Separating coordinates for the generalized Hitchin systems and the classical r-matrices

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Abstract

We exhibit natural Darboux coordinates for the generalized Hitchin systems studied by Bottacin and Markman. These systems are defined on spaces of stable pairs consisting of a vector bundle and a form-valued meromorphic endomorphism of the bundle. In special cases (genus zero, genus one), the bundles are rigid and one has the rational, trigonometric and elliptic Gaudin systems. Explicit formulae are given in these cases.

Résumé

Nous construisons des coordonnées de Darboux naturelles pour les systèmes de Hitchin généralisés, systèmes étudiés par Bottacin et Markman. Ces systèmes sont définis sur l'espace des paires stables qui consistent en un fibré vectoriel holomorphe et une un-forme méromorphe à valeurs dans les endomorphismes du fibré. Dans des cas particuliers (genre zéro, genre un), les fibrés sont rigides et on obtient les systèmes de Gaudin rationnels, elliptiques et trigonométriques. Les formules sont explicitées pour ces trois cas.

1. Introduction

Integrable Hamiltonian systems occur in a wide variety of contexts in mathematical physics, ranging from the very classical problems of 19th century mechanics to the systems occuring in Seiberg-Witten theory. One general class of system which appears in all these guises is the system, due to Markman [Ma] and Bottacin [Bo], which is also known as the generalized Hitchin system. It is defined on a moduli space of pairs (holomorphic vector bundles over a Riemann surface, meromorphic section of the adjoint bundle). Specializing to various cases, mostly over the Riemann sphere, gives the classical examples (tops, geodesics on the ellipsoid, etc.) as well as many interesting and important integrable systems of current interest (Gaudin model, Landau-Lifschitz, and others). More precisely (see the book [FT], the survey [RS2], and the references therein, or the articles [M, AvM, RS1, AHP, HH]):

- Over rational curves, and in some cases, over elliptic curves, and their degenerations into nodal curves, one has that the bundle is rigid, and one is dealing with endomorphisms of a fixed bundle. The systems can then be expressed in terms of classical r-matrices, either rational, elliptic or trigonometric, and the systems one obtains are often referred to as the rational, elliptic or trigonometric Gaudin model.
- Specializing further, one can fix the curve to be rational, fix the rank, and choose special divisors for the poles of the section. One then obtains many of the classical systems: the Neumann oscillator, the various tops, as well as finite gap solutions to the KdV, the NLS, the CNLS and the Boussinesq equations.
- In the elliptic case, one can also further specialize, for example, to the Landau-Lifschitz equation, or the Steklov top.

One natural question in integrable systems is of course solving the equations and finding the flows, and this usually involves some form of separation of variables. This note is devoted to the question of separation of variables for the generalized Hitchin systems, and we will find that there are separating Darboux coordinates which are very natural from a geometric viewpoint, corresponding to the standard algebro-geometric description of these systems in terms of curves and line bundles. This can then of course be specialized to all the cases alluded to above, and in this specialization, one obtains quite detailed formulae.

The coordinates also define a "birational" map between the systems and a symmetric product of a symplectic surface naturally associated to each system. (More properly, rather than a symmetric product, one should be saying a Hilbert scheme of 0-cycles). Other systems with such coordinates ("rank two systems") were studied in [Hu1].

In the special cases of interest to mathematical physics corresponding to when the bundle over the Riemann surface is rigid under deformations, there are, as we mentioned above, three cases. When the Riemann surface is the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, one has the rational r-matrix systems, and the separation of variables was given in [AHH1], as a consequence of a direct calculation. Here we finish the problem and treat the case when the curve is elliptic (elliptic r-matrix) or a nodal rational curve (trigonometric r-matrix), and the explicit coordinates will follow from the general considerations on the Bottacin Markman systems; such a procedure can also be used to give another derivation of the results of [AHH1]. Such coordinates were produced in the rank two elliptic case by Sklyanin [S].

Section two of the paper will begin by recalling some facts about the generalized systems, following [Ma]. In section three, we will show how the coordinates arise, show that they are Darboux coordinates, and explain how they lead to an easy integration of the equations of motion. Section four specializes to the special case of an elliptic curve, and section five to the nodal curve. Finally, there is another context in which the same r-matrices are used, and that is in defining integrable systems over Poisson-Lie groups. In section 6 we will explain how the results of the paper should extend to cover this case.

2. The Bottacin-Markman or generalized Hitchin systems.

Let be a closed Riemann surface of genus γ, D a positive divisor of degree n on . We consider over the moduli spaces $\mathcal{M}(r,D,d)$ of *Higgs pairs* (E,ϕ), where

- E is a degree d rank r holomorphic vector bundle over .
- ϕ is a holomorphic section of the associated adjoint bundle $\mathrm{End}(E)$, twisted by $K(D)$, where K is the canonical bundle of : $\phi \in H^0(,\mathrm{End}(E)\otimes K(D))$. Alternately, ϕ is a meromorphic $\mathrm{End}(E)$-valued 1-form, with poles at the divisor D. The pairs must satisfy an appropriate stability condition; see [Bo],[Ma]. The case considered by Hitchin in [Hi1],[Hi2], is that of $D = 0$.

The first result is that the $\mathcal{M}(r,D,d)$ are Poisson. The Poisson structure can be defined directly ([Bo] or [Ma], section 7), but it is easiest to obtain it by Poisson reduction of a larger space, the cotangent bundle of the moduli space of bundles with level structure at D.

Following [Ma], we consider the moduli space $\mathcal{U}(r,D,d)$ of vector bundles with level structure at D, that is the moduli space of pairs (E,t) where E is rank r vector bundle over , and t is a trivialization of E over the divisor D, that is an isomorphism between $E|_D$ and $\mathcal{O}_D^{\oplus r}$. Again, there is an appropriate stability condition one must impose to get a good moduli space. The tangent space to $\mathcal{U}(r,D,d)$ at (E,t) is canonically isomorphic to $H^1(,\mathrm{End}(E)(-D))$; dually the cotangent space is $H^0(,\mathrm{End}(E)\otimes K(D))$. The cotangent bundle $T^*\mathcal{U}(r,D,d)$ is then identified with a space of triples (E,t,ϕ), with E, t as above and $\phi \in H^0(,\mathrm{End}(E)\otimes K(D))$.

There is a natural action of $Gl(r,D)$, the invertible $\mathcal{O}_D$-valued $r\times r$ matrices, on $\mathcal{U}(r,D,d)$, simply by modification of the trivialization t. The action lifts to a symplectic action on the cotangent bundle, and one has:

PROPOSITION (2.1) [Ma] *1) The action of $Gl(r,D)$ has as moment map*

$$\begin{aligned} \mu : T^*\mathcal{U}(r,D,d) &\to gl(r,D)^* \\ (E,t,\phi) &\mapsto \hat{\phi} \end{aligned} \tag{2.2}$$

where $\hat{\phi}$ is the expression of the polar part of ϕ over D in the t-trivialization, and $gl(r,D)^*$ is identified with $gl(r,D)\otimes_{\mathcal{O}_D}(K)|_D$ by a trace-residue pairing.

2) The quotient $T^*\mathcal{U}(r,D,d)/Gl(r,D)$ is then Poisson, and is naturally identified over an open dense set with $\mathcal{M}(r,D,d)$. Its symplectic leaves are obtained as inverse images under μ of coadjoint orbits.

The next step is to define the integrable system on $\mathcal{M}(r,D,d)$, that is to specify the ring of Hamiltonians. This is given by considering for each pair (E,ϕ), the *spectral curve* S of ϕ. This curve lies in the total space $\mathcal{K}_D$ of the line bundle $K(D)$ over . It is cut out by the equation

$$\det(\phi - \zeta\mathbb{I}) = 0. \tag{2.3}$$

Here ζ represents the tautological section of $\pi^*K(D)$ over $\mathcal{K}_D$, where $\pi : \mathcal{K}_D \to$ is the projection. The adjunction formula tells us that the genus of S is

$$g = r^2(\gamma-1) + \frac{(r-1)rn}{2} + 1. \tag{2.4}$$

We expand (2.3) in powers of ζ:

$$\zeta^r + a_1\zeta^{r-1} + a_2\zeta^{r-2} + \ldots + a_r = 0. \tag{2.5}$$

We have that the $a_i = a_i(E,\phi)$ lie in $H^0(,(K(D))^{\otimes i})$. These spaces have dimension $d_i = (2i+1)(\gamma-1) + in$. Let $v_{1,i},\ldots v_{d_i,i}$ be a basis for $H^0(,(K(D))^{\otimes i})$. Expanding $a_i(E,\phi)$ as

$$a_i(E,\phi) = \sum_{j=1}^{d_i} f_{j,i}(E,\phi)v_{j,i} \tag{2.6}$$

gives one functions $f_{j,i}$ on $\mathcal{M}(r,D,d)$.

PROPOSITION (2.7) [Bo, Ma] *1) The functions $f_{j,i}$ Poisson commute, and define a completely integrable system on $\mathcal{M}(r,D,d)$. Joint level sets of the $f_{j,i}$ are given by fixing the spectral curve S, so that the spectral curve map $\mathcal{M}(r,D,d) \to$ (family of spectral curves) defines a Lagrangian foliation.*

2) The symplectic leaves of the Poisson structure on $\mathcal{M}(r,D,d)$ correspond to fixing the intersection of the spectral curve with the divisor $\pi^{-1}(D)$.

3) The leaf of the Lagrangian foliation at a smooth spectral curve S is a Zariski open set of the Jacobian of S.

The leaf of the Lagrangian foliation at S is thus a family of line bundles on S. The line bundle L corresponding to (E,ϕ) is defined via the exact sequence of sheaves over the surface $\mathcal{K}_D$:

$$0 \to E \otimes K^*(-D) \stackrel{\phi - \zeta \mathbb{I}}{\longrightarrow} E \to L \to 0. \tag{2.8}$$

When the spectral curve is smooth, L is a line bundle supported on the spectral curve.

PROPOSITION (2.9) [Hu1] *One can then reconstruct (E,ϕ) from (S,L):*

- *$E = \pi_*(L)$,*
- *ϕ is the map induced on E by multiplication by the tautological section ζ on L.*

REDUCTION TO $SL(r,\mathbb{C})$

In more generality, one can consider similar structures for arbitrary reductive groups G. The bundle E then gets replaced by a principal G-bundle P, and the bundle End(E) gets replaced by $ad(P)$. We will not consider these structures in such generality; see however [Hu2], and the references therein. We consider the case $G = SL(r,\mathbb{C})$. One then has a moduli space $\mathcal{M}(SL(r,\mathbb{C}),D,d)$ of pairs (E,ϕ), with E a rank r vector bundle with ${}^r(E)$ holomorphically trivial, and ϕ a meromorphic $sl(E)$-valued 1-form, with poles at the divisor D. We now exhibit how these spaces can be obtained from a symplectic reduction, at least up to an r-fold covering.

The group $Pic_0()$ of degree zero line bundles on the base curve acts on $\mathcal{M}(r,D,d)$ by

$$\begin{array}{c} Pic_0() \times \mathcal{M}(r,D,d) \to \mathcal{M}(r,D,d) \\ (V,(E,\phi)) \mapsto (E \otimes \pi^* V, \phi) \end{array}. \tag{2.10}$$

Alternately,

$$(V,(S,L)) \mapsto (S, L \otimes \pi^* V). \tag{2.11}$$

This action is symplectic, and is indeed Hamiltonian, being the flow of the Hamiltonians

$$\mathrm{tr}(\phi) \in H^0(,K(D)). \tag{2.12}$$

If we take the reduction at $0 \in H^0(,K(D))$ with respect to the action of this group, one fixes the trace of ϕ to be zero, then quotients out the action on E of tensoring with a line bundle. Up to an r-th root of the trivial bundle, one can achieve this by fixing the maximal exterior power of E to be a fixed line bundle V, giving:

PROPOSITION (2.13) *The space of pairs*

$$\mathcal{M}_V(r,D,d) = \{(E,\phi) \in \mathcal{M}(r,D,d) | {}^r(E) = V, \mathrm{tr}(\phi) = 0\} \tag{2.14}$$

embeds in $\mathcal{M}(r,D,d)$, symplectically over its smooth locus. It is a branched covering of the quotient $\mathrm{tr}^{-1}(0)/Pic_0()$.

If we consider the case $d = 0$, V trivial, then E is an $SL(r,\mathbb{C})$-bundle, and then $\mathcal{M}_{\mathcal{O}}(r,D,0) = \mathcal{M}(SL(r,\mathbb{C}),D,0)$.

3. Symplectic geometry of $\mathcal{M}(r, D, d)$

We are thus in a situation in which we have two Lagrangian fibrations: the first, on the cotangent bundle $T^*\mathcal{U}(r, D, d)$, is given by projection to $\mathcal{U}(r, D, d)$, and the second, on the reduced space $\mathcal{M}(r, D, d)$, by the integrable system, that is, a map to the space of spectral curves.

Corresponding to the first fibration, we have that the tangents to the fibers are given by elements of $H^0(, \mathrm{End}(E) \otimes K(D))$; on the base, deformations of the bundles, along with the level structure, are given to first order by elements of $H^1(, \mathrm{End}(E)(-D))$. One then has an exact sequence:

$$0 \to H^0(, \mathrm{End}(E) \otimes K(D)) \to T(T^*\mathcal{U}(r, D, d)) \to H^1(, \mathrm{End}(E)(-D)) \to 0. \tag{3.1}$$

We would like to split this sequence in such a way that the image of $H^1(, \mathrm{End}(E)(-D))$ in $T(T^*\mathcal{U}(r, D, d))$ is a Lagrangian subspace, which allows us to write the symplectic form on $T(T^*\mathcal{U}(r, D, d))$ as the natural form on $H^0(, \mathrm{End}(E) \otimes K(D)) \oplus H^1(, \mathrm{End}(E)(-D))$ defined by using Serre duality. The natural way of getting a Lagrangian splitting at (E, t, ϕ) is to take the differential, considered as a section of $T^*\mathcal{U}(r, D, d)$, of a function on $\mathcal{U}(r, D, d)$ near (E, t) whose differential at (E, t) is ϕ. One easily defined function which does the trick is defined as follows. Cover by $n + 1$ open sets, $U_0 = \; -$ support(D) and $U_i, i = 1, .., n$ disjoint discs centered at the points p_i of D. Choose trivializations of E on the U_i compatible with t, and let $F_{0,i}$ be the transition functions of E for these trivializations. Now let V be a subspace of the space of cocycles for $\mathrm{End}(E)(-D)$, mapping isomorphically to $H^1(, \mathrm{End}(E)(-D))$. The (E', t') near (E, t) can be obtained from transition functions $F_{0,i}(1 + v_i), (v_i) = v \in V$. Now if $\phi \in H^0(, \mathrm{End}(E) \otimes K(D))$ is represented in the trivializations by ϕ_0, ϕ_i, define the local function ρ on $\mathcal{U}(r, D, d)$ by $\rho(E', t') = \sum_i \mathrm{res}_{p_i}(\mathrm{tr}(v_i \cdot \phi_i)$. The differential of ρ at $v = 0$ is then ϕ, and the graph of $d\rho$ splits the tangent bundle:

$$T(T^*\mathcal{U}(r, D, d)) \simeq H^0(, \mathrm{End}(E) \otimes K(D)) \oplus H^1(, \mathrm{End}(E)(-D)). \tag{3.2}$$

If (E_s, t_s, ϕ_s) is a curve in $T^*\mathcal{U}(r, D, d)$, with $\frac{d(E,t)}{ds}|_{s=0}$ represented by $v \in V$, and $\frac{d\phi}{ds}|_{s=0}$ represented by ϕ', one can split ϕ' as $\phi'_0 + \phi'_v$, where $\mathrm{res}_p \mathrm{tr}(\phi'_0 v') = 0$ for all $v' \in V$, and ϕ'_v represents a section in $H^0(, \mathrm{End}(E) \otimes K(D))$. With respect to the splitting (3.2), the pair (v, ϕ'_v) represents $\frac{d(E,t,\phi)}{ds}|_{s=0}$.

>From the point of view of the second Lagrangian fibration, the first order deformations of the spectral curve at a fixed spectral curve S are given by sections of the normal bundle N_S, that is, via the adjunction formula, the bundle $K_S \otimes K^*_{\mathcal{K}_D}$. We note that the canonical bundle of $\mathcal{K}_D$ is $\pi^*\mathcal{O}(-D)$, so that $N_S = K_S(D)$. If one is interested in the deformations of the spectral curve which have fixed intersection with $\pi^*(D)$ (so that in $\mathcal{M}(r, D, d)$ one is moving along a symplectic leaf $\mathcal{L}$), we then have that our infinitesimal deformation space for the curves is given by sections of K_S.

In turn, noting that deformations of a line bundle on S are given by the cohomology group $H^1(S, \mathcal{O})$, we have that the tangent spaces at (S, L) to $\mathcal{M}(r, D, d)$ and to the leaf $\mathcal{L}$ in $\mathcal{M}(r, D, d)$ fit into exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & H^1(S, \mathcal{O}) & \to & T(\mathcal{M}(r, D, d)) & \to & H^0(S, K_S(D)) & \to & 0 \\ 0 & \to & H^1(S, \mathcal{O}) & \to & T(\mathcal{L}) & \to & H^0(S, K_S) & \to & 0. \end{array} \tag{3.3}$$

Again, we want to split this last sequence at (S, L): the geometric way of doing this is to extend the line bundle to a neighbourhood of S in $\mathcal{K}_D$, giving us a way of moving the curve while keeping the line bundle fixed. One then has

$$T(\mathcal{L}) \simeq H^1(S, \mathcal{O}) \oplus H^0(S, K_S). \tag{3.4}$$

On this sum there is again a natural skew form $_S$, as the summands are again Serre duals. The extension of the line bundle and the splitting (3.4) it produces are not unique, but the splittings all define the same symplectic form, as a consequence of (3.5) below.

b) $_S = {}_{,red}$

Our first result "abelianises" the symplectic form by lifting to the curve S

PROPOSITION (3.5) *On the leaves $\mathcal{L}$ in $\mathcal{M}(r, D, d)$, over the locus of smooth curves, $_S = {}_{,red}$, the reduction of the form on $T^*\mathcal{U}(r, D, d)$.*

PROOF: It suffices to prove the identity on a dense set, and so we will make the assumption that the spectral curve over the divisor D is unramified. The symplectic reduction by $Gl(r, D)$ from $T^*\mathcal{U}(r, D, d)$ to $\mathcal{L}$ can then be thought of as a two step process: one first restricts to the subset $\mathcal{T}$ in $T^*\mathcal{U}(r, D, d)$ of elements (E, t, ϕ) such that ϕ is diagonal over D in the t-trivialization, then takes the symplectic quotient under the residual action of the torus $T(r, D)$. Let us then take a two parameter family $A(x, y) = (E, t, \phi)(x, y)$ of elements of $\mathcal{T}$ lying in the inverse image of $\mathcal{L}$, and compute the form (A_x, A_y) on this family at $(x, y) = (0, 0)$. Corresponding to $A(x, y)$, there is a family of curves $\pi : S(x, y) \rightarrow$, and line bundles $L(x, y)$ over $S(x, y)$; the trivialization of E at D in an eigenbasis of ϕ gives a trivialization of L at $\pi^{-1}(D)$.

We cover the base curve by open sets $U_0 = -\text{support}(D)$, and $U_i, i = 1, ..., n$ non-intersecting discs around the points p_i in D, so that the curves $S(x, y)$ are unramified over U_i. Let λ_i be coordinates on the U_i. Over U_i, we let the r branches of the curves $S(x, y)$ have coordinates in $\mathcal{K}_D$ given by forms $\zeta_{i,j}(x, y, \lambda_i), j = 1, ..., r$. Choose trivializations of the $L(x, y)$ over the open sets $\pi^{-1}(U_i), i = 0, .., n$, in such a way that they are compatible with the trivializations over $\pi^{-1}(D)$, and let the transition functions for $L(x, y)$ from $\pi^{-1}(U_0)$ to $\pi^{-1}(U_i)$ be given by an r-tuple of functions $f_j(\lambda_i)$, one for each branch of the curve.

Over U_i, given the trivialisations of L, we have a natural basis for $E = \pi_*(L)$, whose i-th element is given by a section which is only non-zero on the i-th branch of S over . In this basis,

$$\phi(x, y, \lambda_i) = \text{diag}(\zeta_{i,j}(x, y, \lambda_i)). \tag{3.6}$$

On the open set U_0, if we choose a non-zero section ρ of $K(D)$, we have an identification

$$L(x, y) \simeq \pi^*(K(D))^*. \tag{3.7}$$

The tautological section ζ of $K(D)$ over $\mathcal{K}_D$ then gets identified with a global section of L over the spectral curve, which identifies sections of $E = \pi_*(L)$ as polynomials in ζ of degree $r - 1$ with coefficients in $\mathcal{O}$, essentially by Lagrange interpolation. In the basis $1, \zeta, \zeta^2, ...$, the matrix of ϕ is in rational canonical form. The transition function for E from this rational canonical basis to the diagonal basis over U_i is then given in terms of the Vandermonde matrix $VD_{j,k} = (\zeta_{i,k}(x, y, \lambda_i))^{j-1}, j, k = 1, .., r$, by

$$F_{0,i} = VD \cdot \text{diag}_j(f_{i,j}) \tag{3.8}$$

where $f_{i,j} = f_j(\lambda_i)$.

Now let us take derivatives. The cocycle representing the variation in the bundle E in the x direction at $(x, y) = (0, 0)$ is given in the U_i trivialization (setting $F = F_{0,i}$ by

$$F^{-1}F_x = \text{diag}_j((\ln(f_{i,j}))_x) + \text{diag}_j(f_{i,j}^{-1}) \cdot VD^{-1} \cdot VD' \cdot \text{diag}_j((\zeta_{i,j}(0, 0, \lambda_i))_x f_{i,j}), \tag{3.9}$$

where $VD'_{j,k} = (j-1)(\zeta_{i,k}(x, y, \lambda_i))^{j-2}$. There is a similar expression for $F^{-1}F_y$. The derivatives of ϕ in the U_i trivializations are given by

$$\phi_x = \text{diag}_j((\zeta_{i,j}(0, 0, \lambda_i))_x). \tag{3.10}$$

As above, one can split ϕ_x as $(\phi_x)_0 + (\phi_x)_v$. There are similar expressions for ϕ_y.

With this in place, the evaluation of (A_x, A_y) is given by

$$\sum_i \mathrm{res}[\mathrm{tr}[(F^{-1}F_x)\cdot(\phi_y)_v - (F^{-1}F_y)\cdot(\phi_x)_v]] \tag{3.11}$$

Using the defining property of the summands $(\phi_x)_0, (\phi_y)_0$, this is equal to

$$\sum_i \mathrm{res}[\mathrm{tr}[F^{-1}F_x\cdot\phi_y - F^{-1}F_y\cdot\phi_x]]. \tag{3.12}$$

Now we can substitute the values of (3.9,3.10), and get

$$\sum_{i,j} \mathrm{res}[(\ln(f_{i,j}))_x(\zeta_{i,j}(0,0,\lambda_i))_y - (\ln(f_{i,j}))_y(\zeta_{i,j}(0,0,\lambda_i))_x] \tag{3.13}$$

>From the explicit version of the Serre duality pairing

$$H^1(S,\mathcal{O})\otimes H^0(S,K_S)\to H^1(S,K_S)\to\mathbb{C},$$

this is, however, exactly $_S(A_x, A_y)$

c) Divisor coordinates for $_S$.

The pairs (curve S of fixed genus g, line bundle L on the curve of fixed degree d) parametrize the symplectic leaves of the moduli space. Let us fix a spectral curve S_0 and a line bundle L_0, and let (S, L) denote a nearby point. Choose a line bundle K_0 of degree $g-d$ on a neighbourhood of S_0 such that the line bundles $\hat{L} = K_0 \otimes L$ (which are then of degree g) on the nearby curves have a one-dimensional space of sections. Corresponding to such generic $\hat{L}$, there is then a well defined divisor $\sum_\mu p_\mu$. These points lie in the curve S, and so in the surface $\mathcal{K}_D$. The point of this section is that when these points are distinct, they can be thought of as providing Darboux coordinates for the varieties $\mathcal{M}$.

Indeed, the surface $\mathcal{K}_D$ comes equipped with a standard meromorphic two-form ω, with poles at the inverse image in $\mathcal{K}_D$ of the divisor D. Choosing again a two parameter family $A(x,y) = (S(x,y), L(x,y))$, with $(S(0,0), L(0,0)) = (S_0, L_0)$, we can take the derivatives $(p_\mu)_x, (p_\mu)_y$ of the corresponding curves $p_\mu(x,y)$ in $\mathcal{K}_D$. We have:

PROPOSITION (3.14)

$$\sum_\mu \omega((p_\mu)_x, (p_\mu)_y) = {}_S(A_x, A_y) \tag{3.15}$$

PROOF: Let us write a local equation for the curves $S(x,y)$ as $g(x,y,\lambda,\zeta) = 0$. With respect to some suitable covering of the curves by open sets, we can suppose that the zeroes of the sections of $\hat{L}(x,y)$ are cut out by $s(x,y,\lambda,\zeta) = 0$, so that the p_μ are given by the simultaneous vanishing of g and s. We note that transition functions for L over S are given by the function s on punctured disks surrounding the zeroes of s. We have:

$$\omega((p_\mu)_x, (p_\mu)_y) = \mathrm{Res}\left((s_y g_x - s_x g_y)\frac{\omega}{g\cdot s}\right), \tag{3.16}$$

where Res denotes the two-dimensional residue; if ω is $f(\lambda,\zeta)d\lambda\wedge d\zeta$, then this expression is simply:

$$f(\lambda,\zeta)\cdot(s_y g_x - s_x g_y)\cdot(s_\zeta g_\lambda - s_\lambda g_\zeta)^{-1}. \tag{3.17}$$

Over the curve S, the "Poincaré residue" reduces this to a residue on the curve:

$$\mathrm{res}_{p_\mu}\left(\frac{s_y}{s}\ P.R(\frac{g_x\omega}{g}) - \frac{s_x}{s}\ P.R(\frac{g_x\omega}{g})\right). \tag{3.18}$$

The terms $\rho_x = P.R(\frac{g_x\omega}{g})$, $\rho_y = P.R(\frac{g_y\omega}{g})$ are simply the expressions of the sections of the normal bundle giving the deformations as a 1-form, under the various identifications which come into play, giving us

$$\sum_\mu(\omega((p_\mu)_x, (p_\mu)_y)) = \sum_\mu \mathrm{res}_{p_\mu}\left(\frac{s_y}{s}\ \rho_x - \frac{s_x}{s}\ \rho_y\right) \tag{3.19}$$

which is the Serre duality form $_S$ on (3.4), applied to A_x, A_y.

>From this, if one chooses Darboux coordinates for the form ω on $\mathcal{K}_D$, one has Darboux coordinates on $\mathcal{M}$.

d) The systems as symmetric products of surfaces

These Darboux coordinates are the concrete manifestation of a more general phenomenon. Indeed, suppose that we have a local integrable system of Jacobians, that is a Lagrangian fibration

$$\mathcal{H} : \mathbb{J} \to U. \tag{3.20}$$

where U is a ball in $\mathbb{C}^g$, and $\mathbb{J}$ is $2g$-dimensional, symplectic (with form). The fibers are Jacobians of smooth genus g curves, and so, corresponding to $\mathbb{J}$ there is a family of curves $\mathbb{S}$, with

$$\mathcal{H}' : \mathbb{S} \to U. \tag{3.21}$$

The Abel map gives us an embedding

$$A : \mathbb{S} \hookrightarrow \mathbb{J}\ . \tag{3.22}$$

This map is not unique, but depends on the choice of a base-point in the fibre J_h for each h in U. One has

THEOREM (3.23) [Hu1]

(i) Let $A^* \wedge A^* = 0$. Under the embedding A, the variety $\mathbb{S}$ is coisotropic. Quotienting by the null foliation, one obtains, restricting U if necessary, a surface Q to which the form A^* projects, defining a symplectic form ω on Q. The curves S_h all embed in Q.

(ii) If A, $\tilde{A}$ are two Abel maps with $A^* \wedge A^* = 0$, $\tilde{A}^* \wedge \tilde{A}^* = 0$, then $A^* = \tilde{A}^*$, when $g \geq 3$, and so Q depends only on $\mathbb{S}$ and not on the particular Abel map chosen. For $g = 2$, $A^* \wedge A^*$ is always zero.

(iii) There is a symplectic isomorphism

$$: \widetilde{SP}^g(Q, \omega) \to \mathbb{J},$$

defined over a Zariski open set, between $\mathbb{J}$ and a desingularisation $\widetilde{SP}^g(Q, \omega) = \widetilde{SP}^g(Q)$ of the g-fold symmetric product $SP^g(Q)$ of Q. The symmetric product $SP^g(S_h)$ of the curves is Lagrangian in $\widetilde{SP}^g(Q)$, and the restriction of to $SP^g(S_h)$ is the Abel map

$$SP^g(S_h) \to J_h\ .$$

$\widetilde{SP}^g(Q)$ is the Hilbert scheme of length g 0-dimensional subschemes of Q.

The case studied here is an example of this phenomenon. Indeed, in our case, the spectral curves are all embedded in the surface $\mathcal{K}_D$, which has a canonical meromorphic two-form ω, with poles along D. On the other hand, the spectral curves on the symplectic leaves also have fixed intersection with D. Blowing up the surface at these intersection points gives a surface $\tilde{\mathcal{K}}_D$ in which the curves move freely, and in which the lift of the ω is holomorphic. Proposition (3.14) is in effect expressing the moduli space as the symmetric product of $\tilde{\mathcal{K}}_D$. Similar theorems can be proven for integrable systems of Prym varieties; see [HM].

4) The elliptic Gaudin system.

a) Elliptic Lie-Poisson structures.

We first recall the elliptic Lie-Poisson structures and the integrable elliptic Gaudin systems, following [RS2]. Let $q = \exp(2\pi i/r)$, and set

$$I_1 = \mathrm{diag}(1, q, q^2, ..., q^{r-1}), I_2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{4.1}$$

Then

$$I_1 I_2 I_1^{-1} = q^{-1} I_2. \tag{4.2}$$

We consider the algebra $\mathcal{L}_\nu$ of semi-infinite Laurent series in the variable $z - \nu$ with values in $sl(r, \mathbb{C})$:

$$\mathcal{L}_\nu = \{\sum_{i=-k}^{\infty} \phi_i (z-\nu)^i, k \in \mathbb{Z},\ \phi_i \in sl(r, \mathbb{C})\} \tag{4.3}$$

Let $\mathcal{L}_\nu^+$ be the subalgebra of series with $\phi_i = 0$ for $i < 0$. Now let D_{red} represent a sum $\nu_1 + \nu_2 + ... + \nu_n$ of distinct points in the fundamental domain of an elliptic curve

$$= \mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}) \tag{4.4}$$

Set

$$\mathcal{L} = \oplus_i \mathcal{L}_{\nu_i}, \quad \mathcal{L}^+ = \oplus_i \mathcal{L}_{\nu_i}^+$$

and define the subalgebra $\mathcal{T} \in \mathcal{L}$ of meromorphic functions with values in $sl(r, \mathbb{C})$ and with poles only at the translates of the ν_i, satisfying the quasiperiodicity relations:

$$\phi(z + \omega_i) = I_i \phi(z) I_i^{-1}, i = 1, 2 \tag{4.5}$$

One can split $\mathcal{L}$ as a sum $\mathcal{L} = \mathcal{L}^+ \oplus \mathcal{T}$. Furthermore, we have on $\mathcal{L}$ a bilinear form given by $(a, b) \mapsto \mathrm{tr}(\sum_i \mathrm{res}_{\nu_i}(ab))$. This identifies $\mathcal{T}$ with the dual of $\mathcal{L}^+$; we equip $\mathcal{T}$ with the canonical Lie Poisson bracket; one has that along symplectic leaves, the order of the poles at D_{red} and the conjugacy class at D_{red} are both fixed.

Following either the theorem of Adler, Kostant and Symes [AKS] , or by using the r-matrix formalism as in [RS2], one has that the functions on $\mathcal{T}$ defined as the coefficients of the equation of the spectral curve:

$$\det(\phi(z) - \zeta\mathbb{I}) = 0$$

Poisson commute on $\mathcal{T}$, and define an integrable system. The flows are given by Lax equations:

$$\dot{\phi}(z) = [P(f(\phi(z), z)), \phi(z)], \tag{4.6}$$

where P is the projection from $\mathcal{L}$ to $\mathcal{T}$, and f is a function depending on the choice of Hamiltonian.

b) Bundles on an elliptic curve

Vector bundles on an elliptic curve were classified by Atiyah [A]. As a consequence of his results, one has:

PROPOSITION (4.7) *Let $(r, d) = 1$.*

(a) The stable bundles E of rank r, degree d on are classified by their top exterior power ${}^r(E)$. One has that $E \otimes L$, $L \in Pic^0()$ is isomorphic to E if and only if $L^r \simeq \mathcal{O}$.

b) For $0 < d < r$, $h^0(, E) = d$.

By the theorem of Narasimhan and Seshadri[NS] , stable bundles correspond to generators of the irreducible representations of a Z-central extension of the fundamental group; the center, for bundles of degree d, rank r , is mapped to q^d. We fix the degree to be one. In our case, that of bundles over an elliptic curve, we are looking at a central extension of $\mathbb{Z} \times \mathbb{Z}$. The generators T_1, T_2 of $\mathbb{Z} \times \mathbb{Z}$ satisfy the relation

$$T_1 T_2 T_1^{-1} = q^{-1} T_2. \tag{4.8}$$

>From this, one sees that T_2^r commutes with T_1, T_2 and so for an irreducible representation, must be central. Similarily, T_1^r is also central.

Multiplication of T_1, T_2 by scalars corresponds to tensoring the vector bundle by a line bundle, and so one might as well begin by classifying irreducible representations satisfying $T_1^r = T_2^r = 1$. One can begin by conjugating T_1 to the diagonals:

$$T_1 = \mathrm{diag}(q^{m_1}, q^{m_2}, ..., q^{m_r}), \tag{4.9}$$

with $0 \leq m_1 \leq m_2 \leq m_3 ... \leq r - 1$. The relation (4.8) tells us then that $(T_2)_{ij} = 0$ unless $m_i - m_j = -1$, modulo r. From this, one sees that the only way to avoid having an invariant non-trivial proper subspace is to have $m_i = i - 1$ (we had arranged the m_i in increasing order) . One can then choose the basis so that

$$T_2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & & . \\ . & . & . & & . \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{4.10}$$

The stable bundle determined by the T_i over an elliptic curve is then unique, up to tensoring by a line bundle. It has a non-zero section, which we will compute. In terms of the matrices T_i, sections will be given by functions on $\mathbb{C}$ with suitable automorphy properties when one translates by a lattice point. Also, as the degree is one, the functions must get multiplied by q as one winds around a fixed puncture in the curve [AB]. We normalize the periods ω_i, so that the elliptic curve be given as $\mathbb{C}/(\frac{1}{r}\mathbb{Z} \oplus \frac{\tau}{r}\mathbb{Z})$, with a projection $\pi : \mathbb{C} \to$. Let p be $\pi((1 + \tau)/2r))$. We have:

PROPOSITION (4.11)*(a) A section of the bundle E is given by an r-tuple of r-valued functions $(\psi_1, ..., \psi_r)$ defined over the inverse image in $\mathbb{C}$ of the punctured curve $- \{p\}$; These functions must satisfy*

$$(z + 1/r) = T_1 \cdot (z), \ (z + \tau/r) = T_2 \cdot (z), \tag{4.12}$$

and be of the form

$$(z - ((1 + \tau)/2r))^{\frac{-1}{r}}(\text{holomorphic}) \tag{4.13}$$

near the puncture.

(b) A section of $\mathrm{End}(E)$ *is given by a matrix* M *of functions on* $\mathbb{C}$*, satisfying*

$$M(z+1/r) = T_1 \cdot M(z) \cdot T_1^{-1},\ M(z+\tau/r) = T_2 \cdot M(z) \cdot T_2^{-1}. \tag{4.14}$$

The sections of $sl(E)$ with poles at D_{red} are then precisely the elements of the subalgebra $\mathcal{T}$ defined above. Now recall that our bundle E is rigid, up to tensoring by a line bundle. If one reduces by the action of $Pic_0() \simeq$, the class of a pair (E,ϕ) is determined by ϕ, which is of trace zero, and corresponds by (4.5) to an element of $\mathcal{T}$. We have the following result of Markman, referring to (2.13):

PROPOSITION (4.15) [Ma] *The symplectic reduction at* 0 *of the open subspace of* $\mathcal{M}(r,D,1)$ *of pairs* (E,ϕ) *is Poisson-isomorphic to the Poisson subspace of* $\mathcal{T}$ *of functions whose polar divisor is bounded by* D*. The isomorphism intertwines the Hamiltonians of the integrable systems defined on the two subspaces.*

The proof proceeds by remarking that for both spaces, there is a well defined Poisson embedding into a product of the duals of finite dimensional Lie algebras, given in both cases by taking polar parts at the divisor D.

c) Sections of the bundles.

Let us take the r-th powers $f_i = {}^r_{i+1}$ of the components of , so that:

$$_i = f_{i-1}^{\frac{1}{r}}. \tag{4.16}$$

We would then like to find an r-tuple F of functions $(f_0,...,f_{r-1})$, which are of the form z^{-1}(holomorphic)$^{\mathrm{r}}$ near the punctures, satisfying

$$F(z+1/r) = F(z),\ F(z+\tau/r) = T_2 \cdot F(z), \tag{4.17}$$

and are such that the r-th roots along the real and imaginary axes satisfy

$$\begin{aligned}(f_i)^{\frac{1}{r}}(z+1/r) &= q^i(f_i)^{\frac{1}{r}}(z)\\ (f_i)^{\frac{1}{r}}(z+\tau) &= (f_i)^{\frac{1}{r}}(z).\end{aligned} \tag{4.18}$$

Since $T_1^r = T_2^r = 1$, one is dealing with functions over the elliptic curve $' = \mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$; let θ be the standard theta function for this curve; recall that it has a simple zero at the points $((1+\tau)/2)+\mathbb{Z}+\tau\mathbb{Z}$, and is otherwise non-zero and holomorphic. We distinguish two cases:

Case 1: r is odd.

Let

$$\theta_{k,j}(z) = \theta(z+\frac{(k+j\tau)}{r}), 0 \le k,j \le (r-1). \tag{4.19}$$

We have the relations:

$$\begin{aligned}\theta_{k,j}(z+m) =& \theta_{k,j}(z),\\ \theta_{k,j}(z+m\tau) =& exp(-\pi i m^2\tau - 2\pi i m(z+\frac{(k+j\tau)}{r}))\theta_{k,j}(z),\\ \theta_{k,j}(z+\frac{1}{r}) =& \theta_{k+1,j}(z),\\ \theta_{k,j}(z+\frac{\tau}{r}) =& \theta_{k,j+1}(z), 0 \le j < (r-1),\\ \theta_{k,r-1}(z+\frac{\tau}{r}) =& \theta_{k,0}(z)exp(-\pi i\tau - 2\pi i(z+\frac{k}{r})).\end{aligned} \tag{4.20}$$

where m is an integer. Now if

$$\rho_j = (\frac{r-1}{2} - j), \tag{4.21}$$

we set

$$f_j(z) = exp\left(2\pi i\tau\left(\frac{-jr(r-1)}{2} + \frac{(r-1)j(j+1)}{2}\right)\right)\prod_{k=0}^{r-1}\left(\frac{\theta_{k,j}^{r-2}(z)\theta_{k,j}(z+\rho_j\tau)}{\prod_{\ell=0,\ell\neq j}^{r-1}\theta_{k,\ell}(z)}\right). \tag{4.22}$$

Using the relations given for the $\theta_{k,j}$, one checks that it has the correct form near the punctures, and that (4.17) holds. Now let τ be imaginary. Let us consider the involutions $f(z) \mapsto f(-z), f(z) \mapsto f(\bar{z})$. Both these involutions preserve the poles and zeros of f_0. From this, one has that f_0 must be even, as $f_0(0) \neq 0$. Using the second involution, one can then multiply f_0 by a constant c so that $cf_0(0)$ is real. The function is then real on both imaginary and real axes, and has no zeros. >From this, one has that (4.18) holds for f_0. From the relations (4.20), (4.18) follows for the other f_i. Deforming, the same then must hold for arbitrary τ.

Case 2: r is even

We then set

$$\xi_{k,j}(z) = \theta(z + \frac{(k+j\tau)}{r} - \frac{(1+\tau)}{2r}), 0 \leq k,j \leq (r-1). \tag{4.23}$$

We have the relations:

$$\begin{aligned}
\xi_{k,j}(z+m) =& \xi_{k,j}(z),\\
\xi_{k,j}(z+m\tau) =& exp(-\pi i m^2\tau - 2\pi i m(z + \frac{(k+j\tau)}{r} - \frac{(1+\tau)}{2r}))\xi_{k,j}(z),\\
\xi_{k,j}(z+\frac{1}{r}) =& \xi_{k+1,j}(z),\\
\xi_{k,j}(z+\frac{\tau}{r}) =& \xi_{k,j+1}(z), 0 \leq j < (r-1),\\
\xi_{k,r-1}(z+\frac{\tau}{r}) =& \xi_{k,0}(z) exp(-\pi i\tau - 2\pi i(z + \frac{k}{r} - \frac{(1+\tau)}{2r})).
\end{aligned} \tag{4.24}$$

where m is an integer. We then define

$$\rho_j = \frac{r}{2} - j \tag{4.25}$$

and set

$$f_j(z) = (-1)^j exp\left(2\pi i\tau\left(\frac{-j(r-1)}{2}\right)\right)\prod_{k=0}^{r-1}\left(\frac{\theta_{k,j}^{r-1}(z)\theta_{k,j}(z+\rho_j\tau)}{\prod_{\ell=0}^{r-1}\xi_{k,\ell}(z)}\right). \tag{4.26}$$

Again, the r-th roots of the f_j define our section.

d) Darboux coordinates and integration of the system

We can now simply apply our theorem of section 3, and obtain Darboux coordinates for our integrable systems. The canonical line bundle of the elliptic curve is trivial, and we can write the total space $\mathcal{K}$ as $\mathbb{C}/(\frac{1}{r}\mathbb{Z} \oplus \frac{\tau}{r}\mathbb{Z}) \times \mathbb{C}$, with corresponding coordinates z, ζ. The symplectic form on $\mathcal{K}$ can then be written as $dz \wedge d\zeta$. We can then use the map $\mathcal{K} \to \mathcal{K}_D$ to transport these coordinates over to $\mathcal{K}_D$, at the same time trivializing the bundle $K(D)$ with a singularity over D. The coordinates (z,ζ) are Darboux coordinates on the blow-up $\tilde{\mathcal{K}}_D$.

Our coordinates were defined as the zeroes (z_μ, ζ_μ) of a suitably normalized section of the line bundle L of (2.8). Recall that L is the cokernel of $(\phi - \zeta\mathbb{I})$; it is then a quotient of E, and so there is a natural map ρ of E to L. Both E and L here have a one-dimensional space of sections, and projecting the section of E gives that of L. The projection $\rho()$ to E vanishes iff lies in the cokernel of $(\phi - \zeta\mathbb{I})$. Let $\tilde{\ }(z, \zeta)$ be the matrix of cofactors of $(\phi - \zeta\mathbb{I})$, so that

$$\tilde{\ }(z, \zeta) \cdot (\phi - \zeta\mathbb{I}) = \det(\phi - \zeta\mathbb{I}) \cdot \mathbb{I}.$$

If the spectral curve is smooth, then all of the eigenspaces are of dimension one [AHH2], and one has that $\rho() = 0$ iff

$$\tilde{\ }(z, \zeta) \cdot (z) = 0. \tag{4.27}$$

There are generically g distinct solutions (z_μ, ζ_μ) to this equation ([AHH1],[AHH2])and these are our Darboux coordinates.

One can then linearize the flows by a standard Liouville generating function technique. Let $C_1, ...C_s$ denote a basis for the Casimir functions amongst the Hamiltonians, and choose a complementary basis $H_1, ...H_g$ for the rest of the Hamiltonians. Fixing C_i determines a symplectic leaf $\mathcal{L}$, and fixing the rest of the H_i determines a spectral curve S, and so defines ζ implicitly in terms of z and the H_i, C_j: $\zeta = \zeta(z, H_i, C_j)$. We set

$$F(z_\mu, H_i, C_j) = \sum_\mu \int^{z_\mu} \zeta(z, H_i, C_j) dz \tag{4.28}$$

Since $\partial F/\partial z_\mu = \zeta_\mu$, the Liouville generating technique tells us that the derivatives $Q_i = \partial F/\partial H_i$ provide linearizing coordinates for the H_i flows. Setting $P(z, \zeta) = \det(\phi(z) - \zeta\mathbb{I})$, we have

$$Q_i = \sum_{\mu=1}^{g} \int^{z_\mu} \frac{(\partial P/\partial H_i)}{(\partial P/\partial z)} dz \tag{4.29}$$

The integrands, as H_i varies, give a basis of the Abelian differentials over the spectral curve. This is a consequence of the Poincaré residue formula and the corresponding exact sequence for differentials over the surface $\tilde{\mathcal{K}}_D$ (see, e.g.[GH]). This gives the linearization one expects from the algebro-geometric picture.

e) A rank two example

Formulae (4.16), (4.22), (4.26) and (4.27) give an explicit way of determining the Darboux coordinates from ϕ. We briefly exhibit the formulae for the case $r = 2$. This case was treated in [S]. It is difficult to see whether the coordinates obtained are the same, though they seem to have common features. In this case, we write the matrix ϕ as

$$\phi = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix}.$$

The periodicity relations work out to:

$$a(z + \frac{1}{2}) = a(z), \quad b(z + \frac{1}{2}) = -b(z), \quad c(z + \frac{1}{2}) = -c(z),$$
$$a(z + \frac{\tau}{2}) = -a(z), \quad b(z + \frac{\tau}{2}) = c(z).$$

Let $D = \sum \nu_i$. The coefficients a, b, c are then linear combinations of, respectively, elliptic functions a_i, b_i, c_i with poles only at $\nu_i, \nu_i + 1/2, \nu_i + \tau/2, \nu_i + (1+\tau)/2$ and their translates. These are fairly straightforward to write out in terms of theta-functions. For example, setting $\rho_i = \nu_i + \frac{1}{4}$

$$a_i = \prod_{k,j=0,1} \frac{\theta_{k,j}(z - rho_i)}{\theta_{k,j}(z - \nu_i)}.$$

The matrix of cofactors of $\phi - \zeta \mathbb{I}$ is given by:

$$\hat{} = \begin{pmatrix} -a(z) - \zeta & -b(z) \\ -c(z) & a(z) - \zeta \end{pmatrix}.$$

The section is given by

$$_1 = \left[\prod_{k=0}^{1} \left(\frac{\theta_{k,0}(z)\theta_{k,0}(z+\tau)}{\xi_{k,0}(z)\xi_{k,1}(z)} \right) \right]^{1/2}$$
$$_2 = -e^{-\pi i \tau} \left[\prod_{k=0}^{1} \left(\frac{\theta_{k,1}(z)\theta_{k,1}(z)}{\xi_{k,0}(z)\xi_{k,1}(z)} \right) \right]^{1/2}.$$

The coordinates (z_μ, ζ_μ) are then the solutions to the equations:

$$\begin{aligned} (-a(z) - \zeta)_1(z) - b(z)_2(z) = 0 \\ -c(z)_1(z) + (a(z) - \zeta)_2(z) = 0 \end{aligned}.$$

5) The trigonometric case

We now exhibit another set of classical systems associated to rigid vector bundles over a curve: those associated to "trigonometric" r- matrices. Our curve here will be a singular nodal rational curve, that is a Riemann sphere $\mathbb{P}^1$ with two points identified. Such curves arise as degeneracies of elliptic curves, and we shall see many common features with the elliptic case.

The rigid bundle E we consider is obtained from the sum of line bundles $E_0 = \mathcal{O}(1) \oplus \mathcal{O} \oplus \mathcal{O} ... \oplus \mathcal{O}$ on $\mathbb{P}^1$ with transition matrix

$$\mathrm{diag(z^{-1}, 1, 1, ..., 1)} \tag{5.1}$$

from $U_0 = \{z \neq \infty\}$ to $U_1 = \{z \neq 0\}$. E is built by identifying the fiber of E_0 over 0 with that over ∞ via a non-singular matrix M. The bundle E_0 is rigid, and we shall see that the glueing matrix M is essentially unique for a stable bundle, once one adjusts by a suitable automorphism of E_0, and fixes the top exterior power of E.

Indeed, consider $E_0 = \mathcal{O}(1) \oplus \mathcal{O}^{r-1}$ on $\mathbb{P}_1$. We would like to find conditions on M to get stability of the bundle. A destabilising bundle is obtained by glueing from $F_0 = \mathcal{O}(1) \oplus \mathcal{O}^k$, and one has a subbundle for the glueing iff the vector e_1 belongs to an M-invariant subspace of dimension less than r. Thus, the bundle is stable iff e_1 is cyclic for M. So if the bundle is stable one can write the matrix M as

$$M = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & 0 & \dots & 0 & a_1 \\ \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & \cdot & & \cdot & \\ & & & & & \\ 0 & 0 & 0 & \dots & 1 & a_{r-1} \end{pmatrix} \qquad (5.2).$$

Now, one can modify M by automorphisms of $\mathcal{O}(1) \oplus \mathcal{O}^{r-1}$, and in particular, by the following automorphism:

$$\begin{array}{l} e_1 \longrightarrow e_1 \\ e_j \longrightarrow e_j + b_j e_1 \end{array}, \quad b_j \in \mathbb{C}. \tag{5.3}$$

Once we compute $\det(M - z\mathbb{I})$, we can see that with a suitable choice of b_i, we finally get $\det(M - z\mathbb{I}) = (-1)^{r+1} a_0$. In other words, by an automorphism, we can set $a_1 = a_2 = ... = a_{r-1} = 0$. So once the determinant is fixed, the bundle is rigid since the bundle $\mathcal{O}(1) \oplus \mathcal{O}^{r-1}$ on $\mathbb{P}_1$ is infinitesimally rigid and the glueing is rigid. The determinant $a_0 \in \mathbb{C}^*$ represents the highest power $\wedge^r E$ of the bundle E in Pic. Let us set $a_0 = 1$; we will take the $Sl(r, \mathbb{C})$ moduli space and so consider ϕ's which are traceless.

We can change trivializations so that the transition matrix becomes:

$$T(z) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & & . \\ . & . & . & & . \\ 0 & 0 & 0 & \dots & 1 \\ z^{-1} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{5.4}$$

instead of the diagonal matrix (5.1). The glueing matrix M is then the identity. The unique (up to scale) section of E is then represented over U_0 by the vector of functions

$$(1 + z, 1, 1, ..., 1)^T.$$

Conjugating T by

$$S = \mathrm{diag}(1, z^{\frac{1}{r}}, z^{\frac{2}{r}}, ..., z^{\frac{r-1}{r}})$$

transforms it to

$$STS^{-1} = z^{\frac{-1}{r}} T_2,$$

with T_2 the matrix of (4.10). This can be thought of as a multi-valued change of trivialisation. The section is then represented by

$$(1 + z, z^{\frac{1}{r}}, z^{\frac{2}{r}}, ..., z^{\frac{r-1}{r}}). \tag{5.5}$$

More generally, meromorphic sections of E are then represented by vectors of functions $F = (f_0, ... f_{r-1})$ on $\mathbb{C}^*$ such that

- $f_i = z^{\frac{i}{r}} \cdot$(meromorphic),
- $\lim_{z \to 0} F(z)$ exists,
- $\lim_{z \to \infty} z^{\frac{-1}{r}} T_2 F(z)$ exists,
- and the two limits coincide.

Changing variables by $z = \exp(2\pi i \ rx)$, one is still dealing with functions $f_i(x)$ satisfying $f_i(x + \frac{1}{r}) = q^i f_i(x)$, that is,

$$F(x + \frac{1}{r}) = T_1 F(x), \tag{5.6}$$

the other matrix T_2 now being used for the boundary conditions at $ix \to \pm\infty$. Similarily, sections of the $\mathrm{End}(E)$ get represented by matrices $M(z)$ with

$$M(x + \frac{1}{r}) = T_1 M(x) T_1^{-1}.$$

As in section 4 , there is a splitting of a loop algebra $\mathcal{L}$ of periodic functions into a $\mathcal{L}^+$ of positive series and a $\mathcal{T}$ of meromorphic functions satisfying only one periodicity condition. Corresponding to this, there is an integrable system, as for the elliptic case. Our construction above shows us that the elements of $\mathcal{T}$ can be interpreted as sections of a rigid bundle E, and so Markman's result (extended to handle singular base curves) gives us a Poisson isomorphism between the coadjoint orbits in $\mathcal{T}$ and the symplectic leaves in the corresponding moduli of Higgs pairs. One then obtains, as in the elliptic case, suitable separating coordinates for these systems.

6) Poisson-Lie groups

There are three main cases of a curve with a rigid bundle, yielding a corresponding splitting of the loop algebra of matrices into a sum of two subalgebras which are dual to each other and so allowing us to define an integrable system, using either the Adler Kostant Symes theorem, or more generally the r-matrix formalism. The curves are either rational, elliptic or nodal rational, and correspond to the rational, elliptic and trigonometric r-matrices respectively. These r-matrices can also be used to define quite different, if related, Poisson structures, the quadratic or Sklyanin bracket. Again these come in three types, rational, elliptic and trigonometric. They are obtained, very roughly, by thinking of (generically invertible) matrix valued functions as elements of a group, rather than an algebra, and applying the formalism for constructing Poisson Lie groups, as in [RS2]. Again one has integrable systems, defined again in terms of spectral curves: the Lagrangian foliations for both the Lie Poisson and Sklyanin structures share the same leaves. The symplectic leaves of the two types of structure are however quite different. In any case, one can ask if there is any analogue of the separating coordinates in the quadratic case.

The answer for the rational quadratic bracket is yes, and can be found in [Sc]. One has, as for the rational Lie Poisson case, a spectral curve, line bundle, and section giving a divisor p_ν on the spectral curve. This curve lies in the same surface as in the Lie Poisson case, that is the total space of the line bundle $\mathcal{K}(D)$ over the curve $\mathbb{P}^1$. The divisor then gives an isomorphism I of the symplectic leaves with a symmetric product of this surface. Let z be a coordinate on $\mathbb{P}_1$, and ζ the corresponding cotangent coordinate, and let $a(z) = 0$ cut out the divisor D. In the Lie Poisson setting, the form $a(z)^{-1}dz \wedge d\zeta$ on $\mathcal{K}(D)$ induce a symplectic form on the symmetric product and turn the isomorphism I into a symplectic one. In the rational quadratic case, one has the same result, but with the form $\zeta^{-1}dz \wedge d\zeta$.

We conjecture that a similar result holds for the quadratic or Sklyanin bracket in both the elliptic and the trigonometric cases. One again has divisor coordinates on a surface $\mathcal{K}(D)$ defined over an elliptic or a nodal curve; instead of taking the symplectic form on $\mathcal{K}(D)$ with poles along $\pi^{-1}(D)$, one chooses the form with a pole along the zero-section. Choosing Darboux coordinates (z, ζ) for this form, and expressing the divisor corresponding to the pair (line bundle, curve) in these coordinates as a sum (z_μ, ζ_μ) should give us separating Darboux coordinates. The validity of this result could be checked with a direct but probably rather difficult calculation, as in [Sc], but does not seem to be feasible with the methods of this paper.

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