) - see [1], 6.1.2, (2); [2], Theorem 1.2.3).
Remark to (C).(ii). For the definition of bounded measure μ as a partial case of measures with polynomial growth at infinity, see [8], V.3, Example 4, $n = 0$; [10], Ch.I, §5.1, $m = 0$. In particular, μ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^{\mathrm{d}}$ iff $g \in \mathcal{F}(L_1(\mathbb{R}^{\mathrm{d}}))$, in which case $||g||_{M_1} = ||g||_{M_\infty} = V(\mu) = ||h||_{L_1}$, where $\check{g} = \mu$, $V(\mu)$ is the total variation of μ, $h \in L_1(\mathbb{R}^{\mathrm{d}})$: $d\mu(x) = h(x)dx$. By the Riemann-Lebesgue lemma, a necessary condition for the absolute continuity of μ is $g \in C_0^0(\mathbb{R}^{\mathrm{d}})$. In this case, one can also write $||g||_{M_1} = ||g||_{M_\infty} = ||\check{g}||_{L_1}$, with a small abuse of notation due to the diversity of the formulae for Fourier transforms of a function and a measure (for $\check{g} = \mu$, $\check{g} * f(x) = \mu * f(x) = \int_{\mathbb{R}^{\mathrm{d}}} f(x - y)d\mu(y)$, $f \in \mathcal{S}(\mathbb{R}^{\mathrm{d}})$).

(D) Young's inequality ([1], 1.2.2; [7], 1.3.3; [9], IX.4, Example 1; [10], Ch.I, §4.1, b)). In the notations of [9], Young's inequality is sharp for $p = 1, \infty$ only, in the sense that then $||g||_{L_1} = ||\hat{g}||_{M_1} = ||\hat{g}||_{M_\infty}$, while for $1 < p < \infty$ one only has $||\hat{g}||_{M_p} \le ||g||_{L_1}$. In particular, $\hat{\mathcal{S}} = \check{\mathcal{S}} = \mathcal{S} \subset L_1$ implies $\mathcal{S} \subset M_p$, $\forall p$: $1 \le p \le \infty$.

(E) $|| \cdot ||_{M_p}$, $1 \le p \le \infty$, is isometrically invariant with respect to linear changes of variables on $\mathbb{R}^{\mathrm{d}}$ ([1], 6.1.3; [2], Theorem 1.2.8).

(F) $|| \cdot ||_{M_p}$, $1 \le p \le \infty$, is isometrically invariant with respect to pointwise multiplication by $e^{i\langle\alpha,\xi\rangle}$, $\alpha, \xi \in \mathbb{R}^{\mathrm{d}}$(i.e., $M_p = e^{i\langle\alpha,\cdot\rangle} \cdot M_p$, and $||g||_{M_p} = ||g(\cdot)e^{i\langle\alpha,\cdot\rangle}||_{M_p}$). The proof of (F) is easy.

(G) Explicit characterization of $||g||_{M_p}$ $p = 1, \infty$, when $g \in \mathcal{S}'(\mathbb{R}^{\mathrm{d}})$ is periodic ([2], section 1.4, p.19). If, in particular, the periodic distribution g is in $C^\infty(\mathbb{R}^{\mathrm{d}})$, then, by 1.2.2.(ii) and (A).(ii), $g \in M_p$, $\forall p$: $1 \le p \le \infty$.

(H) the M_p-norm of a $(2\pi)^d$-periodic multiplier is equivalent to the M_p-norm of a "canonical" non-periodic multiplier in $C_0^\infty(\mathbb{R}^{\mathrm{d}})$ whose support contains $[-\pi, \pi]^d$ and is contained in $[-\frac{5}{4}\pi, \frac{5}{4}\pi]^d$ ([2], Theorem 1.4.1). This property can be generalized for any $g \in C^\infty(\mathbb{R}^{\mathrm{d}})$ which is periodic but not necessarily $(2\pi)^d$-periodic by first applying (E) to obtain a $(2\pi)^d$-periodic multiplier.

Let $1 \le p \le \infty$, $g_t \in M_p$, $t \in \Delta$, where Δ is a fully ordered set (discrete or not). $\{g_t\}_{t\in\Delta}$ is said to be *stable* in M_p iff $\exists C$: $0 < C < \infty$, such that $||g_t||_{M_p} \le C$, $\forall t \in \Delta$.

(I) convergence of stable sequences in M_p ([2], Theorem 1.2.6).

(J) general sufficient conditions for a sufficiently smooth multiplier g to be in M_p together with *upper* bounds for M_p:
(i) the Carlson-Beurling inequality, $1 \le p \le \infty$, ([1], 6.1.5; cf. [2], Theorem 1.3.1);
(ii) the Mihlin-Hörmander multiplier theorem, $1 < p < \infty$,([1], 6.1.6; [7], 1.5.4).
Remarks to (J). In view of (A).(i, iii), it suffices to formulate the Carlson-Beurling inequality for $p = 1, \infty$ only (as in [2]). (J) cannot be directly applied to periodic multipliers. However, it can be applied to such a multiplier indirectly, via (H), and this will be used in the sequel. We do not use explicitly the Mihlin-Hörmander theorem in this paper, but we do use one of the implications of this theorem - [7], 1.5.5, Example 2.

(K) general technique for obtaining sufficient conditions for deriving *lower* bounds for $||\cdot||_{M_p}$. This technique is based on van der Corput's lemma ([2], Lemma 1.5.1). In the proof of Theorem 2.4.1 we shall be using one of the implications of van der Corput's lemma: [2], Corollary 1.5.3.

1.2.4 Function spaces.

For the definition of $\omega_m(f;\delta)_{L_p(\mathbb{R}^d)}$ - the L_p- modulus of smoothness of order m, with step δ, $1 \le p \le \infty$, $m \in \mathbb{N}$, $\delta \ge 0$ - see [1,2,7].

Definition. (Inhomogeneous Sobolev spaces over $\mathbb{R}^d$ - cf. [1,2,7]). Let $m \in \mathbb{N}$, $1 \le p \le \infty$. The (inhomogeneous) Sobolev space

$$W_p^m = W_p^m(\mathbb{R}^d) := \{f \in L_p(\mathbb{R}^d) : ||f||_{W_p^m(\mathbb{R}^d)} < \infty\},$$

$$||f||_{W_p^m(\mathbb{R}^d)} := ||f||_{L_p(\mathbb{R}^d)} + ||f||_{\overset{\bullet}{W}{}_p^m(\mathbb{R}^d)},$$

$$||f||_{\overset{\bullet}{W}{}_p^m(\mathbb{R}^d)} := \sum_{|\alpha|_1=m} ||D^\alpha f||_{L_p(\mathbb{R}^d)}.$$

Let $s \in \mathbb{R}$, $f \in \mathcal{S}'(\mathbb{R}^d)$. The Bessel potential $\mathcal{J}^s f$ of f is defined by

$$\mathcal{J}^s f := \mathcal{F}^{-1}((1+|\cdot|^2)^{s/2}\hat{f})$$

(see [1], 6.2; [7]). We notice that $(1+|\cdot|^2)^{s/2} \in \theta_M$, $\forall s \in \mathbb{R}$, while, e.g., $1+|\cdot|^s \notin \theta_M$ for $s \ne 2k$, $\forall k \in \mathbb{N}$.

Definition. (Inhomogeneous potential spaces - see [1], 6.2.2; cf. [7]). Let $s \in \mathbb{R}$, $1 \le p \le \infty$. The (inhomogeneous) potential space

$$H_p^s = H_p^s(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) ||f||_{H_p^s(\mathbb{R}^d)} := ||\mathcal{J}^s f||_{L_p(\mathbb{R}^d)} < \infty\}.$$

For the definition and properties of Peetre's function φ, see [2], Lemma 2.1.1; [1], 6.1.7. Note that φ is not unique. Throughout the rest of these notes we shall be using the notations (cf. [1]):

$$\varphi_j : \varphi_j(x) := \mathcal{F}^{-1}[\varphi(2^{-j}\cdot)](x), x \in \mathbb{R}^d, j \in \mathbb{Z}, \tag{1}$$

$$\psi_k : \psi_k(x) := \mathcal{F}^{-1}[\sum_{j=-\infty}^{k} \varphi(2^{-j}\cdot)](x), x \in \mathbb{R}^d, k \in \mathbb{Z}, \psi := \hat{\psi}_0, \tag{2}$$

where φ is any fixed Peetre's function. Since $\hat{\varphi}_j$, $\hat{\psi}_k \in C_0^\infty(\mathbb{R}^{\rm d})$ by definition of φ, clearly $\varphi_j, \psi_k \in \mathcal{S}(\mathbb{R}^{\rm d})$ $\backslash C_0^\infty(\mathbb{R}^{\rm d})$ and admit analytic extensions $\varphi_j(\zeta)$, $\psi_k(\zeta)$, $\zeta \in \mathbb{C}^{\rm d}$, which are entire functions of exponential type.

Definition. (The scale of inhomogeneous Besov spaces in the Banach-space range - cf. [1], 6.2.2; [2], 2.2.1, (1.2,3)). Let $1 \le p \le \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$. The (inhomogeneous) Besov space

$$B^s_{pq} = B^s_{pq}(\mathbb{R}^{\rm d}) := \{f \in \mathcal{S}'(\mathbb{R}^{\rm d}) : ||f||_{\mathrm{B^s_{pq}}(\mathbb{R}^{\rm d})} < \infty\},$$

$$||f||_{B^s_{pq}(\mathbb{R}^{\rm d})} = [||\psi_0 * f||^q_{L_p(\mathbb{R}^{\rm d})} + \sum_{j=1}^{\infty}(2^{s_j}||\varphi_j * f||_{L_p(\mathbb{R}^{\rm d})})^q]^{1/q}, 1 \le q < \infty, \tag{3}$$

$$||f||_{B^s_{p\infty}(\mathbb{R}^{\rm d})} = \max\{||\psi_0 * f||_{L_p(\mathbb{R}^{\rm d})}, \sup_{j\in\mathbb{N}}\{2^{s_j}||\varphi_j * f||_{L_p(\mathbb{R}^{\rm d})}\}\}, q = \infty,$$

where ψ_0 and φ_j are defined via (2, 1), respectively.

This definition is a slight modification of the one in [1], 6.2.2. The two are obviously equivalent, but (3) is more convenient to estimate. The definition is independent on the concrete choice of φ, in the sense that for any two different Peetre's functions the respective norms in (3) are equivalent (see, e.g., [1], p.141, after 6.2.2, and also see 6.2.4.(10)).

There follows a list of properties of $\mathcal{J}^\sigma$, W^m_p, H^s_p and B^s_{pq} to be utilized later on.

(A) Lifting property of $\mathcal{J}^\sigma$ in H^s_p and B^s_{pq} - see [1], 6.2.7 and also 6.2.1.

(B) Embeddings within the same scale:
(i)embeddings between different metrics: see [7] and [1], 6.5, p.153 for Sobolev spaces; [1], 6.5.1 for potential and Besov spaces;
(ii) embeddings within the same metric: see [7] and [1], 6.2.3 for Sobolev spaces; [1], 6.2.4.(7), 6.2.3, 6.2.4,(8) and [2], Theorem 2.1 for potential and Besov spaces.

(C) Embeddings between different scales: see [1], 6.2.4.(9), 6.4.4; [7].
Remarks on embeddings. Embeddings about Sobolev spaces are rather classical: for different metrics they are based on the Sobolev embedding theorem, for the same metric - on the inequality about intermediate derivatives. The embedding $B^s_{p,\min\{p,2\}} \hookrightarrow H^s_p \hookrightarrow B^s_{p,\max\{p,2\}}$ ([1], 6.4.4) is more precise than $B^s_{p1} \hookrightarrow H^s_p \hookrightarrow B^s_{p\infty}$ ([1], 6.2.4.(9)). However, the latter is valid for broader range of p: $p \in [1, \infty]$, and thus is the only one available for $p = 1, \infty$. Moreover, if $s \in \mathbb{N}\bigcup\{0\}$, then also $B^s_{p1} \hookrightarrow W^s_p \hookrightarrow B^s_{p\infty}$, $1 \le p \le \infty$, holds true.

(D) Intersections between the three space scales:
(i) Sobolev vs. potential spaces ([1], 6.2.3; [7]): for $m \in \mathbb{N}\bigcup\{0\}$ and $1 < p < \infty$, $W^m_p \rightleftharpoons H^m_p$, for $p = 1$ or $p = \infty$ and if $m \ne 0$, W^m_p and H^m_p are essentially diverse;
(ii) Potential vs. Besov spaces: $H^s_p \rightleftharpoons B^s_{pq}$ iff $p = q = 2$ (cf. [1], 6.4.4); (iii) Sobolev vs. potential vs. Besov spaces: $W^s_p \rightleftharpoons H^s_p \rightleftharpoons B^s_{pq}$ iff $p = q = 2$ and $s \in \mathbb{N}\bigcup\{0\}$.

(E) Equivalent norm in Besov spaces via L_p-moduli of smoothness ([1], 6.2.5, with $N = 0$ in the notations there).

The next result is a kind of "mathematical folklore" - it looks familiar, but cannot be explicitly found in written form in our reference sources. Therefore, we formulate it as a separate lemma and prove it in section 4.

Lemma 1.2.4.1. *(Equivalent Besov norms with different levels of resolution of the inhomogeneous term).*

Let $1 \le p \le \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$, $k_0 \in \mathbb{Z}$. Then,

$$(3') \qquad ||f||_{k_0,p,q,s,d} := [||\psi_{k_0} * f||^q_{L_p(\mathbb{R}^d)} + \sum_{j=k_0+1}^{\infty} (2^{s_j}||\varphi_j * f||_{L_p(\mathbb{R}^d)})^q]^{1/q},$$

(with respective sup-modification for $q = \infty$) is an equivalent norm in $B^s_{pq}(\mathbb{R}^d)$).

Comparing (3, 3') shows that (3) is (3') for $k_0 = 0$. The equivalence constants between the norms in (3) and (3') depend on the concrete value of k_0.

All three space scales considered in this subsection are comprised of *inhomogeneous* spaces. For the purposes of iterative interpolation it could be also of interest to consider the *homogeneous* analogues of these spaces whose theory is quite similar, the role of the Bessel potential $\mathcal{J}^\sigma$ being taken by the Riesz potential ([1], 6.3). However, we expect the difference between the results in the inhomogeneous and homogeneous case to be essential only for $R = \infty$, when the support of the fundamental interpolant is not compact, and when the results in terms of homogeneous spaces can be expected to be the more precise ones.

2 Main results

2.1 Besov regularity of the fundamental interpolant versus M_p- stability of its characteristic function

In this subsection we shall study the regularity of the fundamental interpolant F *when there is no information about zeros of* P. The importance of this situation for applications can be illustrated by the following 1-dimensional example - cf. [3], (7.1.4):

Example 2.1.1. $\Phi_1(\xi) = (1 - e^{-i\xi})^N \cdot (i\xi)^{-N} \cdot \Phi(\xi)$, $\xi \in \mathbb{R}$, where $\Phi(\xi) = \prod_{j=1}^{\infty} P(2^{-j}\xi)$. The factorization multiplier $\lambda_N(\xi) := (1 - e^{-i\xi})^N \cdot (i\xi)^{-N}$ contains all the essential information about the zeros of Φ_1. Typically, this information comes from the requirement that $\check{\Phi}_1$ be the scaling function of a multiresolution analysis ([3], Proposition 5.3.2) and that the respective wavelet should have a certain desired number of vanishing moments. The orthogonality assumption necessarily requires the consideration of an additional mulplier $P(\xi)$ ([3], Corollary 5.5.4). In general, no information about zeros of P is available. In order to obtain upper bounds for the regularity of $\check{\Phi}_1$ (see [3], 7.1.1.), the regularity of $\check{\Phi}$ has to be estimated, and the typical result is $\check{\Phi} \in B^\sigma_{\infty\infty}$ where σ is *negative*.

Our aim in this subsection is to propose a general and refined technique for estimating the negative regularity of $\check{\Phi}$ in Besov spaces, for all $p : 1 \le p \le \infty$, in d dimensions, and potentially admitting straightforward generalization for vector-fields. As we shall see, this technique yields essentially more precise and refined results even in the 1-dimensional case.

Our first main result is a new multiplier theorem.

Theorem 2.1.1. *Let $s \in \mathbb{R}$. Then,*

(i) for $p : 1 \le p \le \infty$,

$$\Phi \in (1 + |\cdot|^2)^{-s/2} M_p \Longrightarrow F \in B^{s-d(1-1/p)}_{p\infty};$$

(ii) for $p : 1 \le p \le \infty$,

$$F \in B^s_{p1} \Longrightarrow \Phi \in (1 + |\cdot|^2)^{-s/2} M_p.$$

Remark 2.1.1. Theorem 2.1.1.(ii) provides a new technique for proving that $\Phi \in M_p$ in cases when Φ is non-smooth. In these cases the usual classical technique - via the Carlson-Beurling inequality or the Mihlin-Hörmander multiplier theorem - fails, because it presumes $\Phi \in W^\nu_2(\mathbb{R}^d)$, where $\nu \in \mathbb{N}$, $\nu > d/2$. While in the theory of difference schemes for ODE and PDE the revelant Φ does have sufficient regularity, with iterative interpolation processes Φ is continuous - but generally does not have to be smooth - on $\mathbb{R}^d$.

Proposition 2.1.1. *Let $-\infty < s_1 \le s_2 < \infty$, $1 < p < \infty$. Then*

$$(1+|\cdot|^2)^{-s_1/2}\cdot M_p \hookleftarrow (1+|\cdot|^2)^{-s_2/2}\cdot M_p.$$

The next two propositions and Theorem 2.1.2 contain basic relationships between the behaviour of $\Phi(\xi)$, $|\xi| \to \infty$, and the respective regularity of F.

Proposition 2.1.2. *(i)Let $s \in I\!R$. Then,*

$$\Phi \in (1+|\cdot|^2)^{-s/2} M_2 \ iff \ \exists C < \infty : |\Phi(\xi)| \le C(1+|\xi|^2)^{-s/2}, \forall \xi \in I\!R^{\mathrm{d}}.$$

(ii) Assume that F is a regular distribution in $\mathcal{S}'$ and $R < \infty$. Then, $\exists C < \infty$:

$$|\Phi(\xi)| \le C, \forall \xi \in I\!R^{\mathrm{d}}, \text{ and } |\Phi(\xi)| = \mathrm{o}(1), |\xi| \to \infty.$$

(iii) If $s \in I\!R$ and $F \in H^s_1$, then $\exists C < \infty$:

$$|\Phi(\xi)|(1+|\xi|^2)^{s/2} \le C, \forall \xi \in I\!R^{\mathrm{d}}, \text{ and } |\Phi(\xi)|(1+|\xi|^2)^{\mathrm{s}/2} = \mathrm{o}(1), |\xi| \to \infty.$$

(iv) If $s \in I\!R$ and $1 < p \le 2$ and $F \in H^s_p$, then

$$|\Phi(\cdot)|(1+|\cdot|^2)^{s/2} \in L_{p'}, p' : 1/p + 1/p' = 1.$$

Proposition 2.1.3. *Let $s \in I\!R$,$1 \le p \le 2$ and $|\Phi(\cdot)|(1+|\cdot|^2)^{s/2} \in L_p$. Then*

$$F \in H^s_{p'}, p' : 1/p + 1/p' = 1.$$

Remark 2.1.2. Proposition 2.1.3 is a generalization and improvement of Lemma 7.1 in [5]. Notice that $H^s_\infty \hookrightarrow B^s_{\infty\infty} = \mathrm{Lip}\ s$ for $s > 0$.

The next theorem is a generalization of Theorem 3.1 in [6].

Theorem 2.1.2. *Under the assumptions of Theorem 3.1 in [6], assume more generally that $R \le \infty$. Then, $\Phi \in \mathcal{S}'(I\!R^{\mathrm{d}})$ is a continuous function such that $\exists a \in I\!R$, $\exists C < \infty$:*

$$|\Phi(\xi)| \le C(1+|\xi|^2)^{a/2}, \xi \in I\!R^{\mathrm{d}}$$

(but in general not for $\xi \in \mathbb{C}^{\mathrm{d}}$!), where

$$a \le ||Q||_{M_2} = ||Q||_{L_\infty(I\!R^{\mathrm{d}})}.$$

Remark 2.1.3. In Theorem 3.1 in [6] the value of a is

$$a = ||Q||_{M_1} = ||Q||_{M_\infty} \ge ||Q||_{M_2} = ||Q||_{L_\infty}.$$

This theorem remains true if the smaller value $a := ||Q||_{L_\infty}$ is chosen.

Remark 2.1.4. $R \le \infty$ in Theorem 2.1.2 means that $\Phi \in \mathcal{S}'(\mathbb{R}^{\mathrm{d}})$ is a continuous function on $\mathbb{R}^{\mathrm{d}}$ which need not necessarily be an entire function of exponential type on $\mathbb{C}^{\mathrm{d}}$ (and, hence, $\operatorname{supp} F$ is not necessarily compact).

In the rest of this section, Besov regularity of F is being measured via growth of $||\Phi_n||_{M_p}$, $n \to \infty$.

Theorem 2.1.3. *Let $s \geq 0$, $1 \leq p \leq \infty$, and assume that $\exists C < \infty$: $||\Phi_n||_{M_p} \leq C2^{sn}$, $\forall n \in \mathbb{N}$.*
(i) If $R \leq \infty$ and $1 < p \leq \infty$, then

$$F \in B_{p\infty}^{-s-d(1-1/p)}. \tag{1}$$

(ii) If $R < \infty$, then (1) is valid for $p : 1 \leq p \leq \infty$.
(iii) If $R \leq \infty$ and $s = 0$, then (1) is valid for $p : 1 \leq p \leq \infty$.

Corollary 2.1.1. *Under the assumptions of Theorem 3.1 in [6] (assuming $T = \frac{1}{2}\mathrm{Id}_{\mathbb{C}^{\mathrm{d}}}$ in the notations there), for $p : 1 \leq p \leq \infty$,*

$$F \in B_{p\infty}^{-\log_2 ||Q||_{M_p} - d(1-1/p)} \hookrightarrow B_{p\infty}^{(1-2|1/2-1/p|)\log_2 ||Q||_{L_\infty} + 2|1/2-1/p| \log_2(\sum_\alpha |c_\alpha|) - d(1-1/p)}.$$

Corollary 2.1.2. *Let $p : 1 \leq p \leq \infty$, $p' : 1/p + 1/p' = 1$, $q \in [\min\{p, p'\}, \max\{p, p'\}]$. Then,*

$$F \in B_{p\infty}^{-||Q||_{M_p} - d(1-1/q)}.$$

Corollary 2.1.3. *Let $p : 1 \leq p \leq \infty$, Then,*

$$F \in B_{p\infty}^{-\log_2(\sum_\alpha |c_\alpha|) - d(1-1/p)}.$$

Theorem 2.1.4. *Let $p : 1 \leq p \leq \infty$, $R < \infty$ and assume*

$$\exists C < \infty : ||\Phi_n||_{M_2} \leq C, \forall n \in \mathbb{N}. \tag{2}$$

Then,

$$\exists C_1 < \infty : ||\Phi_n||_{M_p} \leq C_1 2^{nd|1/2-1/p|}, \forall n \in \mathbb{N}. \tag{3}$$

Corollary 2.1.4. *Assume that $|Q(\xi)| \leq 1$, $\forall \xi \in \mathbb{R}^{\mathrm{d}}$. Then,*

$$F \in B_{p\infty}^{-d(1-1/p+|1/2-1/p|)}, 1 \leq p \leq \infty.$$

Corollary 2.1.4 is in fact true for $R \leq \infty$, assuming $P \in C^\infty(\mathbb{R}^{\mathrm{d}})$. This can be shown by proving a modification of Theorem 2.1.4, when $R < \infty$ is relaxed to $R \leq \infty$, but (2) is replaced by the more stringent

$$||Q||_{M_2} \leq 1. \tag{2'}$$

The proof of this version of Theorem 2.1.4 is essentially the same, but Bernstein's inequality about entire functions of exponential type has to be replaced by a respective bound for the derivatives of $Q(\xi)$ which holds also for $R = \infty$ and C_1 in (3) can be chosen independent of R. The following lemma yields the bound in quest.

Lemma 2.1.1. *Assume $R \leq \infty$, $Q \in C^\infty(\mathbb{R}^{\mathrm{d}})$, $|Q(\xi)| \leq 1$, $\forall \xi \in \mathbb{R}^{\mathrm{d}}$. Then, $\exists C_{\alpha,Q} < \infty$:*

$$||D^\alpha[\Phi_n(2^n \cdot)]||_{L_\infty(\mathbb{R}^{\mathrm{d}})} \leq C_{\alpha,Q} 2^{n|\alpha|_1}, \forall n \in \mathbb{N}, \forall \alpha \in \mathbb{Z}_+^{\mathrm{d}}.$$

Returning to the simple Example 2.1.1, we note that, by Proposition 2.1.3, the result in Lemma 7.1.1 in [3] can be improved from $F \in B_{\infty\infty}^\alpha$ to $F \in H_\infty^\alpha$. The rest of the results in this subsection are new even in the simple context of Example 2.1.1.

2.2 Iterative interpolation preserving positivity

We begin by studying iterative interpolation processes preserving the cone of positive definite distributions in $\mathcal{S}'(\mathbb{R}^{\mathrm{d}})$.

Theorem 2.2.1. *If $P \in C^{\infty}(I\!R^{\mathrm{d}})$ is $(2\pi)^d$- periodic, $R \leq \infty$ and $P \in K_+(\mathcal{S}'(I\!R^{\mathrm{d}}))$, then*

$$2^{-d}||P||_{M_p} = ||Q||_{M_p} \equiv 1, \forall p : 1 \leq p \leq \infty.$$

Corollary 2.2.1. *Under the conditions of Theorem 2.2.1,*

$$F \in B_{p\infty}^{-d(1-1/p)}, \forall p : 1 \leq p \leq \infty.$$

Remark 2.2.1. In view of Theorem 2.2.1, $\check{Q}$ and $F = \check{\Phi}$ can be given a stochastic interpretation. (For simplicity, let us consider in detail the case $d = 1$ only.) $\check{Q}$ and $F = \check{\Phi}$ are cumulative distribution functions. Let $(\mathbb{R}, \mathcal{B}, \mathrm{Pr})$ be a probability measure space, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$, Pr is a probability measure defined on $\mathcal{B}$. Then, $P(2^{-k}\xi)$ is the characteristic function of a discretely distributed random variable η_k such that (1)

$$\mathrm{Pr}\{\eta_k = \alpha\} = c_\alpha, \alpha \in \mathbb{Z}/2^{\mathrm{k}}, \mathrm{k} = 0, 1, \cdots,$$

c_α being the coefficients in the expansion $Q(2^{-k}\xi) = P(-2^{-k}\xi)/2 = \sum_{\alpha \in 2^{-k}\mathbb{Z}} c_\alpha e^{i\alpha\xi}$. Φ_n is the characteristic function of the random variable $\sum_{k=0}^{n} \eta_k$, where η_k are independent, distributed as in (1). $\check{P}$ is understood as a linear combination of Heaviside's functions, and, therefore, $F_{(n)} = \check{\Phi}_n$ is also a linear combination of such functions. Convergence $F_{(n)} \to F$ can be interpreted as weak convergence of cumulative distribution functions. In particular, this implies that $F_{(n)}(x) \to F(x)$ pointwise, $\forall x \in \Omega \subset \mathbb{R}$, where Ω is everywhere dense on $\mathbb{R}$ and, particularly, $F_{(n)}(x) \to F(x)$ pointwise for any $x \in \mathbb{R}$ which is a continuity point of F.

So far, we have been dealing with the case $P \in K_+(\mathcal{S}'(\mathbb{R}^{\mathrm{d}}))$ which implies

$$\Phi \in K_+(\mathcal{S}'(\mathbb{R}^{\mathrm{d}})). \tag{2}$$

Now let us consider the case

$$\Phi \in \mathcal{F}(K_+(\mathcal{S}'(\mathbb{R}^{\mathrm{d}})), \tag{$\hat{2}$}$$

i.e., let us study these iterative interpolation processes which preserve the cone of positive measures with polynomial growth.

Theorem 2.2.2. *Assume that Φ satisfies 1.2.1.(2). Then,*

$$(\hat{2}) \Longleftrightarrow \Phi \in C^0(I\!R^{\mathrm{d}}) \bigcap \mathrm{L}_1(I\!R^{\mathrm{d}}) \text{ and } \Phi(\xi) \geq 0, \forall \xi \in I\!R^{\mathrm{d}}. \tag{3}$$

Corollary 2.2.2.

$$(\hat{2}) \Longrightarrow \bar{F}_1 \in C_0^0(I\!R^{\mathrm{d}}),$$

$$\bar{F}_1 \textit{ is a positive definite function on } I\!R^{\mathrm{d}} \tag{4}$$

and, in particular,

$$\bar{F}_1(0) \geq 0, \tag{5}$$

(6) $$\bar{F}_1(x) = \widetilde{\bar{F}_1(-x)}, \forall x \in I\!R^{\mathrm{d}},$$

($\tilde{z}$ denotes the complex conjugate of $z \in Z\!\!\!Z$),

(7) $$|\bar{F}_1(x)| \leq \bar{F}_1(0), \forall x \in I\!R^{\mathrm{d}},$$

where $\bar{F}_1$ is defined in 1.2.1 .

Remark 2.2.2. The results in [6], section 4, imply that, if $\Phi(\xi) \geq 0$, $\forall \xi \in \mathbb{R}^{\mathrm{d}}$, then $\bar{F}_1 \in C_0^0(\mathbb{R}^{\mathrm{d}})$. Our Theorem 2.2.2 and Corollary 2.2.2 show additionally that $\bar{F}_1$ also satisfies (4-7).

2.3 Approximation error of iterative interpolation in Besov spaces of singular distributions

Let us consider the problem of estimating $||F_{(n)} - F||$ in an appropriate norm $||\cdot||$, under natural assumptions about P. Since $F_{(n)}$ and F are in $\mathcal{S}'(\mathbb{R}^{\mathrm{d}})$, $||\cdot||_X$ would be appropriate if X is a Banach space continuously embedded in $\mathcal{S}'(\mathbb{R}^{\mathrm{d}})$ and $F_{(n)} \in X$, $n \in \mathbb{N}$, is a Cauchy sequence in X. The requirement $F_{(n)} \in X$ implies that the most common choice $X = L_p$, $1 \leq p \leq \infty$, or any other space $X \subset L_{1,\mathrm{loc}}(\mathbb{R}^{\mathrm{d}})$ is inappropriate here, because $F_{(n)}$ is a singular distribution. Now we shall find all possible choices of X within the Besov scale.

Proposition 2.3.1. *For any $p : 1 \leq p \leq \infty$, the minimal (i.e., with largest norm) Besov space containing the δ-function is $B_{p\infty}^{-d(1-1/p)}$.*

Corollary 2.3.1. *Let $R < \infty$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Then,*

$$F_{(n)} \in B_{pq}^s \; iff \; either \; s < -d(1-1/p), \; or \; s = -d(1-1/p) \; and \; q = \infty.$$

Definition 2.3.1. Let (cf. [1], 7.4.1; [2], (3.1)). Let $\mu > 0$. $P(\xi)$ is said to be accurate of order μ, if (for $Q(\xi) = 2^{-d}P(-\xi)$) $\exists \epsilon_0 > 0$, $\exists d_0 > 0$:

$$|Q(\xi)| \geq d_0 \; and \; \ln Q(\xi) = |\xi|^\mu H_0(\xi), \forall \xi \in \bar{\mathcal{U}}_{\epsilon_0} := \{\eta \in \mathbb{R}^{\mathrm{d}} : |\eta| \leq \epsilon_0\},$$

where

$$H_0 \in C^\infty(\bar{\mathcal{U}}_{\epsilon_0} \setminus \{0\}), D^\alpha H_0 \; is \; bounded \; on \; \bar{\mathcal{U}}_{\epsilon_0} \setminus \{0\}, \forall \alpha \in Z\!\!\!Z_+^{\mathrm{d}},$$

and

$$\exists h_0 > 0 : |H_0(\xi)| \geq h_0, \forall \xi \in \bar{\mathcal{U}}_{\epsilon_0} \setminus \{0\}.$$

Remark 2.3.1. In general, Q, $\ln Q$ and H_0 are $\mathbb{C}$-valued (note that $|\cdot|^\mu$ is $\mathbb{R}_+$- valued). $Q(0) = 1 \neq 0$, $Q \in C^0(\mathbb{R}^{\mathrm{d}})$, so ϵ_0 always exists.

Theorem 2.3.1. *Let $R \leq \infty$, $1 < p < \infty$, $\mu > 0$, $\sigma : 0 \leq \sigma \leq \mu$. Assume that P is accurate of order μ and $\exists C_1 < \infty$: $||\Phi_n||_{M_p} \leq C_1$, $\forall n \in I\!N$. Then, $\exists C = C(p, \sigma, \mu, \epsilon_0, C_1) < \infty$:*

(1) $$||F_{(n)} - F||_{B_{p\infty}^{-\sigma-d(1-1/p)}} \leq C2^{-\sigma n}, \forall n \in I\!N.$$

Remark 2.3.2. If $R < \infty$, Theorem 2.3.1 can be shown to hold for $p : 1 \leq p \leq \infty$.

Here is a model corollary.

Corollary 2.3.2. *If P is accurate of order $\mu > 0$ and $c_\alpha \geq 0$, $\forall \alpha \in Z\!\!\!Z_+^{\mathrm{d}}$, if $\sigma : 0 \leq \sigma \leq \mu$, and if either $R \leq \infty$, $1 < p < \infty$, or $R < \infty$, $1 \leq p \leq \infty$, then $\exists C = C(p, \sigma, \mu, \epsilon_0) < \infty$: (1) holds true.*

2.4 A note on factorization

An important question arising with iterative interpolation is for which characteristic polynomials $P(\xi)$ the corresponding fundamental interpolant $F(x)$ has high regularity. For such a "good" $P(\xi)$, given any $P_1(\xi)$ and respective $F_1(\xi)$, the product polynomial

$$P_{[n]}(\xi) := P(\xi)^n P_1(\xi), \xi \in \mathbb{R}^{\mathrm{d}}, \tag{1}$$

generates the fundamental interpolant

$$F_{[n]} := \overbrace{F * \cdots * F * F_1}^{n-\text{times}}, \tag{2}$$

whose regularity improves with the increase of n.

While the study of such "good" polynomials $P(\xi)$ has drawn much attention in the univariate case (cf. Example 2.1.1, $\lambda_N(\xi)$), less is known in the case of iterative interpolation of multivariate functions and very little in the more general case of iterative interpolation of multivariate vector fields.

In this subsection we give a precise definition of "good" polynomials P and prove some new necessary conditions which impose natural restrictions on $|P(\xi)|$ and $\arg P(\xi)$. We would like to stress that the present results, valid in the context of iterative interpolation of multivariate functions, can be extended to the more general situation of treating *multivariate vector fields*. We intend to consider this general situation in a separate study later on.

Definition 2.4.1. The trigonometric polynomial $P(\xi)$, $\xi \in \mathbb{R}^{\mathrm{d}}$, $P(0) = 2^d$, is said to be an elementary factorization multiplier, if P and the corresponding characteristic function Φ satisfy the following conditions:

(i) $\exists n \in \mathbb{N}$: $\Phi(\xi)^n$ is integrable on $\mathbb{R}^{\mathrm{d}}$;

(ii) $\forall p : 1 \le p \le \infty$, $\forall f \in L^p(\mathbb{R}^{\mathrm{d}})$, $\limsup_{n\to\infty} ||f * \mathcal{F}^{-1}(Q^n)||_{L_p(\mathbb{R}^{\mathrm{d}})} < \infty$;

(iii) $|P(\xi)|^2$ is accurate of order $\mu = 2\cdot \deg[|P(\xi)|^2]$ (in the sense of Definition 2.3.1);

(iv) $P \in K_+(\mathcal{S}'(\mathbb{R}^{\mathrm{d}}))$;

(v) P does not admit a non-trivial decomposition into a product of other trigonometric polynomials, at least one of which has properties (i-iv).

Let us discuss this definition in a sequence of remarks.

Remark 2.4.1. (i) is a (rather weak) condition ensuring that the factor P^n, $n \in \mathbb{N}$, improves the regularity of $F_{[n]}$ corresponding to $P_{[n]}(\xi)$ - see (1, 2). In particular, (i) is fulfilled if *any* of the following two conditions holds true:

(i') $\exists \alpha > 0$, $\exists C < \infty$: $|\Phi(\xi)| \le C(1 + |\xi|^2)^{-\alpha/2}$, $\forall \xi \in \mathbb{R}^{\mathrm{d}}$;

(i") $\Phi(\xi) \equiv \Phi_1(\xi) e^{i\langle\beta,\xi\rangle}$, $\xi \in \mathbb{R}^{\mathrm{d}}$,

where $\Phi_1(\xi) \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^{\mathrm{d}}$, and $\beta \in \bigcup_{\nu=0}^{\infty}(2^{-\nu}\mathbb{Z})^{\mathrm{d}}$ ($= G_\infty$ in the notations of [6]). Case (i") happens whenever $P(\xi) = P_1(\xi)e^{i\langle\beta_1,\xi\rangle}$, $\xi \in \mathbb{R}^{\mathrm{d}}$, where P_1 is real and $\beta_1 \in G_\infty$.

Remark 2.4.2. (ii) is equivalent to:

(ii') $\forall p : 1 \le p \le \infty$, $\forall f \in L^p$, $\exists C = C_p(f) < \infty$: $||f * \mathcal{F}^{-1}(Q^n)||_{L_p} \le C_p(f)$.

The meaning of (ii') is that, the multiplicity n increasing, the L_p- regularity of g_n:

$$g_n(x) := \sum_{k\in\mathbb{Z}^{\mathrm{d}}} c_k^{(n)} f(2x - k), x \in \mathbb{R}^{\mathrm{d}},$$

is preserved for any $p : 1 \leq p \leq \infty$, $f \in L_p$ and $n \in \mathbb{N}$. Here $c_k^{(n)}$ are the coefficients of the expansion

$$Q(\xi)^n = \sum_{k \in \mathbb{Z}^{\mathrm{d}}} C_k^{(n)} e^{i\langle k,\xi\rangle}, \xi \in \mathbb{R}^{\mathrm{d}}.$$

Remark 2.4.3. We note that $|P|^2 = P\tilde{P}$ is a trigonometric polynomial such that $c_{-k} \neq 0$ iff $c_k \neq 0$. Condition (iii) means that, in a neighbourhood of $\xi = 0$, $\ln[|P(\xi)|^2]$ approximates with the maximal possible order $\mu = 2\deg[|P(\xi)|^2]$. Indeed, if $\mu > 2\deg[|P(\xi)|^2]$, then $|P(\xi)|^2 \equiv 1$, $\xi \in \mathbb{R}^{\mathrm{d}}$.

Remark 2.4.4. (iv) ensures that the fundamental interpolant F corresponding to P is a positive definite distribution in $\mathcal{S}'$ and, in view of (i), $F^{[n]} := \overbrace{F * \cdots * F}^{n-\text{times}}$ is a non-negative function. Because it is also compactly supported (see Theorem 3.1 in [6]), $F^{[n]} \in L_1 \bigcap L_\infty$.

Remark 2.4.5. (v) is the condition that justifies the use of the word "elementary" in Definition 2.4.1. It is included in the definition in order to eliminate multiplicities, as well as additional factors that do not necessarily improve the regularity of the fundamental interpolant (e.g., factors ensuring other important properties like, say, orthogonality - cf. Example 2.1.1).

In the sequel of this subsection we consider several examples of $P(\xi)$ satisfying various subsets of (i-v) in Definition 2.4.1 and prove a necessary condition about $\ln[|P(\xi)|^2]$ and $\arg P(\xi)$ for condition (ii) to hold true.

Example 2.4.1. Let $d = 1$ and $P(\xi) = 1 + e^{i\xi}$, i.e., $Q(\xi) = (1 + e^{-i\xi})/2$. Then, it is well-known that (see, e.g., [3], p.211; Example 2.1.1) $\Phi(\xi) = (1 - e^{-i\xi})(i\xi)^{-1}$ and (i) holds true. Moreover, it is easy to verify that both (i') (with $\alpha = 1$) and (i") hold true; (ii) is fulfilled with $C_p(f) = ||f||_{L_p}$ (see also Remarks 2.4.2, 4); (iii) is fulfilled with $\deg[|P|^2] = 1$; (iv, v) are also fulfilled; so $P(\xi)$ is an elementary factorization multiplier.

The condition $Q(0) = 1$ for the trigonometric polynomial $Q(\xi) = |Q(\xi)|e^{i\cdot\arg Q(\xi)}$, $\xi \in \mathbb{R}^{\mathrm{d}}$, implies that there exist a neighbourhood $\mathcal{U}$ of $\xi = 0$ in $\mathbb{R}^{\mathrm{d}}$, such that

(a) $|Q(\xi)| = e^{\ln|Q(\xi)|}$, $\xi \in \mathcal{U}$;

(b) $\ln|Q(0)| = \arg Q(0) = 0$;

(c) $\ln|Q(\xi)|$ and $\arg Q(\xi)$ are analytic on $\mathcal{U}$.

Utilizing (b,c) and expanding $\ln|Q(\xi)|$ and $\arg Q(\xi)$ in local Taylor expansion yields

$$\ln[|Q(\xi)|]^2] = \varrho_\mu(\xi)(1 + o(1)), \xi \to 0; \tag{3}$$

$$\arg Q(\xi) = \langle\gamma, \xi\rangle + \sigma_\nu(\xi)(1 + o(1)), \xi \to 0; \tag{4}$$

where: $\mu, \nu \in \mathbb{N}$, $\nu \geq 2$, $\gamma \in \mathbb{R}^{\mathrm{d}}$; ϱ_μ and σ_ν are *homogeneous* algebraic polynomials of total degree μ, ν, respectively (coinciding for homogeneous polynomials with their respective order of homogeneity); $\varrho_\mu(\xi) \not\equiv 0$ (to exclude the trivial case $|Q(\xi)| \equiv 1$); $\xi \in \mathcal{U}$.

The next theorem, which is the main result in this subsection, yields a necessary condition, in terms of $\ln|P(\xi)|$ and $\arg Q(\xi)$, for $P(\xi)$ to satisfy (ii) in Definition 2.4.1.

Theorem 2.4.1. *Assume that $P(\xi)$ is such that*

(i) $\arg Q(\xi)$ *is* not *a linear function of $\xi \in \mathcal{U}$ (i.e., in (4) $\sigma_\nu|_{\mathcal{U}} \not\equiv 0$);*

(ii) $\mu > \nu$ *(i.e., in (3,4) $\varrho_\mu(\xi) = o(\sigma_\nu(\xi))$, $\xi \to 0$).*

Then, $P(\xi)$ does not satisfy Definition 2.4.1.(ii).

In other words, a necessary condition for $P(\xi)$ to satisfy Definition 2.4.1.(ii) is that at least one of the following two conditions holds true:

(i_1) $\arg Q(\xi)$ is a linear function on $\mathcal{U}$ (i.e., $\exists\gamma \in \mathbb{R}^{\mathrm{d}}$: $\arg Q(\xi) \equiv \langle\gamma, \xi\rangle$, $\forall\xi \in \mathcal{U}$);

(i_2) $\mu \leq \nu$ in (3,4).

Let us consider two examples illustrating Definition 2.4.1.(ii) and Theorem 2.4.1.

Example 2.4.2. $P(\xi) := (1-\tau)e^{-i\xi} + 1 + \tau e^{i\xi}$, $0 \leq \tau \leq 1$. In this case

$$Q(\xi) = [\cos(\xi/2) + i(1-2\tau)\sin(\xi/2)]\cos(\xi/2);$$

$$|Q(\xi)|^2 = \cos^2(\xi/2)[1 - 4\tau(1-\tau)\sin^2(\xi/2)] \geq \cos^4(\xi/2)$$

for $\tau \in [0,1]$;

$$\ln[|Q(\xi)|^2] = -(4\tau^2 - 4\tau - 1)\xi^2 + (16\tau^2 - 16\tau - 1)\xi^4/48 + O(\xi^6), \xi \to 0;$$

$$\arg Q(\xi) = \arctan[(1-2\tau)\tan(\xi/2)] = (1-2\tau)\{\xi/2 + \tau(1-\tau)[\xi^3/6 + O(\xi^5)]\}, \xi \to 0.$$

It can be seen that $\arg Q(\xi)$ is linear iff $\tau = 0, 1/2, 1$. Let $\tau \in (0,1/2)\bigcup(1/2,1)$. Then, if $\mu > \nu$ in (3,4), Theorem 2.4.1 would imply that Definition 2.4.1.(ii) is not fulfilled for P. However, in view of $\tau \in [0,1]$, $P \in K_+(\mathcal{S}'(\mathbb{R}^{\mathrm{d}}))$ and, by Theorem 2.2.1,

$$||f * \mathcal{F}^{-1}(Q^n)||_{L_p} \leq ||Q^n||_{M_p}||f||_{L_p} \leq ||Q||^n_{M_p}||f||_{L_p} = ||f||_{L_p}, \tag{5}$$

so $P(\xi)$ satisfies Definition 2.4.1.(ii) for any $\tau \in [0,1]$. This means that for $\tau \in (0,1/2)\bigcup(1/2,1)$ $\mu \leq \nu$ holds. This can be verified from the above explicit computations of the analytic expansions of $\ln[|Q(\xi)|^2]$ and $\arg Q(\xi)$ in a neighbourhood of $\xi = 0$, and it turns out that $\mu = 2 < 3 = \nu$.

For which $\tau \in [0,1]$ is $P(\xi)$ an elementary factorization multiplier? It can be verified that Definition 2.4.1.(iii) holds true for $P(\xi)$ iff $\tau = 0, 1/2, 1$. However, for $\tau = 1/2$, $P(\xi) = 2e^{i\xi}[(1+e^{-i\xi})/2]^2$, so in this case Definition 2.4.1.(v) is not fulfilled.

Thus, the only values of $\tau \in [0,1]$, for which Definition 2.4.1.(i-v) is satisfied, are 0 and 1.

Example 2.4.3. Let us now consider a simple instance for which Definition 2.4.1.(ii) fails.

$$P(\xi) = (\lambda^2 + \lambda)e^{-i\xi} + 2(1-\lambda^2) + (\lambda^2 - \lambda)e^{i\xi}, 0 < |\lambda| < 1, \lambda \in \mathbb{R}. \tag{6}$$

The corresponding $Q(\xi)$ for (6) is exactly the characteristic polynomial of the so-called Lax-Wendroff scheme applied in numerical analysis of PDE (see, e.g., [2], pp.101,114). It satisfies Hypothesis 1.3 in [6] iff $\lambda = \pm 2^{-1/2}$. For $P(\xi)$ given by (6) (see also [2], p.101),

$$Q(\xi) = 1 - 2\lambda^2\sin^2(\xi/2) + 2i\lambda\sin(\xi/2)\cdot\cos(\xi/2);$$

$$|Q(\xi)|^2 = 1 - 4\lambda^2(1-\lambda^2)\sin^4(\xi/2) \geq [1 + \cos^2(\xi/2)]\cos^2(\xi/2),$$

for $\lambda \in \mathbb{R}$, $|\lambda| \in (0,1)$;

$$\arg Q(\xi) = \arctan[\frac{\lambda\sin\xi}{(1-\lambda^2) + \lambda^2\cos\xi}];$$

$$\ln[|Q(\xi)|^2] = -2\lambda^2(1-\lambda^2)[\xi^4 + O(\xi^6)], \xi \to 0;$$

$$\arg Q(\xi) = \lambda\{\xi - (1-\lambda^2)[\xi^3/6 + O(\xi^5)]\}, \xi \to 0;$$

so $\mu = 4 > 3 = \nu$, and, by Theorem 2.4.1, Definition 2.4.1.(ii) fails for $P(\xi)$.

Remark 2.4.6. Hypothesis 1.3 in [6] reduces $P(\xi)$ in (6) to $P(\xi)$defined in Example 2.4.2, with $\tau = \tau_1$ or $\tau = \tau_2$, where $\tau_{1,2}$ are the roots of $4\tau^2 - 4\tau - 1 = 0$, $\tau_{1,2} \notin [0,1]$.

3 Concluding remarks

As we have already mentioned in the introduction, we consider this paper only as a first step in our research. Because of this, we have included in it mainly results in whose proofs the basics of the theory of Fourier multipliers on scalar-valued L_p-spaces are revealed with only a minimum of additional technical involvement. It must be stressed here that all essential techniques discussed in the preliminaries do admit appropriate generalizations for vector fields; in particular, the theory of Fourier multipliers can be extended to vector-valued L_p-spaces (see, e.g., [1]). Here we mention some topics of interest for further research (which would also be more technically involved).

• we can obtain more refined results in all four subsections of section 2 by using more detailed (but still very general) information about the characteristic polynomial. For example, in Corollary 2.1.4 we obtained

$$F \in B_{pq}^{-d(1-1/p+|1/2-1/p|)}(\mathbb{R}^{\mathrm{d}}), 1 \leq \mathrm{p} \leq \infty, \tag{1}$$

under the assumption $|Q(\xi)| \leq 1$, $\forall \xi \in [-\pi, \pi]^d$. If it is additionally assumed that

$$|Q(\xi)| < 1, \forall \xi \in [-\pi, \pi]^d \setminus \{0\} \tag{2}$$

and 2.4.(3,4) hold in a neighbourhood of $\xi = 0$, with $\mu, \nu \in \mathbb{N}$ and μ-necessarily even in view of (2), then our methods for proving (1) show that in this case the more refined

$$F \in B_{p\infty}^{-d(1-1/p+|1/2-1/p|\cdot\max\{0,1-\nu/\mu\})}(\mathbb{R}^{\mathrm{d}}), 1 \leq \mathrm{p} \leq \infty, \tag{1'}$$

holds. However, compared to (1), the proof of (1') requires greater technical involvement.

• in this work we applied only the theory of Fourier multipliers on L_p, i.e., from L_p to L_p. A richer variety of applications can be addressed if more general Fourier multipliers from L_{p_0} to L_{p_1} are considered. A further refinement of the results can be obtained if Fourier multipliers between Lorentz spaces are considered, the respective regularity estimates being in terms of the four-indexed refinement of the Besov scale (see, e.g., [1]).

• the case $0 < p < 1$, $0 < q \leq \infty$, when Besov spaces are quasi-Banach spaces, is of considerable interest in its own. However, in order to study this case via an extension of our method, one needs to consider also maximal functions.

• dyadic refinement - corresponding to the operator $\frac{1}{2}\mathrm{Id}_{\mathbb{R}^{\mathrm{d}}}$ - can be replaced (as is done in [4, 6]) by an arbitrary invertible $T \in L(\mathbb{R}^{\mathrm{d}})$ whose spectral radius is less than 1. $\mathbb{Z}^{\mathrm{d}}$ can also be replaced by a general group with properties discussed in [4, 6].

• the final and ultimate goal of further research based on the theory of Fourier multipliers on vector-valued L_p-spaces, however, is the study of iterative interpolation of multivariate vector fields. Recent results on multiwavelets (which are univariate vector fields) display considerable advantages (from computational point of view) of the utilization of vector fields. In view of the high complexity of available methods when applied to multivariate vector fields, there are very few, if any, results available at this level of generality, to the best of our knowledge. One of the big new challenges is that commutativity of matrices plays an important role. The theory of Fourier multipliers on vector-valued L_p-spaces makes it possible to obtain sharp results also with respect to commutativity (e.g., see Theorem 5.1.1 in [2] for a result related to a system of first-order hyperbolic PDE). We believe that, together with other potential applications of Fourier multipliers to vector- field refinement, they may prove to be one of the key tools in deriving and classifying all possible factorizations leading to orthonormal multivariate multiwavelets.

4 Proofs

Representation of $f \in \mathcal{S}'(\mathbb{R}^{\mathrm{d}})$ as a pointwise multivalued mapping (see subsection 1.2.1.)

It is possible to establish a bijective correspondence between the elements of $\mathcal{S}'(\mathbb{R}^{\mathrm{d}})$ and a certain class of multivalued mappings defined pointwise on $\mathbb{R}^{\mathrm{d}}$. Indeed (see [10], Ch.I, §5.4, (4.1); cf. [8], Theorem V.10), for any $f \in \mathcal{S}'(\mathbb{R}^{\mathrm{d}})$ there exists a unique $g \in C^0(\mathbb{R}^{\mathrm{d}})$, such that:

(i) $g(x)$ has polynomial growth as $|x| \to \infty$;

(ii) $f = D^\alpha g$ (in the weak sense) where

$$\alpha \in \mathbb{Z}^{\mathrm{d}}, |\alpha|_1 = \mathrm{md}, \alpha = (\overbrace{m, \cdots, m}^{\mathrm{d-times}}),$$

$m = m_f + 2$, m_f being the order of f (see Schwartz' theorem about the finite order of $f \in \mathcal{S}'(\mathbb{R}^{\mathrm{d}})$ -[10], §5.2, (2.1)). This general result suggests the following representation for f

$$f \in \mathcal{S}'(\mathbb{R}^{\mathrm{d}}) \longmapsto f_1(\mathrm{x}), \mathrm{x} \in \mathbb{R}^{\mathrm{d}}$$

where $\forall x \in \mathbb{R}^{\mathrm{d}}$ $(y \in f_1(x))$ iff

$$[\exists h^{(\nu)} \in \mathbb{R}^{\mathrm{d}} : \mathrm{h}^{(\nu)} = (\mathrm{h}_1^{(\nu)}, \cdots, \mathrm{h}_{\mathrm{d}}^{(\nu)}), \mathrm{h}_{\mathrm{k}}^{(\nu)} \neq 0, \mathrm{k} = 1, \cdots, \mathrm{d},$$

$$\lim_{\nu \to \infty} h^{(\nu)} = 0, (h_1^{(\nu)} \cdots h_d^{(\nu)})^{-m} \Delta^m_{h_1^{(\nu)}} \cdots \Delta^m_{h_d^{(\nu)}} g(x) = h^{-\alpha} \Delta^\alpha g(x) \to y, \nu \to \infty)],$$

g, m, α being as defined above with respect to f, Δ_h^m being the univariate finite difference of order m with step h - see, e.g. [1,2,7].

Proof of Proposition 1.2.1 (i) Since $G_\infty \subset \mathbb{R}^{\mathrm{d}}$ and $F_1 = \bar{F}_1|_{G_\infty}$, "$\Longrightarrow$" is obvious. In the opposite direction, G_∞ is dense in $\mathbb{R}^{\mathrm{d}}$, F_1 is defined on G_∞ and is continuous on G_∞. Therefore by a standard argument about extension of a densely defined continuous function onto the closure of its domain, $\bar{F}_1$ is well-defined and continuous on $\bar{G}_\infty = \mathbb{R}^{\mathrm{d}}$.

(ii) By (i), $\bar{F}_1$ is the continuous extension on $\mathbb{R}^{\mathrm{d}}$ of F_1 defined on G_∞. By Theorem 3.1 in [6], the characteristic function of F is $\Phi \in C^0(\mathbb{R}^{\mathrm{d}})$. By continuity of $\bar{F}_1$ and, hence, of F_1 on G_∞, and by $R < \infty$, it follows that $\bar{F}_1$ satisfies 1.2.1.(6), $\forall x \in \mathbb{R}^{\mathrm{d}}$. Again by $R < \infty$, it then follows that $\hat{\bar{F}}_1$ satisfies 1.2.1.(2). By Theorem 3.1 in [6], the solution of 1.2.1.(2) is unique. Since Φ also satisfies 1.2.1.(2), it follows that $\hat{\bar{F}}_1(\xi) = \Phi(\xi)$, $\forall \xi \in \mathbb{R}^{\mathrm{d}}$. On the other hand, by definition, $\check{\Phi} = F$, therefore, $\Phi(\xi) = \hat{F}(\xi)$, $\forall \xi \in \mathbb{R}^{\mathrm{d}}$. Hence, $\mathcal{F}(\bar{F}_1 - F)(\xi) = \hat{\bar{F}}_1(\xi) - \hat{F}(\xi) = \Phi(\xi) - \Phi(\xi) = 0$, $\forall \xi \in \mathbb{R}^{\mathrm{d}}$. Therefore, $\bar{F} - F = 0$ Lebesgue-a.e. on $\mathbb{R}^{\mathrm{d}} \Longrightarrow$ 1.2.1.(8). $\bar{F}_1 \in C^0(\mathbb{R}^{\mathrm{d}})$, 1.2.1.(8) $\Longrightarrow F$ is continuous Lebesgue-a.e. on $\mathbb{R}^{\mathrm{d}} \Longleftrightarrow F$ is locally Riemann-integrable on $\mathbb{R}^{\mathrm{d}}$. Theorem 3.1 in [6] $\Longrightarrow$ supp F is compact, therefore F is Riemann-integrable on $\mathbb{R}^{\mathrm{d}}$. □

Proof of Lemma 1.2.4.1. We shall show that, given $k_1, k_2 \in \mathbb{Z}$, the norms $||\cdot||_{k_\nu,p,q,s,d}$, $\nu = 1, 2$, defined in 1.2.4.(3'), are equivalent, with equivalence constants depending on k_1, k_2 and q. We shall consider the case $1 \leq q < \infty$; the case $q = \infty$ is analogous, but technically simpler. The case $k_1 = k_2$ is trivial. With no loss of generality, let $k_1 < k_2$. First we prove that

$$(1') \qquad \exists C' = C'(k_1, k_2, s, q) < \infty : ||\cdot||_{k_2,p,q,s,d} \leq C'||\cdot||_{k_1,p,q,s,d}.$$

By 1.2.4.(1,2) and $k_1 < k_2$,

$$\psi_{k_2} = \psi_{k_1} + \sum_{j=k_1+1}^{k_2} \varphi_j.$$

Therefore,

$$||f||_{k_2,p,q,s,d} = [||(\psi_{k_1} + \sum_{j=k_1+1}^{k_2} \varphi_j) * f||^q_{L_p} + \sum_{j=k_2+1}^{\infty} (2^{sj}||\varphi_j * f||_{L_p})^q]^{1/q} \leq$$

$$\leq [(||\psi_{k_1} * f||_{L_p} + \sum_{j=k_1+1}^{k_2} ||\varphi_j * f||_{L_p})^q + \sum_{j=k_2+1}^{\infty} (2^{sj}||\varphi_j * f||_{L_p})^q]^{1/q} \leq$$

$$\leq [(||\psi_{k_1} * f||_{L_p} + C_1'(k_1, k_2, s) \sum_{j=k_1+1}^{k_2} (2^{sj}||\varphi_j * f||_{L_p})^q + \sum_{j=k_2+1}^{\infty} (2^{sj}||\varphi_j * f||_{L_p})^q]^{1/q} \leq$$

$$\leq \{C_2'(k_0, k_1, s, q)[||\psi_{k_1} * f||_{L_p}^q + \sum_{j=k_1+1}^{k_2} (2^{sj}||\varphi_j * f||_{L_p})^q] + \sum_{j=k_2+1}^{\infty} (2^{sj}||\varphi_j * f||_{L_p})^q\}^{1/q} \leq$$

$$\leq C'(k_0, k_1, s, q)||f||_{k_1,p,q,s,d} \Longrightarrow (1').$$

Next we prove that

$$(1") \qquad \exists C'' = C''(k_1, k_2, s, p, q) < \infty : ||\cdot||_{k_1,p,q,s,d} \leq C''||\cdot||_{k_2,p,q,s,d}.$$

By 1.2.4.(1, 2),

$$\delta = \psi_{k_2} + \sum_{j=k_2+1}^{\infty} \varphi_j.$$

Besides, by 1.2.4.(1,2) and $k_1 < k_2$,

$$\varphi_{k_1} * \varphi_{j_1} \equiv 0, j_1 \in k_2 + \mathbb{N};$$

$$\varphi_j * \varphi_{j_1} \equiv 0, j = k_1 + 1, \cdots, k_2 - 1, j_1 \in k_2 + \mathbb{N}, \text{ or } \mathrm{j} = \mathrm{k}_2, \mathrm{j}_1 \in \mathrm{k}_2 + 1 + \mathbb{N}.$$

Therefore,

$$\Lambda_1 := ||\psi_{k_1} * f||_{L_p}^q + \sum_{j=k_1+1}^{k_2} (2^{sj}||\varphi_j * f||_{L_p})^q =$$

$$= ||\psi_{k_1} * (\psi_{k_2} + \sum_{j=k_2+1}^{\infty} \varphi_{j_1}) * f||_{L_p}^q + \sum_{j=k_1}^{k_2} [2^{sj}||\varphi_j * (\psi_{k_2} + \sum_{j=k_2+1}^{\infty} \varphi_{j_1}) * f||_{L_p}]^q =$$

$$= ||\psi_{k_1} * \psi_{k_2} * f||_{L_p}^q + \sum_{j=k_1+1}^{k_2-1} (2^{sj}||\varphi_j * \psi_{k_2} * f||_{L_p})^q + (2^{sk_2}||\varphi_{k_2} * \psi_{k_2} * f + \varphi_{k_2} * \varphi_{k_2+1} * f||_{L_p})^q \leq$$

$$\leq ||\psi_{k_1} * \psi_{k_2} * f||_{L_p}^q + \sum_{j=k_1+1}^{k_2-1} (2^{sj}||\varphi_j * \psi_{k_2} * f||_{L_p})^q +$$

$$+C_1''(q)[(2^{sk_2}||\varphi_{k_2} * \psi_{k_2} * f||_{L_p})^q + (2^{sk_2}||\varphi_{k_2} * \varphi_{k_2+1} * f||_{L_p})^q] \leq$$

$$\leq ||\psi_{k_1} * \psi_{k_2} * f||_{L_p}^q + C_2''(k_1, k_2, s, q) \sum_{j=k_1+1}^{k_2} ||\varphi_j * \psi_{k_2} * f||_{L_p}^q + C_3''(s, q)(2^{s(k_2+1)}||\varphi_{k_2} * \varphi_{k_2+1} * f||_{L_p})^q \leq$$

$$\leq [||\hat{\psi}_{k_1}||_{M_p}^q + C_2''(k_1, k_2, s, q) \sum_{j=k_1+1}^{k_2} ||\varphi(2^{-j}\cdot)||_{M_p}^q]||\psi_{k_2} * f||_{L_p}^q +$$

$$+C_3''(s, q)||\varphi(2^{-k_2}\cdot)||_{M_p}^q [2^{s(k_2+1)}||\varphi_{k_2+1} * f||_{L_p}]^q =: \Lambda_2.$$

$$1.2.3.(\mathrm{E}) \Longrightarrow ||\varphi(2^{-j}\cdot)||_{M_p} = ||\varphi||_{M_p}, \forall j \in \mathbb{Z} \Longrightarrow$$

$$\Lambda_2 = [||\hat{\psi}_{k_1}||_{M_p}^q + (k_2 - k_1)C_2''(k_1, k_2, s, q)||\varphi||_{M_p}^q]||\psi_{k_2} * f||_{L_p}^q +$$

$$+C_3''(s, q)||\varphi||_{M_p}^q [2^{s(k_2+1)}||\varphi_{k_2+1} * f||_{L_p}]^q =$$

$$= C_4''(k_1,k_2,s,p,q)||\psi_{k_2} * f||_{L_p}^q + C_5''(s,p,q)[2^{s(k_2+1)}||\varphi_{k_2+1} * f||_{L_p}]^q \leq$$

$$\leq C_6''(k_1,k_2,s,p,q)[||\psi_{k_2} * f||_{L_p}^q + \sum_{j=k_2+1}^{\infty} (2^{sj}||\varphi_j * f||_{L_p})^q] =: \Lambda_3 \Longrightarrow$$

$$||f||_{k_1,p,q,s,d} = [\Lambda_1 + \sum_{j=k_2+1}^{\infty} (2^{sj}||\varphi_j * f||_{L_p})^q]^{1/q} \leq$$

$$\leq [\Lambda_3 + \sum_{j=k_2+1}^{\infty} (2^{sj}||\varphi_j * f||_{L_p})^q]^{1/q} \leq C''(k_1,k_2,p,q,s)||f||_{k_2,p,q,s,d}. \ \square$$

Proof of Theorem 2.1.1. (i). 1.2.4.(A) $\Longrightarrow$

$$||F||_{B_{p\infty}^{s-d(1-1/p)}} = \max\{||\psi_0 * F||_{L_p}, \sup_{k\in\mathbb{N}}\{2^{k[s-d(1-1/p)]}||\varphi_k * F||_{L_p}\}\} \sim$$

$$\sim \max\{||\psi_0 * F||_{L_p}, \sup_{k\in\mathbb{N}}\{2^{-kd(1-1/p)}||\varphi_k * (\mathcal{J}^s F)||_{L_p}\}\} =: \Lambda_1.$$

Denote $\Phi_s(\xi) := (1+|\xi|^2)^{s/2}\Phi(\xi)$, $\xi \in \mathbb{R}^{\mathrm{d}}$. For $x \in \mathbb{R}^{\mathrm{d}}$,

$$[\varphi_k * (\mathcal{J}^s F)](x) = \mathcal{F}^{-1}[\varphi(2^{-k}\cdot)\Phi_s(\cdot)](x) = 2^{kd}\{\check{\varphi} * \mathcal{F}^{-1}[\Phi_s(2^k\cdot)]\}(2^k x) \Longrightarrow$$

$$||\varphi_k * (\mathcal{J}^s F)||_{L_p} = 2^{kd(1-1/p)}||\check{\varphi} * \mathcal{F}^{-1}[\Phi_s(2^{-k}\cdot)]||_{L_p} \leq$$

$$\leq 2^{kd(1-1/p)}||\Phi_s(2^{-k}\cdot)||_{M_p}\cdot||\check{\varphi}||_{L_p} = 2^{kd(1-1/p)}||\Phi_s||_{M_p}||\check{\varphi}||_{L_p}.$$

$$||\psi_0 * (\mathcal{J}^s F)||_{L_p} = ||\check{\psi} * \check{\Phi}_s||_{L_p} \leq ||\Phi_s||_{M_p}\cdot||\check{\psi}||_{L_p} \Longrightarrow$$

$$\Lambda_1 \leq \max\{||\check{\varphi}||_{L_p}, ||\check{\psi}||_{L_p}\}||\Phi_s||_{M_p} \Longrightarrow \text{ (i).}$$

(ii) Consider $||\Phi_s||_{M_p}$. By definition of $||\cdot||_{M_p}$, $\exists f \in L_p$: $||f||_{L_p} = 1$ and

$$||\Phi_s||_{M_p} \leq 2||\check{\Phi}_s * f||_{L_p}. \tag{2}$$

$p < \infty \Longrightarrow C_0^\infty(\mathbb{R}^{\mathrm{d}})$ is dense in $L_p(\mathbb{R}^{\mathrm{d}}) \Longrightarrow$ with no loss of generality in (2) $f \in L_p \bigcap L_1 \Longrightarrow$

$$||\Phi_s||_{M_p} \leq 2||(\mathcal{J}^s F) * f||_{L_p} =$$

$$= 2||(\psi_0 + \sum_{k=1}^{\infty}\varphi_k) * (\mathcal{J}^s F) * f||_{L_p} \leq 2(||\psi_0 * (\mathcal{J}^s F) * f||_{L_p} + \sum_{k=1}^{\infty}||\varphi_k * (\mathcal{J}^s F) * f||_{L_p} \leq$$

$$\leq 2(||\psi_0 * (\mathcal{J}^s F)||_{L_p} + \sum_{k=1}^{\infty}||\varphi_k * (\mathcal{J}^s F)||_{L_p})||f||_{L_1} \sim ||F||_{B_{p1}^s}\cdot||f||_{L_1} < \infty \Longrightarrow \text{ (ii). } \square$$

Proof of Proposition 2.1.1. $(1+|\cdot|^2)^{-\alpha/2} \in M_p$, $\forall p \in (1,\infty), \forall \alpha > 0$ (see, e.g., [7], 1.5.5.2) $\Longrightarrow$

$$||\Phi_{s_1}||_{M_p} = ||(\Phi_{s_1})_{s_2-s_1}||_{M_p} \leq ||(1+|\cdot|^2)^{-(s_2-s_1)/2}||_{M_p}\cdot||\Phi_{s_2}||_{M_p}. \ \square$$

Proof of Proposition 2.1.2. (i). 1.2.3.(C).(i) $\Longrightarrow ||\Phi_s||_{M_2} = ||\Phi_s||_{L_\infty} \Longrightarrow$ (i).

(ii) By assumption, $F \in L_{1,\mathrm{loc}}$; $R < \infty \Longrightarrow \operatorname{supp} F$ is compact $\Longrightarrow F \in L_1$. By Riemann-Lebesgue's lemma $\Phi \in C^0(\mathbb{R}^{\mathrm{d}})$ and $|\Phi(\xi)| = o(1)$, $|\xi| \to \infty \Longrightarrow \exists C : |\Phi(\xi)| \leq C$, $\forall \xi \in \mathbb{R}^{\mathrm{d}} \Longrightarrow$ (ii).

(iii). Same idea as in (ii).

(iv). The idea is the same as in (ii, iii), with application of the Hausdorff-Young inequality or Plancherel's theorem instead of Riemann-Lebesgue's lemma. □

Proof of Proposition 2.1.3. Application of Riemann-Lebesgue/Hausdorff-Young/Plancherel for $p = 1, p \in (1,2), p = 2$, respectively. □

Proof of Theorem 2.1.2. A direct check that the idea of the proof of Theorem 3.1 in [6] can be carried through under the assumptions of the present theorem, too. □

The following lemma seems to be also of independent interest: it provides an important example when Theorem 2.1.1.(ii) is applicable, while Carlson- Beurling and Mihlin-Hörmander theorems are not.

Lemma 4.1. $\mathcal{S}.L_\infty \subset M_p$, $\forall p : 1 < p < \infty$.

Proof: By 1.2.3.(A).(i), it suffices to consider $p : 2 \leq p < \infty$.
$h_1 \in \mathcal{S}, h_2 \in L_\infty \subset \mathcal{S}' \Longrightarrow \check{h}_1 \in \mathcal{S}, \check{h}_2 \in \mathcal{S}' \Longrightarrow \check{h}_1 * \check{h}_2 \in \theta_M$.
For $p = 2$, by Plancherel's theorem, $||\check{h}_1 * \check{h}_2||_{L_2} = ||h_1 h_2||_{L_2} < \infty$. Moreover,

$$||D^\alpha(\check{h}_1 * \check{h}_2)||_{L_2} = ||[(\cdot)^\alpha h_1(\cdot)]h_2(\cdot)||_{L_2} < \infty, \forall \alpha \in \mathbb{Z}_+^{\mathrm{d}} \Longrightarrow$$

$$\mathcal{F}^{-1}(h_1 h_2) = \check{h}_1 * \check{h}_2 \in W_2^\nu, \forall \nu \in \mathbb{N} \bigcup \{0\}, \Longrightarrow \mathcal{F}^{-1}(\mathrm{h}_1\mathrm{h}_2) \in \mathrm{B}_{22}^{\mathrm{s}}, \forall \mathrm{s} \geq 0$$

(by 1.2.4.(D)). Let $\epsilon > 0$. Then, $p \in (2, \infty)$, 1.2.4.(B).(i) $\Longrightarrow$

$$B_{22}^{\epsilon + d(1/2 - 1/p)} \hookrightarrow B_{p1}^0 \Longrightarrow \mathcal{F}^{-1}(h_1 h_2) \in B_{p1}^0.$$

Hence, Theorem 2.1.1.(ii) $\Longrightarrow h_1 h_2 \in M_p$. □

Proof of Theorem 2.1.3. (i). By 1.2.1. (2-5), and by the theorem's premises,

$$||F||_{B_{p\infty}^{-s-d(1-1/p)}} = \max\{||\mathcal{F}^{-1}(\psi\Phi)||_{L_p}, \sup_{k \in \mathbb{N}} \{2^{-k[s+d(1-1/p)]} ||\mathcal{F}^{-1}[\varphi(2^{-k}\cdot)\Phi(\cdot)]||_{L_p}\}\} =$$

$$= \max\{||\mathcal{F}^{-1}(\psi\Phi)||_{L_p}, \sup_{k \in \mathbb{N}} \{2^{-k[s+d(1-1/p)]} ||\mathcal{F}^{-1}[\varphi(2^{-k}\cdot)\Phi_k(\cdot)\Phi(2^{-k}\cdot)]||_{L_p}\}\} \leq$$

$$\leq C \max\{||\mathcal{F}^{-1}(\psi\Phi)||_{L_p}, \sup_{k \in \mathbb{N}} \{2^{-kd(1-1/p)} ||\mathcal{F}^{-1}[\varphi(2^{-k}\cdot)\Phi(2^{-k}\cdot)]||_{L_p}\}\} =$$

$$= C \max\{||\mathcal{F}^{-1}(\psi\Phi\chi)||_{L_p}, \sup_{k \in \mathbb{N}} \{2^{-kd(1-1/p)} ||\mathcal{F}^{-1}[\varphi(2^{-k}\cdot)\Phi(2^{-k}\cdot)\chi(2^{-k}\cdot)]||_{L_p}\}\} =: \Lambda_1,$$

where

$$\chi \in C_0^\infty(\mathbb{R}^{\mathrm{d}}), \operatorname{supp} \chi \supset\supset \operatorname{supp} \psi \bigcup \operatorname{supp} \varphi, \chi|_{\operatorname{supp} \psi \bigcup \operatorname{supp} \varphi} \equiv 1.$$

We continue the above chain of inequalities with

$$\Lambda_1 \leq C \max\{||\psi\Phi||_{M_p} ||\check{\chi}||_{L_p}, \sup_{k \in \mathbb{N}} \{2^{-kd(1-1/p)} ||\varphi(2^{-k}\cdot)\Phi(2^{-k}\cdot)||_{M_p} ||\mathcal{F}^{-1}[\chi(2^{-k}\cdot)]||_{L_p}\}\} =: \Lambda_2.$$

By change of variables and by 1.2.3.(E),

$$||\mathcal{F}^{-1}[\chi(2^{-k}\cdot)]||_{L_p} = 2^{-kd(1-1/p)} ||\check{\chi}||_{L_p}, ||\varphi(2^{-k}\cdot)\Phi(2^{-k}\cdot)||_{M_p} = ||\varphi\Phi||_{M_p} \Longrightarrow$$

$$\Lambda_2 = C. \max\{||\psi\Phi||_{M_p}, ||\varphi\Phi||_{M_p}\} ||\check{\chi}||_{L_p} \leq C_p. \max\{||\psi\Phi||_{M_p}, ||\varphi\Phi||_{M_p}\}.$$

$||\psi\Phi||_{M_p}$ and $||\varphi\Phi||_{M_p}$ are being dealt with analogously. Consider $\psi\Phi$. $\Phi \in C^0(\mathbb{R}^{\mathrm{d}})$, $\psi \in C^0(\mathbb{R}^{\mathrm{d}})$, $\Longrightarrow \psi\Phi_1 \equiv \psi\Phi$, where $\Phi_1 := \Phi I_{\operatorname{supp} \psi}$, $I_{\operatorname{supp} \psi}(\xi)$ being the respective indicator function. Because $\operatorname{supp} \psi$ is compact, $\Phi_1 \in L_\infty(\mathbb{R}^{\mathrm{d}})$. Lemma 4.1 $\Longrightarrow \psi\Phi \in M_p \Longrightarrow$ (i) for $1 < p < \infty$.

Let now $2 \le p \le \infty$. By using Riemann-Lebesgue/Hausdorff-Young/Plancherel argument for $p = \infty$, $p \in (2, \infty)$, $p = 2$, respectively,

$$||\mathcal{F}^{-1}[\varphi(2^{-k}\cdot)\Phi(2^{-k}\cdot)]||_{L_p} = 2^{kd}||(\check{\varphi} * F)(2^k\cdot)||_{L_p} = 2^{kd(1-1/p)}||\check{\varphi} * F||_{L_p} \le C_p 2^{kd(1-1/p)}||\varphi\Phi||_{L_{p'}},$$

and $\varphi\Phi \in L_{p'}$, because $\varphi, \Phi \in C^0(\mathbb{R}^{\mathrm{d}})$ and $\operatorname{supp}\varphi$ is compact. Analogously for $\psi\Phi$. $\Longrightarrow$ (i) for $p = \infty \Longrightarrow$ (i), $1 \le p \le \infty$.

(ii) $R < \infty$, $1 \le p \le \infty$. It is enough to prove that there exists $C < \infty$, such that

$$||\mathcal{F}^{-1}[\varphi(2^{-k}\cdot)\Phi(2^{-k}\cdot)]||_{L_p} \le C 2^{kd(1-1/p)}. \tag{3}$$

First proof of (3).

$$||\mathcal{F}^{-1}[\varphi(2^{-k}\cdot)\Phi(2^{-k}\cdot)]||_{L_p} = 2^{kd(1-1/p)}||\mathcal{F}^{-1}(\varphi\Phi)||_{L_p} =$$
$$= 2^{kd(1-1/p)}||\mathcal{F}^{-1}[(1+|\cdot|^2)^{(\alpha+d+\epsilon)/2}\varphi(\cdot)\Phi_{-(\alpha+d+\epsilon)}(\cdot)]||_{L_p} =$$
$$= 2^{kd(1-1/p)}||(\mathcal{J}^{\alpha+d+\epsilon}\check{\varphi}) * (\mathcal{J}^{-\alpha-d-\epsilon}F)||_{L_p},$$

where

$$\alpha = ||Q||_{M_1} = 2^{-d}||P||_{M_1} = 2^{-d}\sum_{\alpha}|c_\alpha|,$$

$\epsilon > 0$. By Theorem 3.1 in [6],

$$\Phi_{-(\alpha+d+\epsilon)}(\zeta) = (1+|\zeta|^2)^{-(\alpha+d+\epsilon)/2}\Phi(\zeta), \zeta \in \mathbb{C}^{\mathrm{d}},$$

is an entire function of exponential type ($(1+|\zeta|^2)^{-(\alpha+d+\epsilon)/2}$ is entire analytic on $\mathbb{C}^{\mathrm{d}}$), such that $\Phi_{-(\alpha+d+\epsilon)}|_{\mathbb{R}^{\mathrm{d}}}$ is integrable. Therefore, $\mathcal{J}^{-(\alpha+d+\epsilon)}F \in C^0(\mathbb{R}^{\mathrm{d}})$ and $\operatorname{supp}\mathcal{J}^{-(\alpha+d+\epsilon)}F$ is compact. This and $\mathcal{J}^{\alpha+d+\epsilon}\check{\varphi} \in \mathcal{S}$ imply

$$\check{\varphi} * F = (\mathcal{J}^{\alpha+d+\epsilon}\check{\varphi}) * (\mathcal{J}^{-(\alpha+d+\epsilon)}F) \in L_p, \forall p : 1 \le p \le \infty \Longrightarrow (3).$$

Second proof of (3). As earlier,

$$||\mathcal{F}^{-1}[\varphi(2^{-k}\cdot)\Phi(2^{-k}\cdot)]||_{L_1} \le 2^{kd(1-1/p)}||\check{\chi}||_{L_1}||\varphi\Phi||_{M_1}.$$

For $\epsilon > 0$, by the embedding $B^{\epsilon}_{12} \hookrightarrow B^0_{11}$ (see 1.2.4.(B).(ii) and the remarks on embeddings), by 1.2.4.(E) and the bound (based on Hölder's inequality):

$$\omega_k(f;\delta)_{L_{P_1}} \le C(d,p,p_1,k)\cdot R^{d(1/p_1-1/p)}\omega_k(f;\delta)_{L_p}, 1 \le p_1 \le p \le \infty, k \in \mathbb{N},$$

valid for $f \in L_{p_1}$ with $\operatorname{supp} f \subset \{x \in \mathbb{R}^{\mathrm{d}} : |x| \le R\}$, applied for $p = 2$ and $p_1 = 1$, it follows that

$$||\check{\varphi} * F||_{B^0_{11}} \le C_{d,\epsilon,R}\cdot ||\check{\varphi} * F||_{B^{\epsilon}_{22}} < \infty.$$

Therefore, by Theorem 2.1.1.(ii), $||\varphi\Phi||_{M_1} < \infty$.
By the considerations already made in (i, ii), it suffices to prove $\psi\varphi \in M_p$, $\varphi\Phi \in M_p$ under the assumptions $R \le \infty$, $s = 0$. Consider, e.g., $\psi\Phi$. $\psi \in C^\infty_0 \Longrightarrow \psi \in M_p, \forall p : 1 \le p \le \infty$; $s = 0 \Longrightarrow \exists C < \infty : ||\Phi_n||_{M_p} \le C$, $\forall n \in \mathbb{N}$.
Theorem 3.1 in [6] $\Longrightarrow \Phi_n \to \Phi$ uniformly on compact subsets of $\mathbb{R}^{\mathrm{d}}$. Hence, 1.2.3.(I) $\Longrightarrow \Phi \in M_p$ and $||\Phi||_{M_p} \le C$.

1.2.3.(B) $\Longrightarrow ||\psi\Phi||_{M_p} \le ||\psi||_{M_p}||\Phi||_{M_p} \le C||\psi||_{M_p} < \infty \Longrightarrow$ (iii) $\square$

Proof of Corollary 2.1.1. 1.2.3.(B, E) $\Longrightarrow$

$$||\Phi_n||_{M_p} = ||\prod_{k=0}^{n-1}Q(2^{-k}\cdot)||_{M_p} \le \prod_{k=0}^{n-1}||Q(2^{-k}\cdot)||_{M_p} = ||Q||^n_{M_p} = 2^{n\log_2||Q||_{M_p}}.$$

According to the assumptions of Theorem 3.1 in [6], $R < \infty$.
Theorem 2.1.3.(ii), $s = \log_2 ||Q||_{M_p} \Longrightarrow$

$$F \in B_{p\infty}^{-\log_2 ||Q||_{M_p} - d(1-1/p)}.$$

1.2.3.(A). (i, ii) $\Longrightarrow$

$$||Q||_{M_p} \leq ||Q||_{M_2}^{1-2|1/2-1/p|} ||Q||_{M_1}^{2|1/2-1/p|} = ||Q||_{M_2}^{1-2|1/2-1/p|} ||Q||_{M_\infty}^{2|1/2-1/p|}.$$

The proof is finalized by utilizing $B_{pq}^{s_1} \hookrightarrow B_{pq}^{s_2}$, $-\infty < s_2 \leq s_1 < \infty$, (see 1.2.4(B).(ii)).□

Proof of Corollary 2.1.2. Follows from 1.2.3.(A). (i, iii). □

Proof of Corollary 2.1.3. Follows from $M_1 = M_\infty \hookrightarrow M_p$ (by 1.2.3.(A)), 1.2.4.(B).(ii) and 1.2.3.(G). □

Proof of Theorem 2.1.4. P is $(2\pi)^d$-periodic $\Longrightarrow \Phi_n(\cdot) = \prod_{k=1}^n Q(2^{-k}\cdot)$ is $(2^{n+1}\pi)^d$-periodic $\Longrightarrow$ $\Phi_n(2^n\cdot)$ is $(2\pi)^d$-periodic. By 1.2.3.(E), $||\Phi_n(\cdot)||_{M_p} = ||\Phi_n(2^n\cdot)||_{M_p}$.
By 1.2.3.(H), $\exists \eta : C_0^\infty \ni \eta : \mathbb{R}^{\mathrm{d}} \to [0,1]$, with $\operatorname{supp}\eta \supset\supset [-\pi,\pi]^d$ and $\eta \equiv 1$ in a neighbourhood of $[-\pi,\pi]^d$, such that, if $a(\xi)$ is $(2\pi)^d$-periodic and $a \in C^\infty \bigcap M_p$, for some $p : 1 \leq p \leq \infty$, then $||a||_{M_p} \sim ||a\eta||_{M_p}$. Hence,

$$||\Phi_n||_{M_p} \sim ||\Phi_n(2^n\cdot)\eta(\cdot)||_{M_p}, \tag{4}$$

with equivalence constants depending on p only.

We begin preparing to apply the Carlson-Beurling inequality. First of all, we notice that Q and Φ_n are in $C^\infty(\mathbb{R}^{\mathrm{d}})$. Next, we shall make use of the fact that $R < \infty$. By Bernstein's inequality about entire functions of exponential "rectangular" type (see 1.2.2 and [7], 3.2.2(8)) and the relationship between entire functions of exponential "rectangular" and "spherical" type (1.2.2; [7], 3.2.2.(8)), by 1.2.3.(C).(ii) and the assumption about stability of Φ_n in M_2,

$$||D^\alpha \Phi_n||_{L_\infty(\mathbb{R}^{\mathrm{d}})} \leq C_{|\alpha|_1,d} R_n^{|\alpha|_1} ||\Phi_n||_{L_\infty(\mathbb{R}^{\mathrm{d}})} \tag{5}$$

$$\leq C_{|\alpha|_1,d} R^{|\alpha|_1} ||\Phi_n||_{L_\infty(\mathbb{R}^{\mathrm{d}})} = C_{|\alpha|_1,d} R^{|\alpha|_1} ||\Phi_n||_{M_2} \leq C_{|\alpha|_1,d} R^{|\alpha|_1} C,$$

where

$$R_n = (\sum_{k=1}^n 2^{-k}) R \nearrow_{n\to\infty} R,$$

and $C_{|\alpha|_1,d}$ is the factor corresponding to the equivalence between the "rectangular" and the "spherical" type (see also 1.2.2). From (5) we get

$$||D^\alpha[\Phi_n(2^n\cdot)]||_{L_\infty} = 2^{n|\alpha|_1} ||(D^\alpha \Phi_n)(2^n\cdot)||_{L_\infty} = \tag{6}$$

$$= 2^{n|\alpha|_1} ||D^\alpha \Phi_n||_{L_\infty} \leq C_{|\alpha|_1,d} R^{|\alpha|_1} C 2^{n|\alpha|_1}, \alpha \in \mathbb{Z}_+^{\mathrm{d}}.$$

By 1.2.3.(C).(ii) and stability of Φ_n in M_2, Hölder's inequality yields (7)

$$||\Phi_n(2^n\cdot)\eta(\cdot)||_{L_2(\mathbb{R}^{\mathrm{d}})} \leq C||\eta||_{L_2(\mathbb{R}^{\mathrm{d}})} = C_d. \tag{7}$$

(6,7) $\Longrightarrow$

$$\sum_{|\alpha|_1=d} ||D^\alpha[\Phi_n(2^n\cdot)\eta(\cdot)]||_{L_2(\mathbb{R}^{\mathrm{d}})} \leq C_{d,R} 2^{nd} ||\eta||_{W_2^d(\mathbb{R}^{\mathrm{d}})} = C'_{d,R} 2^{nd}.$$

Now we apply the Carlson-Beurling inequality to obtain

$$||\Phi_n(2^n\cdot)\eta(\cdot)||_{M_1} = ||\Phi_n(2^n\cdot)\eta(\cdot)||_{M_\infty} \leq C 2^{nd/2}.$$

(4,6) $\Longrightarrow$ Theorem 2.1.4 for $p = 1$ and $p = \infty$. The general case follows from $p = 1, \infty$ and $p = 2$ ($||\Phi_n||_{M_2} \leq C$), by 1.2.3.(A).(ii). $\square$

Proof of Corollary 2.1.4. Application of Theorem 2.1.3.(ii) with s defined in Theorem 2.1.4. $\square$

For the proof of Lemma 2.1.1, the following lemma will be needed.

Lemma 4.2. *Let $\alpha \in \mathbb{Z}_+^{\mathrm{d}}$, $|\alpha|_1 = m \in \mathbb{N} \bigcup \{0\}$. Then,*

$$D^\alpha \Phi_n(2^n \xi) = \sum_{\nu_1=0}^{n-1} \cdots \sum_{\nu_m=0}^{n-1} 2^{\sum_{\mu=1}^m \nu_m} [\prod_{k=0}^{n-1} G_{\alpha,\nu_1,\cdots,\nu_m,k}(2^k \xi)], \forall \xi \in \mathbb{R}^{\mathrm{d}}, \forall \mathrm{n} \in \mathrm{m}+1+\mathbb{N},$$

where $G_{\alpha,\nu_1,\cdots,\nu_m,k}$ are such that $\forall(\nu_1, \cdots, \nu_m, k)$ $\exists \alpha_1 \in \mathbb{Z}_+^{\mathrm{d}}$: $|\alpha_1|_1 \leq m$, so that

$$G_{\alpha,\nu_1,\cdots,\nu_m,k} \equiv D^{\alpha_1} Q,$$

and, for every fixed $(\nu_1, \cdots, \nu_m)$,

$$|\alpha_1|_1 = |\alpha_1(\nu_1, \cdots, \nu_m, k)|_1 \neq 0$$

for at most m values of k in the set $I_n = \{0, 1, \cdots, n-1\}$.

Proof: Induction in $|\alpha|_1$. For $|\alpha|_1 = 0, 1$, the conclusion of the lemma is easy to verify by direct computation. Assume that the lemma is true for every $\alpha \in \mathbb{Z}_+^{\mathrm{d}}$: $0 \leq |\alpha|_1 \leq m$. Now consider any $\alpha \in \mathbb{Z}_+^{\mathrm{d}}$: $|\alpha|_1 = m+1$. Let $\beta \in \mathbb{Z}_+^{\mathrm{d}}$: $|\beta|_1 = 1$, $\gamma \in \mathbb{Z}_+^{\mathrm{d}}$: $|\gamma|_1 = m$ be such that $D^\alpha = D^\beta D^\gamma$. By the induction hypothesis for $D^\alpha[\Phi_n(2^n \xi)]$,

$$D^\alpha[\Phi_n(2^n \xi)] = D^\alpha\{\sum_{\nu_1=0}^{n-1} \cdots \sum_{\nu_m=0}^{n-1} 2^{\sum_{\mu=1}^m \nu_\mu} [\prod_{k \in I_n} G_{\gamma,\nu_1,\cdots,\nu_m,k}(2^k \xi)]\} =$$

$$= \sum_{\nu_1=0}^{n-1} \cdots \sum_{\nu_m=0}^{n-1} 2^{\sum_{\mu=1}^m \nu_\mu} \sum_{\nu_{m+1}=0}^{n-1} \{[\prod_{k \in I_n \setminus \{\nu_{m+1}\}} G_{\gamma,\nu_1,\cdots,\nu_m,k}(2^k \xi)] 2^{\nu_{m+1}} D^\beta G_{\gamma,\nu_1,\cdots,\nu_m,\nu_{m+1}}(2^{\nu_{m+1}} \xi)\} =$$

$$= \sum_{\nu_1=0}^{n-1} \cdots \sum_{\nu_{m+1}=0}^{n-1} 2^{\sum_{\mu=1}^{m+1} \nu_\mu} \prod_{k=0}^{n} G_{\alpha,\nu_1,\cdots,\nu_m,\nu_{m+1},k}(2^k \xi),$$

where

$$G_{\alpha,\nu_1,k}(\eta) = \begin{cases} Q(\eta), & k \neq \nu_1 \\ D^\beta Q(\eta), & k = \nu_1, \end{cases}$$

$$G_{\alpha,\nu_1,\cdots,\nu_m,\nu_{m+1},k}(\eta) = \begin{cases} G_{\gamma,\nu_1,\cdots,\nu_m,k}(\eta), & k \neq \nu_{m+1} \\ D^\beta G_{\gamma,\nu_1,\cdots,\nu_m,k}(\eta), & k = \nu_{m+1}. \end{cases}$$

Utilizing the induction hypothesis for $G_{\alpha,\nu_1,\cdots,\nu_m,k}$, it is easily seen from the expressions obtained for $D^\alpha[\Phi_n(2^n \xi)]$ and $G_{\alpha,\nu_1,\cdots,\nu_m,\nu_{m+1},k}(\eta)$ that the lemma's statement holds for $|\alpha|_1 = m+1$. $\square$

Proof of Lemma 2.1.1. By Lemma 4.2 and the triangle inequality,

$$||D^\alpha[\Phi_n(2^n \cdot)]||_{L_\infty} \leq \sum_{\nu_1=0}^{n-1} \cdots \sum_{\nu_{|\alpha|_1}=0}^{n-1} 2^{\sum_{\mu=1}^{|\alpha|_1} \nu_\mu} \prod_{k=0}^{n-1} ||G_{\alpha,\nu_1,\cdots,\nu_m,k}(2^k \cdot)||_{L_\infty} =: \Lambda_1.$$

Again by Lemma 4.2, utilizing $||Q||_{L_\infty} \leq 1$, $m_1 \leq |\alpha|_1$, m_1 being the number of admissible values of $\alpha_1 : |\alpha_1|_1 \neq 0$, as defined in Lemma 4.2, one obtains

$$\prod_{k=0}^{n-1} ||G_{\alpha,\nu_1,\cdots,\nu_m,k}(2^k \cdot)||_{L_\infty} \leq (\max_{0 \leq |\beta|_1 \leq |\alpha|_1} ||D^\beta Q||_{L_\infty})^{m_1} \leq$$

$$\leq \max\{1, \max_{0\leq|\beta|_1\leq|\alpha|_1} ||D^\beta Q||_{L_\infty}\}^{m_1} =: C_\alpha(Q).$$

Therefore,

$$\Lambda_1 \leq C_\alpha(Q) \sum_{\nu_1=0}^{n-1} \cdots \sum_{\nu_{|\alpha|_1}=0}^{n-1} 2^{\sum_{\mu=1}^{|\alpha|_1}\nu_\mu} = C_\alpha(Q)(2^n-1)^{|\alpha|_1} \leq C_\alpha(Q)\cdot 2^{n|\alpha|_1}.$$

The proof is completed by noticing that, in view of 1.2.2.(iii), $C_\alpha(Q) < \infty$. □

Proof of Theorem 2.2.1. By periodicity of P and 1.2.2.(i), $P \in K_+(\mathcal{S}'(\mathbb{R}^d))$ iff $c_\alpha = c_\alpha(P) \geq 0$, $\forall \alpha \in \mathbb{Z}^d$. $Q(0) = 2^{-d}P(0) = 1$, therefore, $1 \leq ||Q||_{L_\infty} = ||Q||_{M_2}$. On the other hand,

$$||Q||_{M_1} = ||Q||_{M_\infty} = 2^{-d}\sum_{\alpha\in\mathbb{Z}^d} |c_\alpha| = 2^{-d}\sum_\alpha c_\alpha = Q(0) = 1.$$

Therefore, by 1.2.3.(A).(iii, i),

$$1 \leq ||Q||_{M_2} \leq ||Q||_{M_p} \leq ||Q||_{M_\infty} = 1, \forall p: \ 1 \leq p \leq \infty. \ \square$$

Proof of Corollary 2.2.1. Application of Theorem 2.1.3.(iii) with $s = 0$, which is justified by Theorem 2.2.1. □

Proof of Theorem 2.2.2. 2.2.(3) $\Longrightarrow \mu : d\mu(\xi) = \Phi(\xi)d\xi$ is a bounded positive measure on $\mathbb{R}^d$. This, together with Bochner's theorem about positive definite functions (see 1.2.2), implies "$\Longleftarrow$".
In the other direction, in view of 2.2.($\hat{2}$), Bochner-Schwartz' theorem implies

(8) $$\exists \mu = \mu_\Phi - \text{ a positive measure on } \mathbb{R}^d \text{ with polynomial growth.}$$

1.2.2.(2), Theorem 2.1.2 $\Longrightarrow$

(9) $$\Phi \in C^0(\mathbb{R}^d), \text{ and } \Phi \text{ has polynomial growth at infinity.}$$

In particular,

(10) $$\Phi \in L_{1,\text{loc}}(\mathbb{R}^d).$$

(8, 10) $\Longrightarrow \mu$ is locally absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, i.e.,

$$\int_{\mathbb{R}^d} \chi_\Omega(\xi)d\mu(\xi) = \int_{\mathbb{R}^d} \chi_\Omega(\xi)\Phi(\xi)d\xi, \forall \Omega \subset\subset \mathbb{R}^d,$$

where χ_Ω is the indicator function of Ω. Hence,

(11) $$\Phi(\xi) \geq 0, \forall \xi \in \mathbb{R}^d : \xi \text{ is a Lebesgue point of } \Phi.$$

(10) $\Longrightarrow$ the Lebesgue points of Φ are a.e. on $\mathbb{R}^d$, therefore, dense on $\mathbb{R}^d$. Hence, (9, 11) $\Longrightarrow \Phi(\xi) \geq 0$, $\forall \xi \in \mathbb{R}^d$. This , together with Theorem 4.3 in [6], implies $\Phi \in L_1(\mathbb{R}^d)$, i.e., "$\Longrightarrow$". □

Proof of Corollary 2.2.2. 2.2.(3), Riemann-Lebesgue's lemma, 1.2.1(8) $\Longrightarrow \bar{F}_1 \in C_0^0(\mathbb{R}^d)$. Bochner's theorem about positive-definite functions, 1.2.1.(8) and IX.2.(1-3) in [9] (see also [10], Ch.II, §8.1, (1.1)) together imply 2.2.(4-7). □

Proof of Proposition 2.3.1. Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $s \in \mathbb{R}$. $\delta \in \mathcal{S}' \Longrightarrow ||\delta||_{B^s_{pq}}$ is always well-defined but may be $+\infty$. So in this context "meaningful" is equivalent to "finite". Moreover, since Besov spaces are translation invariant, it is sufficient to reduce the consideration to the delta-function supported at 0.

$$||\delta||_{B^s_{pq}} = \{||\mathcal{F}^{-1}[\psi(\cdot)\cdot 1]||^q_{L_p} + \sum_{k=1}^\infty 2^{skq}||\mathcal{F}^{-1}[\varphi(2^{-k}\cdot)\cdot 1]||^q_{L_p}\}^{1/q}$$

(with the sup-modification for $q=\infty$.) Hence,

$$||\delta||_{B^s_{pq}} = [||\check{\psi}||^q_{L_p} + \sum_{k=1}^{\infty} 2^{skq} 2^{kd(1-1/p)} ||\check{\varphi}||^q_{L_p}]^{1/q} = \{||\check{\psi}||^q_{L_p} + ||\check{\varphi}||^q_{L_p} \sum_{k=1}^{\infty} 2^{[s-d(1-1/p)]kq}\}^{1/q},$$

which is finite iff either $s-d(1-1/p)=0$, $q=\infty$, or $s-d(1-1/p)=0$, $q=\infty$, or $s-d(1-1/p)<0$, $1\le q\le\infty$. The embedding $B^{s_1}_{pq_1} \hookrightarrow B^{s_2}_{pq_2}$, $s_1>s_2$, $1\le q_1\le\infty$, $1\le q_2\le\infty$ (see 1.2.4.(B).(ii)), shows that, indeed, $B^{-d(1-1/p)}_{p\infty}$ is the minimal Besov space containing δ, in the sense that for fixed p, if $s\in\mathbb{R}$, $q: 1\le q\le\infty$, and $\delta\in B^s_{pq}$, then $B^{-d(1-1/p)}_{p\infty} \hookrightarrow B^s_{pq}$. □

Proof of Corollary 2.3.1. $R<\infty \Longrightarrow F_{(n)} \in B^s_{pq}$ iff $\delta\in B^s_{pq}$. The proof is completed by applying Proposition 2.3.1. □

In the proof of Theorem 2.3.1 the following two lemmata will be invoked.

Lemma 4.3. *$\exists\epsilon_0=\epsilon_0(P)>0$: $\Phi\in C^\infty(\bar{\mathcal{U}}_{\epsilon_0})$, where $\mathcal{U}_{\epsilon_0}=\{\xi\in\mathbb{R}^{\mathrm{d}} : |\xi|<\epsilon_0\}$.*

Proof: $Q(0)=2^{-d}P(0)=1\neq 0$ and $Q\in C^\infty(\mathbb{R}^{\mathrm{d}})$. Moreover, Q is periodic. Therefore, by 1.2.2.(iii), all derivatives of Q are bounded on $\mathbb{R}^{\mathrm{d}}$. The above properties of Q imply:
$\exists\epsilon_0>0, \exists\delta_0>0$: $|Q||_{\mathcal{U}_{\epsilon_0}}\ge\delta_0$, therefore, $\ln Q\in C^\infty(\bar{\mathcal{U}}_{\epsilon_0})$ and all its derivatives are bounded on $\bar{\mathcal{U}}_{\epsilon_0}$. This, together with $\Phi_n\to\Phi$, $n\to\infty$, uniformly on compacts, implies

$$\sum_{k=0}^{n} \ln[Q(2^{-k}\cdot)] \to \ln\Phi(\cdot)$$

uniformly on $\bar{\mathcal{U}}_{\epsilon_0}$,

$$D^\alpha[\ln Q(2^{-k}\cdot)]\in C^0(\bar{\mathcal{U}}_{\epsilon_0}), \forall\alpha\in\mathbb{Z}^{\mathrm{d}},$$

and

$$\sum_{k=0}^{\infty} |D^\alpha[\ln Q(2^{-k}\xi)]| \le C\sum_{k=0}^{\infty} 2^{-k|\alpha|_1} < \infty, \forall\xi\in\bar{\mathcal{U}}_{\epsilon_0}.$$

By the Weierstrass criterion, $\ln\Phi\in C^\infty(\bar{\mathcal{U}}_{\epsilon_0})$, therefore, $\Phi\in C^\infty(\bar{\mathcal{U}}_{\epsilon_0})$. □

Lemma 4.4. *Assume that P is accurate of order $\mu>0$ Then,*

$$\ln\Phi(\xi) = |\xi|^\mu\Gamma_0(\xi), \xi\in\bar{\mathcal{U}}_{\epsilon_0},$$

where $\Gamma_0\in C^\infty(\bar{\mathcal{U}}_{\epsilon_0}\setminus\{0\})$, $D^\alpha\Gamma_0$ is bounded on $\bar{\mathcal{U}}_{\epsilon_0}\setminus\{0\}$, $\forall\alpha\in\mathbb{Z}^{\mathrm{d}}_+$, ϵ_0 and $\bar{\mathcal{U}}_{\epsilon_0}$ are defined in Definition 2.3.1.

Proof: From the proof of Lemma 4.3 and from Definition 2.3.1 it follows that

$$\Gamma_0(\xi) = \sum_{k=0}^{\infty} 2^{-k\mu} H_0(2^{-k}\xi), \forall\xi\in\bar{\mathcal{U}}_{\epsilon_0}, \tag{12}$$

where H_0 and $\bar{\mathcal{U}}_{\epsilon_0}$ are defined in Definition 2.3.1 and the equality in (12) is in the sense of uniform convergence on $\bar{\mathcal{U}}_{\epsilon_0}$. This implies that $\Gamma_0\in C^0(\bar{\mathcal{U}}_{\epsilon_0})$. The additional properties of Γ_0 follow from the respective properties of H_0 and the Weierstrass criterion. □

Remark 4.1. From the proof of Lemmas 4.3,4 it follows that the value of $\epsilon_0>0$ in Definition 2.3.1 and Lemmas 4.3,4 can be chosen to be the same. We shall be implicitly assuming this in the sequel.

Proof of Theorem 2.3.1. In this proof it will be convenient to use the k_0-equivalent norm in Besov spaces (see Lemma 1.2.4.1), where k_0 is the biggest integer such that $2^{k_0+1} \le \epsilon_0$. In order to simplify the notations within this proof, we assume that the Peetre's function φ (see 1.2.4) satisfies

$$\operatorname{supp} \varphi \subset \{\xi \in \mathbb{R}^{\mathrm{d}} : 2^{\mathrm{k}_0-1} \le |\xi| \le 2^{\mathrm{k}_0+1}\},$$

and the respective ψ_0:

$$\operatorname{supp} \psi_0 \subset \{\xi \in \mathbb{R}^{\mathrm{d}} : |\xi| \le 2^{\mathrm{k}_0}\}.$$

With these notations 1.2.4.(3') becomes 1.2.4.(3). Throughout the sequel of the proof of this theorem, $||\cdot||_{B^s_{pq}}$ denotes the so-defined equivalent norm.

$s = 0$, Theorem 2.1.3.(iii) $\Longrightarrow F \in B^{-d(1-1/p)}_{p\infty}$.

If $R < \infty$, then, by Corollary 2.3.1, $F_{(n)} \in B^{-d(1-1/p)}_{p\infty}$. If $R = \infty$, we cannot use Corollary 2.3.1 but $||\Phi_n||_{M_p} < \infty$ and Theorem 2.1.1.(i), $s = 0$, still imply $F_{(n)} \in B^{-d(1-1/p)}_{p\infty}$. Therefore, in all cases $F_{(n)} - F \in B^{-d(1-1/p)}_{p\infty}$, hence, by 1.2.4.(B).(ii), $||F_{(n)} - F||_{B^{-\sigma-d(1-1/p)}_{p\infty}}$, is meaningful, $\forall \sigma \ge 0$.

Let $\sigma : 0 \le \sigma \le \mu$. By 1.2.1.(3,4) and the theorem's assumption about stability of $\{\Phi_n\}_{n=1}^{\infty}$ in M_p,

$$||F_{(n)} - F||_{B^{-\sigma-d(1-1/p)}_{p\infty}} = \max\{||\mathcal{F}^{-1}\{\psi(\cdot)\Phi_n(\cdot)[1 - \Phi(2^{-n}\cdot)]\}||_{L_p},$$

$$\sup_{k\in\mathbb{N}}\{2^{-k[\sigma+d(1-1/p)]}||\mathcal{F}^{-1}\{\varphi(2^{-k}\cdot)\Phi_n(\cdot)[1 - \Phi(2^{-n}\cdot)]\}||_{L_p}\}\} \le$$

$$\le ||\Phi_n||_{M_p}\cdot \max\{||\mathcal{F}^{-1}\{\psi(\cdot)[1 - \Phi(2^{-n}\cdot)]\}||_{L_p},$$

$$\sup_{k\in\mathbb{N}}\{2^{-k[\sigma+d(1-1/p)]}||\mathcal{F}^{-1}\{\varphi(2^{-k}\cdot)[1 - \Phi(2^{-n}\cdot)]\}||_{L_p}\}\} \le$$

$$\le C_1\cdot \max\{||\mathcal{F}^{-1}\{\psi(\cdot)[1 - \Phi(2^{-n}\cdot)]\}||_{L_p},$$

$$\sup_{k\in\mathbb{N}}\{2^{-k[\sigma+d(1-1/p)]}||\mathcal{F}^{-1}\{\varphi(2^{-k}\cdot)[1 - \Phi(2^{-n}\cdot)]\}||_{L_p}\}\} =$$

$$= C_1\cdot \max\{||\mathcal{F}^{-1}\{\psi(\cdot)[1 - \Phi(2^{-n}\cdot)]\chi(\cdot)\}||_{L_p},$$

$$\sup_{k\in\mathbb{N}}\{2^{-k[\sigma+d(1-1/p)]}||\mathcal{F}^{-1}\{\varphi(2^{-k}\cdot)[1 - \Phi(2^{-n}\cdot)]\chi(2^{-k}\cdot)\}||_{L_p}\}\} =: \Lambda_1,$$

where

$$\chi \in C_0^{\infty}(\mathbb{R}^{\mathrm{d}}), \operatorname{supp}\chi \supset\supset (\operatorname{supp}\psi \bigcup \operatorname{supp}\varphi), \chi|_{\operatorname{supp}\psi \bigcup \operatorname{supp}\varphi} \equiv 1.$$

By $||h(2^{-k}\cdot)||_{L_p(\mathbb{R}^{\mathrm{d}})} = 2^{kd(1-1/p)}||h||_{L_p(\mathbb{R}^{\mathrm{d}})}$, and by 1.2.3.(E),

$$\Lambda_1 \le C_1\cdot \max\{||\psi(\cdot)[1 - \Phi(2^{-n}\cdot)]||_{M_p}\cdot ||\check{\chi}||_{L_p},$$

$$\sup_{k\in\mathbb{N}}\{2^{-k[\sigma+d(1-1/p)]}||\varphi(2^{-k}\cdot)[1 - \Phi(2^{-n}\cdot)]||_{M_p}\cdot 2^{kd(1-1/p)}||\check{\chi}||_{L_p}\}\} =$$

$$= C_p\cdot \max\{||\psi(\cdot)[1 - \Phi(2^{-n}\cdot)]||_{M_p}, \sup_{k\in\mathbb{N}}\{2^{-k\sigma}||\varphi(2^{-k}\cdot)[1 - \Phi(2^{-n}\cdot)]||_{M_p}\}\} =$$

$$= C_p\cdot \max\{||\psi(\cdot)[1 - \Phi(2^{-n}\cdot)]||_{M_p}, \sup_{k\in\mathbb{N}}\{2^{-k\sigma}||\varphi(\cdot)[1 - \Phi(2^{k-n}\cdot)]||_{M_p}\}\} =: \Lambda_2.$$

Consider $\chi_{\epsilon_0} \in C_0^{\infty}(\mathbb{R}^{\mathrm{d}})$, such that $\operatorname{supp}\chi_{\epsilon_0} \subseteq \bar{\mathcal{U}}_{\epsilon_0}$, $\operatorname{supp}\chi_{\epsilon_0} \supset\supset \frac{1}{2}\bar{\mathcal{U}}_{\epsilon_0}$ and $\chi_{\epsilon_0}|_{\frac{1}{2}\bar{\mathcal{U}}_{\epsilon_0}} \equiv 1$.

$$\Lambda_2 = C_p\cdot \max\{||\psi(\cdot)[1 - \Phi(2^{-n}\cdot)][\chi_{\epsilon_0}(\cdot) + 1 - \chi_{\epsilon_0}(\cdot)]||_{M_p},$$

$$\sup_{k\in\mathbb{N}}\{2^{-k\sigma}||\varphi(\cdot)[1 - \Phi(2^{k-n}\cdot)][\chi_{\epsilon_0}(2^{k-n}\cdot) + 1 - \chi_{\epsilon_0}(2^{k-n}\cdot)]||_{M_p}\}\} \le$$

$$\leq C_p\cdot \max\{||\psi(\cdot)[1-\Phi(2^{-n}\cdot)][\chi_{\epsilon_0}(\cdot)||_{M_p}+||\psi(\cdot)[1-\Phi(2^{-n}\cdot)][1-\chi_{\epsilon_0}(\cdot)]||_{M_p},$$

$$\sup_{k\in\mathbb{N}}\{2^{-k\sigma}||\varphi(\cdot)[1-\Phi(2^{k-n}\cdot)]\chi_{\epsilon_0}(2^{k-n}\cdot)||_{M_p}+||\varphi(\cdot)[1-\Phi(2^{k-n}\cdot)][1-\chi_{\epsilon_0}(2^{k-n}\cdot)]||_{M_p}\}\}=:\Lambda_3.$$

Consider first the terms containing φ (i.e., $k\in\mathbb{N}$). The simple identity

$$e^{\beta}-e^{\alpha}=(\beta-\alpha)e^{\alpha}\int_0^1 e^{\varrho(\beta-\alpha)}d\varrho,$$

for $\beta=\ln\Phi(\xi)$, $\alpha=0$, yields

$$\Phi(\xi)-1=|\xi|^{\mu}\Gamma_0(\xi)\int_0^1 e^{\varrho|\xi|^{\mu}\Gamma_0(\xi)}d\varrho=|\xi|^{\sigma}\Gamma_{\mu-\sigma}(\xi)\int_0^1\Phi(\xi)^{\varrho}d\varrho=:|\xi|^{\sigma}G_{\sigma}(\xi), \tag{13}$$

where: $\xi\in\bar{\mathcal{U}}_{\epsilon_0}$ (recall that $|\Phi||_{\mathcal{U}_{\epsilon_0}}$ is bounded away from zero);

$$G_{\sigma}(\xi)=\Gamma_{\mu-\sigma}(\xi)\int_0^1\Phi(\xi)^{\varrho}d\varrho, G_{\sigma}(0)=0,\forall\sigma\in[0,\mu);\Gamma_{\mu-\sigma}(\xi):=|\xi|^{\mu-\sigma}\Gamma_0(\xi),\forall\xi\in\bar{\mathcal{U}}_{\epsilon_0}.$$

By homogeneity of $|\cdot|^{\sigma}$, $0\leq\sigma\leq\mu$, and by 1.2.3.(E),

$$2^{-k\sigma}||\varphi(\cdot)[1-\Phi(2^{k-n}\cdot)]\chi_{\epsilon_0}(2^{k-n}\cdot)||_{M_p}=$$

$$=2^{-k\sigma}||\varphi(\cdot)|2^{k-n}\cdot|G_{\sigma}(2^{k-n}\cdot)\chi_{\epsilon_0}(2^{k-n}\cdot)||_{M_p}=2^{-k\sigma+(k-n)\sigma}||\cdot|^{\sigma}\varphi(\cdot)G_{\sigma}(2^{k-n}\cdot)\chi_{\epsilon_0}(2^{k-n}\cdot)||_{M_p}\leq$$

$$\leq 2^{-n\sigma}||\cdot|^{\sigma}\varphi(\cdot)||_{M_p}||G_{\sigma}(2^{k-n}\cdot)\chi_{\epsilon_0}(2^{k-n}\cdot)||_{M_p}=2^{-n\sigma}||\cdot|^{\sigma}\varphi(\cdot)||_{M_p}||G_{\sigma}(\cdot)\chi_{\epsilon_0}(\cdot)||_{M_p}=:\Lambda_4.$$

$\varphi\in C_0^{\infty}(\mathbb{R}^{\mathrm{d}})$ and $0\notin\operatorname{supp}\varphi\Longrightarrow$

$$|\cdot|^{\sigma}\varphi(\cdot)\in C_0^{\infty}(\mathbb{R}^{\mathrm{d}})\subset\mathcal{S}(\mathbb{R}^{\mathrm{d}})\Longrightarrow|\cdot|^{\sigma}\varphi(\cdot)\in\mathrm{M_p},1\leq\mathrm{p}\leq\infty.$$

(13), Lemma 4.3, Remark 4.1, Lemma 4.4, $\operatorname{supp}\chi_{\epsilon_0}$ - compact $\Longrightarrow\bar{G}_{\sigma}\in L_{\infty}$, where

$$\bar{G}_{\sigma}(\xi):=G_{\sigma}(\xi)\cdot I_{\operatorname{supp}\chi_{\epsilon_0}}(\xi),$$

$I_{\operatorname{supp}\chi_{\epsilon_0}}$ being the indicator function of $\operatorname{supp}\chi_{\epsilon_0}$, $\xi\in\mathbb{R}^{\mathrm{d}}$. Lemma 4.1 $\Longrightarrow G_{\sigma}\chi_{\epsilon_0}\in M_p$, $1<p<\infty$. Therefore, $\Lambda_4\leq C(p,\sigma,\epsilon_0,d)2^{-n\sigma}$.

Consider now

$$\Lambda_5:=2^{-k\sigma}||\varphi(\cdot)[1-\Phi(2^{k-n}\cdot)][1-\chi_{\epsilon_0}(2^{k-n}\cdot)]||_{M_p}.$$

Case $k\geq n$: $2^{-k\sigma}\leq 2^{-k\sigma}\cdot 2^{(k-n)\sigma}=2^{-n\sigma}$, and it is enough to prove that

$$\Lambda_6:=||\varphi(\cdot)[1-\Phi(2^{k-n}\cdot)][1-\chi_{\epsilon_0}(2^{k-n}\cdot)]||_{M_p}\leq K<\infty,$$

K being independent of k and n. By the triangle inequality and 1.2.3.(B, E),

$$\Lambda_6\leq||\varphi||_{M_p}+||\varphi(\cdot)\Phi(2^{k-n}\cdot)||_{M_p}+||\varphi(\cdot)\chi_{\epsilon_0}(2^{k-n}\cdot)]||_{M_p}+||\varphi||_{M_p}||\varphi(2^{k-n}\cdot)\chi_{\epsilon_0}(2^{k-n}\cdot)]||_{M_p}\leq$$

$$\leq||\varphi||_{M_p}[1+||\Phi(2^{k-n}\cdot)||_{M_p}+||\chi_{\epsilon_0}(2^{k-n}\cdot)]||_{M_p}+||\Phi(2^{k-n}\cdot)\chi_{\epsilon_0}(2^{k-n}\cdot)||_{M_p}]=$$

$$=||\varphi||_{M_p}(1+||\Phi||_{M_p}+||\chi_{\epsilon_0}||_{M_p}+||\Phi\chi_{\epsilon_0}||_{M_p})\leq||\varphi||_{M_p}(1+||\Phi||_{M_p})(1+||\chi_{\epsilon_0}||_{M_p})=:\Lambda_7.$$

By construction of χ_{ϵ_0}, $\chi_{\epsilon_0}\in C_0^{\infty}(\mathbb{R}^{\mathrm{d}})\subset M_p$, $\forall p:1\leq p\leq\infty$, therefore, $\exists C'=C'(p,\epsilon_0)<\infty$, $||\chi_{\epsilon_0}||_{M_p}=C'(p,\epsilon_0)$. By stability assumption about $\{\Phi_n\}_{n=1}^{\infty}$, $||\Phi_n||_{M_p}\leq C_1$, $\forall n\in\mathbb{N}$; by Theorem 3.1 in [6], $\Phi_n\to\Phi$ uniformly on compacts in $\mathbb{R}^{\mathrm{d}}$; hence 1.2.3.(I) implies $\Phi\in M_p$ and $||\Phi||_{M_p}\leq C_1$, $\Longrightarrow$ $\Lambda_6\leq C''(1+C_1)(1+C')=K(p,\epsilon_0,C_1)<\infty$, $\Longrightarrow\Lambda_6\leq K=K(p,\epsilon_0,C_1)$, $\Longrightarrow\Lambda_5\leq K\cdot 2^{-n\sigma}$.

Case $k < n$: $\Lambda_5 = 0$. (This was the reason to select the k_0 - equivalent norm in Besov spaces with $2^{k_0+1} \leq \epsilon_0$. The choice of k_0 to be the *largest possible* ensures that, of all appropriate k_0 - equivalent norms, we select the one with equivalence constants closest to 1.)

The proof for the terms containing ψ (i.e., $k = 0$) is the same, with one exception: it cannot be claimed that $|\cdot|^\sigma \psi(\cdot) \in \mathcal{S}$, since $0 \in \operatorname{supp} \psi$ and $|\cdot|^\sigma$ is non-smooth at 0 when $\sigma \notin 2\mathbb{N}$. Nevertheless, $|\cdot|^s \psi(\cdot) \in M_p$, $1 < p < \infty$, follows from Lemma 4.1 in the same way as for $G_\sigma \chi_{\epsilon_0}$. □

Proof of Corollary 2.3.2. Follows from Theorem 2.2.1 and Theorem 2.3.1. □

The following lemma, related to the results in subsection 2.4, is of independent interest in itself. It is a generalization of Theorem 4.3 in [6] (with $T = (1/2)\mathrm{Id}_{\mathbb{R}^{\mathrm{d}}}$, in the notations of [6]).

Lemma 4.5. *If there exists* $\gamma \in G_\infty$ *such that* $\arg \Phi(\xi) \equiv \langle \gamma, \xi \rangle$, $\xi \in \mathbb{R}^{\mathrm{d}}$, *then* Φ *is integrable.*

Proof: $\gamma \in G_\infty \Longrightarrow \exists n_0 \in \mathbb{N} : \gamma \in G_{n_0}$ (for the notations G_∞ and G_n, see [4,6]). The proof is along the same lines as the one of Theorem 4.3 in [6], taking $n > n_0$ and $F(\gamma)$ instead of $F(0)$ in the estimate of $\mathcal{J}_n$ as defined in [6]. □

Proof that 2.4.(i") $\Longrightarrow$ 2.4.(i) in Remark 2.4.1. Assume that 2.4.(i") holds for $\Phi(\xi)$. Then,

$$\Phi(\xi)^2 = \Phi_1(\xi)^2 e^{i\langle 2\beta, \xi \rangle}, \Phi_1(\xi) \in \mathbb{R}, \beta \in \mathrm{G}_\infty.$$

Therefore,

$$|\Phi(\xi)^2| = \Phi_1(\xi)^2, \arg[\Phi(\xi)^2] = \langle \gamma, \xi \rangle, \gamma = 2\beta \in G_\infty.$$

By Lemma 4.5, $\Phi(\xi)^2$ is integrable $\Longrightarrow$ 2.4.(i). □

Proof that $C_p(f) = ||f||_{L_p}$ in Example 2.4.1. See 2.4.(5). □

Proof of Theorem 2.4.1. As noted in Remark 2.4.2, Definition 2.4.1.(ii) is equivalent to the requirement that $\{||f * \mathcal{F}^{-1}(Q^n)||_{L_p} : n \in \mathbb{N}\}$ be a bounded set in $\mathbb{R}$ for any $p : 1 \leq p \leq \infty$ and any $f \in L_p$. Fix $p : 1 \leq p \leq \infty$. Since L_p is a Banach space, the uniform-boundedness (Banach-Steinhaus) principle implies that there exists $C = C_p < \infty$, such that

$$||Q^n||_{M_p} \leq C_p, \forall n \in \mathbb{N}. \tag{14}$$

The idea of the rest of this proof is to show that, under the conditions of Theorem 2.4.1, (14) fails for $p \neq 2$. Indeed, assume that (14) is fulfilled for some $p \in [1, 2) \bigcup (2, \infty]$. Then, by 2.4.(a-c), 2.4.(3,4), and by the assumed conditions (i, ii) of Theorem 2.4.1,

$$Q(\xi) = e^{i\{\langle \alpha, \xi \rangle + \sigma_\nu(\xi)[1+o(1)]\}}, \xi \to 0. \tag{15}$$

Consider $Q_\alpha(\xi) := Q(\xi) e^{-i\langle \alpha, \xi \rangle}$, $\xi \in \mathbb{R}^{\mathrm{d}}$, $\alpha \in \mathbb{R}^{\mathrm{d}}$. By 1.2.3.(F), (14) implies

$$||Q_\alpha^n||_{M_p} = ||Q^n||_{M_p} \leq C_p, \forall n \in \mathbb{N}. \tag{16}$$

$\sigma_\nu(\xi)$ being homogeneous of order $\nu \geq 2$, it follows from (15) that

$$\lim_{n \to \infty} \{[Q_\alpha(n^{-1/\nu}\xi)]^n\} = \lim_{n \to \infty} e^{ni\sigma_\nu(n^{-1/\nu}\xi)} = e^{i\sigma_\nu(\xi)}, \tag{17}$$

uniformly on compacts in $\mathbb{R}^{\mathrm{d}}$.

Therefore, by 1.2.3.(E, I), utilizing (17, 16), one obtains

$$e^{i\sigma_\nu(\cdot)} \in M_p. \tag{18}$$

However, (18) contradicts to Corollary 1.5.3 in [2] (see 1.2.3.(K)). □

References

[1] J. Bergh, J. Löfström. *Interpolation Spaces. An Introduction.* Grundl. der Math. Wiss., 233. Berlin, Heidelberg, New York: Springer, 1976.

[2] P. Brenner, V. Thomée, L.B. Wahlbin. *Besov spaces and Applications to Difference Methods for Initial-Value Problems.* Lect. Notes in Math., 434. Berlin, Heidelberg, New York: Springer, 1975.

[3] I. Daubechies. *Ten Lectures on Wavelets.* Philadelphia: SIAM, 1992.

[4] G. Deslauriers, J. Dubois, S. Dubuc. Multidimensional iterative interpolation. *Can. J. Math.* **43** (1991), 297-312.

[5] G. Deslauriers, S. Dubuc. Symmetric iterative interpolation processes. *Constr. Approx.* **5** (1989) 49-68.

[6] G. Deslauriers, S. Dubuc. Continuous iterative interpolation processes. In: *Iteration Theory* (Ch. Mira, N. Netzer, C. Simó, Gy. Targonski Eds.). Singapore: World Scientific, 1991.

[7] S.M. Nikol'skiĭ. *Approximation of Functions of Several Variables and Imbedding Theorems.* Grundl. der Math. Wiss., 205. Berlin: Springer, 1975.

[8] M. Reed, B. Simon. *Methods of Modern Mathematical Physics, v.1: Functional Analysis.* New York, London: Academic Press, 1972.

[9] M. Reed, B. Simon. *Methods of Modern Mathematical Physics, v.2: Fourier Analysis, Self Adjointness.* New York: Academic Press, 1975.

[10] V.S. Vladimirov. *Generalized Functions in Mathematical Physics.* Moscow: Nauka, 1979. (In Russian.)