An integrable chain and bi-orthogonal polynomials

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Abstract

An integrable chain connected to the isospectral evolution of the polynomials of type $R-I$ introduced by Ismail and Masson is presented. The equations of motion of this chain generalize the corresponding equations of the relativistic Toda chain introduced by Ruijsenaars. We study simple self-similar solutions to these equations that are obtained through separation of variables. The corresponding polynomials are expressed in terms of the Gauss hypergeometric function. It is shown that these polynomials are stable (up to shifts of the parameters) against Darboux transformations of the generalized chain.

1. Introduction

It is well known that the equations of motion for the ordinary Toda chain can be obtained from prescribing a time evolution upon the ordinary orthogonal polynomials that satisfy the three-term recurrence relation

$$P_{n+1}(x) + v_n P_n(x) + u_n P_{n-1}(x) = xP_n(x), \quad P_0 = 1,\ P_1(x) = x - b_0. \tag{1.1}$$

Indeed, assume that the recurrence coefficients b_n, u_n and hence the polynomials $P_n(x;t)$ themselves depend on an additional parameter t (a "time" in the integrable systems terminology). The polynomials $P_n(x)$ are monic, meaning that $P_n(x) = x^n + O(x^{n-1})$. Hence $\dot{P}_n(x;t) \equiv \frac{dP_n(x,t)}{dt}$ will be a polynomial of the degree $n-1$. The simplest possible Ansatz

$$\dot{P}_n(x) = -u_n P_{n-1}(x) \tag{1.2}$$

leads to the following equations for the recurrence coefficients

$$\dot{u}_n = u_n(b_n - b_{n-1}), \quad \dot{b}_n = u_{n+1} - u_n \tag{1.3}$$

which coincide with the ordinary Toda chain equations [9].

In [5], the same Ansatz was exploited in order to derive the *relativistic* Toda chain equations starting from the spectral problem associated to the Laurent biorthogonal plynomials (LBP) that satisfy the recurrence relation [2]

$$P_{n+1}(z) + d_n P_n(z) = z(P_n(z) + b_n P_{n-1}(z)), \quad P_0 = 1, P_1(z) = z - d_0. \tag{1.4}$$

The LBP obtained from (1.4) are also monic and one can hence similarly try the following Ansatz

$$\dot{P}_n(z) = \frac{b_n}{d_n} P_{n-1}(z). \tag{1.5}$$

This Ansatz leads to the following equations [5]

$$\begin{aligned} \dot{d}_n &= \frac{b_n}{d_{n-1}} - \frac{b_{n+1}}{d_{n+1}}, \\ \dot{b}_n &= b_n \left(\frac{1}{d_{n-1}} - \frac{1}{d_n} \right). \end{aligned} \tag{1.6}$$

For reference, let us point out that another possible Ansatz is [5]

$$\dot{P}_n(z) = -b_n(P_n(z) - zP_{n-1}(z)). \tag{1.7}$$

It leads to the equations

$$\begin{aligned} \dot{d}_n &= -d_n(b_{n+1} - b_n) \\ \dot{b}_n &= -b_n(b_{n+1} - b_{n-1} + d_{n-1} - d_n). \end{aligned} \tag{1.8}$$

In spite of the apparent difference between equations (1.6) and (1.8), it has been shown in [5] that these two systems are both equivalent to the relativistic Toda chain equations.

It is natural to consider more general systems of polynomials satisfying more complicated recurrence relations and to seek new integrable chains connected with them.

In this paper we present one such chain related to the $R-I$ biorthogonal polynomials introduced in [3] and satisfying the recurrence relation

$$P_{n+1}(z) + d_n P_n(z) = z P_n(z) + b_n(z-\beta_n) P_{n-1}(z) \tag{1.9}$$

with initial conditions

$$P_0 = 1,\ P_1(z) = z - d_0 \tag{1.10}$$

where d_n, b_n, β_n are some given coefficients. Note that when $\beta_n = const$ we return (after shifting the argument $z - \beta \to z$) to the LBP.

In the next section we recall some basic results about the polynomials of $R-I$ type. In Section 3 we derive the associated integrable chain and construct its Darboux-Bäcklund transformations. In section 4 we obtain special solutions to our chain by separating the time and spatial variables.

2. Polynomials of $R-I$ type and their simplest properties

The polynomials of $R-I$ type obtained through the recurrence relation (1.9) are obviously monic: $P_n(z) = z^n + O(z^{n-1})$. Ismail and Masson established the following orthogonality property for these polynomials [3].

Assume that the roots of the polynomials $P_n(z)$ do not coincide with the points β_n and moreover that $b_n \neq 0$. There then exists a linear functional $\mathcal{L}$ defined on the space of all possible rational functions $A(z)/B_n(z)$ with prescribed denominators $B_n(z) = (z-\beta_1)(z-\beta_2)\dots(z-\beta_n),\ n = 1, 2, \dots$ such that

$$\mathcal{L}\left\{\frac{P_n(z)z^j}{B_n(z)}\right\} = 0, \quad 0 \le j < n\,. \tag{2.1}$$

The relation (2.1) is the basic orthogonality property for the polynomials of $R-I$ type. In contrast to the standard situation, (2.2) implies that instead of being orthogonal the polynomials of $R-I$ type are *biorthogonal* with respect to some other rational functions. Under some conditions, the linear functional can be presented as integral over some contour in the complex domain with some measure μ (see, e.g. [3])

$$\int_C \frac{P_n(z)z^j}{B_n(z)} d\mu(z) = 0, \quad 0 \le j < n \tag{2.2}$$

Note that the linear functional $\mathcal{L}$ is uniquely defined by the recurrence coefficients b_n, d_n, β_n.

The orthogonality relation (2.1) leads to the folowing bi-orthogonality relation (for details see [11]). Introduce the rational functions

$$R_n(z) = (-1)^n \frac{P_n(z)}{b_1 \dots b_n (z-\beta_1)\dots(z-\beta_n)}. \tag{2.3}$$

These functions satisfy the recurrence relation

$$R_{n-1}(z) + d_n\, R_n(z) + b_{n+1}\beta_{n+1}\, R_{n+1}(z) = z(R_n(z) + b_{n+1}\, R_{n+1}(z)) \tag{2.4}$$

which is formally the transposed of the recurrence relation (1.9). We can now construct another set of rational functions

$$H_n(z) = R_n(z) + b_{n+1} R_{n+1}(z) \tag{2.5}$$

which possesses the bi-orthogonality property

$$\mathcal{L}\{P_n(z) H_m(z)\} = 0, \quad n \neq m. \tag{2.6}$$

The rational functions $H_n(z)$ admit the representation

$$H_n(z) = \frac{Q_n(z)}{(z-\beta_1)\dots(z-\beta_{n+1})} \tag{2.7}$$

where the monic polynomials $Q_n(z)$ are

$$Q_n(z) = \sigma_n\left(P_{n+1}(z) - (z-\beta_{n+1})\, P_n(z)\right) \tag{2.8}$$

and

$$\sigma_n = 1/(b_n - d_n + \beta_{n+1}). \tag{2.9}$$

It turns out that the polynomials $Q_n(z)$ also belong to the $R-I$ class. They satisfy the recurrence relation

$$Q_{n+1}(z) + (\tilde{d}_n - z)Q_n(z) - \tilde{b}_n(z-\beta_{n+1})Q_{n-1}(z) = 0 \tag{2.10}$$

where

$$\tilde{b}_n = b_n\frac{\sigma_n}{\sigma_{n-1}},\ \tilde{d}_n = \beta_{n+1} + \frac{\sigma_n}{\sigma_{n-1}}(d_{n+1} - \beta_{n+2}) \tag{2.11}$$

It is naturally to call $Q_n(z)$ the $R-I$ polynomials dual to $P_n(z)$.

The polynomials $P_n(z)$ admit the following spectral transformations.

Assume that μ is an arbitrary parameter which does not coincide with any of the zeros of $P_n(z)$. Introduce the new polynomials

$$\tilde{P}_n(z) = \frac{P_{n+1}(z) - A_n\, P_n(z)}{z-\mu} \tag{2.12}$$

where $A_n = P_{n+1}(\mu)/P_n(\mu)$. Then it is easily shown that the polynomials $\tilde{P}_n(z)$ are again the $R-I$ polynomials satisfying the recurrence relation

$$\tilde{P}_{n+1}(z) + \tilde{d}_n\tilde{P}_n(z) = z\tilde{P}_n(z) + \tilde{b}_n(z-\beta_n)\tilde{P}_{n-1}(z) \tag{2.13}$$

where

$$\begin{aligned} \tilde{b}_n &= b_n\frac{A_n + b_{n+1}}{A_{n-1} + b_n} \\ \tilde{d}_n &= -\frac{\tilde{b}_n A_{n-1}}{b_n} + d_{n+1} + A_{n+1}. \end{aligned} \tag{2.14}$$

It is natural to call Christoffel transformation, the transformation from the polynomials $P_n(z)$ to $\tilde{P}_n(z)$ because in the case of the ordinary orthogonal polynomials, the formula (2.12) is precisely the Christoffel transformation ([1], [8]).

Note that the parameters β_n remain the same under the transformation (2.12). The new linear functional $\tilde{\mathcal{L}}$ (corresponding to the polynomials $\tilde{P}_n(z)$) is expressed as

$$\tilde{\mathcal{L}} = (z-\mu)\mathcal{L} \tag{2.15}$$

which is in complete accordance with the ordinary case ([1]).

Reciprocal to the Christoffel transformation (CT) is the Geronimus transformation (GT) defined by

$$\tilde{P}_n(z) = u_n P_n(z) + v_n(z-\beta_n)P_{n-1}(z), \tag{2.16}$$

where u_n, v_n are coefficients related by the condition $u_n + v_n = 1$ (which obviously follows from the monic property of the polynomials $P_n(z)$). Explicitly, we have

$$v_n = \frac{\phi_n/\phi_{n-1}}{\beta_n - \mu + \phi_n/\phi_{n-1}} \tag{2.17}$$

where ϕ_n is an arbitrary *non-polynomial* solution of the difference equation

$$\phi_{n+1} + (d_n - \mu)\phi_n = b_n(\mu - \beta_n)\phi_{n-1}, \tag{2.18}$$

and μ is an arbitrary spectral parameter.

It is easily verified that the polynomials $\tilde{P}_n(z)$ defined through (2.16) do satisfy the recurrence relation of the $R - I$ type with coefficients

$$\begin{aligned} \tilde{b}_n &= b_{n-1} \frac{v_n}{v_{n-1}} \\ \tilde{d}_n &= d_n \frac{u_{n+1}}{u_n} + \beta_{n+1} \frac{v_{n+1}}{u_n} - \beta_n \frac{v_n}{u_n}. \end{aligned} \tag{2.19}$$

The GT is reciprocal to the CT in the following sense. Let $\tilde{P}_n(z)$ be the LBP obtained from $P_n(z)$ by the CT (2.12). Using the recurrence relations, it is then easily verified that the polynomials $P_n(z)$ are expressed in terms of $\tilde{P}_n(z)$ as $P_n(z) = u_n \tilde{P}_n(z) + v_n P_{n-1}(z)$ with some appropriate coefficients u_n, v_n, i.e. just as in the GT.

3. Equations of the generalized relativistic Toda chain

In this section we derive the dynamical equations of the integrable chain connected with the polynomials of $R - I$ type. This chain is a natural extension of the relativistic Toda chain.

To that end, assume that the polynomials $P_n(z; t)$ depend on some additional parameter (time) t. Obviously the recurrence coefficients $d_n(t), b_n(t)$ are also assumed to depend on time.

We choose the following Ansatz:

$$\dot{P}_n(z; t) = \kappa_n \left((z - \gamma_n) P_{n-1}(z; t) - P_n(z; t)\right) \tag{3.1}$$

where $\kappa_n(t)$ and $\gamma_n(t)$ are some functions to be determined. It is clear that the rhs of (3.1) is a polynomial of order $n - 1$.

The compatibility of the Ansatz (3.1) and the recurrence relation (1.9) demands the following restrictions:

1)$\gamma_n = \beta_n$;

2)β_n does not depend on t;

3)$\kappa_n(t) = b_n(t)\alpha(t)$,

where $\alpha(t)$ can be an arbitrary function of t. This arbitrariness is superficial, for it simply entails a change of the time variable.

Choosing $\alpha(t) = 1$, we get

$$\dot{P}_n(z; t) = b_n \left((z - \beta_n) P_{n-1}(z; t) - P_n(z; t)\right) = -\frac{b_n}{\sigma_{n-1}} Q_{n-1}(z) \tag{3.2}$$

where σ_n is given by (2.9) and $Q_n(z)$ are polynomials given by (2.8).

We then obtain the following equations for the coefficients $b_n(t)$ and $d_n(t)$

$$\begin{aligned} \dot{b}_n &= b_n(b_{n-1} - b_{n+1} + d_n - d_{n-1}) \\ \dot{d}_n &= \beta_{n+1}b_{n+1} - \beta_n b_n + d_n(b_n - b_{n+1}). \end{aligned} \tag{3.3}$$

We shall refer the equations (3.3) as the generalized relativistic Toda chain equations (GRTC). If $\beta_n = 0$, we recover the ordinary relativistic Toda chain (RTC) equations [5]. (The case $\beta_n = const$ is also reduced to the RTC by shifting the coefficients d_n.)

Note that in contrast to the ordinary integrable chains (like the one of Toda and its relativistic version), the chain (3.3) contains *arbitrary external* parameters β_n.

An important property of the GRTC is its covariance with respect to the Darboux transformations introduced in the previous section. Let $b_n(t), d_n(t)$ be some solution of the GRTC with $P_n(z;t)$ the corresponding $R-I$ polynomials. The new coefficients constructed by the formulas (2.14) and (2.19) are also seen to satisfy the GRTC equations.

Note that there are two possible type of the Darboux transformations for the GRTC.

It is worth mentioning that the Ansatz (3.2) (that leads to the equations (3.3)) can be obtained as some "infinitesimal" limit of the Christoffel transformation (2.12).

Indeed, consider the Christoffel transformation (2.12) for sufficiently large values of the spectral parameter μ. From the recurrence relation

$$P_{n+1}(\mu) + (d_n - \mu)\,P_n(\mu) - b_n(\mu - \beta_n)\,P_{n-1}(\mu) = 0$$

we easily find

$$A_n = \frac{P_{n+1}(\mu)}{P_n(\mu)} = \mu + b_n - d_n + O(1/\mu) \tag{3.4}$$

Hence from (2.12), one obtains

$$\begin{aligned} \tilde{P}_n(z) - P_n(z) &= \frac{(z + b_n - d_n)P_n(z) - P_{n+1}(z)}{\mu} + O(1/\mu^2) = \\ &\frac{b_n(P_n(z) - (z-\beta_n)P_{n-1}(z))}{\mu} + O(1/\mu^2) \end{aligned} \tag{3.5}$$

Now it is sufficient to take the following parametrization of the time variable

$$\tilde{P}_n(z) - P_n(z) = -\frac{\dot{P}_n(z)}{\mu} + O(1/\mu^2) \tag{3.6}$$

to get the Ansatz (3.2).

In this respect the Christoffel transformation (2.12) can be considered as defining a "discrete-time evolution"with respect to some discrete variable τ in the sense that $\tilde{P}_n(z;\tau) = P_n(z;\tau - 1/\mu)$ In other words, the Christoffel transformation can be interpreted as giving the outcome of shifts in the variable τ). In the limit $\mu \to \infty$ the discrete variable τ becomes the continuous variable t.

4. A special solution with separated variables

We shall consider here a class of solutions of the GRTC.

Choose the following separation of variables $b_n(t) = n\chi(t)$ where the function $\chi(t)$ depends on t only. Substituting in (3.3), we get the following solution

$$\begin{aligned} d_n(t) &= \xi\, n + \eta(t) \\ \beta_n &= a_1 n + a_2 + a_3/n \end{aligned} \tag{4.1}$$

where a_1, a_2, a_3 are arbitrary constants and the relations between the functions $\chi(t), \xi(t), \eta(t)$ are given by the equations

$$\begin{aligned}\dot{\xi} &= -\chi\xi + 2a_1\chi \\ \dot{\eta} &= -\chi\eta + (a_1 + a_2)\chi \\ \dot{\chi} &= \chi(\xi - 2\chi).\end{aligned} \tag{4.2}$$

In general the constant a_1, a_2, a_3 are all arbitrary. However if we want the relation (3.2) to be valid for $n = 0$ we should put $a_3 = 0$.

We then get the recurrence relation for the corresponding polynomials $P_n(z)$:

$$P_{n+1}(z) + (\xi\, n + \eta - z)P_n(z) - n\,\chi\,(z - a_1 n - a_2)P_{n-1}(z) = 0. \tag{4.3}$$

The difference equation (4.3) belongs to the class of those having solutions in terms of the Gauss hypergeometric functions (see, e.g. [6]). In particular, for the polynomial solutions we have

$$P_n(z) = (c)_n\,(-\gamma f)^{-n}\,{}_2F_1(-n, b; c; 1 + f) \tag{4.4}$$

where

$$\begin{aligned}b &= \frac{\chi\gamma^2(z - a_1 - a_2) + \gamma(z - \eta)}{2 + \gamma\xi} \\ c &= \frac{\chi\gamma^2(z - a_1 - a_2)}{1 + \gamma\xi} \\ f &= \frac{1}{\gamma\xi + 1}\end{aligned} \tag{4.5}$$

and γ is a root of the quadratic equation $a_1\chi\gamma^2 + \xi\gamma + 1 = 0$.

The system (4.2) leads to the following equation for $\zeta = \xi - 2a_1$

$$\dot{\zeta} = \zeta(C\,\zeta^2 - a_1 - \zeta) \tag{4.6}$$

where C is an integration constant. We then have for $\chi(t)$ and $\eta(t)$

$$\begin{aligned}\chi(t) &= a_1 + \zeta(t) - C\,\zeta^2(t) \\ \dot{\eta} &= \chi(t)(a_1 + a_2 - \eta(t))\end{aligned}$$

In general, the functions $\chi(t), \eta(t), \xi(t)$ cannot be expressed in terms of elementary ones. However if one imposes the restriction $C = 0$ we then have

$$\xi(t) = a_1 + \chi(t) \tag{4.7}$$

and the system (4.2) has the following general solution

$$\begin{aligned}\chi(t) &= \frac{a_1}{1 + C_1 e^{-a_1 t}} \\ \xi(t) &= a_1\,\frac{2 + C_1 e^{-a_1 t}}{1 + C_1 e^{-a_1 t}} \\ \eta(t) &= \frac{(a_1 + a_2)\,e^{a_1 t} + C_2}{C_1 + e^{a_1 t}}\end{aligned} \tag{4.8}$$

where $C_{1,2}$ are arbitrary constants.

Let us consider the Darboux transformations of the solutions (4.8). It is easily verified that for $\mu = C_2/C_1$ one has

$$A_n = P_{n+1}(\mu)/P_n(\mu) = \frac{\mu - a_1 n - a_1 - a_2}{1 + C_1 e^{-a_1 t}}. \tag{4.9}$$

Using formulas (2.14), we find for the transformed recurrence coefficients

$$\begin{aligned} \tilde{b}_n &= b_n \\ \tilde{d}_n &= d_n + \frac{a_1 C_1}{C_1 + e^{a_1 t}}. \end{aligned} \tag{4.10}$$

Hence the Darboux transformations preserve the class of special solutions (4.8). Only the parameter C_2 is changed under such transformations: $C_2 \to C_2 + a_1 C_1$, whereas all other parameters a_1, a_2, C_1 are unaffected.

The expression (4.4) for the polynomials themselves also simplifies. We now have for the parameters

$$b = \frac{C_2 - zC_1}{C_1\, a_1},\ c = \frac{a_1 + a_2 - z}{a_1}$$
$$f = -1 - C_1 e^{-a_1 t},\ \gamma = -1/a_1.$$

Hence the Darboux transformation (2.14) leads to a shift in the parameter $b \to b+1$ whereas all other parameters remain unchanged.

Using well known transformation formulas for the Gauss hypergeometric function, it is convenient to rewrite the expression for the polynomials $P_n(z)$ in one of the following two form

$$P_n(z) = (-a_1)^n \,(1 + (a_2 - z)/a_1)_n \,{}_2F_1\left({-n, \frac{C_1 a_1 + C_1 a_2 - C_2}{C_1\, a_1} \atop \frac{a_1 + a_2 - z}{a_1}} ; \frac{C_1}{C_1 + e^{a_1 t}} \right). \tag{4.11}$$

or

$$P_n(z) = \left(\frac{-a_1\, e^{a_1 t}}{C_1 + e^{a_1 t}} \right)^n \left(\frac{C_1 a_1 + C_1 a_2 - C_2}{C_1\, a_1} \right)_n {}_2F_1\left({-n, \frac{C_2 - C_1 z}{C_1\, a_1} \atop \frac{C_2 - C_1 a_2 - n a_1 C_1}{C_1 a_1}} ; 1 + C_1 e^{-a_1 t} \right). \tag{4.12}$$

Then, upon putting $a_2 = -1, C_1 = -1, C_2 = a_1\,\rho$ and taking the limit $a_1 \to 0$, one obtains

$$P_n(z) = z^n \,{}_2F_0\left({-n, 1+\rho \atop -} ; 1/(zt) \right) \tag{4.13}$$

The polynomials (4.13) are seen to coincide with those obtained in [5] where the separation of variables for the Laurent biorthogonal polynomials was considered.

Let us look finally at the polynomials $Q_n(z)$ (2.8) under the restriction $C = 0$. It is easily seen that $\sigma_n = 1/(a_1 + a_2 - \eta(t))$. Hence, from the formulas (2.11) we get the recurrence coefficients: $\tilde{b}_n = b_n, \tilde{d}_n = d_n + \chi(t)$. This means that the polynomials $Q_n(z)$ are obtained from $P_n(z)$ by a simple shift of the parameter $a_2 \to a_2 + a_1$. We thus have

$$Q_n(z) = a_1^n \,(2 + (a_2 - z)/a_1)_n \,{}_2F_1\left({-n, \frac{2C_1 a_1 + C_1 a_2 - C_2}{C_1\, a_1} \atop \frac{2a_1 + a_2 - z}{a_1}} ; \frac{C_1}{C_1 + e^{a_1 t}} \right). \tag{4.14}$$

We see therefore that the polynomials $Q_n(z)$ belong to the same class of solutions as the polynomials $P_n(z)$.

5. The orthogonality measure

We shall now determine the orthogonality measure for the polynomials (4.12) obtained in the previous section.

For this we first consider the ordinary Meixner polynomials $M_n(x;\gamma,c)$ defined by [4]

$$M_n(x;\gamma,c) = {}_2F_1\left({-n, -x \atop \gamma}; 1-1/c\right) \tag{5.1}$$

where $\gamma > 0$ and $0 < c < 1$. These polynomials are orthogonal with respect to a discrete measure [4]

$$\sum_{k=0}^{\infty} w_k M_n(k) k^j = 0, \quad j = 0, 1, \dots, n-1 \tag{5.2}$$

where the weight function w_k is the Pascal distribution

$$w_k(\gamma, c) = \frac{(\gamma)_k c^k}{k!}. \tag{5.3}$$

Observe now that for arbitrary positive integer j

$$w_k(\gamma - j, c) = const \frac{w_k(\gamma, c)}{(\gamma + k - 1)(\gamma + k - 2)\dots(\gamma + k - j)}. \tag{5.4}$$

Hence, we also have the orthogonality relation

$$\sum_{k=0}^{\infty} \frac{(\gamma)_k c^k \, {}_2F_1\left({-n,-x \atop \gamma-n}; 1-1/c\right) k^j}{k!(k+\gamma-1)(k+\gamma-2)\dots(k+\gamma-n)} = 0, \quad j = 0, 1, \dots, n-1. \tag{5.5}$$

Introduce the polynomials

$$T_n(x) = M_n(x; \gamma - n, c) = {}_2F_1\left({-n, -x \atop \gamma - n}; 1 - 1/c\right) \tag{5.6}$$

which differ from the ordinary Meixner polynomials by a shift in the parameter $\gamma \to \gamma - n$. From (5.5) we obtain that the polynomials $T_n(x)$ are in fact the $R-I$ polynomials satisfying the orthogonality relation of the type (2.1) where the measure is located at the points $z = 0, 1, 2, \dots$ and

$$\beta_k = k - \gamma, \; k = 1, 2, \dots \tag{5.7}$$

More explicitly, we have

$$\sum_{k=0}^{\infty} \frac{(\gamma)_k c^k \, T_n(k) \, k^j}{k!(k - \beta_1)\dots(k - \beta_n)} = 0, \quad j = 0, 1, \dots, n-1 \tag{5.8}$$

with β_k given by (5.7).

Returning to our case, we notice that if one identifies

$$\gamma = \frac{C_2 - C_1 a_2}{C_1 a_1}, \quad x = -\frac{C_2 - C_1 z}{C_1 a_1}, \quad c = -e^{a_1 t}/C_1, \tag{5.9}$$

one obtains (up to an unnecessary factor not depending on x) that

$$P_n(z) = T_n(x) \tag{5.10}$$

with $T_n(x)$ as defined in (5.6).

Our polynomials $P_n(z)$ are hence orthogonal with respect to a discrete measure. It should be noted however that such measure cannot exists at all values of the time t. Indeed assume that $a_1 < 0, C_1 < 0$. Then the requirement $0 < c < 1$ (needed for the convergence of the weight function) is fulfilled only for $t > -\frac{\ln(-c_1)}{a_1}$. Moreover we should demand that $\gamma \neq N, \quad N = 0, \pm 1, \pm 2, \ldots$ as is clear from (5.5).

What happens when these restrictions are violated is an interesting open problem.

6. The operator form of the evolution equation

In this section we show that the evolution equations (3.3) admit a remarkable operator form which can be considered as some generalization of the Lax representation of the ordinary Toda-chain evolution equation.

Introduce two operators A, B which act on some discrete basis $|n\rangle, \ n = 0, 1, \ldots$ according to

$$\begin{aligned} A|0\rangle &= d_0|0\rangle + b_1\,\beta_1\,|1\rangle \\ A|n\rangle &= |n-1\rangle + d_n|n\rangle + b_{n+1}\,\beta_{n+1}\,|n+1\rangle \\ B|n\rangle &= |n\rangle + b_{n+1}\,|n+1\rangle, \quad n = 0, 1, 2 \ldots \end{aligned} \tag{6.1}$$

Let $\Phi = \sum_{n=0}^{\infty} \psi_n |n\rangle$ be an eigenstate of the generalized eigenvalue problem

$$A\,\Phi = \lambda\,B\,\Phi \tag{6.2}$$

from (6.2) and (6.1) we get the recurrence relations for the expansion coefficients

$$\psi_{n+1} + (d_n - \lambda)\psi_n - b_n(\lambda - \beta_n)\psi_{n-1} = 0 \tag{6.3}$$

This relation coincides with equation (1.9) for the polynomials $P_n(\lambda)$. Assume that the oprators A, B (as well as the state Φ) are functions of t. It can then be shown that if a pair of operators L, M exists such that the generalized Lax representation [10]

$$\dot{A} = AL - MA, \quad \dot{B} = BL - MB \tag{6.4}$$

is realized, we then have *an isospectral flow*

$$A(t)\Phi(t) = \lambda B(t)\Phi(t) \tag{6.5}$$

i.e. the eigenvalue λ does not depend on t if the evolution of the state $\Phi(t)$ is governed by the equation

$$\dot{\Phi} = -L\Phi. \tag{6.6}$$

For our purpose it is sufficient to choose both L, M as two-diagonal operators

$$\begin{aligned} L|n\rangle &= u_n^{(1)}|n\rangle + u_n^{(2)}|n-1\rangle \\ M|n\rangle &= v_n^{(1)}|n\rangle + v_n^{(2)}|n-1\rangle \end{aligned} \tag{6.7}$$

Substituting (6.7) into (6.4) we arrive at the restriction

$$u_n^{(2)} = v_n^{(2)} = const. \tag{6.8}$$

Without loss of generality we can put $u_n^{(2)} = v_n^{(2)} = -1$. The remaining coefficients are then of the form

$$u_n^{(1)} = b_n - d_n + \alpha(\lambda), \quad v_n^{(1)} = b_{n+1} - d_n + \alpha(\lambda) \tag{6.9}$$

where $\alpha(\lambda)$ is an arbitrary function of λ (not depending on t).

The operator system (6.4) is equivalent to our evolution equations (3.3) and the equation (6.6) is equivalent to

$$\dot{\psi}_n = -u_n^{(1)}\psi_n - u_{n+1}^{(2)}\psi_{n+1} = \psi_{n+1} - (b_n - d_n + \alpha(\lambda))\psi_n \tag{6.10}$$

Choosing $\alpha(\lambda) = \lambda$ we recover the Ansatz (3.2).

This provides the generalized Lax representation for our chain. It should be mentioned that the basic formalism (6.4) was developed in [10] in the context of isospectral flows connected with generalized eigenvalue problems. For the case of the relativistic Toda chain this Lax representation was proposed in [7] We have here another (slightly more general) example of such an isospectral flow.

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